The rational corona-Bezout equation and Toeplitz operators

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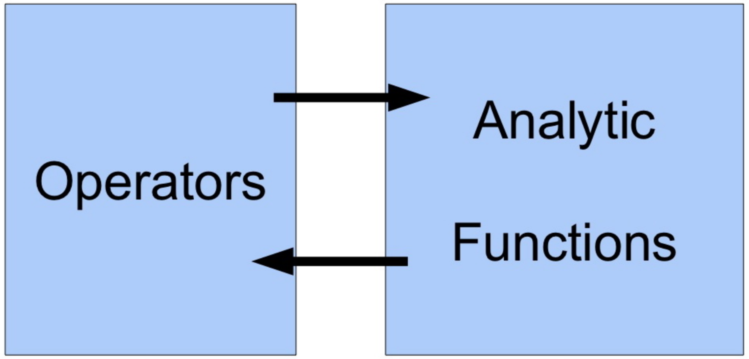
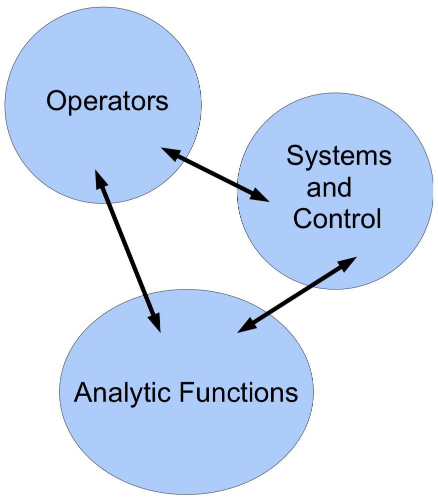
Operators and analytic functions

Past 60 years:



Operators, analytic functions, systems and control

Since the mid seventies:

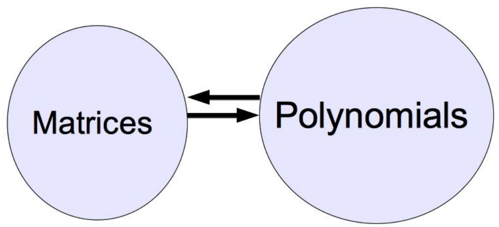


Matrices and polynomials

Étienne Bézout, 1730 - 1783

James Joseph Sylvester

1814 - 1897



Outline of the talk

1. The classical Bezout equation and interplay with matrices
2. The corona-Bezout equation
   - H^∞ functions
   - Carleson's corona theorem
   - Fuhrmann's matrix version
3. Connections with operator theory
   - Analytic Toeplitz operators
   - The Toeplitz operator corona theorem
4. Rational corona-Bezout equation
   - The Smith McMillan form
   - State space representations
5. A state space description of all solutions to the rational corona-Bezout equation
   - The least squares solution
   - Parameterization of all solutions

The classical Bezout equation

$$(*) \qquad x(\lambda)p(\lambda) + y(\lambda)q(\lambda) = 1 \qquad \text{[Bezout equation]}$$

Here p and q are scalar polynomials $p(\lambda) = \sum_{j=0}^{\ell} \lambda^j p_j$, $q(\lambda) = \sum_{j=0}^{m} \lambda^j q_j$.

LEMMA. *The Bezout equation has polynomial solutions x and y if and only if p and q are coprime, that is, p and q have no common zero.*

Interplay with matrices

$$\mathbf{R} = \left[\begin{array}{cccccccc} p_0 & \cdots & \cdots & p_\ell & & & & \\ & p_0 & \cdots & \cdots & p_\ell & & & \\ & & \ddots & & & & \ddots & \\ & & & p_0 & & \cdots & \cdots & p_\ell \\ q_0 & \cdots & \cdots & q_{m-1} & q_m & & & \\ & \ddots & & & & & \ddots & \\ & & q_0 & \cdots & & \cdots & \cdots & q_m \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \ell \end{array} \qquad \text{[Resultant]}$$

$$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}_{\ell + m}$$

Sylvester 1853: *The number of common zeros of p and q, multiplicities taken into account, is equal to* $\dim \operatorname{Ker} \mathbf{R}$.

Why is the resultant interesting?

$$x(\lambda)p(\lambda) + y(\lambda)q(\lambda) = h(\lambda), \quad h(\lambda) = \sum_{j=0}^{\ell+m-1} \lambda^j h_j$$

$$\uparrow\downarrow$$

$$\begin{bmatrix} x_0 & \cdots & x_{m-1} & y_0 & \cdots & y_{\ell-1} \end{bmatrix} \mathbf{R} = \begin{bmatrix} h_0 & \cdots & h_{\ell+m-1} \end{bmatrix}$$

COR. *The following are equivalent:*

(a) *the polynomials p and q are copime;*

(b) *the resultant is non-singular;*

(c) *the Bezout equation has polynomial solutions x and y satisfying the degree constraints* $\deg x \leq m-1$, $\deg y \leq \ell - 1$.

Remark. Not true for matrix polynomials (Gohberg-Heinig, 1975).

Counterexample

Consider

$$P(\lambda) = \begin{bmatrix} \lambda - 1 & 0 \\ 1 & \lambda - 1 \end{bmatrix}, \quad Q(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda - 2 \end{bmatrix}.$$

Note that both P and Q are monic, and $\det P$ and $\det Q$ have no common zeros. Hence P and Q have no common zeros, and we expect $\mathbf{R}(P, Q)$ to be invertible. However

$$\mathbf{R}(P,Q) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \operatorname{Ker} \mathbf{R}(P,Q) = \langle \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} \rangle.$$

Corona-Bezout equation

Polynomials are replaced by H^∞-functions, i.e, by functions that are analytic and uniformy bounded on the unit disc $\mathbb{D}$:

$$g_1(z)x_1(z) + g_2(z)x_2(z) + \cdots + g_q(z)x_q(z) = 1, \quad z \in \mathbb{D}.$$

THM [Carleson 1962]. *For the corona-Bezout equation to be solvable it is necessary and sufficient that there exists $\delta > 0$ such that*

$$\sum_{j=1}^{q} |g_j(z)|^2 \geq \delta, \quad z \in \mathbb{D}.$$

Important issue: $\sup_{z\in\mathbb{D}} \sum_{j=1}^{q} |x_j(z)|^2$ as small as possible; δ^{-1} is a lower bound. The value $\delta^{-1} + \delta^{-2}[7(\log \delta^{-1})^{1/2} + 20 \log \delta^{-1}]$ is reachable.

Corona-Bezout equation – matrix version (1)

$$G(z) = \begin{bmatrix} g_1(z) & \cdots & g_q(z) \end{bmatrix}, \quad X(z) = \begin{bmatrix} x_1(z) \\ \vdots \\ x_q(z) \end{bmatrix}$$

Using G and X as above, we rewrite the corona-Bezout equation as a matrix function equation:

$$G(z)X(z) = 1, \quad z \in \mathbb{D}$$

Throughout a $p \times q$ matrix M is identified with the linear operator from the Euclidean space $\mathbb{C}^q$ into the Euclidean space $\mathbb{C}^p$ defined by the canonical action of M relative to the standard bases, and $\|M\|$ is the corresponding operator norm.

Corona-Bezout equation – matrix version (2)

$$(*) \qquad G(z)X(z) = I_p, \quad z \in \mathbb{D}.$$

Here $G \in H^\infty_{p\times q}$, where $p \leq q$, and we seek $X \in H^\infty_{q\times p}$ such that $(*)$ holds.

Notation: $H^\infty_{p\times q}$, the space of $p \times q$ matrices of which the entries are bounded analytic function on $\mathbb{D}$.

$$\|\Phi\|_\infty = \sup_{z\in\mathbb{D}} \|\Phi(z)\|, \ \Phi \in H^\infty_{m\times p} \qquad \text{[H-infinity norm]}$$

THM [Fuhrmann 1968]. *For the matrix corona-Bezout equation to be solvable it is necessary and sufficient that there exists $\delta > 0$ such that*

$$G(z)G(z)^* \geq \delta\, I_p, \quad z \in \mathbb{D}.$$

N.B. Does not generalize to infinite matrices.

Connection with operators – preliminaries

$$\ell^2_+(\mathbb{C}^k) = \{x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \mid x_j \in \mathbb{C}^k, \quad \sum_{j=0}^{\infty} \|x_j\|^2 < \infty\},$$

inner product: $\langle x, y\rangle = \sum_{j=0}^{\infty} y_j^* x_j$; norm: $\|x\| = (\sum_{j=0}^{\infty} \|x_j\|^2)^{1/2}$.

$$\mathcal{F} : \ell^2_+(\mathbb{C}^k) \to H^2_k, \quad \mathcal{F} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = \hat{x}, \quad \hat{x}(z) = \sum_{j=0}^{\infty} z^j x_j$$

$\mathcal{F}$: Fourier transform *H^2_k: Hardy space of $\mathbb{C}^k$-functions*

Connection with operators

With $G \in H^\infty_{p\times q}$ we associate the Toeplitz operator T_G defined by G:

$$T_G = \begin{bmatrix} G_0 & & & \\ G_1 & G_0 & & \\ G_2 & G_1 & G_0 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^q) \to \ell^2_+(\mathbb{C}^p).$$

$$G(z)X(z) = I_m \Rightarrow T_G T_X = T_{GX} = I_{\ell^2_+(\mathbb{C}^m)} \Rightarrow T_G \text{ right invertible}$$

Inverse problem [Arveson 1975]. If an analytic Toeplitz operator is right invertible, is it right invertible in the algebra of analytic Toeplitz operators?

Corona theorem for analytic Toeplitz operators

$(*)$ $$G(z)X(z) = I_p, \quad z \in \mathbb{D}.$$

THM [Sz-Nagy-Foias, 1976]. *Let $G \in H^\infty_{p\times q}$. Then there exists X in $H^\infty_{q\times p}$ satisfying $(*)$ if and only if T_G has a right inverse, and in this case*

$$\inf\{\|X\|_\infty \mid X \in H^\infty_{q\times p} \text{ is a solution of } (*)\} = \|T_G^+\|,$$

where $T_G^+ = T_G^*(T_G T_G^*)^{-1}$.

N.B.1. The operator T_G^+ is the *Moore-Penrose right inverse* of T_G, that is,

$$T_G T_G^+ = I \quad \text{and} \quad \mathrm{Im}\ T_G^+ = (\mathrm{Ker}\ T_G)^\perp.$$

N.B.2. Proof uses the commutant lifting theorem, and carries over to operator-valued functions. Hence the operator corona theorem is not equivalent with the function theory version.

Rational corona-Bezout equation

$(*)$ $$G(z)X(z) = I_p$$

Given: $G \in RH^\infty_{p\times q}$ [$p \times q$ rational H^∞ matrix function]

Problem: Find $X \in RH^\infty_{q\times p}$ such that $(*)$ holds.

Condition for solvability: $G(z)$ has full row rank for each $|z| \le 1$ ($\Leftrightarrow$ Carleson's condition).

Necessity obvious. Sufficiency: use Smith-McMillan form

$$G(z) = U(z) \begin{bmatrix} \rho_1(z) & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \rho_p(z) & 0 & \cdots & 0 \end{bmatrix} V(z)$$

Smith-McMillan form:

$$G(z) = U(z)\begin{bmatrix} \rho_1(z) & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \rho_p(z) & 0 & \cdots & 0 \end{bmatrix} V(z)$$

$$X(z) = V(z)^{-1}\begin{bmatrix} \rho_1(z)^{-1} & & \\ & \ddots & \\ & & \rho_p(z)^{-1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} U(z)^{-1}$$

Then $X \in RH^\infty_{q\times p}$ and $G(z)X(z) = I_p$. [From the computional point of view the Smith-McMillan is difficult to work with.]

Intermezzo about state space representation

$G \in H^\infty_{p\times q}$

$$H_G = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & G_4 & \cdots \\ G_3 & G_4 & G_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^q) \to \ell^2_+(\mathbb{C}^p) \quad \text{[Hankel operator]}$$

Characteristic property: $S_p^* H_G = H_G S_q$ [S_k forward shift on $\ell^2_+(\mathbb{C}^k)$]

$G \in RH^\infty_{p\times q} \iff \text{rank } H_G < \infty$

$G \in RH^\infty_{p\times q}$ admits a state space representation:

$$G(z) = D + zC(I_n - zA)^{-1}B$$

A, B, C, D matrices of appropriate sizes, A stable.

Constructing a state space representation

Given $G \in RH^\infty_{p\times q}$, define:

$$\mathcal{X} = \mathrm{Im}\, H_G, \quad A = S_p^*|_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}, \qquad [\text{Use } S_p^* H_G = H_G S_q]$$

$$B : \mathbb{C}^q \to \mathcal{X}, \quad Bu = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \end{bmatrix} u,$$

$$C = \begin{bmatrix} I_p & 0 & 0 & \cdots \end{bmatrix}|_{\mathcal{X}} : \mathcal{X} \to \mathbb{C}^p, \quad D = G_0.$$

Then $G(z) = D + zC(I_n - zA)^{-1}B$ with $n = \dim \mathcal{X} = \text{rank } H_G$. Moreover

$$H_G = W_{obs}W_{con}, \; W_{obs} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}, \; W_{con} = \begin{bmatrix} B & AB & A^2 & \cdots \end{bmatrix}.$$

State space version of the rational corona problem

$$(*) \qquad G(z)X(z) = I_p$$

Given: $G \in RH^\infty_{p\times q}$ in state space form: $G(z) = D + zC(I_n - zA)^{-1}B$.

Problem: Derive all solutions $X \in RH^\infty_{p\times q}$ of $(*)$ in terms of the matrices A, B, C, D.

Assume T_G is right invertible. Consider the $p \times q$ matrix function Φ defined by

$$\Phi(z)u = \mathcal{F}_{\mathbb{C}^p} T_G^*(T_G T_G^*)^{-1}\begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \end{bmatrix} u \quad u \in \mathbb{C}^p \quad (z \in \mathbb{D})$$

Remark: $\Phi \in H^2_{q\times p}$ and $G(z)\Phi(z) = I_p$ for $z \in \mathbb{D}$.

$$(G\Phi)(\cdot) = G(\mathcal{F}T_G^*(T_GT_G^*)^{-1}\begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \end{bmatrix}) = \mathcal{F}T_GT_G^*(T_GT_G^*)^{-1}\begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \end{bmatrix} = I_p.$$

This H^2 solution Φ has two special properties: (1) Φ is the least squares solution, and (2) Φ is rational. Property (1) means that for any other solution V we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it})^*\Phi(e^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} V(e^{it})^*V(e^{it})\,dt.$$

First goal: To compute Φ in terms of the matrices A, B, C, D in the state space representation G?

How to invert $T_GT_G^*$?

Set $R(z) = G(z)G^*(z)$, where $G^*(z) = G(\bar{z}^{-1})^*$, and put

$$T_R = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \qquad [T_R \neq T_GT_G^*]$$

But $T_R = T_GT_G^* + H_GH_G^*$, and thus $T_R \ge T_R - H_GH_G^* = T_GT_G^* >> 0$.

LEMMA. *$T_GT_G^*$ invertible* $\Leftrightarrow$ (i) *T_R invertible and* (ii) *$I - H_G^*T_R^{-1}H_G$ invertible.*

In that case

$$(T_GT_G^*)^{-1} = T_R^{-1} + T_R^{-1}H_G(I - H_G^*T_R^{-1}H_G)^{-1}H_G^*T_R^{-1}.$$

First step: inverting T_R [a classical topic]

$R(z) = G(z)G^*(z)$, where $G(z) = D + zC(I_n - zA)^{-1}B$. Thus

$$R_0 = DD^* + CPC^*, \quad R_j = R_{-j}^* = CA^{j-1}\Gamma \text{ for } j = 1, 2, 3, \ldots$$

$$P - APA^* = BB^* \text{ and } \Gamma = BD^* + APC^*.$$

THM *The Toeplitz operator T_R is invertible if and only if the discrete algebraic Riccati equation*

$$\text{(ARE)} \quad Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$$

has a (unique) stabilizing solution Q, that is, $R_0 - \Gamma^*Q\Gamma$ is positive definite, Q is an $n \times n$ matrix satisfying $(*)$ and

the matrix $A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$ is stable

First step: inverting T_R – continued

THM *The Toeplitz operator T_R is invertible if and only if*

$$\text{(ARE)} \quad Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$$

has a (unique) stabilizing solution Q. In that case

$$T_R^{-1} = T_\Psi T_\Psi^*, \ \Psi(z) = \Big(I_m - zC_0(I_n - zA_0)^{-1}\Gamma\Big)\Delta^{-1}, \text{ where}$$

$$C_0 = (R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA), \quad A_0 = A - \Gamma C_0,$$

$$\Delta = (R_0 - \Gamma^*Q\Gamma)^{1/2}.$$

N.B.: $Q = W_{obs}^*T_R^{-1}W_{obs}$, where $W_{obs} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$.

Second step: computing $T_R^{-1}H_G(I - H_G^* T_R^{-1}H_G)^{-1}H_G^* T_R^{-1}$

THM *Let Q be the stabilizing solution of the Riccati equation*

$$\text{(ARE)} \qquad Q = A^*QA + (C - \Gamma^* QA)^*(R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA).$$

Then $I - H_G^* T_R^{-1} H_G$ is invertible if and only if $I_n - PQ$ is non-singular, where $P - APA^* = B$. In that case

$$T_R^{-1}H_G(I - H_G^* T_R^{-1}H_G)^{-1}H_G^* T_R^{-1} = K(I_n - PQ)^{-1}PK^*,$$

where

$$K = \begin{bmatrix} C_0 \\ C_0A_0 \\ C_0A_0^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell_+^2(\mathbb{C}^m).$$

Conclusion: $(T_G T_G^*)^{-1} = T_\Psi T_\Psi^* + K(I_n - PQ)^{-1}PK^*.$

State space formula for the least squares solution

THM 1 [Frazho-K-Ran, 2010]. *Equation $G(z)X(z) = I_m$ has a stable rational matrix solution if and only if the corresponding Riccati equation (ARE) has a stabilizing solution Q, and $I_n - PQ$ is non-singular, where $P - APA^* = B$. In that case the least squares solution Φ is given by*

$$\Phi(z) = \Big(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)D_1, \text{ where}$$

$$A_0 = A - \Gamma(R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA),$$
$$C_1 = D^*C_0 + B^*QA_0,$$
$$\text{with } C_0 = (R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA),$$
$$D_1 = (D^* - B^*Q\Gamma)(R_0 - \Gamma^* Q\Gamma)^{-1} + C_1(I_n - PQ)^{-1}PC_0^*.$$

In particular, Φ is rational, and the McMillan degree of $\Phi \leq$ the McMillan degree of G.

The null space of T_G

Assume T_G is surjective. General H^∞ theory tells us:

- Beurling-Lax: $\mathrm{Ker}\, T_G = T_\Theta \ell_+^2(\mathbb{C}^k)$, Θ inner, $q \times k$.
- $k = q - m$.
- $\Theta(0)$ one to one.
- $T_\Theta T_\Theta^* = I_{\ell_+^2(\mathbb{C}^q)} - T_G^*(T_G T_G^*)^{-1}T_G$

Hence

$$\Theta(\cdot)\Theta(0)^* = \mathcal{F} T_\Theta T_\Theta^* \begin{bmatrix} I_q \\ 0 \\ 0 \\ \vdots \end{bmatrix} = I_q - \mathcal{F} T_G^*(T_G T_G^*)^{-1}T_G \begin{bmatrix} I_q \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$\Theta(0)^*\Big(\Theta(0)\big(\Theta(0)^*\Theta(0)\big)^{-1}\Big) = I_k$$

Parameterization of all rational solutions

THM 2 [Frazho-K-Ran, 2010]. *Assume T_G is surjective. Then all solutions of the rational corona problem are given by $X = \Phi + \Theta F$. Here Φ is the least squares solution, the free parameter F is an arbitrary function in $RH^\infty_{(q-p)\times p}$, and $\Theta \in RH^\infty_{q\times(q-p)}$ is given by*

$$\Theta(z) = \Big(I_q - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)\hat{D}.$$

Here A_0 and C_1 are as in **THM 1**, *and $\hat{D}$ is any one-to-one $q \times (q-p)$ matrix such that*

$$\hat{D}\hat{D}^* = I_q - (D^* - B^*Q\Gamma)(R_0 - \Gamma^* Q\Gamma)^{-1}(D - \Gamma^* QB) + \\ - B^*QB - C_1(I_n - PQ)^{-1}PC_1^*.$$

Furthermore, $\hat{D}$ is uniquely determined up to a constant unitary matrix on the right, Θ is inner and rational, and the McMillan degree of Θ is less than or equal to the McMillan degree of G.

Example

$G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$. Obviously, $\begin{bmatrix} 1+z & -z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$.

A minimal realization of G is given by

$$A = 0, \quad B = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad C = 1, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Solution Stein equation: $P = 2$. Since $G(z)G^*(z) = 3 + z + z^{-1}$, the corresponding Riccati equation is

$$Q = \frac{1}{3 - Q},$$

and the stabilizing solution is given by $Q = \frac{1}{2}(3 - \sqrt{5})$. We see that $1 - PQ = \sqrt{5} - 2 \neq 0$, as expected.

Example – continued

$G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$.

Least squares solution of $G(z)X(z) = 1$:

$$X(z) = \frac{Q}{1 - 2Q}(1 + zQ)^{-1} \begin{bmatrix} 1 - Q \\ Q \end{bmatrix}, \quad \text{where } Q = \tfrac{1}{2}(3 - \sqrt{5}).$$

All stable rational 2×1 matrix solutions of $G(z)V(z) = 1$ are given by $V(z) = X(z) + \Theta(z)\varphi(z)$, where φ is any scalar stable rational function and

$$\Theta(z) = \sqrt{Q}(1 + zQ)^{-1} \begin{bmatrix} z \\ 1 + z \end{bmatrix}, \quad \text{where } Q = \tfrac{1}{2}(3 - \sqrt{5}).$$

A few final remarks

- A state space version of Tolokonnikov's lemma. The corona problem viewed as a completion problem
- Continuous analogue [work in progress]. Role of Toeplitz operators is taken over by Wiener-Hopf integral operators.
- For a rational G we computed the optimal solution (when $m = 1$) and for the suboptimal case the maximum entropy solution (joint work in progress with Art Frazho and Sanne ter Horst).
- The corona-Bezout equation remains a source of inspiration. See the recent papers of Tavan Trent, Sergei Treil, and Sergei Treil and Brett Wick (several variables) and others.

Thank you for your attention!