Analytic Properties of Laguerre-type Orthogonal Polynomials

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1. Abstract

In the last years some attention has been paid to the so called nontrivial probability measures supported either on the real line or on the unit circle. The aim of this work is to provide a summary of analytic properties of orthogonal polynomials associated with the Uvarov's canonical linear spectral transformation of the Laguerre's classical measure (the gamma probability measure) supported on the positive real semiaxis.

2. Laguerre MOPS. Definitions and some Basic Properties

The Laguerre orthogonal polynomials are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_\alpha : \mathbb{P}\times\mathbb{P}\to\mathbb{R}$, with

$$\langle p,q\rangle_\alpha = \int_0^\infty pqx^\alpha e^{-x}dx,\ \alpha>-1,\ p,q\in\mathbb{P}.$$

- $\mathbb{P}$ linear space of polynomials with real coefficients.
- $\|p\|_\alpha^2 = \int_0^\infty |p(x)|^2 x^\alpha e^{-x}dx,\ \alpha>-1,\ p\in\mathbb{P}.$

Let $\{L_n^\alpha(x)\}_{n\geq 0}$ be the Monic Orthogonal Polynomial Sequence (**MOPS**) with respect to the above inner product. They satisfy the following three-term recurrence relation (**TTRR**)

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + \beta_n L_n^\alpha(x) + \gamma_n L_{n-1}^\alpha(x), \quad n\geq 1,$$

with $L_0^\alpha(x)=1,\ L_1^\alpha(x) = x-(\alpha+1),\ \beta_n = 2n+\alpha+1$ and $\gamma_n = n(n+\alpha)$.

Laguerre nth-Kernel

$$K_n(x,y) = \sum_{j=0}^n \frac{L_j^\alpha(y)L_j^\alpha(x)}{\|L_j^\alpha\|_\alpha^2},\ \forall n\in\mathbb{N}.$$

Christoffel-Darboux formula

$$K_n(x,y) = \frac{1}{\|L_n^\alpha\|_\alpha^2}\frac{L_{n+1}^\alpha(x)L_n^\alpha(y) - L_n^\alpha(x)L_{n+1}^\alpha(y)}{x-y},\ \forall n\in\mathbb{N}.$$

Confluent form of K_n

$$K_n(x,x) = \frac{\left[L_{n+1}^\alpha(x)\right]' L_n^\alpha(x) - \left[L_n^\alpha(x)\right]' L_{n+1}^\alpha(x)}{\|L_n^\alpha\|_\alpha^2},\ \forall n\in\mathbb{N}.$$

3. Asymptotic Properties of the Laguerre MOPS

Let $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$ be the Laguerre OPS with leading coefficients $(-1)^n/n!$. They satisfy the Perron's asymptotics formula on $\mathbb{C}\setminus\mathbb{R}_+$.

$$\widehat{L}_n^\alpha(x) = \tfrac{1}{2}\pi^{-1/2}e^{x/2}(-x)^{-\alpha/2-1/4}n^{\alpha/2-1/4}\exp\{2(-nx)^{1/2}\} \times\left\{\sum_{k=0}^{p-1} C_k(x)n^{-k/2} + \mathcal{O}(n^{-p/2})\right\}.$$

Here $C_k(x)$ is independent of n. This relation holds for x in the complex plane with a cut along the positive real semiaxis. ([1] Szegő, Theorem 8.22.3).

4. Uvarov Perturbation of the Laguerre Measure

Let $\{\widetilde{L}_n^\alpha(x)\}_{n\geq 0}$ be the MOPS associated with the measure

$$d\tilde{\mu} = x^\alpha e^{-x}dx + M\delta(x-a),$$

with $M\in\mathbb{R}_+$, $\delta(x-a)$ the Dirac delta function in $x=a$, and $a\notin[0,\infty)$.

The polynomials of the MOPS $\{\widetilde{L}_n^\alpha(x)\}_{n\geq 0}$ are called *Laguerre-type polynomials*, and they are orthogonal with respect to the inner product

$$\langle p,q\rangle = \int_0^\infty pqx^\alpha e^{-x}dx + Mp(a)q(a).$$

5. Uvarov Perturbed Laguerre MOPS

5.1. The Connection Formula.

Let $\{\widetilde{L}_n^\alpha(x)\}_{n\geq 0}$ be the Laguerre-type MOPS. Then we can expand $\widetilde{L}_n^\alpha(x)$ in terms of the classical Laguerre polynomials. Using some previous results from the recent work [2], we obtain the following *Connection Formula.*

Let $\{\widetilde{L}_n^\alpha\}_{n\geq 0}$ be the sequence of monic Laguerre-type MOPS, then

$$(x-a)\widetilde{L}_n^\alpha(x) = L_{n+1}^\alpha(x) + A_nL_n^\alpha(x) + B_nL_{n-1}^\alpha(x), \quad n\geq 1,$$

where

$$A_n = -a_n - \gamma_n\frac{b_n}{a_{n-1}},\ B_n = \gamma_n b_n,$$

with

$$a_n = \frac{L_{n+1}^\alpha(a)}{L_n^\alpha(a)}, \quad b_n = \frac{1+MK_n(a,a)}{1+MK_{n-1}(a,a)}, \quad n\geq 1.$$

5.2. The Hypergeometric Representation

Using the well known representation of $L_n^\alpha(x)$ as an hypergeometric function, and the former Connection Formula, we have the Hypergeometric Representation of Laguerre-type MOPS,

$$\widetilde{L}_n^\alpha(x) = \left(\frac{C_{n,\alpha}}{x-a}\right) {}_3F_3\left(1-e_1, 1-e_2, -n-1; -e_0, -e_1, \alpha+1; x\right),$$

with $C_{n,\alpha}$, e_0, e_1, and e_2 depending on n and α.

5.3. The Three Term Recurrence Formula

The measure $d\tilde{\mu}$ is standard, therefore the MOPS $\{\widetilde{L}_n^\alpha(x)\}_{n\geq 0}$ satisfies the following TTRR

$$x\widetilde{L}_n^\alpha(x) = \widetilde{L}_{n+1}^\alpha(x) + \widetilde{\beta}_n\widetilde{L}_n^\alpha(x) + \widetilde{\gamma}_n\widetilde{L}_{n-1}^\alpha(x), \quad n\geq 1,$$

where the recurrence coefficients (and their asymptotic behavior) are

$$\widetilde{\beta}_n = \beta_n + a_n\left(1-\tfrac{1}{b_n}\right) - a_{n-1}\left(1-\tfrac{1}{b_{n-1}}\right), \quad \frac{\widetilde{\beta}_n}{\beta_n} = 1 - \frac{\sqrt{|a|}}{2}n^{-3/2} + \mathcal{O}(n^{-5/2}),$$

$$\widetilde{\gamma}_n = \tfrac{b_n}{b_{n-1}}\gamma_n, \quad \frac{\widetilde{\gamma}_n}{\gamma_n} = 1 + 2\sqrt{|a|}n^{-1/2} + \mathcal{O}(n^{-1}).$$

5.4. Ladder Operators and 2nd Order Diferential Equation

Using the Connection Formula, and the former TTRR, we obtain the Laguerre-type Lowering ($\mathcal{L}_n$) and Raising ($\mathcal{R}_n$) Operators,

$$\mathcal{L}_n = x(x-a)^2D - nx^2 + D_nx + E_n,$$
$$\mathcal{R}_n = -\widetilde{\gamma}_n\mathcal{L}_n + \left(x-\widetilde{\beta}_n\right)u(x;n),$$

where $u(x;n)$ is the quadratic polynomial $u(x;n) = F_nx^2 + G_nx + H_n$, and the coefficients D_n, E_n, F_n, G_n, H_n depend on a_n, b_n, n and α. Combining the action of these *ladder operators*,

$$\mathcal{L}_{n+1}\left[\frac{1}{u(x;n)}\mathcal{R}_n\right]\left(\widetilde{L}_n^\alpha(x)\right) = u(x;n+1)\widetilde{L}_n^\alpha(x),$$

the second order differential equation

$$\mathcal{A}(x;n)\left[\widetilde{L}_n^\alpha(x)\right]'' + \mathcal{B}(x;n)\left[\widetilde{L}_n^\alpha(x)\right]' + \mathcal{C}(x;n)\widetilde{L}_n^\alpha(x) = 0$$

follows immediately.

5.5. Electrostatic Interpretation for the Zeros

Let $\tilde{x}_{n,k}, 1\leq k\leq n$ be the k-th zero of $\widetilde{L}_n^\alpha(x)$. We have an equilibrium position for these zeros under the presence of the external potential

$$V_{a<0}(x) = \ln(x-a)^2x^{\alpha+1}e^{-x} + \ln Q_n(x).$$

The first term represents a *long range potential* associated with the weight function, and the other one is a *short range potential*, corresponding to two unit charges located at the two real zeros of the quadratic polynomial $Q_n(x)$.

6. Asymptotic Properties of Laguerre-type MOPS

6.1. Outer Strong Asymptotics

Using the the Connection Formula and the Perron's Formula, we obtain

$$\frac{\widehat{\widetilde{L}}_n^\alpha(x)}{\widehat{L}_n^\alpha(x)} \sim \frac{\sqrt{-x}-\sqrt{|a|}}{\sqrt{-x}+\sqrt{|a|}}$$

with the convergence locally uniformly on $\mathbb{C}\setminus\mathbb{R}_+$.

6.2. Mehler-Heine Formula

Using the former proposition, we also prove that

$$\frac{\widehat{\widetilde{L}}_n^\alpha(x/n)}{n^\alpha} \sim \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} - 2\frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} = -x^{-\alpha/2}J_\alpha(2\sqrt{x}),$$

where J_α is the Bessel function of the first kind.

Remark: According to the Hurwitz theorem, the above result shows that the point $x=a$ attracts one zero of $\widetilde{L}_n^\alpha(x)$ for n large enough.

6.3. Rescaled Asymptotics

$$\lim_{n\to\infty}\frac{\widehat{\widetilde{L}}_n^\alpha(nx)}{\widehat{L}_n^\alpha(nx)} \rightrightarrows 1,$$

uniformly on compacts subsets of $\mathbb{C}\setminus[0,4]$.

References

[1] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. **23**, fourth edition, Providence RI, 1975.

[2] R. Álvarez-Nodarse, F. Marcellán, and J. Petronilho, *WKB Approximation and Krall-type Orthogonal Polynomials*, Acta Appl. Math. **54** (1998), 27–58.

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