Number Theory meets Approximation Theory

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- Irrationality
- Hermite-Padé approximation
- Apéry's proofs for $\zeta(2)$ and $\zeta(3)$
- Rivoal's proof: infinitely many $\zeta(2n+1)$ are irrational
- Hata's upper bound for the irrationality measure of π
- q-extensions: $\zeta_q(1)$ and $\zeta_q(2)$
- Still open: $\zeta(5)$, $\zeta_q(3)$, Catalan, Euler

Most real numbers are irrational.

Only a few remarkable mathematical constants are known to be irrational

- $\sqrt{2}$ (Pythagoras)

- e (Euler?)

- π (Lambert 1761), π^2 (Legendre 1794)

- $\zeta(3)$ (Apéry, 1977)

Proof of irrationality is usually by contradiction.

Theorem 1. $\sqrt{2}$ *is irrational.*

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Proof. Suppose $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Then

$$p^2 = 2q^2$$

and hence p^2 is even. But this is only possible if p is even, $p = 2n$. Then

$$4n^2 = 2q^2$$

and hence q^2 is even, and also q is even, $q = 2m$. But then $\gcd(p, q) \geq 2$. Contradiction. □

Theorem 2. *The number e is irrational.*

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Proof. Take the Taylor series with remainder

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^\xi}{(n+1)!}, \qquad 0 < \xi < x.$$

For $x = 1$ this gives

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{(n+1)!}.$$

Suppose $e = \frac{p}{q}$, then

$$0 < n!\frac{p}{q} - \sum_{k=0}^{n} \frac{n!}{k!} < \frac{e}{n+1}.$$

Contradiction if $n \geq q$. $\square$

Lemma 1 (main tool). *Let $x \in \mathbb{R}$. If there exist integers p_n and q_n such that*

1. $q_n x - p_n \neq 0$ *for every* $n \in \mathbb{N}$
2. $\lim_{n\to\infty}(q_n x - p_n) = 0$

then x is irrational

This allows a constructive proof of irrationality: given $x \in \mathbb{R}$, construct integer sequences $\{p_n, q_n : n \in \mathbb{N}\}$ satisfying (1) and (2), i.e., rational approximants p_n/q_n for x for which

$$x - \frac{p_n}{q_n} = o\left(\frac{1}{q_n}\right), \qquad n \to \infty$$

(better than order 1).

measure of irrationality

$$\mu(x) = \text{sup}\{r > 0 : \left|x - \frac{a}{b}\right| < \frac{1}{b^r} \text{ has infinitely many solutions } (a,b) \in \mathbb{Z}\}$$

- if $x \in \mathbb{Q}$ then $\mu(x) = 1$
- if x is irrational, then $\mu(x) \geq 2$
- if x is algebraic, then $\mu(x) = 2$ (Roth)

Lemma 2. *If there exist integers p_n and q_n such that $q_n \leq q_{n+1} \leq q_n^{1+o(1)}$ and*

$$x - \frac{p_n}{q_n} = \mathcal{O}\left(\frac{1}{q_n^{1+r}}\right)$$

for $0 < r < 1$, then $2 \leq \mu(x) \leq 1 + \frac{1}{r}$.

Continued fractions: every $x \in \mathbb{R}$ can be written as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} = [a_0; a_1, a_2, a_3, \ldots]$$

with $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$.

Truncate to find rational approximants for x

$$[a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n}$$

with

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 3. *A number is rational if and only if its continued fraction terminates. If x is irrational, then*

$$0 < (-1)^{n-1}\left(x - \frac{p_n}{q_n}\right) < \frac{1}{a_{n+1}q_n^2}.$$

Hence every irrational number can be approximated to order 2 by convergents of a continued fraction.

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{\ddots}}}}}}}}}$$

$$a_0 = 2, \quad a_{3n-2} = a_{3n} = 1, \quad a_{3n-1} = 2n, \qquad n \geq 1.$$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}$$

Convergents

$$\frac{p_1}{q_1} = \frac{22}{7}, \qquad , \frac{p_3}{q_3} = \frac{355}{113}$$

No formula is known for the continued fraction coefficients a_n of π.

Hermite-Padé approximation

r functions $f_1, \ldots, f_r$ given by

$$f_k(z) = \int_{-\infty}^{\infty} \frac{d\mu_k(x)}{z-x}, \qquad z \notin \mathbb{R}$$

multi-index $\vec{n} = (n_1, n_2, \ldots, n_r)$ and $|\vec{n}| = n_1 + n_2 + \cdots + n_r$

Type I Hermite-Padé approximation

Find polynomials $A_{\vec{n},1}, \ldots, A_{\vec{n},r}$ with degree $A_{\vec{n},k} \leq n_k$ such that

$$\sum_{k=1}^{r} A_{\vec{n},k}(z) f_k(z) - R_{\vec{n}}(z) = \mathcal{O}(\frac{1}{z^{|\vec{n}|}}), \qquad z \to \infty,$$

$$R_{\vec{n}}(z) = \sum_{k=1}^{r} \int \frac{A_{\vec{n},k}(z) - A_{\vec{n},k}(x)}{z-x} \, d\mu_k(x)$$

Hermite-Padé approximation

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$$R_{\vec{n}}(z) = \sum_{k=1}^{r} \int \frac{A_{\vec{n},k}(z) - A_{\vec{n},k}(x)}{z-x}\, d\mu_k(x)$$

Hermite-Padé approximation

r functions $f_1, \ldots, f_r$ given by

$$f_k(z) = \int_{-\infty}^{\infty} \frac{d\mu_k(x)}{z-x}, \qquad z \notin \mathbb{R}$$

Type II Hermite-Padé approximation

Find a polynomial $P_{\vec{n}}$ of degree $\leq |\vec{n}| = n_1 + n_2 + \cdots + n_r$ such that

$$P_{\vec{n}}(z) f_k(z) - Q_{\vec{n},k}(z) = \mathcal{O}(\frac{1}{z^{n_k+1}}), \qquad z \to \infty,$$

$$Q_{\vec{n},k}(z) = \int \frac{P_{\vec{n}}(z) - P_{\vec{n}}(x)}{z-x} \, d\mu_k(x)$$

Hermite-Padé approximation

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Find a polynomial $P_{\vec{n}}$ of degree $\leq |\vec{n}| = n_1 + n_2 + \cdots + n_r$ such that

$$P_{\vec{n}}(z) f_k(z) - Q_{\vec{n},k}(z) = \int \frac{P_{\vec{n}}(x)}{z-x} d\mu_k(x)$$

$$Q_{\vec{n},k}(z) = \int \frac{P_{\vec{n}}(z) - P_{\vec{n}}(x)}{z-x} d\mu_k(x)$$

Irrationality of $\zeta(2) = \pi^2/6$

$$f_1(z) = \int_0^1 \frac{dx}{z-x} = \sum_{k=0}^{\infty} \frac{1}{k+1}\frac{1}{z^{k+1}}, \qquad f_2(z) = \int_0^1 \frac{-\log x}{z-x}\,dx = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\frac{1}{z^{k+1}}$$

Find polynomials A_n, B_n of degree $\leq n$ and a polynomial C_n such that

$$\begin{aligned} A_n(z) - B_n(z)\log z &= \mathcal{O}\left((1-z)^{n+1}\right), \qquad z \to 1, \\ A_n(z)f_1(z) + B_n(z)f_2(z) - C_n(z) &= \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \qquad z \to \infty. \end{aligned}$$

- Type I Hermite-Padé for 1 and $\log z$ at $z = 1$
- Type I Hermite-Padé for f_1 and f_2 near infinity
- Type II because one uses the same polynomials (A_n, B_n) for both problems

Evaluate this approximation at $z = 1$ because

$$f_2(1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \zeta(2) = \frac{\pi^2}{6},$$

beware that $f_1(1) = \infty$ (harmonic series), but also $A_n(1) = 0$ so that

$$\lim_{z \to 1} A_n(z) f_1(z) = 0.$$

$$B_n(1)\zeta(2) - C_n(1) = \int_0^1 \frac{F_n(x)}{1-x}\, dx$$

Type I Hermite-Padé for 1 and $\log z$ gives

$$F_n(x) = A_n(x) - B_n(x) \log x = \int_x^1 (1 - x/t)^n P_n(t) \, \frac{dt}{t}$$

Type I Hermite-Padé for f_1 and f_2 gives orthogonality conditions

$$\int_0^1 F_n(x)x^k\,dx = 0, \qquad k = 0,1,\ldots,n-1$$

which is equivalent to

$$\int_0^1 P_n(t)t^k\,dt = 0, \qquad k = 0,1,\ldots,n-1.$$

Hence P_n is the Legendre polynomial on $[0,1]$

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^k x^k.$$

All the polynomials can now be obtained explicitely

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} x^k$$

$$A_n(x) = \sum_{k \neq j} \binom{n}{k} \binom{n+k}{k} \binom{n}{j} (-1)^{k+j} \frac{x^j - x^k}{k-j}$$

$$\begin{aligned} C_n(1) \; &= \; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \sum_{j=0}^{k-1} \frac{1}{(j+1)^2} \\ &\quad - \sum_{k \neq j} \binom{n}{k} \binom{n+k}{k} \binom{n}{j} (-1)^{k+j} \frac{H_k - H_j}{k-j} \end{aligned}$$

with $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

Hence $B_n(1)$ is an integer, but $C_n(1)$ is rational. The denominators in $C_n(1)$ can be cancelled if we multiply by d_n^2, with $d_n = \mathrm{lcm}(1, 2, \ldots, n)$.

$$\underbrace{d_n^2 B_n(1)}_{q_n}\zeta(2) - \underbrace{d_n^2 C_n(1)}_{p_n} = d_n^2 \int_0^1 \frac{F_n(x)}{1-x}\,dx$$

Rodrigues formula for Legendre polynomials gives

$$q_n\zeta(2) - p_n = d_n^2 \int_0^1 \int_0^1 \frac{y^n t^n (1-y)^n (1-t)^n}{(1-yt)^{n+1}}\,dy\,dt$$

Theorem 4. *If* $q_n = d_n^2 B_n(1)$ *and* $p_n = d_n^2 C_n(1)$*, then*

$$\lim_{n\to\infty} (q_n\zeta(2) - p_n)^{1/n} \leq e^2 \left(\frac{\sqrt{5}-1}{2}\right)^5 < 1.$$

Hence $\zeta(2)$ *is irrational.*

$\zeta(3)$ is irrational

Hermite-Padé for

$$f_1(z) = \int_0^1 \frac{1}{z-x}\,dx, \quad f_2(z) = -\int_0^1 \frac{\log x}{z-x}\,dx, \quad f_3(z) = \frac{1}{2}\int_0^1 \frac{\log^2 x}{z-x}\,dx$$

Find polynomials A_n, B_n of degree $\leq n$ and polynomials C_n and D_n such that

$$\begin{aligned} A_n(z)f_1(z) + B_n(z)f_2(z) - C_n(z) &= \mathcal{O}(1/z^{n+1}), \qquad z \to \infty \\ A_n(z)f_2(z) + 2B_n(z)f_3(z) - D_n(z) &= \mathcal{O}(1/z^{n+1}), \qquad z \to \infty \end{aligned}$$

$$A_n(1) = 0.$$

$$f_1(1) = \infty, \quad f_2(1) = \zeta(2), \quad f_3(1) = \zeta(3)$$

Evaluated at $z = 1$ we find

$$2B_n(1)\zeta(3) - D_n(1) = -\int_0^1 \frac{F_n(x)}{1-x} \log x \, dx$$

with

$$F_n(x) = A_n(x) - B_n(x) \log x = \int_x^1 P_n(x/t) P_n(t) \frac{dt}{t}$$

and P_n the Legendre polynomial on $[0, 1]$.

$$\begin{aligned} 2d_n^3 B_n(1)\zeta(3) - d_n^3 D_n(1) & \\ &= d_n^3 \int_0^1 \int_0^1 \int_0^1 \frac{x^n y^n z^n (1-x)^n (1-y)^n (1-z)^n}{[1-(1-xy)z]^{n+1}} \, dx \, dy \, dz \end{aligned}$$

Theorem 5 (Apéry). *If* $q_n = 2d_n^3 B_n(1)$ *and* $p_n = d_n^3 D_n(1)$, *then*

$$\lim_{n\to\infty} (q_n \zeta(3) - p_n)^{1/n} \leq e^3(\sqrt{2}-1)^4 < 1.$$

Hence $\zeta(3)$ *is irrational.*

Infinitely many $\zeta(2n+1)$ are irrational (Ball and Rivoal, 2001)

Lemma 3 (Nesterenko). *Let $x_1, \ldots, x_r$ be real numbers. Suppose there exist r sequences of integers $(p_{k,n}, n \in \mathbb{N}\}$, $1 \leq k \leq r$ such that for $0 < \alpha_1 \leq \alpha_2 < 1$*

$$\alpha_1 \leq \liminf \left| \sum_{k=1}^{r} p_{k,n} x_k \right|^{1/n} \leq \limsup \left| \sum_{k=1}^{r} p_{k,n} x_k \right|^{1/n} \leq \alpha_2$$

and for some $\beta > 1$

$$\lim_{n \to \infty} |p_{k,n}|^{1/n} \leq \beta, \qquad k = 1, 2, \ldots, r.$$

Then the dimension of the space spanned by $x_1, \ldots, x_r$ over $\mathbb{Q}$ is at least

$$\frac{\log \beta - \log \alpha_1}{\log \beta - \log \alpha_1 + \log \alpha_2}.$$

$$f_j(z) = \frac{(-1)^{j-1}}{(j-1)!} \int_0^1 \frac{\log^{j-1} x}{z-x}\, dx, \qquad f_j(1) = \zeta(j)$$

Find r polynomials $A_{n,1}, \ldots, A_{n,r}$ of degree n and a polynomial B_n such that ($\ell < r/2$)

$$\begin{aligned}
\sum_{j=1}^{r} (j-1)!(-1)^{j-1} A_{n,j}(z) f_j(z) - B_n(z) &= \mathcal{O}(1/z^{\ell n+1}), \qquad z \to \infty \\
\sum_{j=1}^{r} (j-1)! A_{n,j}(z) f_j(1/z)/z + B_n^*(z) &= \mathcal{O}(z^{(\ell+1)n}), \qquad z \to 0 \\
\sum_{j=1}^{r} A_{n,j}(x) \log^{j-1} x &= \mathcal{O}\left((1-x)^{r(n+1)-2\ell n-1}\right), \quad x \to 1.
\end{aligned}$$

Theorem 6 (Rivoal). *At least one of the numbers* $\zeta(5), \zeta(7), \zeta(9), \zeta(11), \zeta(13), \zeta(15), \zeta(17), \zeta(19), \zeta(21)$ *is irrational.*

Theorem 7 (Zudilin). *At least one of the numbers* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational.*

Linear independence over $\mathbb{Q}$

Lemma 4. *Let $x_1, \ldots, x_r$ be real numbers.*
Suppose that for all $a_0, a_1, \ldots, a_r \in \mathbb{Z}$ there exist infinite sequences of integers p_n and $q_{n,1}, \ldots, q_{n,r}$ such that

1. $a_0 p_n + a_1 q_{n,1} + \cdots + a_r q_{n,r} \neq 0$ *for every* $n \in \mathbb{N}$

2. $\lim_{n\to\infty}(p_n x_k - q_{n,k}) = 0$ *for* $k = 1, 2, \ldots, r$.

Then $1, x_1, \ldots, x_r$ are linearly independent over $\mathbb{Q}$.

1, $\log 2$ and π are linearly independent over $\mathbb{Q}$ Hata, 1993

Type II Hermite-Padé approximation to

$$f_1(z) = \int_0^1 \frac{1}{z-x}\,dx, \qquad f_2(z) = \int_0^{-i} \frac{1}{z-x}\,dx$$

$$f_1(i) = -\frac{1}{2}\log 2 - \frac{i\pi}{4}, \qquad f_2(i) = \log 2$$

$$\begin{aligned} P_{n,n}(z)f_1(z) - Q_{n,n}(z) &= \mathcal{O}(1/z^{n+1}), \qquad z \to \infty \\ P_{n,n}(z)f_2(z) - R_{n,n}(z) &= \mathcal{O}(1/z^{n+1}), \qquad z \to \infty \end{aligned}$$

$$P_{n,n}(i)\underbrace{[2f_1(i) + f_2(i)]}_{-i\pi/2} - [2Q_{n,n}(i) + R_{n,n}(i)] = \int_\Gamma \frac{P_{n,n}(x)}{i-x}\,dx$$

The polynomial $P_{n,n}$ is determined by

$$\begin{aligned} \int_0^1 P_{n,n}(x) x^k \, dx &= 0, \qquad k = 0, \ldots, n-1, \\ \int_0^{-i} P_{n,n}(x) x^k \, dx &= 0, \qquad k = 0, \ldots, n-1 \end{aligned}$$

and is a Jacobi-Angelesco polynomial.

Asymptotic behaviour of $P_{n,n}(i)$ and $\int \frac{P_{n,n}(x)}{i-x} \, dx$, together with some elementary number theory (to cancel denominators) gives

Theorem 8 (Hata). *The numbers* 1, $\log 2$ *and* π *are linearly independent over* $\mathbb{Q}$. *Furthermore, the measure of irrationality for* π *is bounded by*

$$\mu(\pi) \leq 8.016$$

A q-extension of the zeta-function

We define the following q-zeta function

$$\zeta_q(s) = \sum_{k=1}^{\infty} \frac{k^{s-1} q^k}{1-q^k}, \qquad 0 < q < 1.$$

This is a q-analog which is relevant in number theory and for which

$$\lim_{q \to 1} (1-q)^s \zeta_q(s) = \Gamma(s)\zeta(s), \qquad \Re s > 1.$$

- $\zeta_q(1)$ is irrational when $1/q$ is an integer ≥ 2. Bézivin (1988), Borwein (1991), Bundschuh an Väänänen (1994);

- $\zeta_q(2)$ is irrational when $1/q$ is an integer ≥ 2. Duverney (1995), Nesterenko (1996);

- $1, \zeta_q(1), \zeta_q(2)$ are linearly independent over $\mathbb{Q}$ when $1/q$ is an integer ≥ 2. Bundschuh an Väänänen (2005), Zudilin (2005), Postelmans, WVA (2007)

- Infinitely many $\zeta_q(2n+1)$ are irrational. Krattenthaler, Rivoal, Zudilin (2006);

- There is at least one irrational number in $\{\zeta_q(3), \zeta_q(5), \zeta_q(7), \zeta_q(9)\}$ when $1/q$ is an integer ≥ 2.

Irrationality of $\zeta_q(2)$

Theorem 9 (C. Smet, WVA). *Let $q = 1/p$ with $p \in \mathbb{N} \setminus \{0,1\}$. Then $\zeta_q(2)$ is irrational and*

$$2 \leq \mu(\zeta_q(2)) \leq \frac{10\pi^2}{5\pi^2 - 24} = 3.8936\ldots$$

Ingredients: q-integral

$$\int_0^1 f(x)\, d_q(x) = \sum_{k=0}^{\infty} q^k f(q^k)$$

Hermite-Padé approximation to

$$f_1(z) = \int_0^1 \frac{d_q x}{z-x}, \qquad f_2(z) = \int_0^1 \frac{\log_q x}{z-x}\, d_q(x)$$

$$f_1(p^n) = \zeta_q(1) - \sum_{k=1}^{n-1} \frac{1}{p^k - 1}, \qquad f_2(p^n) = \zeta_q(2) - \sum_{k=1}^{n-1} \frac{k}{p^k - 1} - n f_1(p^n)$$

Find polynomials A_n and B_n of degree n and a polynomial C_n such that

$$A_n(x) + B_n(x) \log_q(x) = 0, \qquad x = 1, p, p^2, \ldots, p^n,$$

$$A_n(z) f_1(z) + B_n(z) f_2(z) - C_n(z) = \mathcal{O}(1/z^{n+1}), \qquad z \to \infty$$

In this case we have for $x \in \{q^k, k \in \mathbb{Z}\}$

$$F_n(x) = A_n(x) + B_n(x) \log_q x = \int_x^1 P_n(x/t)(qt;q)_n \frac{d_q t}{t},$$

where P_n is a little q-Legendre polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1};q)_k}{(q;q)_k (q;q)_k} x^k$$

Explicit formulas

$$A_n(p^n) - nB_n(p^n) = 0$$

$$B_n(x) = \sum_{k=0}^{n} p^{-2kn+k^2} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \begin{bmatrix} n+k \\ k \end{bmatrix}_p x^k$$

$$\begin{aligned} C_n(x) \;\; &= \;\; \sum_{k \neq j} (-1)^{k+j} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ j \end{bmatrix}_p \begin{bmatrix} n+k \\ k \end{bmatrix}_p \times \text{more stuff} \\ & \quad + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \begin{bmatrix} n+k \\ k \end{bmatrix}_p p^{-2kn+k^2} \sum_{j=0}^{k-1} \frac{p^{k-j} x^j}{(p^{k-j}-1)^2} \end{aligned}$$

We need integer values for our approximants:

$$p^{\lfloor n^2/4 \rfloor} B_n(p^n) \in \mathbb{Z}$$

$$p^{\lfloor n^2/4 \rfloor} C_n(p^n) \in \mathbb{Q}$$

but the denominators contain only $p^j - 1$ and $(p^j - 1)^2$ with $1 \leq j \leq n$. Hence we multiply by an extra factor $d_n^2(p)$ with

$$d_n(p) = \mathrm{lcm}(p-1, p^2-1, \ldots, p^n-1) = \prod_{k=1}^{n} \Phi_k(p)$$

with Φ_k the cyclotomic polynomials, i.e., the irreducible factors of the polynomials $x^n - 1$:

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

$$\underbrace{d_n^2(p) p^{\lfloor n^2/4 \rfloor} B_n(p^n)}_{q_n} \zeta_q(2) - \underbrace{d_n^2(p) p^{\lfloor n^2/4 \rfloor} C_n(p^n)}_{p_n}$$

$$= d_n^2(p) p^{\lfloor n^2/4 \rfloor} q^{n+1} \int_0^1 \int_0^1 \frac{(qx;q)_n (qy;q)_n x^n y^n}{\prod_{j=0}^{n} (p^{n+j} - qxy)} \, d_qx \, d_qy$$

Theorem 10. *Let p be an integer ≥ 2, then*

$$\lim_{n\to\infty} d_n(p)^{1/n^2} = p^{3/\pi^2},$$

$$\lim_{n\to\infty} q_n^{1/n^2} = p^{6/\pi^2+5/4},$$

and

$$\lim_{n\to\infty} |q_n\zeta_q(2) - p_n|^{1/n^2} \leq p^{6/\pi^2-5/4} = p^{-0.6421}.$$

1, $\zeta_q(1)$ and $\zeta_q(2)$ are linearly independent over $\mathbb{Q}$ Postelmans, WVA

$$\zeta_q(s) = \sum_{k=1}^{\infty} \frac{k^{s-1} q^k}{1 - q^k}$$

Type II Hermite-Padé approximation to f_1, f_2: find a polynomial $P_{n,n-1}$ of degree $2n-1$ and polynomials $Q_{n,n-1}$, $R_{n,n-1}$ such that

$$P_{n,n-1} f_1(z) - Q_{n,n-1}(z) = \mathcal{O}(1/z^{n+1}), \qquad z \to \infty$$

$$P_{n,n-1} f_2(z) - R_{n,n-1}(z) = \mathcal{O}(1/z^{n}), \qquad z \to \infty$$

$P_{n,n-1}$ is a multiple little q-Jacobi polynomial

last slide

Interesting open problems:

- Is $\zeta(5)$ irrational?

- Is $\zeta_q(3)$ irrational?

- Irrationality of Catalan's constant

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

- Irrationality of Euler's constant

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = -\int_0^\infty e^{-x} \log x \, dx$$

very last slide

Theorem 11 (Aptekarev). *Suppose*

$$\delta = \int_0^\infty \frac{e^{-x}}{1+x}\,dx.$$

At least one of the two numbers γ *or* δ *is irrational.*