Combinatorics, Logic and Probability

Logical limit laws for planar graphs and graphs on surfaces

Marc Noy
Universitat Politècnica de Catalunya, Barcelona
Barcelona Graduate School of Mathematics

A zero-one law

$\mathcal{G}$ class of labelled graphs
$\mathcal{G}_n$ graphs in $\mathcal{G}$ with n vertices
Uniform distribution on $\mathcal{G}_n$

$$\mathbf{P}(G \in \mathcal{G}_n) = \frac{1}{2^{\binom{n}{2}}}$$

Graph properties expressible in first-order logic

Example G contains a triangle $\exists x \exists y \exists z\ (x \sim y) \wedge (y \sim z) \wedge (z \sim x)$

Theorem For every first order property A

$$\lim_{n \to \infty} \mathbf{P}(G \in \mathcal{G}_n \text{ satisfies } A) \in \{0, 1\}$$

A holds in $\mathcal{G}$ with high probability (whp) if

$$\lim_{n \to \infty} \mathbf{P}(G \text{ satisfies } A \colon G \in \mathcal{G}_n) = 1$$

Whp every object satisfies ϕ or whp no object satisfies ϕ

Outline

1. First order and second order logic. Ehrenfeucht-Fraïssé games
2. Logical limit laws: planar graphs an related classes of graphs
3. Graphs on surfaces

Based on joint work with

- Peter Heinig, Anusch Taraz (Hamburg),
  Tobias Müller (Utrecht)
- Albert Atserias (Barcelona), Stephan Kreutzer (Berlin)

First order logic (FO)

Quantifiers: $\forall, \exists$
Variables: $x, y, z, \ldots$
Boolean connectives and syntax: $\vee, \wedge, \neg, \rightarrow, (), =$

For a given class of structures we add relations of any given arity
Graphs: $E(x, y)$ adjacency relation, written $x \sim y$
Ordered structures: $x < y$
Abelian groups: $x + y = z$

Some examples in graphs

- Existence of an isolated vertex: $\exists x, \forall y \ \neg(x \sim y)$
- Existence of a triangle: $\exists x \exists y \exists z \ (x \sim y) \wedge (y \sim z) \wedge (z \sim x)$
- Existence of fixed H as a subgraph (or induced subgraph)
- Existence of a connected component is isomorphic to H

If G satisfies ϕ we say G is a model of ϕ and write

$$G \models \phi$$

Graph connectivity

A graph (V, E) is connected if

$$\forall x \forall y \, \neg(x = y) \to \exists x_1 \ldots \exists x_k \text{ distinct from } x \text{ and } y$$
$$(x \sim x_1) \wedge (x_1 \sim x_2) \wedge \cdots \wedge (x_k \sim y)$$

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Not in FO!

But diameter $\leq k$ (for fixed k) is in FO

Another attempt at expressing connectivity

$$\forall A \subset V, A \neq \emptyset, A \neq V \; \exists x \in A, \exists y \notin A \; (x \sim y)$$

This is a second order formula: quantification over relations

Monadic Second Order (MSO) logic is a fragment of SO
MSO = FO + quantification over sets of vertices (unary relations)

MSO = FO + quantification over sets of vertices

- Being connected is in MSO
- Being acyclic is in MSO
- 3-colorability is in MSO
- Hamiltonian is not in MSO but it is in MSO_2: quantification over sets of vertices and sets of edges
- For planar graphs MSO and MSO_2 are equally powerful

Remark FO is ‘local’ and MSO is highly ‘non-local’
We’ll make precise locality of FO later

Theorem Graph connectivity is not expressible in FO

First attempt: analyze each FO formula and show it cannot express connectivity

$$\forall x \exists y \forall z\ ((x \sim z) \wedge \neg(y \sim z)) \vee (\exists w(z \sim w) \vee (\neg y \sim w))\ ???$$

Strategy: analyze simultaneously all formulas of a given complexity
Depth of formula ϕ = maximum number of nested quantifiers in ϕ

- depth$(\phi) = 0$ if ϕ is quantifier free
- depth$(\psi) + 1$ if $\phi = \forall x \psi(x)$
- depth$(\psi) + 1$ if $\phi = \exists x \psi(x)$

Logical equivalence of graphs
$G \equiv_k H$ if G and H satisfy exactly the same formulas of depth $\leq k$
Finitely many equivalence classes

Suppose for each $k \geq 1$ we find graphs G_k, H_k such that

- G_k is connected and H_k is not
- $G_k \equiv_k H_k$

If ϕ expresses connectivity and k = depth(ϕ), then contradiction!

Logic through combinatorial games

Ehrenfeucht-Fraïssé game $\mathrm{Ehr}_k(G, H)$

- Spoiler and Duplicator play k rounds on two graphs G, H
- At each round Spoiler picks a vertex (from any graph) and Duplicator picks a vertex from the other graph

$(a_1, \dots, a_k)$ vertices selected from G
$(b_1, \dots, b_k)$ vertices selected from H

Duplicator wins iff $(a_1, \dots, a_i) \leftrightarrow (b_1, \dots b_i)$ isomorphism for all i

Theorem (Ehrenfeucht-Fraïssé)
$G \equiv_k H \iff$ Duplicator has a winning strategy for $\mathrm{Ehr}_k(G, H)$

Provides a purely combinatorial characterization of FO logic

Proofs of non-expressability in FO

Connectivity

$G_k = C_{3^k}, \quad H_k = C_{3^k} \cup C_{3^k}$

Claim: $G_k \equiv_k H_k$
Proof by induction on k

Additional properties not in FO

- Acyclic
- 3-colorable
- Hamiltonian
- Eulerian
- Planar
- Rigid (no non-trivial automorphism)

Zero-one laws

$\mathcal{G}$ class of (labelled) graphs
$\mathcal{G}_n$ graphs in $\mathcal{G}$ with n vertices
Probability distribution on $\mathcal{G}_n$ for each n

The zero-one law holds in $\mathcal{G}$ if for every formula ϕ in FO

$$\lim_{n\to\infty} \mathbf{P}(G \models \phi : \; G \in \mathcal{G}_n) \in \{0, 1\}$$

Whp every object satisfies ϕ or whp no object satisfies ϕ

The classical example

$\mathcal{G}$ class of all labelled graphs $\quad |\mathcal{G}_n| = 2^{\binom{n}{2}}$

Uniform distribution $\quad \mathbf{P}(G) = \frac{1}{2^{\binom{n}{2}}}, \qquad G \in \mathcal{G}_n$

Theorem Glebski, Kogan, Liagonkii, Talanov (1969) Fagin (1976)
The zero-one law holds for labelled graphs

The $G(n,p)$ model

- Class: Labelled graph with n vertices
- Every possible edge xy independently with probability p

$$\mathbf{P}(G) = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$$

$G(n,1/2)$ is the uniform distribution

Extenson Property E_r

For all disjoint $A, B \subset \{1,\ldots,n\}$ with $|A| = |B| = r$

$$\exists z \notin A \cup B \qquad (\forall x \in A \quad z \sim x) \quad \wedge \quad (\forall y \in B \quad z \not\sim y)$$

Lemma $G(n,p)$ satisfies E_r whp for constant p

$$\mathbf{P}(G_n \not\models E_r) \leq \binom{n}{r}\binom{n-r}{r}(1-p^r(1-p)^r)^{n-2r} \to 0, \qquad \text{as } n \to \infty$$

Theorem The 0-1 law holds in $G(n,p)$ for constant p

Assume $(a_1,\ldots,a_i) \leftrightarrow (b_1,\ldots,b_i)$ and Spoiler plays a_{i+1}
Let $A_1 = \{a_j | a_{i+1} \sim a_j, 1 \leq j \leq i\}$, $A_2 = \{a_j | a_{i+1} \not\sim a_j, 1 \leq j \leq i\}$
Duplicator plays $b_{i+1} = z$ as in E_r for the sets A_1 and A_2
Hence Duplicator wins whp

The 0-1 law does not hold in $G\left(n, p = \frac{1}{n}\right)$
$p = 1/n$ is the threshold for the appearance of a triangle

Number of triangles in $G(n, p = 1/n)$ tends to a Poisson$(1/6)$

Shelah, Spencer 1988
The 0-1 law holds in $G(n, p = n^{-\alpha})$ for $\alpha \in [0, 1]$ irrational

Trees

Theorem McColm (2002)
The zero-one law holds for trees in FO and MSO

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$\mathcal{T}$ labelled trees $\quad |\mathcal{T}_n| = n^{n-2}$ Cayley's formula

Typical properties of a random tree

- Height is $\Theta(\sqrt{n})$
- Maximum degree is $\Theta\left(\dfrac{\log n}{\log\log n}\right)$
- Has $\sim e^{-1}n$ leaves (vertices of degree 1)
- Has αn pendant copies of any fixed rooted tree H

T has H as a pendant copy if T has a subtree $\cong H$
joined to T through an edge incident with the root of H

FO zero-one law for trees

Theorem (McColm)
The zero-one law in FO holds for trees

Sketch of proof
For each $k \geq 1$
$T_1, \ldots, T_m$ representatives of all $\equiv_k$ types of trees
'Universal' tree U_k: k copies of each T_i glued with a new root

- A random tree contains a pendant copy of U_k w.h.p.
- If T, T' both contain a pendant copy of U_k then $T \equiv_k T'$

Duplicator wins $\mathrm{Ehr}_k(T, T')$ by playing in suitable subtrees of U_k
Hence T and T' satisfy the same formulas of depth $\leq k$ whp

Remark We play on rooted trees for defining the winning strategy but the root is not part of the language

MSO Ehrenfeucht-Fraïssé games

MSO $\mathrm{Ehr}_k(G,H)$ games: vertex moves and set moves
Duplicator must respond with the same kind of move as Spoiler
$(a_1,\dots,a_r),(b_1,\dots,b_r)$ vertex moves
$(A_1,\dots,A_s),(B_1,\dots,B_s)$ set moves

Duplicator wins if $(a_1,\dots,a_r)\leftrightarrow(b_1,\dots,b_r)$ and
$a_i\in A_j \iff b_i\in B_j$

- $G\equiv_k^{MSO} H$ if satisfy the same MSO formulas of depth $\le k$
- k–MSO types are the equivalence classes

$G\equiv_k^{MSO} H \iff$ Duplicator has winning strategy $\mathrm{Ehr}_k^{MSO}(G,H)$

McColm The MSO zero-one law holds for trees

Proof idea Define U_k as before with 2^k copies of each type of tree
Pigeonhole argument

What follows is joint work with
Tobias Müller, Peter Heinig, Anusch Taraz

- Extension to forests (acyclic graphs)
- Extension to more general classes of graphs

Forests

There is no zero-one law in the class of forests

$$\mathbf{P}(\text{Random forest has an isolated vertex}) \to e^{-1}, \quad n \to \infty$$

Properties of random forests

- Connected with probability $\to e^{-1/2} \approx 0.607$
- The largest component has expected size $n - O(1)$
- Fragment = complement of largest component
  H unlabelled forest, $\quad \mathbf{P}(\text{Fragment} \simeq H) \to \mu_H$

Theorem

Each MSO property has a limiting probability for random forests (Convergence law)

Sketch of proof

- Type of the components determines type of the forest
- Largest component has a fixed type (by 0-1 law for trees)
- Sum over fragments $\mathcal{A}(\phi)$ that make ϕ hold:

$$\lim_{n\to\infty} \mathbf{P}(\text{Random forest } \models \phi) = \sum_{H \in \mathcal{A}(\phi)} \mu_H$$

Planar graphs

For each k there exists a planar graph U_k such that

- If G, G' planar contain a pedant copy of U_k then $G \equiv_k G'$
- W.h.p. a random planar graph contains a pendant copy of U_k
  McDiarmid, Steger, Welsh 2005 Giménez, N. 2009

Theorem

The zero-one MSO law holds for connected planar graphs

The convergence MSO law holds for arbitrary planar graphs

Minor-closed classes of graphs

H is a minor of G if it can be obtained from a subgraph of G by contracting edges

$\mathcal{G}$ is minor-closed if

$$G \in \mathcal{G}, \quad H \text{ minor of } G \Rightarrow H \in \mathcal{G}$$

Forests, Planar, Graphs embeddable in a fixed surface S
Outerplanar, Series-Parallel, Bounded tree-width, ΔY-reducible

$\mathcal{G}$ addable if it is closed under disjoint unions and adding bridges between different components

Theorem (McDiarmid 2009)
$\mathcal{G}$ addable and minor-closed, H fixed graph in $\mathcal{G}$
A random graph in $\mathcal{G}$ contains a pendant copy of H w.h.p.

Theorem
A zero-one MSO law holds for connected graphs in $\mathcal{G}$
A convergence MSO law holds for arbitrary graphs in $\mathcal{G}$

The set of limiting probabilities

$$L = \{\lim \mathbf{P}(G_n \models \phi) : \ \phi \text{ MSO formula}\}$$

$L \subseteq [0,1]$ is countable and symmetric with respect to $1/2$

Theorem If $\mathcal{G}$ addable minor-closed class, then $\overline{L}$ is a finite union of closed intervals

Forests

$$\overline{L} = [0, 0.1703] \cup [0.2231, 0.3935] \cup [0.6065, 0.7769] \cup [0.8297, 1]$$

$$0.6065\cdots = e^{-1/2} = \lim \mathbf{P}(\text{Random forest is connected})$$

ϕ is true for trees whp $\Rightarrow \lim \mathbf{P}(\phi) \geq 0.6065$

ϕ is false for trees whp $\Rightarrow \lim \mathbf{P}(\phi) \leq 1 - 0.6065 = 0.3935$

For planar graphs $\overline{L}$ = union of 108 intervals, of length $\approx 10^{-6}$

Lemma (Guthrie-Nymann 1988)
$p_1 \geq p_2 \geq \cdots \geq p_n \cdots > 0$ and $\sum p_n < +\infty$

If $p_n \leq \sum_{k>n} p_k$ for $n \geq n_0$ then

$$\left\{ \sum_{i \in A} p_i : \ A \subset \mathbb{N} \right\}$$

is a finite union of closed intervals

In our case the p_i are the probabilities of the possible fragments

- Same $\overline{L}$ for FO and MSO
- At least two intervals since

$$\mathcal{G} \text{ addable} \implies \lim \mathbf{P}(\text{connectivity}) \geq e^{-1/2} \approx 0.6065$$

Graphs on surfaces

$\mathcal{G}_S$ class of graphs embeddable in S
Minor-closed but not addable: K_5 embeds in the torus not $K_5 \cup K_5$

$$B(x, r) = \{y \colon d(x, y) \leq r\}$$

A random graph in $\mathcal{G}_S$ satisfies w.h.p.

- All balls $B(x, R)$ are planar for fixed $R > 0$
  Chapuy-Fusy-Giménez-Mohar-N., Bender-Gao 2011
- Contains a pendant copy of any fixed connected planar graph
  McDiarmid 2008 CFGMN

Gaifman's locality theorem

Every FO formula is equivalent to a Boolean combination of basic local sentences

$$\exists x_1, \ldots, \exists x_s \left(\bigwedge_{i \neq j} d(x_i, x_j) > 2r \right) \wedge \left(\bigwedge_{i} \psi_{B_r(x_i)}(x_i) \right)$$

Theorem

A zero-one FO law holds for connected graphs in $\mathcal{G}_S$

A convergence FO law holds for arbitrary graphs in $\mathcal{G}_S$

$$p(\phi) = \lim \mathbf{P}(G_n \models \phi) \text{ independent of } S$$

Theorem (Albert Atserias, Stephan Kreutzer, M.N.)

- No Zero-One MSO law for connected graphs of genus $g > 0$
- No convergence MSO law for graphs of genus $g > 0$

Proofs use several facts

1. CFGMN 2011

   A random graph of genus $g > 0$ has w.h.p. a unique non-planar 3-connected component

   - 3-connected components are MSO definable
   - Minors are MSO definable, hence planarity too

2. Ellingham 1996

   A 3-connected graph of genus g has a spanning tree with maximum degree $\leq 4g$ [$\Delta \leq 3$ for planar Barnette 1966]

3. Courcelle 2003

   For bounded genus MSO $\equiv$ MSO_2 (quantification over vertices and edges)

4. Giménez-Noy-Rué 2013

   Local limit law for $X_n = |\text{3-connected component of genus } g|$

   $$\mathbf{P}(X_n = \alpha n + x n^{2/3}) \sim n^{-2/3} f(x), \quad f \text{ density of a stable law}$$

Theorem

The probability that X_n is even is MSO expressible and

$$\mathbf{P}(X_n \text{ even}) \to 1/2$$

Sketch of proof

Because of spanning tree of bounded degree, parity is MSO expressible

Because of local limit law for X_n, $\mathbf{P}(X_n \text{ even}) \to 1/2$
(Not enough convergence in distribution)

Note $L = \{\lim \mathbf{P}(G_n \models \phi) : \phi \text{ MSO formula}\}$

$$\overline{L} = [0, 1]$$

Non-convergence for $g > 0$

We can produce an MSO formula ϕ such that $\mathbf{P}(G_n \models \phi)$ does not converge for random graphs of genus $g > 0$

Claim The 3-connected component of genus g contains w.h.p. an MSO definable grid minor M with

$$\log \log n \leq |M| \leq n$$

Inspired on encoding Turing machine computations in a grid one can capture parity of the iterated logarithm $\log^* |M|$ and produce a formula without limiting probability

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