Some problems in the theory of characters of finite groups

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ICMAT

April 2014

Let be G a finite group.

A complex **representation** of G is a group homomorphism

$$\mathcal{X} : G \to \mathrm{GL}(n, \mathbb{C})\,.$$

The integer $1 \leq n$ is the **degree** of the representation.

A representation $\mathcal{X}$ of G of degree n induces a linear action of G on $V = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n}$ via

$$(v_1, \ldots, v_n) \cdot g = (v_1, \ldots, v_n) \cdot \mathcal{X}(g)\,,$$

where $(v_1, \ldots, v_n) \in V$ and $g \in G$.

Conversely, if G acts on a finite dimensional $\mathbb{C}$-vector space V, then representing the linear maps

$$\begin{array}{rccc} f_g : & V & \longrightarrow & V \\ & v & \mapsto & v \cdot g \end{array} \qquad \forall g \in G \, ,$$

in matrix form with respect to a fixed base for V, induces a representation of G.

If $\mathcal{X}$ is a representation of G, then the map

$$\begin{array}{rccc} \chi : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \mathrm{tr}(\mathcal{X}(g)) \end{array}$$

is the **character** of $\mathcal{X}$.

There exist bijections:

$$\begin{matrix}\text{Isomorphism classes} \\ \text{of modules of } G\end{matrix} \Longleftrightarrow \begin{matrix}\text{Similarity classes of} \\ \text{representations of } G\end{matrix} \Longleftrightarrow \text{Characters of } G$$

If $\mathcal{X}, \mathcal{Y}$ are representations of G, then

$$\begin{pmatrix} \mathcal{X} & 0 \\ 0 & \mathcal{Y} \end{pmatrix} (g) = \begin{pmatrix} \mathcal{X}(g) & 0 \\ 0 & \mathcal{Y}(g) \end{pmatrix}, \quad \forall g \in G$$

defines a representation of G. Thus finite sums of characters are characters.

A character χ of G is **irreducible**, denoted $\chi \in \mathrm{Irr}(G)$, if χ cannot be expressed as the sum of two characters of G.

Theorem

Let G be a finite group. Then $|\mathrm{Irr}(G)| = |\mathrm{Cl}(G)|$.

Recall that for $g \in G$, its conjugacy class in G is

$$\mathrm{Cl}_G(g) = \{x^{-1}gx \mid x \in G\}, \text{ and}$$

$$\mathrm{Cl}(G) = \{\mathrm{Cl}_G(g) \mid g \in G\}.$$

Character Table of a Group

$$G = PSL(2, 11)$$
$$|G| = 660 = 2^2 \times 3 \times 5 \times 11$$

	1	**2**	**3**	**6**	$\mathbf{5_1}$	$\mathbf{5_2}$	$\mathbf{11_1}$	$\mathbf{11_2}$
χ_1	1	1	1	1	1	1	1	1
χ_2	10	2	1	-1	0	0	-1	-1
χ_3	10	-2	1	1	0	0	-1	-1
χ_4	11	-1	-1	-1	1	1	0	0
χ_5	12	0	0	0	α_1	α_2	1	1
χ_6	12	0	0	0	α_2	α_1	1	1
χ_7	5	1	-1	1	0	0	β	$\overline{\beta}$
χ_8	5	1	-1	1	0	0	$\overline{\beta}$	β

$\alpha_1 = (-1 + \sqrt{5})/2$
$\alpha_2 = (-1 - \sqrt{5})/2$

$\beta = (-1 + i\sqrt{11})/2$

What information on the structure of a finite group can be read from its character table?

order of G	$\|G\| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$
normal subgroups	$N = \cap_{\chi} \ker(\chi)$
center of G	$\mathbf{Z}(G) = \cap_{\chi \in \mathrm{Irr}(G)} \mathbf{Z}(\chi)$
derived subgroup of G	$G' = \cap_{\lambda(1)=1} \ker(\lambda)$
sizes of conjugacy classes	orthogonality relations
primes dividing orders of elements	G. Higman
$\vdots$	

In particular, whether the group is simple, nilpotent, solvable, abelian, prime power order...

There exist non-isomorphic groups with the same character table:

Character table of a nonabelian group of order 8

	$\mathbf{k_1}$	$\mathbf{k_2}$	$\mathbf{k_3}$	$\mathbf{k_4}$	$\mathbf{k_5}$
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	1	-1	1	1	-1
χ_4	1	1	-1	1	-1
χ_5	2	0	0	-2	0

Fields of Values

The **field of values** of a character χ of G is defined as

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(x) \mid x \in G) \, .$$

$$G = \langle a, b, c \,|\, a^2 = b^2 = c^8 = 1,\; b^a = bc^4,\; c^a = c^3,\; c^b = c \rangle$$
$$|G| = 32$$

	$\mathbf{1}$	$\mathbf{2_1}$	$\mathbf{2_2}$	$\mathbf{2_3}$	$\mathbf{2_4}$	$\mathbf{4_1}$	$\mathbf{4_2}$	$\mathbf{4_3}$	$\mathbf{4_4}$	$\mathbf{4_5}$	$\mathbf{8_1}$	$\mathbf{8_2}$	$\mathbf{8_3}$	$\mathbf{8_4}$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	
χ_2	1	-1	1	1	1	1	-1	1	1	1	-1	-1	-1	-1
χ_3	1	1	-1	1	1	1	1	-1	1	1	-1	-1	-1	-1
χ_4	1	-1	-1	1	1	1	-1	-1	1	1	1	1	1	1
χ_5	1	1	1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1
χ_6	1	-1	1	1	-1	1	1	-1	-1	-1	-1	1	-1	1
χ_7	1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	-1	1
χ_8	1	-1	-1	1	-1	1	1	1	-1	-1	1	-1	1	-1
χ_9	2	0	0	2	-2	-2	0	0	2	2	0	0	0	0
χ_{10}	2	0	0	2	2	-2	0	0	-2	-2	0	0	0	0
χ_{11}	2	0	0	-2	0	0	0	0	$2i$	$-2i$	$-\sqrt{2}$	$-\sqrt{2}i$	$\sqrt{2}$	$\sqrt{2}i$
χ_{12}	2	0	0	-2	0	0	0	0	$2i$	$-2i$	$\sqrt{2}$	$\sqrt{2}i$	$-\sqrt{2}$	$-\sqrt{2}i$
χ_{13}	2	0	0	-2	0	0	0	0	$-2i$	$2i$	$-\sqrt{2}$	$\sqrt{2}i$	$\sqrt{2}$	$-\sqrt{2}i$
χ_{14}	2	0	0	-2	0	0	0	0	$-2i$	$2i$	$\sqrt{2}$	$-\sqrt{2}i$	$-\sqrt{2}$	$\sqrt{2}i$

Theorem (Isaacs, Navarro, Sangroniz)

A finite 2-group G has 5 rational irreducible characters if and only if G is dihedral, semidihedral or generalized quaternion.

Corollary

There are finitely many 2-groups with 5 real irreducible characters.

Theorem (Sangroniz, T.)

There are finitely many 2-groups with 7 real irreducible characters.

Work in progress with A. Jaikin-Zapirain:

Improve the bound for the smallest odd number r such that there exist infinitely many finite 2-groups with r real irreducible characters, using techniques from the theory of pro-p groups.