# INVERSE FINITE-TYPE RELATIONS BETWEEN SEQUENCES OF POLYNOMIALS

by

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**Abstract**

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Let $\phi$ be a monic polynomial, with $\deg \phi = t \geq 0$. We say that there is a finite-type relation between two monic polynomial sequences $\{B_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ with respect to $\phi$, if there exists $(s, r) \in \mathbb{N}^2,\ r \geq s$, such that

$$\phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_\nu(x),\ n \geq s,\ \text{with } \lambda_{r,r-s} \neq 0. \quad (*)$$

The corresponding inverse finite-type relation of $(*)$ consists in a finite-type relation as follows:

$$\Omega_s^*(x;n)B_n(x) = \sum_{\nu=n-t}^{n+s} \theta^*_{n,\nu} Q_\nu(x),\ n \geq t,\ \text{with } \theta^*_{r+t,r} \neq 0,$$

where $\deg \Omega_s^*(x;n) = s,\ n \geq t$. When the orthogonality of the two previous sequences is assumed, the inverse finite-type relation is always possible [11]. This work essentially studies the case when only the sequence $\{B_n\}_{n\geq 0}$ is orthogonal. In fact, we find necessary and sufficient conditions leading to inverse finite-type relations. In particular, the structure relation characterizing semi-classical sequences is a special case of the general situation. Some examples will be analyzed.

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**Key words**: Finite-type relations, recurrence relations, orthogonal polynomials, semi-classical polynomials.

**Resumen**

Sea $\phi$ un polinomio mónico, con $\deg \phi = t \geq 0$. Decimos que hay relación de tipo finito entre dos sucesiones de polinomios mónicos $\{B_n\}_{n\geq 0}$ y $\{Q_n\}_{n\geq 0}$ con respecto a $\phi$, si existe $(s,r) \in \mathbb{N}^2,\ r \geq s$, tal que

$$\phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_\nu(x),\ n \geq s, \text{ with } \lambda_{r,r-s} \neq 0. \quad (*)$$

La correspondiente relación de tipo finito de $(*)$ consiste en una relación de tipo finito como sigue:

$$\Omega_s^*(x;n)B_n(x) = \sum_{\nu=n-t}^{n+s} \theta_{n,\nu}^* Q_\nu(x),\ n \geq t, \text{ with } \theta_{r+t,r}^* \neq 0,$$

donde $\deg \Omega_s^*(x;n) = s,\ n \geq t$. Cuando se supone la ortogonalidad de las dos sucesiones previas, la relación de tipo finito inversa siempre es posible [11]. En este trabajo se estudia el caso en que solo la sucesión $\{B_n\}_{n\geq 0}$ es ortogonal. De hecho, encontramos condiciones necesarias y suficientes que conducen a relaciones de tipo finito inversas. En particular, la la relación de estructura que caracteriza a las sucesiones semiclásicas es un caso especial de la situación general. Se estudian varios ejemplos.

**Palabras clave**: Relaciones de tipo finito, relaciones de recurrencia, polinomios ortogonales, polinomios semi clásicos.

## 1. Introduction and background

Let $\mathbb{P}$ be the linear space of complex polynomials in one variable and $\mathbb{P}'$ its topological dual space. We denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_n := \langle u, x^n\rangle,\ n \geq 0$, the moments of $u$ with respect to the polynomial sequence $\{x^n\}_{n\geq 0}$.

We will introduce some useful operations in $\mathbb{P}'$. For any linear functional $u$ and any polynomial $h$, let $Du = u'$ and $hu$ be the linear functionals defined by duality

$$\begin{aligned} \langle u', f\rangle &:= -\langle u, f'\rangle, && f \in \mathbb{P}, \\ \langle hu, f\rangle &:= \langle u, hf\rangle, && f,\ h \in \mathbb{P}. \end{aligned}$$

Let $\{B_n\}_{n\geq 0}$ be a monic polynomial sequence (MPS), $\deg B_n = n,\ n \geq 0$, and $\{u_n\}_{n\geq 0}$ its dual sequence, $u_n \in \mathbb{P}',\ n \geq 0$, defined by $\langle u_n, B_m\rangle := \delta_{n,m},\ n,\ m \geq 0$, where $\delta_{n,m}$ is the Kronecker symbol.

Let recall the following results [11].

**Lemma 1.1.** *For any $u \in \mathbb{P}'$ and any integer $m \geq 1$, the following statements are equivalent.*

i) $\langle u, B_{m-1}\rangle \neq 0, \quad \langle u, B_n\rangle = 0,\ n \geq m.$

ii) *There exist $\lambda_\nu \in \mathbb{C},\ 0 \leq \nu \leq m-1,\ \lambda_{m-1} \neq 0$, such that $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.*

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of the sequence $\{B_n^{[1]}\}_{n\geq 0}$, where $B_n^{[1]}(x) = (n+1)^{-1}B'_{n+1}(x)$, $n \geq 0$, satisfies

$$(u_n^{[1]})' = -(n+1)u_{n+1},\ n \geq 0. \quad (1.1)$$

**Definition 1.2.** The linear functional $u$ is said to be regular if there exists a monic polynomial sequence $\{B_n\}_{n\geq 0}$ such that

$$\langle u, B_nB_m\rangle = b_n\delta_{n,m}, \quad n,\ m \geq 0, \quad (1.2)$$

where

$$b_n = \langle u, B_n^2\rangle \neq 0,\ n \geq 0. \quad (1.3)$$

Then the sequence $\{B_n\}_{n\geq 0}$ is said to be orthogonal (MOPS) with respect to $u$.

As a straightforward consequence we get

- The linear functional can be represented by $u = (u)_0u_0$, and the following relations hold

$$u_n = b_n^{-1}B_nu,\ n \geq 0. \quad (1.4)$$

•• The sequence $\{B_n\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$\begin{aligned} B_{n+2}(x) &= (x-\beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geq 0, \\ B_1(x) &= x-\beta_0, \quad B_0(x) = 1, \end{aligned} \tag{1.5}$$

where $\gamma_{n+1} \neq 0,\ n \geq 0$ (see [4]).

In the sequel and under the assumption of the previous definition, we need to put

$$b^{\nu}_{n,m} = b_m^{-1}\langle u, x^{\nu} B_m B_n\rangle, \quad (n,\nu,m) \in \mathbb{N}^3. \tag{1.6}$$

In particular, one has

$$b^{\nu}_{n,m} = \begin{cases} 0, & \text{if } \nu+m<n, \quad 0 \leq m < n,\ \nu \geq 0, \\ (b_n/b_m), & \text{if } \nu = n-m, \quad 0 \leq m \leq n. \end{cases}$$

Let $\phi$ be a monic polynomial, with $\deg \phi = t \geq 0$. For any MPS $\{B_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ with dual sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ respectively, the following formula always holds

$$\phi(x)Q_n(x) = \sum_{\nu=0}^{n+t} \lambda_{n,\nu} B_\nu(x),\ n \geq 0, \tag{1.7}$$

where $\lambda_{n,\nu} = \langle u_\nu, \phi Q_n\rangle,\ 0 \leq \nu \leq n+t,\ n \geq 0$.

**Definition 1.3.** ([12]) If there exists an integer $s \geq 0$ such that

$$\phi(x)Q_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_\nu(x),\ n \geq s, \tag{1.8}$$

and

$$\exists\, r \geq s,\ \lambda_{r,r-s} \neq 0, \tag{1.9}$$

then, we shall say that (1.8) – (1.9) gives a finite-type relation between $\{B_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with respect to $\phi$.

When instead of (1.9), we take

$$\lambda_{n,n-s} \neq 0,\ n \geq s, \tag{1.9'}$$

we shall say that (1.8) – (1.9′) is a strictly finite-type relation.

The corresponding inverse finite-type relation of (1.8) – (1.9) consists in establishing, whenever it is possible, a finite-type relation between $\{Q_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$, as follows

$$\Omega^*_s(x;n)B_n(x) = \sum_{\nu=n-t}^{n+s} \theta^*_{n,\nu} Q_\nu(x),\ n \geq t, \tag{1.10}$$

$\theta^*_{r+t,r} \neq 0$, where $\{\Omega^*_s(x;n)\}_{n\geq t}$ is a MPS, $\deg \Omega^*_s(x;n) = s,\ n \geq t$, and

$$(\theta^*_{n,\nu})_{\nu=n-t}^{n+s}, n \geq t, \tag{1.11}$$

a system of complex numbers (SCN), with $\theta^*_{n,n+s} = 1,\ n \geq t$.

When both two sequences are orthogonal, the inverse relation is always possible. In this case, the polynomials $\Omega^*_s(x;n),\ n \geq 0$, are independent of $n$, (see [12], Proposition 2.4). As a current example, we can mention the two structure relations characterizing the classical polynomials, (**Hermite**, **Laguerre**, **Bessel**, **Jacobi**, see [11]), which could solely be two inverse finite-type relations.

In other studies, we find several situations where one of the two sequences is orthogonal. For example, the structure relations characterizing semi-classical sequences associated with Hahn's operators $L_{q,\omega}$, with parameters $q$ and $\omega$, [9]. The Coherent pairs and Diagonal sequences are also examples of finite type-relations [7, 12, 13, 14]. But the inverse relations corresponding to other finite-type relations are not yet considered.

The paper essentially gives a necessary and sufficient condition allowing the existence of the inverse finite-type relations when the orthogonality of the sequence $\{B_n\}_{n\geq 0}$ is assumed. From now on, it would be necessary to study the case where the sequence $\{Q_n\}_{n\geq 0}$ is orthogonal. It would be very useful to deal with many other situations like General Coherent pairs, see [6, 8] in the framework of Sobolev inner products.

## 2. A basic result

We use this section to introduce some auxiliary result for the proof of the main theorem in section 3.

**Lemma 2.1.** *Suppose $\{B_n\}_{n\geq 0}$ is a MOPS and $\{Q_n\}_{n\geq 0}$ fulfils (1.8) – (1.9), where $t = 0$ and $s \geq 1$. For any SCN $(\theta_{n,\nu})_{\nu=n}^{n+s}$, $n \geq 0$, where $\theta_{n,n+s} = 1$, $n \geq 0$, and $\theta_{r,r} \neq 0$, there exist a unique MPS $\{\Omega_s(x;n)\}_{n\geq 0}$, $\deg \Omega_s(x;n) = s$, $n \geq 0$, and a SCN $(\zeta^{[0]}_{n,\nu})_{\nu=n-s}^{n-1}$, $n \geq 0$, such that*

$$\begin{aligned} \sum_{\nu=n}^{n+s} \theta_{n,\nu} Q_\nu(x) &= \sum_{i=n-s}^{n+s} \theta^{[0]}_{n,i} B_i(x) \\ &= \Omega_s(x;n)B_n(x) + \sum_{\nu=n-s}^{n-1} \zeta^{[0]}_{n,\nu} B_\nu(x),\ n \geq 0\,, \end{aligned} \tag{2.1}$$

*where*

$$\theta^{[0]}_{n,i} = \sum_{\nu=\max(n,i)}^{\min(n,i)+s} \theta_{n,\nu}\lambda_{\nu,i}, \quad n-s \le i \le n+s\,, n \ge 0\,, \tag{2.2}$$

$$\theta^{[0]}_{r,r-s} = \theta_{r,r}\lambda_{r,r-s} \neq 0, \tag{2.3}$$

$$\sum_{\nu=n}^{m+s} \theta_{n,\nu}\lambda_{\nu,m} = b_m^{-1}\langle u, \Omega_s(x;n)B_nB_m\rangle + \zeta^{[0]}_{n,m}\,, \quad n-s \le m \le n-1, n \ge 0, \tag{2.4}$$

$$\sum_{\nu=m}^{n+s} \theta_{n,\nu}\lambda_{\nu,m} = b_m^{-1}\langle u, \Omega_s(x;n)B_nB_m\rangle, \quad n \le m \le n+s-1, \ \ n \ge 0. \tag{2.5}$$

*Proof.* Let $(\theta_{n,\nu})_{\nu=n}^{n+s}$, $n \ge 0$, where $\theta_{n,n+s} = 1$, $n \ge 0$, and $\theta_{r,r} \neq 0$, be a SCN. From (1.8) – (1.9), with $t = 0$ and $s \ge 1$, we get

$$\sum_{\nu=n}^{n+s} \theta_{n,\nu}Q_\nu(x) = \sum_{\nu=n}^{n+s}\theta_{n,\nu}\sum_{i=\nu-s}^{\nu}\lambda_{\nu,i}B_i(x) = \sum_{\nu=n}^{n+s}\theta_{n,\nu}\sum_{i=n-s}^{n+s}\chi_{i,\nu}\lambda_{\nu,i}\, B_i(x), \ n \ge 0,$$

where, for each pair of integers $(i, \nu)$ such that $n-s \le i \le n+s$ and $n \le \nu \le n+s$, we took

$$\chi_{i,\nu} = \begin{cases} 1, & \text{if } \ \nu - s \le i \le \nu, \\ 0, & \text{otherwise.}\end{cases}$$

The permutation of these two sums yields

$$\sum_{\nu=n}^{n+s} \theta_{n,\nu}Q_\nu(x) = \sum_{i=n-s}^{n+s}\theta^{[0]}_{n,i}B_i(x),$$

where

$$\theta^{[0]}_{n,i} = \sum_{\nu=\max(n,i)}^{\min(i+s,n+s)} \theta_{n,\nu}\lambda_{\nu,i}\,, \quad n-s \le i \le n+s, \ n \ge 0\,, \tag{2.6}$$

$$\theta^{[0]}_{r,r-s} = \ \theta_{r,r}\lambda_{r,r-s} \neq 0.$$

Hence, (2.2) and (2.3) are valid.

The Euclidean division by $B_n(x)$ in the right hand side in (2.6) gives

$$\sum_{i=n-s}^{n+s}\theta^{[0]}_{n,i}B_i(x) = \Omega_s(x;n)B_n(x) + \sum_{\nu=0}^{n-1}\zeta^{[0]}_{n,\nu}B_\nu(x), \ n \ge 0.$$

Multiplying by $B_m(x)$ and using the orthogonality of $\{B_n\}_{n\ge0}$,

$$\sum_{i=n-s}^{n+s}\theta^{[0]}_{n,i}\delta_{m,i} = b_m^{-1}\langle u, \Omega_s(x;n)B_nB_m\rangle + \sum_{\nu=0}^{n-1}\zeta^{[0]}_{n,\nu}\delta_{m,\nu}.$$

In particular, for $0 \le m \le n-s-1$ and $n \ge s+1$, it follows that $\zeta^{[0]}_{n,m} = 0$. Hence, (2.1) holds. Moreover, for $n-s \le m \le n-1$ and $n \ge s$, we recover (2.4).

Finally, for $n \le m \le n+s-1$ and $n \ge 0$, we deduce (2.5). ☐

**Proposition 2.2.** *Assume $\{B_n\}_{n\ge0}$ is a MOPS and $\{Q_n\}_{n\ge0}$ fulfils (1.8) – (1.9), with $t \ge 1$. For any SCN $(\theta_{n,\nu})_{\nu=n-t}^{n+s}$, $n \ge 0$, where $\theta_{n,n+s} = 1$, $n \ge 0$ and $\theta_{r+t,r} \neq 0$, there exist a unique MPS $\{\Omega_{s+t}(x;n)\}_{n\ge0}$, where $\deg\Omega_{s+t}(x;n) = s+t$, $n \ge 0$, and a SCN $(\zeta^{[t]}_{n,\nu})_{\nu=n-s-t}^{n-1}$, $n \ge 0$, such that for every integer $n \ge 0$*

$$\phi(x)\sum_{\nu=n-t}^{n+s}\theta_{n,\nu}Q_\nu(x) = \sum_{i=n-s-t}^{n+s+t}\theta^{[t]}_{n,i}B_i(x) = \Omega_{s+t}(x;n)B_n(x) + \sum_{\nu=n-s-t}^{n-1}\zeta^{[t]}_{n,\nu}B_\nu(x), \tag{2.7}$$

*where*

$$\theta^{[t]}_{n,i} = \sum_{\nu=\max(n,i)}^{\min(n,i)+s+t}\theta_{n,\nu-t}\lambda_{\nu-t,i}\,, \quad n-s-t \le i \le n+s+t, \tag{2.8}$$

$$\theta^{[t]}_{r+t,r-s} = \theta_{r+t,r}\lambda_{r,r-s} \neq 0, \tag{2.9}$$

$$\sum_{\nu=n}^{m+s+t}\theta_{n,\nu-t}\lambda_{\nu-t,m} = b_m^{-1}\langle u, \Omega_{s+t}(x;n)B_nB_m\rangle + \zeta^{[t]}_{n,m}\,, \quad n-s-t \le m \le n-1, \tag{2.10}$$

$$\sum_{\nu=m}^{n+s+t}\theta_{n,\nu-t}\lambda_{\nu-t,m} = b_m^{-1}\langle u, \Omega_{s+t}(x;n)B_nB_m\rangle\,, \quad n \le m \le n+s+t-1. \tag{2.11}$$

*Proof.* The case $t = 0$ was analyzed in Lemma 2.1. Let us take $t \ge 1$. Consider the MPS $\{P_n\}_{n\ge t}$ defined by

$$P_{n+t}(x) = \phi(x)Q_n(x), \ n \ge 0, \tag{2.12}$$

From (1.8) – (1.9), we have

$$P_n(x) = \sum_{\nu=n-t-s}^{n} \tilde{\lambda}_{n,\nu} B_\nu(x),\ n \geq t,$$

where $\tilde{\lambda}_{n,\nu} = \lambda_{n-t,\nu},\ n-t-s \leq \nu \leq n,\ n \geq t$, and $\tilde{\lambda}_{t+r,r-s} \neq 0$. Now, let $(\theta_{n,\nu})_{\nu=n-t}^{n+s}$, $n \geq 0$, where $\theta_{n,n+s} = 1,\ n \geq 0$, and $\theta_{r+t,r} \neq 0$, be a SCN. One has

$$\begin{aligned}\phi(x) \sum_{\nu=n-t}^{n+s} \theta_{n,\nu} Q_\nu(x) &= \sum_{\nu=n-t}^{n+s} \theta_{n,\nu} P_{\nu+t}(x) \\ &= \sum_{\nu=n}^{n+t+s} \tilde{\theta}_{n,\nu} P_\nu(x),\ n \geq 0, \quad (2.13)\end{aligned}$$

where $\tilde{\theta}_{n,\nu} = \theta_{n,\nu-t},\ n \leq \nu \leq n+t+s,\ n \geq 0$. Obviously, $(\tilde{\theta}_{n,\nu})_{\nu=n}^{n+s+t}$, $n \geq 0$, is a SCN such that

$$\tilde{\theta}_{n,n+s+t} = \theta_{n,n+s} = 1,\ n \geq 0, \quad \tilde{\theta}_{r+t,r+t} = \theta_{r+t,r} \neq 0.$$

But from Lemma 2.1, there exist a unique MPS $\{\Omega_{t+s}(x;n)\}_{n\geq 0}$ and a SCN $(\zeta_{n,\nu}^{[t]})_{\nu=n-t-s}^{n-1}$, $n \geq 0$, such that

$$\begin{aligned}\sum_{\nu=n}^{n+t+s} \tilde{\theta}_{n,\nu} P_\nu(x) &= \sum_{i=n-t-s}^{n+t+s} \theta_{n,i}^{[t]} B_i(x) \\ &= \Omega_{t+s}(x;n) B_n(x) + \sum_{\nu=n-t-s}^{n-1} \zeta_{n,\nu}^{[t]} B_\nu(x), \quad (2.14)\end{aligned}$$

for every integer $n \geq 0$, where

$$\theta_{n,i}^{[t]} = \sum_{\nu=\max(n,i)}^{\min(n,i)+t+s} \tilde{\theta}_{n,\nu} \tilde{\lambda}_{\nu,i},\ n-t-s \leq i \leq n+t+s,$$

$$\theta_{r+t,r-s}^{[t]} = \tilde{\theta}_{r+t,r+t} \tilde{\lambda}_{r+t,r-s} \neq 0,$$

$$\sum_{\nu=n}^{m+t+s} \tilde{\theta}_{n,\nu} \tilde{\lambda}_{\nu,m} = b_m^{-1} \langle u, \Omega_{t+s}(x;n) B_n B_m \rangle + \zeta_{n,m}^{[t]},$$
$$n-t-s \leq m \leq n-1,$$

$$\sum_{\nu=m}^{n+t+s} \tilde{\theta}_{n,\nu} \tilde{\lambda}_{\nu,m} = b_m^{-1} \langle u, \Omega_{t+s}(x;n) B_n B_m \rangle,$$
$$n \leq m \leq n+t+s-1.$$

Finally, by using (2.13), (2.14), and taking into account the expressions of $\tilde{\lambda}_{n,\nu}$ and $\tilde{\theta}_{n,\nu}$, we find the desired results. □

## 3. **A matrix approach and main results**

In this section, we will work under the assumptions of the Proposition 2.2 and we will give a matrix approach to our problem.

If $\Omega_{t+s}(x;n) = \sum_{\nu=0}^{t+s} v_{n,\nu} x^\nu,\ n \geq 0$, where $v_{n,t+s} = 1$, then relation (2.10) reads

$$\sum_{\nu=n}^{m+s+t} \lambda_{\nu-t,m} \theta_{n,\nu-t} = \sum_{\nu=0}^{t+s-1} b_{n,m}^{\nu} v_{n,\nu} + \zeta_{n,m}^{[t]} + b_{n,m}^{s+t},$$
$$n-s-t \leq m \leq n-1,$$

or, alternatively,

$$\begin{aligned}&\sum_{j=1}^{m+s+t-n+1} \lambda_{j+n-t-1,m} \theta_{n,j+n-t-1} \\ &= \sum_{j=1}^{t+s} b_{n,m}^{j-1} v_{n,j-1} + \zeta_{n,m}^{[t]} + b_{n,m}^{s+t},\end{aligned}$$

for every $n-s-t \leq m \leq n-1$..

Replacing $m$ by $i+n-s-t-1$, we get

$$\begin{aligned}\sum_{j=1}^{i} k_{i,j}^n \Theta_{n,j} = \sum_{j=1}^{t+s} t_{i,j}^n V_{n,j} + \zeta_{n,i+n-s-t-1}^{[t]} + \\ b_{n,i+n-s-t-1}^{s+t},\ 1 \leq i \leq s+t,\end{aligned}$$

where for $i,j = 1,2,\ldots,s+t$,

$$k_{i,j}^n = \begin{cases} \lambda_{j+n-t-1,i+n-s-t-1}, & 1 \leq j \leq i \\ 0, & \text{otherwise}, \end{cases}$$

and $t_{i,j}^n = b_{n,i+n-s-t-1}^{j-1}$,

$$\Theta_{n,j} = \theta_{n,j+n-t-1}, \text{ and } \quad V_{n,j} = v_{n,j-1}.$$

So we can write it as

$$\mathbf{K}_n \Theta_n = \mathbf{T}_n V_n + W_n + E_n,\ n \geq 0, \quad (3.1)$$

where

$$\mathbf{K}_n = \left(k_{i,j}^n\right)_{1\leq i,j\leq s+t}, \quad \mathbf{T}_n = \left(t_{i,j}^n\right)_{1\leq i,j\leq s+t},$$

$$\Theta_n = \left(\Theta_{n,1}, \Theta_{n,2}, \ldots, \Theta_{n,s+t}\right)^T,$$

$$V_n = \left(V_{n,1}, V_{n,2}, \ldots, V_{n,s+t}\right)^T,$$

$$W_n = \left(\zeta_{n,n-s-t}^{[t]}, \zeta_{n,n+1-s-t}^{[t]}, \ldots, \zeta_{n,n-1}^{[t]}\right)^T, \quad \text{and}$$

$$E_n = \left(b_{n,n-s-t}^{s+t}, b_{n,n+1-s-t}^{s+t}, \ldots, b_{n,n-1}^{s+t}\right)^T.$$

In the same way, using $\theta_{n,n+s} = 1$, (2.11) can be written as

$$\sum_{\nu=m}^{n+s+t-1} \lambda_{\nu-t,m}\theta_{n,\nu-t} = \sum_{\nu=0}^{t+s-1} b_{n,m}^{\nu} v_{n,\nu} + b_{n,m}^{s+t} - \lambda_{n+s,m},$$
$$n \leq m \leq n+s+t-1\,,$$

or, equivalently,

$$\sum_{j=m-n+1}^{s+t} \lambda_{j+n-t-1,m}\theta_{n,j+n-t-1} =$$
$$\sum_{j=1}^{t+s} b_{n,m}^{j-1} v_{n,j-1} + b_{n,m}^{s+t} - \lambda_{n+s,m},$$

for every $n \leq m \leq n+s+t-1$. Replacing $m$ by $i+n-1$, we get

$$\sum_{j=i-1}^{t+s} m_{i,j}^{n}\Theta_{n,j} = \sum_{j=1}^{t+s} s_{i,j}^{n} V_{n,j} + b_{n,i+n-1}^{s+t} - \lambda_{n+s,i+n-1},$$
$$1 \leq i \leq s+t,$$

where for $i, j = 1, 2, ..., s+t$,

$$m_{i,j}^{n} := \begin{cases} \lambda_{j+n-t-1,i+n-1},\ 1 \leq i \leq j \\ 0, \quad \text{otherwise}\,, \end{cases}$$

$$s_{i,j}^{n} := \begin{cases} b_{n,i+n-1}^{j-1},\ 1 \leq i \leq j \\ 0, \quad \text{otherwise}. \end{cases}$$

Thus, we can use the matrix representation

$$\mathbf{M}_n\Theta_n = \mathbf{S}_n V_n + F_n,\ n \geq 0, \tag{3.2}$$

where

$$\mathbf{M}_n = \left(m_{i,j}^{n}\right)_{1\leq i,j\leq s+t} \quad , \quad \mathbf{S}_n = \left(s_{i,j}^{n}\right)_{1\leq i,j\leq s+t},$$

and

$$F_n = \big(b_{n,n}^{s+t} - \lambda_{n+s,n}, b_{n,n+1}^{s+t} - \lambda_{n+s,n+1},$$
$$\ldots, b_{n,n+s+t-1}^{s+t} - \lambda_{n+s,n+s+t-1}\big)^{T}.$$

Our data are $\Theta_n$, $E_n$, $F_n$, $\mathbf{M}_n$, $\mathbf{S}_n$, $\mathbf{T}_n$, $\mathbf{K}_n$ and our unknowns are $V_n$ and $W_n$.

From (3.2), we get

$$V_n = \mathbf{S}_n^{-1}(\mathbf{M}_n\Theta_n - F_n).(3.3)$$

Thus, substituting in (3.1) we get $\mathbf{K}_n\Theta_n - W_n - E_n = \mathbf{T}_n\mathbf{S}_n^{-1}(\mathbf{M}_n\Theta_n - F_n)$, i.e,

$$W_n = (\mathbf{K}_n - \mathbf{T}_n\mathbf{S}_n^{-1}\mathbf{M}_n)\Theta_n + \mathbf{T}_n\mathbf{S}_n^{-1}F_n - E_n.$$

As a consequence, for every choice of $\Theta_n$, we get $W_n$. From (3.3), we deduce $V_n$.

On the other hand, there exists a one-to-one correspondence between the vectors $W_n$ and $\Theta_n$ if and only if the matrix of dimension $s+t$, $\mathbf{K}_n - \mathbf{T}_n\mathbf{S}_n^{-1}\mathbf{M}_n$, is nonsingular.

Under such a condition, there exists a unique choice for $\Theta_n$ such that $W_n = 0$. Thus, we get

$$\Theta_n = \left(\mathbf{K}_n - \mathbf{T}_n\mathbf{S}_n^{-1}\mathbf{M}_n\right)^{-1}(E_n - \mathbf{T}_n\mathbf{S}_n^{-1}F_n),$$

and from (3.3), $V_n = \mathbf{S}_n^{-1}\mathbf{M}_n\Theta_n - \mathbf{S}_n^{-1}F_n$. Then,

$$V_n = \left(\mathbf{K}_n\mathbf{M}_n^{-1}\mathbf{S}_n - \mathbf{T}_n\right)^{-1} E_n$$
$$- \left[\left(\mathbf{K}_n\mathbf{M}_n^{-1}\mathbf{S}_n - \mathbf{T}_n\right)^{-1}\mathbf{T}_n + \mathbf{I}_{s+t}\right]\mathbf{S}_n^{-1}F_n,$$

where $\mathbf{I}_{s+t}$ is the unit matrix. Hence, the polynomial $\Omega_{s+t}(x;n)$ is explicitly given.

Let introduce

$$\Delta_n(t,s) = \det(\mathbf{K}_n - \mathbf{T}_n\mathbf{S}_n^{-1}\mathbf{M}_n),\ n \geq 0.$$

Thus, we have proved the following result

**Proposition 3.1.** *Assume $\{B_n\}_{n\geq 0}$ is a MOPS and $\{Q_n\}_{n\geq 0}$ fulfils (1.8)−(1.9). For a fixed integer $p \geq t+1$, the following statements are equivalent.*

i) $\Delta_n(t,s) \neq 0,\ n \geq p$.
ii) *There exist a unique SCN $(\theta_{n,\nu}^{*})_{\nu=n-t}^{n+s}$, $n \geq p$, with $\theta_{n,n+s}^{*} = 1$, $n \geq p$, and $\theta_{r+t,r}^{*} \neq 0$, if $p \leq r+t$, and a unique MPS $\{\Omega_{s+t}^{*}(x;n)\}_{n\geq p}$, $\deg \Omega_{s+t}^{*}(x;n) = s+t$, $n \geq p$, such that*

$$\Omega_{s+t}^{*}(x;n)B_n(x) = \phi(x)\sum_{\nu=n-t}^{n+s}\theta_{n,\nu}^{*}Q_\nu(x)\,, \tag{3.4}$$

*for $n \geq p$.*

Our main result is

**Theorem 3.2.** *Let $\{B_n\}_{n\geq 0}$ be a MOPS and $\{Q_n\}_{n\geq 0}$ be the MPS satisfying (1.8) − (1.9). For each fixed integer $p \geq t+1$, if we suppose that $\phi(x)$ and $B_n(x)$ are coprime for every $n \geq p$, then the following statements are equivalent.*

i) $\Delta_n(t,s) \neq 0,\ n \geq p$.
ii) *There exist a unique SCN $(\theta_{n,\nu}^{*})_{\nu=n-t}^{n+s}$, $n \geq p$, where $\theta_{n,n+s}^{*} = 1$, $n \geq p$, and $\theta_{r+t,r}^{*} \neq 0$ if $p \leq r+t$, and a unique MPS $\{\Omega_{s}^{*}(x;n)\}_{n\geq p}$, $\deg \Omega_{s}^{*}(x;n) = s$, $n \geq p$, such that*

$$\Omega_{s}^{*}(x;n)B_n(x) = \sum_{\nu=n-t}^{n+s}\theta_{n,\nu}^{*}Q_\nu(x),\ n \geq p\,. \tag{3.5}$$

*Proof.* Taking into account $\phi(x)$ and $B_n(x)$ are coprime for every $n \geq p$, from (3.4) we deduce that $\phi$ divides $\Omega^*_{s+t}(x;n)$, $n \geq p$. So, $\Omega^*_{s+t}(x;n) = \phi(x)\Omega^*_s(x;n)$, $n \geq p$. Hence, the desired result follows. □

The orthogonal polynomial sequence $\{B_n\}_{n\geq 0}$ and the polynomial sequence $\{Q_n\}_{n\geq 0}$ can be related by a general finite-type relation (see [1]). It reads as follows

$$F(Q_n, ..., Q_{n-l}) = G(B_n, ..., B_{n-s}),$$

where $F$ and $G$ are fixed functions.

When $F$ and $G$ are linear functions, some situations dealing with the inverse problem have been analyzed in [1,2]. There, necessary and sufficient conditions in order to $\{Q_n\}_{n\geq 0}$ be orthogonal are obtained.

This kind of linear relations reads as follows.

There exists $(l, s, r) \in \mathbb{N}^3$, with $r \geq \tilde{s} = \max(l, s)$ such that

$$\sum_{\nu=n-l}^{n} \zeta_{n,\nu} Q_\nu(x) = \sum_{\nu=n-s}^{n} \lambda_{n,\nu} B_\nu(x),\ n \geq \tilde{s}, \tag{3.6}$$

with $\zeta_{r,r-l}\lambda_{r,r-s} \neq 0$. Here, $\zeta_{n,n} = \lambda_{n,n} = 1$, $n \geq \tilde{s}$.

More recently, in [5], **A. M. Delgado** and **F. Marcellán** exhaustively describe all the set of pairs of quasi-definite (regular) linear functionals such that their corresponding sequences of monic polynomials $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are related by a differential expression

$$P_n(x) + s_n P_{n-1}(x) = R^{[1]}_n(x) + t_n R^{[1]}_{n-1}(x),\ n \geq 1,$$

where $t_n \neq 0$, for every $n \geq 1$, and with the technical condition $t_1 \neq s_1$.

Notice that in general $\{R^{[1]}_n\}_{n\geq 0}$ is not a MOPS.

In the same context of our contribution, we show that the corresponding inverse finite-type relation between two sequences satisfying (3.6) is possible under certain conditions.

Indeed, let consider the MPS $\{C_n\}_{n\geq \tilde{s}}$ given by

$$C_n(x) = \sum_{\nu=n-s}^{n} \lambda_{n,\nu} B_\nu(x), \qquad n \geq \tilde{s}\,. \tag{3.7}$$

With the finite-type relation between the sequences $\{C_n\}_{n\geq \tilde{s}}$ and $\{B_n\}_{n\geq \tilde{s}}$, we can associate the determinants $\Delta_n(0, s)$, $n \geq \tilde{s}$. So, we have.

**Corollary 3.3.** *Let $\{B_n\}_{n\geq 0}$ be a MOPS and $\{Q_n\}_{n\geq 0}$ be the MPS satisfying (3.6). For each fixed integer $p \geq \max(s, l, 1)$, if $\Delta_n(0, s) \neq 0$, $n \geq p$, then there exist a unique SCN $(\zeta^*_{n,\nu})^{n+s}_{\nu=n-l}$, $n \geq p$, where $\zeta^*_{n,n+s} = 1$, $n \geq p$, and $\zeta^*_{r,r-l} \neq 0$ if $p \leq r$, and a unique MPS $\{\Omega^*_s(x;n)\}_{n\geq p}$, $\deg \Omega^*_s(x;n) = s$, $n \geq p$, such that*

$$\Omega^*_s(x;n)B_n(x) = \sum_{\nu=n-l}^{n+s} \zeta^*_{n,\nu} Q_\nu(x),\ n \geq p. \tag{3.8}$$

*Proof.* From Theorem 3.2, with $t = 0$, there exists the corresponding inverse finite-type relation associated with the relation (3.7) if and only if $\Delta_n(0, s) \neq 0$, $n \geq p$. Equivalently, there exist a unique SCN $(\theta^*_{n,\nu})^{n+s}_{\nu=n}$, $n \geq p$, where $\theta^*_{n,n+s} = 1$, $n \geq p$, and $\theta^*_{r,r} \neq 0$, if $p \leq r$, and a unique MPS $\{\Omega^*_s(x;n)\}_{n\geq p}$, $\deg \Omega^*_s(x;n) = s$, $n \geq p$, such that

$$\Omega^*_s(x;n)B_n(x) = \sum_{\nu=n}^{n+s} \theta^*_{n,\nu} C_\nu(x),\ n \geq p\,. \tag{3.9}$$

But from (3.6) and (3.7), the above expression becomes

$$\begin{aligned}\Omega^*_s(x;n)B_n(x) &= \sum_{\nu=n}^{n+s} \theta^*_{n,\nu} \sum_{i=\nu-l}^{\nu} \zeta_{\nu,i} Q_i(x) \\ &= \sum_{\nu=n}^{n+s} \theta^*_{n,\nu} \sum_{i=n-l}^{n+s} \tilde{\chi}_{i,\nu}\zeta_{\nu,i}\, Q_i(x),\ n \geq p,\end{aligned}$$

where, for each pair of integers $(i, \nu)$ such that $n - l \leq i \leq n + s$ and $n \leq \nu \leq n + s$, we took

$$\tilde{\chi}_{i,\nu} = \begin{cases} 1, & \text{if,}\ \ \nu - l \leq i \leq \nu, \\ 0, & \text{otherwise.} \end{cases}$$

The permutation inside these two sums yields

$$\Omega^*_s(x;n)B_n(x) = \sum_{i=n-l}^{n+s} \zeta^*_{n,i} Q_i(x),$$

where

$$\zeta^*_{n,i} = \sum_{\nu=\max(n,i)}^{\min(i+l,n+s)} \theta^*_{n,\nu}\zeta_{\nu,i}\,,$$

if $n - l \leq i \leq n + s$, $n \geq p$, and

$$\zeta^*_{r,r-l} = \theta^*_{r,r}\zeta_{r,r-l} \neq 0,$$

if $p \leq r$. □

## 4. The case: $(\mathbf{t}, \mathbf{s}) = (\mathbf{0}, \mathbf{1})$

Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional $u$ and satisfying the three-term recurrence relation (1.5).

Consider the following finite-type relation between $\{B_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with index $s=1$, with respect to $\phi(x)=1$,

$$Q_n(x) = B_n(x) + \lambda_{n,n-1}B_{n-1}(x),\ n\geq 0, \tag{4.1}$$

$$\exists r\geq 1, \qquad \lambda_{r,r-1}\neq 0. \tag{4.2}$$

From Lemma 2.1, for every set of complex numbers, $\theta_{n,n}$, $n\geq 0$, with $\theta_{r,r}\neq 0$, there exists a unique MPS $\{\Omega_1(x;n)\}_{n\geq 0}$, where $\Omega_1(x;n)=x+v_{n,0}$, $n\geq 0$, and a unique set of complex numbers, $\zeta^{[0]}_{n,n-1}$, $n\geq 0$, such that

$$Q_{n+1}(x)+\theta_{n,n}Q_n(x) = \Omega_1(x;n)B_n(x) + \zeta^{[0]}_{n,n-1}B_{n-1}(x),\ n\geq 0, \tag{4.3}$$

where

$$\begin{cases} \lambda_{n,n-1}\theta_{n,n} &= \zeta^{[0]}_{n,n-1}+\gamma_n,\ n\geq 1, \\ \theta_{n,n}-v_{n,0} &= -\lambda_{n+1,n}+\beta_n,\ n\geq 0. \end{cases} \tag{4.4}$$

The determinants associated with $(4.1)-(4.2)$ are given by

$$\Delta_0(0,1)=0, \quad \Delta_n(0,1)=\lambda_{n,n-1},\ n\geq 1, \tag{4.5}$$

where $\Delta_r(0,1)=\lambda_{r,r-1}\neq 0$. As a consequence of Theorem 3.2, when $t=0$ and $s=1$, we have the following result

**Proposition 4.1.** *Let $\{B_n\}_{n\geq 0}$ be a MOPS and $\{Q_n\}_{n\geq 0}$ be the MPS satisfying $(4.1)-(4.2)$. For every fixed integer $p\geq 1$, the following statements are equivalent*

i) $\lambda_{n,n-1}\neq 0,\ n\geq p$.
ii) *There exist a unique set of complex numbers $\theta^*_{n,n}\neq 0$, $n\geq p$, and a unique MPS $\{\Omega^*_1(x;n)\}_{n\geq p}$, $\deg\Omega^*_1(x;n)=1$, $n\geq p$, such that*

$$\Omega^*_1(x;n)B_n(x) = Q_{n+1}(x)+\theta^*_{n,n}Q_n(x),\ n\geq p\,. \tag{4.6}$$

We write

$$\theta^*_{n,n} = \frac{\gamma_n}{\lambda_{n,n-1}},\ n\geq p, \tag{4.7}$$

$$\Omega^*_1(x;n) = x+v^*_{n,0},$$

where

$$v^*_{n,0} = \frac{\gamma_n}{\lambda_{n,n-1}}+\lambda_{n+1,n}-\beta_n,\ n\geq p\,. \tag{4.8}$$

*Example.* In order to illustrate the result of Proposition 4.1, we study the structure relation characterizing a semi-classical polynomial sequence, $\{B_n\}_{n\geq 0}$, orthogonal with respect to the linear functional $u$ solution of the functional equation

$$u'+\psi u = 0, (4.9)$$

where $\psi(x)=-ix^2+2x-i(\alpha-1)$ and with regularity condition $\alpha\notin\bigcup_{n\geq 0}E_n$, where $E_0=\{\alpha\in\mathbb{C}\ :\ F(\alpha)=0\}$, $F(\alpha)=\int_{-\infty}^{+\infty}e^{\frac{ix^3}{3}-x^2+i(\alpha-1)x}dx$, and for each integer $n\geq 1$, $E_n=\{\alpha\in\mathbb{C}\ :\ \Xi_n(\alpha)=0\}$. Here, $\Xi_n(\alpha)$ is the Hankel determinant associated with $u$. Notice that $u$ is a semi-classical linear functional of class one [10].

The recurrence coefficients $\beta_n$ and $\gamma_{n+1}$, $n\geq 0$, of the sequence $\{B_n\}_{n\geq 0}$ are determined by the system [10] :

$$\begin{cases} \dfrac{n+1}{\gamma_{n+1}} &= 2-i(\beta_n+\beta_{n+1}),\ n\geq 0, \\ i(\gamma_{n+2}+\gamma_{n+1}) &= \psi(\beta_{n+1}),\ n\geq 0, \\ \gamma_1=-i\psi(\beta_0), & \beta_0=-i\dfrac{F'(\alpha)}{F(\alpha)}. \end{cases} \tag{4.10}$$

The sequence $\{B_n\}_{n\geq 0}$ is characterized by the following structure relation [10] :

$$B^{[1]}_n(x) = B_n(x)-\frac{i\gamma_n\gamma_{n+1}}{n+1}B_{n-1}(x),\ n\geq 1. \tag{4.11}$$

Thus, taking into account $\lambda_{n,n-1}=-\dfrac{i\gamma_n\gamma_{n+1}}{n+1}\neq 0$, $n\geq 1$, we deduce a strictly finite-type relation between the sequences $\{B_n\}_{n\geq 0}$ and $\{B^{[1]}_n\}_{n\geq 0}$ with index $s=1$, with respect to $\phi(x)=1$,

From Proposition 4.1, we get the following inverse relation, for $n\geq 1$,

$$(x+v^*_{n,0})B_n(x) = B^{[1]}_{n+1}(x)+\frac{i(n+1)}{\gamma_{n+1}}B^{[1]}_n(x), \tag{4.12}$$

where $v^*_{n,0}=\dfrac{i(n+1)}{\gamma_{n+1}}-\dfrac{i\gamma_{n+1}\gamma_{n+2}}{n+2}-\beta_n$, $n\geq 1$. The sequence $\{B_n\}_{n\geq 0}$ could be characterized by a relation as (4.12). It is the aim of the following result.

**Proposition 4.2.** *Let $\{B_n\}_{n\geq 0}$ be a MOPS satisfying (1.5). Then the following statements are equivalent.*

i) *There exists a set of non-zero complex numbers $\{\lambda_{n,n-1}\}_{n\geq 1}$ such that, for $n\geq 1$,*

$$B^{[1]}_n(x) = B_n(x)+\lambda_{n,n-1}B_{n-1}(x)\,. \tag{4.13}$$

ii) *There exists a set of complex numbers* $\{\varrho_n\}_{n\geq 0}$, *with* $\varrho_n \neq 0$, $n \geq 1$, *and* $\varrho_0 = 0$, *such that for* $n \geq 0$,

$$(x+\frac{\gamma_{n+1}}{\varrho_{n+1}}+\varrho_n-\beta_n)B_n(x) = B^{[1]}_{n+1}(x)+\varrho_n B^{[1]}_n(x)\,. \quad (4.14)$$

*Proof.* Assume that i) holds. From Proposition 4.1, we get

$$(x+\frac{\gamma_{n+1}}{\varrho_{n+1}}+\varrho_n-\beta_n)B_n(x) = B^{[1]}_{n+1}(x)+\varrho_n B^{[1]}_n(x),\ n \geq 1,$$

where $\varrho_n = \gamma_n \lambda^{-1}_{n,n-1}$, $n \geq 1$. For $n = 1$, in (4.13), we obtain $\lambda_{1,0} = \dfrac{\beta_0-\beta_1}{2}$. Then, $\dfrac{\gamma_1}{\varrho_1} = \dfrac{\beta_0-\beta_1}{2}$. Hence,

$$(x+\frac{\gamma_1}{\varrho_1}-\beta_0)B_0(x) = x-\frac{\beta_0+\beta_1}{2} = B^{[1]}_1(x)+\varrho_0 B^{[1]}_0(x),$$

*i.e.* $\varrho_0 = 0$. Thus, ii) holds. Conversely, let us take $\lambda_{n,n-1} = \dfrac{\gamma_n}{\varrho_n}$, $n \geq 1$, and consider the MPS $\{A_n\}_{n\geq 0}$ defined by

$$A_n(x) = B_n(x) + \lambda_{n,n-1}B_{n-1}(x),\ n \geq 1. \quad (4.15)$$

From Proposition 4.1, we get

$$(x + v^*_{n,0})B_n(x) = A_{n+1}(x) + \theta^*_{n,n}A_n(x),\ n \geq 1,$$

where $v^*_{n,0} = \dfrac{\gamma_{n+1}}{\varrho_{n+1}} + \varrho_n - \beta_n$, $n \geq 1$, and $\theta^*_{n,n} = \dfrac{\gamma_n}{\lambda_{n,n-1}} = \varrho_n$, $n \geq 1$. From the assumption ii) and the previous relation, it follows that

$$A_{n+1}(x) + \varrho_n A_n(x) = B^{[1]}_{n+1}(x) + \varrho_n B^{[1]}_n(x),\ n \geq 1.$$

Equivalently,

$$A_n(x)-B^{[1]}_n(x) = (\prod_{\nu=1}^{n}\varrho_\nu)\big(A_1(x)-B^{[1]}_1(x)\big) = 0,\ n \geq 1.$$

But, from (4.15) for $n = 1$ we get $A_1(x) = x - \beta_0 + \dfrac{\gamma_1}{\varrho_1}$. From (4.14), with $n = 0$, we get $B^{[1]}_1(x) = x - \beta_0 + \dfrac{\gamma_1}{\varrho_1}$. Hence, $A_n(x) = B^{[1]}_n(x)$, $n \geq 0$. Thus according to (4.15), i) holds. ☐

## 5. The case $(\mathbf{t}, \mathbf{s}) = (\mathbf{0}, \mathbf{2})$

Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional $u$ and satisfying (1.5). Consider the following finite-type relation between $\{B_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, with index $s = 2$, with respect to $\phi(x) = 1$, for $n \geq 0$,

$$Q_n(x) = B_n(x)+\lambda_{n,n-1}B_{n-1}(x)+\lambda_{n,n-2}B_{n-2}(x), \quad (5.1)$$

$$\exists r \geq 2, \qquad \lambda_{r,r-2} \neq 0. \quad (5.2)$$

From Lemma 2.1, for every system of complex numbers $(\theta_{n,\nu})^{n+2}_{\nu=n}$, $n \geq 0$, where $\theta_{n,n+2} = 1$, $n \geq 0$ and $\theta_{r,r} \neq 0$, there exists a unique MPS $\{\Omega_2(x; n)\}_{n\geq 0}$, where $\Omega_2(x; n) = x^2 + v_{n,1}x + v_{n,0}$, $n \geq 0$, and a unique system of complex numbers, $(\zeta^{[0]}_{n,\nu})^{n-1}_{\nu=n-2}$, $n \geq 0$, such that

$$\sum_{\nu=n}^{n+2}\theta_{n,\nu}Q_\nu(x) = \Omega_2(x; n)B_n(x) + \zeta^{[0]}_{n,n-1}B_{n-1}(x) + \zeta^{[0]}_{n,n-2}B_{n-2}(x)\,,\ n \geq 0, \quad (5.3)$$

where

$$\begin{cases} \lambda_{n+2,n+1} + \theta_{n,n+1} & = \beta_{n+1} + \beta_n + v_{n,1},\ n \geq 0, \\ \lambda_{2,0} + \theta_{0,1}\lambda_{1,0} + \theta_{0,0} & = \gamma_1 + \beta_0(\beta_0 + v_{0,1}) + v_{0,0}, \\ \lambda_{n+2,n} + \theta_{n,n+1}\lambda_{n+1,n} + \theta_{n,n} & = \gamma_{n+1} + \gamma_n + \beta_n(\beta_n + v_{n,1}) + v_{n,0},\ n \geq 1, \\ \theta_{n,n+1}\lambda_{n+1,n-1} + \theta_{n,n}\lambda_{n,n-1} & = \gamma_n(\beta_n + \beta_{n-1} + v_{n,1}) + \zeta^{[0]}_{n,n-1},\ n \geq 1, \\ \theta_{n,n}\lambda_{n,n-2} & = \gamma_n\gamma_{n-1} + \zeta^{[0]}_{n,n-2},\ n \geq 2, \end{cases} \quad (5.4)$$

The determinants associated with (5.1) – (5.2) are

$$\Delta_0(0, 2) = \Delta_1(0, 2) = 0,$$
$$\Delta_n(0, 2) = \lambda_{n,n-2}(\lambda_{n+1,n-1} - \gamma_n),\ n \geq 2. \quad (5.5)$$

As a consequence of Theorem 3.2, where $t = 0$ and $s = 2$, we have the following result

**Proposition 5.1.** *Let* $\{B_n\}_{n\geq 0}$ *be a MOPS and* $\{Q_n\}_{n\geq 0}$ *be the MPS satisfying* (5.1) – (5.2). *For every fixed integer* $p \geq 2$, *the following statements are equivalent*

i) $\lambda_{n,n-2}(\lambda_{n+1,n-1} - \gamma_n) \neq 0$, $n \geq p$.

ii) *There exist a unique SCN* $(\theta^*_{n,\nu})_{\nu=n}^{n+2}$, $n \geq p$, *with* $\theta^*_{n,n+2} = 1$, $n \geq p$, *and* $\theta^*_{r,r} \neq 0$, *if* $p \leq r$, *and there exists a unique MPS* $\{\Omega_2^*(x;n)\}_{n\geq p}$, *where* $\deg \Omega_2^*(x;n) = 2$, $n \geq p$, *such that, for* $n \geq p$,

$$\Omega_2^*(x;n)B_n(x) = Q_{n+2}(x) + \theta^*_{n,n+1}Q_{n+1}(x) + \theta^*_{n,n}Q_n(x)\,. \tag{5.6}$$

We write

$$\theta^*_{n,n+1} = \frac{\left[\lambda_{n,n-2}(\beta_{n-1} - \beta_{n+1} + \lambda_{n+2,n+1}) - \lambda_{n,n-1}\gamma_{n-1}\right]\gamma_n}{\lambda_{n,n-2}(\lambda_{n+1,n-1} - \gamma_n)},$$
$$\theta^*_{n,n} = \frac{\gamma_n\gamma_{n-1}}{\lambda_{n,n-2}},$$

$$\Omega_2^*(x;n) = x^2 + v^*_{n,1}x + v^*_{n,0},\ n \geq p, \tag{5.7}$$

where

$$\begin{aligned} v^*_{n,0} = {} & \theta^*_{n,n} + (\lambda_{n+1,n} - \beta_n)\theta^*_{n,n+1} - \gamma_{n+1} - \gamma_n + \\ & \lambda_{n+2,n} + \beta_n(\beta_{n+1} - \lambda_{n+2,n+1}), \\ v^*_{n,1} = {} & \theta^*_{n,n+1} - \beta_{n+1} - \beta_n + \lambda_{n+2,n+1}. \end{aligned}$$

*Example.* Let $\{B_n\}_{n\geq 0}$ be the sequence of monic polynomials, orthogonal with respect to the linear functional $u$ such that

$$\langle u, p\rangle = \int_{-\infty}^{+\infty} p(x)e^{-x^4}dx.$$

This sequence of polynomials was introduced by **P. Nevai** (see [15]) in the framework of the so-called Freud measures. These polynomials satisfy the three-term recurrence relation (1.5), with coefficients $\beta_n = 0$, $n \geq 0$, and where $\gamma_{n+1}$, $n \geq 0$, are given by a non-linear recurrence relation (see [3] and [15])

$$n = 4\gamma_n(\gamma_{n+1} + \gamma_n + \gamma_{n-1}),\ n \geq 1,$$

with $\gamma_0 = 0$ and $\gamma_1 = \Gamma(3/4)\Gamma(1/4)$.

The sequence $\{B_n\}_{n\geq 0}$ satisfies the following structure relation ( see [3])

$$B_n^{[1]}(x) = B_n(x) + \lambda_{n,n-2}B_{n-2}(x),\ n \geq 2, \tag{5.8}$$

where

$$\lambda_{n,n-2} = \frac{4}{n+1}\gamma_{n+1}\gamma_n\gamma_{n-1} \neq 0,\ n \geq 2.$$

From (5.3), with $Q_n(x) = B_n^{[1]}(x)$, $n \geq 0$, and the fact that the polynomial sequences $\{B_n\}_{n\geq 0}$ and $\{B_n^{[1]}\}_{n\geq 0}$ are symmetric, i.e, $B_n(-x) = (-1)^nB_n(x)$, $n \geq 0$, we get, for $n \geq 0$,

$$\begin{aligned} B^{[1]}_{n+2}(x) + \theta_{n,n}B^{[1]}_n(x) = {} & (x^2 + v_{n,0})B_n(x) \\ & + \zeta^{[0]}_{n,n-2}B_{n-2}(x), \end{aligned} \tag{5.9}$$

where

$$\begin{cases} \lambda_{2,0} + \theta_{0,0} & = \gamma_1 + v_{0,0}, \\ \lambda_{n+2,n} + \theta_{n,n} & = \gamma_{n+1} + \gamma_n + v_{n,0},\ n \geq 1, \\ \theta_{n,n}\lambda_{n,n-2} & = \gamma_n\gamma_{n-1} + \zeta^{[0]}_{n,n-2},\ n \geq 2. \end{cases} \tag{5.10}$$

Since we have $\lambda_{n,n-2}$, $n \geq 2$, the choice $\zeta^{[0]}_{n,n-2} = 0$, $n \geq 2$, is possible and yields the inverse relation

$$(x^2 + v^*_{n,0})B_n(x) = B^{[1]}_{n+2}(x) + \theta^*_{n,n}B^{[1]}_n(x),\ n \geq 0, \tag{5.11}$$

where

$$\theta^*_{n,n} = \frac{n+1}{4\gamma_{n+1}},$$

$$v^*_{n,0} = \frac{n+1}{4\gamma_{n+1}} - \gamma_n - \gamma_{n+1} + \frac{4}{n+3}\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}.$$

Here, the determinants associated with (5.8) are

$$\Delta_n(0,2) = \frac{4}{n+1}\gamma_{n+1}\gamma_n^2\gamma_{n-1}\left[\frac{4}{n+2}\gamma_{n+2}\gamma_{n+1} - 1\right], \tag{5.12}$$

$n \geq 2$, with $\Delta_0(0,2) = \Delta_1(0,2) = 0$.

From Proposition 5.1, we deduce that the uniqueness of the previous inverse relation requires that $\lambda_{n+1,n-1} - \gamma_n = \gamma_n\left[\frac{4}{n+2}\gamma_{n+2}\gamma_{n+1} - 1\right] \neq 0$, $n \geq 2$. Equivalently, $4\gamma_{n+2}\gamma_{n+1} \neq n+2$, $n \geq 2$. Indeed, by using (5.8), where $n$ is replaced by $n+1$ and taking into account the orthogonality of the polynomial sequence $\{B_n\}_{n\geq 0}$, we get $B^{[1]}_{n+1}(x) = xB_n(x) + (\lambda_{n+1,n-1} - \gamma_n)B_{n-1}(x)$, $n \geq 1$. On the other hand, if we suppose that there exists an integer $n_0 \geq 2$ such that $\lambda_{n_0+1,n_0-1} - \gamma_{n_0} = 0$, then $B^{[1]}_{n_0+1}(x) = xB_{n_0}(x)$. In this case (5.11), with $n = n_0$ will be written as $(x^2 + \alpha x + v^*_{n,0})B_n(x) = B^{[1]}_{n+2}(x) + \alpha B^{[1]}_{n+1}(x) + \theta^*_{n,n}B^{[1]}_n(x)$, for all $\alpha \in \mathbb{C}$. This contradicts the uniqueness of the inverse relation.

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