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Asymptotic Properties of Orthogonal Polynomials with Respect to a Non-discrete Jacobi-Sobolev Inner Product

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Abstract Let $\{Q_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ denote the sequence of polynomials orthogonal with respect to the non-discrete Sobolev inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)d\mu_{\alpha,\beta}(x) + \lambda \int_{-1}^{1} f'(x)g'(x)d\nu_{\alpha,\beta}(x)$$

where $\lambda > 0$ and $d\mu_{\alpha,\beta}(x) = (x-a)(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$, $d\nu_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$ with $a < -1$, $\alpha, \beta > 0$. Their inner strong asymptotics on $(-1, 1)$, a Mehler-Heine type formula as well as some estimates of the Sobolev norms of $Q_n^{(\alpha,\beta)}$ are obtained.

Keywords Jacobi orthogonal polynomials · Jacobi-Sobolev type orthogonal polynomials · Asymptotics

Mathematics Subject Classification (2000) 33C45 · 42C05

1 Introduction

For a nontrivial probability measure σ, supported on $[-1, 1]$, we define the normed linear space $L^p(\sigma)$ of all measurable functions f on $[-1, 1]$ such that $\|f\|_{L^p(\sigma)} < \infty$, where

$$\|f\|_{L^p(\sigma)} = \begin{cases} (\int_{-1}^{1} |f(x)|^p d\sigma(x))^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \text{ess sup}_{-1<x<1} |f(x)| & \text{if } p = \infty. \end{cases}$$

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Let us now introduce the weighted Sobolev spaces (see, for instance, [2, Chap. III] in a more general framework)

$$W^{1,p} = \{f \,:\, \|f\|_{W^{1,p}}^p = \|f\|_{L^p(\mu_{\alpha,\beta})}^p + \lambda \|f'\|_{L^p(\nu_{\alpha,\beta})}^p < \infty\}, \quad 1 \le p < \infty,$$

$$W^{1,\infty} = \{f \,:\, \|f\|_{W^{1,\infty}} = \max\{\|f\|_{L^\infty(\mu_{\alpha,\beta})}, \lambda \|f'\|_{L^\infty(\nu_{\alpha,\beta})}\} < \infty\},$$

where λ is a nonnegative real number and the pair of measures defining the weighted Sobolev space are $d\mu_{\alpha,\beta}(x) = (x-a)(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$, $d\nu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$ with $a < -1$, $\alpha, \beta > 0$.

Let f and g in $W^{1,2}$. We can introduce the Sobolev inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)d\mu_{\alpha,\beta}(x) + \lambda \int_{-1}^{1} f'(x)g'(x)d\nu_{\alpha,\beta}(x) \tag{1}$$

where $\lambda > 0$. Let $\{Q_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ denote the sequence of polynomials orthogonal with respect to (1), such that $Q_n^{(\alpha,\beta)}$ is a polynomial of degree n and it has the same leading coefficient as the classical Jacobi polynomial of degree n $P_n^{(\alpha-1,\beta-1)}(x) = \frac{1}{2}^n\binom{2n+\alpha+\beta-2}{n}x^n +$lower degree terms. The existence and uniqueness of such a sequence of orthogonal polynomials follows in a straightforward way from the application of the Gram-Schmidt orthogonalization method to the canonical basis $(x^n)_{n=0}^\infty$ when the inner product (1) is considered. Notice that, as an equivalent fact, the corresponding monic orthogonal polynomial minimizes the norm associated with the above inner product in the convex set of monic polynomials of degree n.

In other words

$$\langle Q_m^{(\alpha,\beta)}, Q_n^{(\alpha,\beta)}\rangle = \int_{-1}^{1} Q_n^{(\alpha,\beta)}(x)Q_m^{(\alpha,\beta)}(x)d\mu_{\alpha,\beta}(x)$$
$$+ \lambda \int_{-1}^{1} \left(\hat{Q}_m^{(\alpha,\beta)}(x)\right)' \left(\hat{Q}_n^{(\alpha,\beta)}(x)\right)' d\nu_{\alpha,\beta}(x) = 0, \quad \text{if } n \ne m. \tag{2}$$

These polynomials constitute a particular case of the so-called coherent pairs of measures introduced in [6] and completely described in [10]. In the literature (see [9] among others), they are called Jacobi-Sobolev polynomials of type 1. In [11], [14], and [15] the authors deduced the asymptotics of the distribution of their zeros as well as the outer relative asymptotics of the Jacobi- Sobolev polynomials $\{Q_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ with respect to the standard Jacobi polynomials $\{P_n^{(\alpha-1,\beta-1)}(x)\}_{n=0}^\infty$. In [12] the inner and the outer asymptotics for the difference $Q_n^{(\alpha,\beta)} - P_n^{(\alpha-1,\beta-1)}$ are given.

The interest of sequences of polynomials orthogonal with respect to Sobolev inner products defined by a coherent pair of measures was pointed out in [7] when you wish to approximate a function f by its projection into polynomials and, simultaneously, to approximate its derivative by the derivative of the polynomial approximant. Given that the derivative of f is steep, it is only to be expected that the quality of the projection in the conventional $L^2(\sigma)$ norm (the standard Fourier projection) deteriorates. Several computational examples show that the Sobolev projection is better. In particular, the standard Fourier projection is poor near the end points, whereas the Sobolev-Fourier projection displays reasonably good behavior throughout the interval.

The main goal of this contribution is to obtain some pointwise estimates as well as the asymptotic behavior of the polynomials $\{Q_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ and the sequence of their first derivatives $\{Q_n'^{(\alpha,\beta)}(x)\}_{n=0}^\infty$ in $[-1,1]$. In such a way, we complete the study of the asymptotic properties presented by the above mentioned authors. As a consequence, Cohen type inequalities for Fourier-Sobolev expansions in terms of such sequences of Jacobi-Sobolev orthogonal polynomials can be deduced (see [5]).Thus, the study of necessary conditions for the norm convergence of such expansions as well as the study of the divergence a.e. can be done.

The structure of the paper is as follows. In Sect. 2 we summarize the main properties of Jacobi polynomials to be used in the sequel. In Sect. 3 we use the connection between the sequences $\{Q_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ and $\{P_n^{(\alpha-1,\beta-1)}(x)\}_{n=0}^{\infty}$ in order to get upper bounds for the Jacobi-Sobolev polynomials and their first derivatives in the interval $[-1,1]$. The Mehler-Heine type formula is deduced in Proposition 6. Finally, in Proposition 8 the behavior of the $W^{1,p}$ norms of Jacobi-Sobolev orthonormal polynomials is given.

Throughout this paper positive constants are denoted by $c,\ c_1,\dots$ and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and the notation $u_n \sim v_n$ means $c_1 u_n \le v_n \le c_2 u_n$ for sufficiently large n.

2 Jacobi Polynomials

For $\alpha,\beta > -1$, we denote by $P_n^{(\alpha,\beta)}$ the usual Jacobi polynomial of degree n, which is orthogonal on $[-1,1]$ to every polynomial of degree at most $n-1$ with respect to the measure $\nu_{\alpha,\beta}$. They are normalized in such a way that $P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$. We denote the monic Jacobi polynomial of degree n by

$$\hat{P}_n^{(\alpha,\beta)}(x) = (h_n^{\alpha,\beta})^{-1} P_n^{(\alpha,\beta)}(x), \tag{3}$$

where (see [1, formula (22.3.1)])

$$h_n^{\alpha,\beta} = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n}.$$

Notice that monic Jacobi polynomial of degree n minimizes the norm associated with the measure $\nu_{\alpha,\beta}$ among all the monic polynomials of degree n (see [3, Exercise 3.4]).

Next we list some properties of the Jacobi polynomials which we will use in the sequel. The Jacobi monic polynomials satisfy the orthogonality condition (see (2) and [1, formula (22.2.1)])

$$\begin{aligned}\int_{-1}^{1} &\hat{P}_n^{(\alpha,\beta)}(x)\hat{P}_m^{(\alpha,\beta)}(x)d\nu_{\alpha,\beta}(x)\\ &= 2^{2n+\alpha+\beta+1}\frac{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)}\delta_{m,n}.\end{aligned} \tag{4}$$

For the first derivative we get the expression (see [17, formula (4.21.7)])

$$\frac{d}{dx}\hat{P}_n^{(\alpha,\beta)}(x) = n\hat{P}_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{5}$$

We have the following estimate for the polynomials $P_n^{(\alpha,\beta)}$ due to S. Bernstein (see [4, Theorem 1], [13], and [17, formula (7.32.6)])

$$|P_n^{(\alpha,\beta)}(x)| \le cn^{-1/2}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} \tag{6}$$

for $x \in [-1, 1]$, $\alpha, \beta \geq -1/2$ and $n \geq 1$, where c is a nonnegative real constant independent of x and n as well as (see [17, 7.32.2])

$$|P_n^{(\alpha,\beta)}(x)| \leq \begin{cases} cn^{\alpha} & \text{for } x \in [0,1],\ \alpha \geq -1/2, \\ cn^{\beta} & \text{for } x \in [-1,0],\ \beta \geq -1/2, \\ cn^{-1/2} & \text{for } x \in [-1,1],\ \alpha \leq -1/2,\ \beta \leq -1/2, \end{cases} \tag{7}$$

for $n \geq 1$.

The Mehler-Heine type formula for Jacobi orthogonal polynomials is (see [17, Theorem 8.1.1])

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = (z/2)^{-\alpha} J_\alpha(z), \tag{8}$$

where $J_\alpha(z)$ is the Bessel function. This formula holds uniformly for $|z| \leq R$, for R a given positive real number.

The inner strong asymptotics of $P_n^{(\alpha,\beta)}$, for $\theta \in [\epsilon, \pi - \epsilon]$ and $\epsilon > 0$, is given by (see [17, Theorem 8.21.8])

$$\begin{aligned} P_n^{(\alpha,\beta)}(\cos\theta) &= \pi^{-1/2} n^{-1/2} \\ &\quad \times \left[\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\cos(k\theta+\gamma) + O(n^{-1})\right], \end{aligned} \tag{9}$$

where $k = n + (\alpha+\beta+1)/2$, $\gamma = -(\alpha+1/2)\pi/2$.

For $\alpha, \beta, \mu > -1$, and $1 \leq q \leq \infty$ (see [8, (2.2)], [17, p. 391. Exercise 91])

$$\left(\int_0^1 (1-x)^{\mu} |P_n^{(\alpha,\beta)}(x)|^p dx\right)^{1/p} \sim \begin{cases} n^{-1/2} & \text{if } 2\mu > p\alpha - 2 + p/2, \\ n^{-1/2}(\log n)^{1/p} & \text{if } 2\mu = p\alpha - 2 + p/2, \\ n^{\alpha - \frac{2\mu+2}{p}} & \text{if } 2\mu < p\alpha - 2 + p/2. \end{cases}$$

In particular, for $\alpha, \beta > -1$ and $\tau = \max\{\alpha, \beta\}$

$$\|P_n^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})} \sim \begin{cases} n^{-1/2} & \text{if } 2\tau > p\tau - 2 + p/2, \\ n^{-1/2}(\log n)^{1/p} & \text{if } 2\tau = p\tau - 2 + p/2, \\ n^{\tau - \frac{2\tau+2}{p}} & \text{if } 2\tau < p\tau - 2 + p/2. \end{cases} \tag{10}$$

3 Asymptotics of Jacobi-Sobolev Orthogonal Polynomials

Let us denote by $\hat{Q}_n^{(\alpha,\beta)}$ the monic Jacobi-Sobolev polynomial of degree n i.e. $\hat{Q}_n^{(\alpha,\beta)}(x) = (h_n^{\alpha-1,\beta-1})^{-1} Q_n^{(\alpha,\beta)}(x)$. We have the following relation between the Jacobi-Sobolev and Jacobi monic orthogonal polynomials:

Proposition 1 *For* $\alpha, \beta > 0$

$$\hat{P}_n^{(\alpha-1,\beta-1)}(x) = \hat{Q}_n^{(\alpha,\beta)}(x) + \hat{d}_{n,n-1}(\lambda)\hat{Q}_{n-1}^{(\alpha,\beta)}(x), \qquad n \geq 1,$$

where $\hat{d}_{n,n-1}(\lambda) \cong \frac{c}{n^2}$.

Proof Expanding $\hat{P}_{n+1}^{(\alpha-1,\beta-1)}(x)$ in terms of the basis $\{\hat{Q}_k^{(\alpha,\beta)}\}_{k=0}^{n+1}$ of the linear space of polynomials with degree at most $n+1$, we get

$$\hat{P}_{n+1}^{(\alpha-1,\beta-1)}(x) = \hat{Q}_{n+1}^{(\alpha,\beta)}(x) + \sum_{k=0}^{n} \hat{d}_{n+1,k}(\lambda)\hat{Q}_k^{(\alpha,\beta)}(x),$$

where

$$\hat{d}_{n+1,k}(\lambda) = \frac{\langle \hat{P}_{n+1}^{(\alpha-1,\beta-1)}, \hat{Q}_k^{(\alpha,\beta)}\rangle}{\langle \hat{Q}_k^{(\alpha,\beta)}, \hat{Q}_k^{(\alpha,\beta)}\rangle}. \tag{11}$$

For $k = 0, \dots, n$

$$\begin{aligned}
&\langle \hat{P}_{n+1}^{(\alpha-1,\beta-1)}, \hat{Q}_k^{(\alpha,\beta)}\rangle \\
&\quad = \int_{-1}^{1} \hat{P}_{n+1}^{(\alpha-1,\beta-1)}(x)\hat{Q}_k^{(\alpha,\beta)}(x)(x-a)(1-x)^{\alpha-1}(1+x)^{\beta-1}dx \\
&\qquad + (n+1)\lambda \int_{-1}^{1} \hat{P}_n^{(\alpha,\beta)}(x)\left(\hat{Q}_k^{(\alpha,\beta)}(x)\right)'(1-x)^{\alpha}(1+x)^{\beta}dx \\
&\quad = \int_{-1}^{1} \hat{P}_{n+1}^{(\alpha-1,\beta-1)}(x)x\hat{Q}_k^{(\alpha,\beta)}(x)(1-x)^{\alpha-1}(1+x)^{\beta-1}dx.
\end{aligned}$$

Therefore

$$\langle \hat{P}_{n+1}^{(\alpha-1,\beta-1)}, \hat{Q}_k^{(\alpha,\beta)}\rangle = 0, \quad k = 0, \dots, n-1.$$

On the other hand

$$\langle \hat{P}_{n+1}^{(\alpha-1,\beta-1)}, \hat{Q}_n^{(\alpha,\beta)}\rangle = \|\hat{P}_{n+1}^{(\alpha-1,\beta-1)}\|^2_{L^2(\nu_{\alpha-1,\beta-1})}. \tag{12}$$

But from the extremal property of the Jacobi monic orthogonal in terms of the corresponding norm (see [17])

$$\|\hat{P}_n^{(\alpha,\beta)}\|^2_{L^2(d\nu_{\alpha,\beta})} = \inf\{\|P\|^2_{L^2(\nu_{\alpha,\beta})} : \deg P = n,\ P \text{ monic}\}$$

we get

$$\begin{aligned}
\|\hat{Q}_n^{(\alpha,\beta)}\|^2_{W^{1,2}} &= \|\hat{Q}_n^{(\alpha,\beta)}\|^2_{L^2(\mu_{\alpha,\beta})} + \lambda\|\hat{Q}'^{(\alpha,\beta)}_n\|^2_{L^2(\nu_{\alpha,\beta})} \\
&\geq (-1-a)\|\hat{Q}_n^{(\alpha,\beta)}\|^2_{L^2(\nu_{\alpha-1,\beta-1})} + \lambda n^2\|\hat{P}_{n-1}^{(\alpha,\beta)}\|^2_{L^2(\nu_{\alpha,\beta})} \\
&\geq (-1-a)\|\hat{P}_n^{(\alpha-1,\beta-1)}\|^2_{L^2(\nu_{\alpha-1,\beta-1})} + \lambda n^2\|\hat{P}_{n-1}^{(\alpha,\beta)}\|^2_{L^2(\nu_{\alpha,\beta})}.
\end{aligned} \tag{13}$$

On the other hand, from the extremal property of the monic polynomial $\hat{Q}_n^{(\alpha,\beta)}$ in terms of the norm associated with the inner product (1) and using (4) we have

$$\begin{aligned}
\|\hat{Q}_n^{(\alpha,\beta)}\|^2_{W^{1,2}} &\leq \|\hat{P}_n^{(\alpha-1,\beta-1)}\|^2_{W^{1,2}} \\
&= \|\hat{P}_n^{(\alpha-1,\beta-1)}\|^2_{L^2(\mu_{\alpha,\beta})} + \lambda n^2\|\hat{P}_{n-1}^{(\alpha,\beta)}\|^2_{L^2(\nu_{\alpha,\beta})} \\
&\leq (1-a)\|\hat{P}_n^{(\alpha-1,\beta-1)}\|^2_{L^2(\nu_{\alpha-1,\beta-1})} + \lambda n^2\|\hat{P}_{n-1}^{(\alpha,\beta)}\|^2_{L^2(\nu_{\alpha,\beta})}.
\end{aligned} \tag{14}$$

Thus, from (3), (12), and (13)

$$\|\hat{Q}_n^{(\alpha,\beta)}\|_{W^{1,2}}^2 \cong \lambda n^2 \|\hat{P}_{n-1}^{(\alpha,\beta)}\|_{L^2(\nu_{\alpha,\beta})}^2. \tag{15}$$

Finally, from (10), (11), and (14) we deduce

$$\hat{d}_{n,n-1}(\lambda) \cong \frac{c}{n^2}.$$

□

Corollary 1 *For* $\alpha, \beta > 0$

$$P_n^{(\alpha-1,\beta-1)}(x) = Q_n^{(\alpha,\beta)}(x) + d_{n-1}(\lambda) Q_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1, \tag{16}$$

where

$$d_{n-1}(\lambda) = \hat{d}_{n,n-1}(\lambda) \frac{h_n^{\alpha-1,\beta-1}}{h_{n-1}^{\alpha-1,\beta-1}} \cong \frac{c}{n^2}, \quad n \geq 1. \tag{17}$$

Using (15) in a recursive way we get the representation of the polynomial $Q_n^{(\alpha,\beta)}$ in terms of the elements of the sequence $\{P_n^{(\alpha-1,\beta-1)}(x)\}_{n=0}^{\infty}$

$$Q_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} (-1)^k b_k^{(n)}(\lambda) P_{n-k}^{(\alpha-1,\beta-1)}(x), \tag{18}$$

where $b_k^{(n)}(\lambda) = \prod_{j=1}^{k} d_{n-j}(\lambda)$ and $b_0^{(n)}(\lambda) = 1$.

Proposition 2 *There exists a constant* $d > 1$ *such that the coefficients* $b_k^{(n)}(\lambda)$ *in* (17) *satisfy* $b_k^{(n)}(\lambda) < d\frac{1}{n2^k}$ *for every* $n \geq 1$ *and* $1 \leq k \leq n$.

Proof From (16) we have $\lim_{n\to\infty} 2(n+1)d_n(\lambda) = 0$, then there exist $n_0 \in \mathbf{N}$ and a constant $d > 1$ such that $2(n+1)d_n(\lambda) < 1$ for all $n \geq n_0$ and $2(n+1)d_n(\lambda) < d$ for $n = 1, \ldots, n_0 - 1$. Therefore, for $1 \leq k \leq n - n_0$

$$b_k^{(n)}(\lambda) = \prod_{j=1}^{k} d_{n-j}(\lambda) < \frac{1}{n2^k},$$

and for $n - n_0 \leq k \leq n$,

$$\begin{aligned} b_k^{(n)}(\lambda) &= \prod_{j=1}^{n-n_0} d_{n-j}(\lambda) \prod_{j=n-n_0+1}^{k} d_{n-j}(\lambda) \\ &\leq \frac{1}{n2^{n-n_0}} \left(\frac{d}{2}\right)^{k-n+n_0} = d^{k-n+n_0} \frac{1}{n2^k} \leq d^{n_0} \frac{1}{n2^k}. \end{aligned}$$

□

Proposition 3 (i) *For the polynomials* $Q_n^{(\alpha,\beta)}$ *we get the following estimate*

$$|Q_n^{(\alpha,\beta)}(x)| \leq cn^{-1/2}(1-x)^{-\alpha/2+1/4}(1+x)^{-\beta/2+1/4}$$

for $x \in [-1, 1]$, $\alpha, \beta \geq 1/2$ *and* $n \geq 1$.

(ii) *For the polynomials* ${Q'}_n^{(\alpha,\beta)}$ *we get*

$$|{Q'}_n^{(\alpha,\beta)}(x)| \le cn^{1/2}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}$$

for $x \in [-1,1]$, $\alpha, \beta > 0$ *and* $n \ge 1$.

Proof (i) Using (17) we have

$$|Q_n^{(\alpha,\beta)}(\cos\theta)| \le \sum_{k=0}^{n} b_k^{(n)}(\lambda)|P_{n-k}^{(\alpha-1,\beta-1)}(\cos\theta)|.$$

From (5)

$$|P_{n-k}^{(\alpha,\beta)}(\cos\theta)| \le c\sqrt{\frac{n}{n-k}}n^{-1/2}\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}$$

for $\alpha, \beta \ge -1/2$ and $0 \le k \le n-1$. Thus, according to Proposition 2

$$\begin{aligned} |Q_n^{(\alpha,\beta)}(\cos\theta)| &\le \sum_{k=0}^{n} b_k^{(n)}(\lambda)|P_{n-k}^{(\alpha-1,\beta-1)}(\cos\theta)| \\ &\le cb_n^{(n)}(\lambda) + cn^{-1/2}\theta^{-\alpha+1/2}(\pi-\theta)^{-\beta+1/2}\sum_{k=0}^{n-1}\frac{1}{2^k} \\ &\le cn^{-1/2}\theta^{-\alpha+1/2}(\pi-\theta)^{-\beta+1/2}. \end{aligned}$$

On the other hand, taking into account [17, formula (4.21.7)] and (17) the proof of the case (ii) can be done in a similar way. □

Proposition 4 *For* $n \ge 1$

$$|Q_n^{(\alpha,\beta)}(x)| \le \begin{cases} cn^{\alpha-1} & \text{for } x \in [0,1],\ \alpha \ge 1/2, \\ cn^{\beta-1} & \text{for } x \in [-1,0],\ \beta \ge 1/2, \\ cn^{-1/2} & \text{for } x \in [-1,1],\ \alpha \le 1/2,\ \beta \le 1/2, \end{cases}$$

and

$$|{Q'}_n^{(\alpha,\beta)}(x)| \le \begin{cases} cn^{\alpha+1} & \text{for } x \in [0,1],\ \alpha > 0, \\ cn^{\beta+1} & \text{for } x \in [-1,0],\ \beta > 0. \end{cases}$$

Proof From (6), for $0 \le k \le n-1$

$$|P_{n-k}^{(\alpha,\beta)}(x)| \le \begin{cases} c\left(\frac{n-k}{n}\right)^{\alpha} n^{\alpha} & \text{for } x \in [0,1],\ \alpha \ge -1/2, \\ c\left(\frac{n-k}{n}\right)^{\beta} n^{\beta} & \text{for } x \in [-1,0],\ \beta \ge -1/2, \\ c\left(\frac{n-k}{n}\right)^{-1/2} n^{-1/2} & \text{for } x \in [-1,1],\ \alpha \le -1/2,\ \beta \le -1/2. \end{cases}$$

Now, we can conclude the proof in the same way as we did in Proposition 3. □

Corollary 2 *For* $\alpha, \beta \ge 1/2$

$$|Q_n^{(\alpha,\beta)}(\cos\theta)| \le cA(n, \alpha-1, \beta-1, \theta),$$

and for $\alpha, \beta > 0$

$$|Q'^{(\alpha,\beta)}_n(\cos\theta)| \le cA(n,\alpha,\beta,\theta),$$

where

$$A(n,\alpha,\beta,\theta) = \begin{cases} n^{-1/2}(\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}) & \text{if } c/n \le \theta \le \pi - c/n, \\ n^{\alpha} & \text{if } 0 \le \theta \le c/n, \\ n^{\beta} & \text{if } \pi - c/n \le \theta \le \pi. \end{cases}$$

Proof The inequality

$$n^{\alpha} \le cn^{-1/2}\theta^{-\alpha-1/2}$$

holds for $\theta \in (0, c/n]$ and the inequality

$$n^{\beta} \le cn^{-1/2}(\pi-\theta)^{-\beta-1/2}$$

holds for $\theta \in [\pi - c/n, \pi)$. Therefore, from Proposition 3 and Proposition 4, the statement follows immediately. □

Next we show that, like for the classical Jacobi polynomials, the polynomial $Q^{(\alpha,\beta)}_n(x)$ attains its maximum in $[-1, 1]$ at the end-points, more precisely:

Proposition 5 (i) *For* $\alpha, \beta \ge 1/2$, *and* $q = \max\{\alpha, \beta\}$

$$\max_{-1\le x\le 1} |Q^{(\alpha,\beta)}_n(x)| = |Q^{(\alpha,\beta)}_n(b)| \sim n^{q-1},$$

where either $b = 1$ *when* $q = \alpha$ *or* $b = -1$ *when* $q = \beta$.

(ii) *For* $\alpha, \beta > 0$ *and* $q = \max\{\alpha, \beta\}$

$$\max_{-1\le x\le 1} |Q'^{(\alpha,\beta)}_n(x)| = |Q'^{(\alpha,\beta)}_n(b)| \sim n^{q+1},$$

where either $b = 1$ *when* $q = \alpha$ *or* $b = -1$ *when* $q = \beta$.

Proof (i) We will prove only the case $q = \alpha$. If $q = \beta$ it can be done in a similar way. From (15), (16), and Proposition 4

$$Q^{(\alpha,\beta)}_n(x) = P^{(\alpha-1,\beta-1)}_n(x) - d_{n-1}(\lambda)Q^{(\alpha,\beta)}_{n-1}(x) = P^{(\alpha-1,\beta-1)}_n(x) - O(n^{\alpha-3}).$$

Now, from [17, Theorem 7.32.1] the result follows.

Taking into account [17, formula (4.21.7)] the case (ii) can be proved in a similar way. □

Next, we deduce a Mehler-Heine type formula for $Q^{(\alpha,\beta)}_n$ and $(Q^{(\alpha,\beta)}_n)'$.

Proposition 6 *Let* $\alpha, \beta > 0$. *Uniformly on compact subsets of* **C**

(i)

$$\lim_{n\to\infty} n^{-\alpha+1} Q^{(\alpha,\beta)}_n\left(\cos\frac{z}{n}\right) = (z/2)^{-\alpha+1}J_{\alpha-1}(z), \tag{19}$$

(ii)

$$\lim_{n\to\infty} n^{-\alpha-1} Q_n'^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = (z/2)^{-\alpha} J_\alpha(z). \tag{20}$$

Proof (i) Multiplying in (15) by $(n+1)^{-\alpha+1}$ we obtain

$$V_n(z) = Y_n(z) + D_{n-1}(\lambda) Y_{n-1}(z),$$

where $Y_n(z) = (n+1)^{-\alpha+1} Q_n^{(\alpha,\beta)}(\cos\frac{z}{n})$, $V_n(z) = (n+1)^{-\alpha+1} P_n^{(\alpha-1,\beta-1)}(\cos\frac{z}{n})$ and $D_{n-1}(\lambda) = d_{n-1}(\lambda)(\frac{n}{n+1})^{\alpha-1} \cong \frac{c}{n^2}$ according to (16).

Using above relation in a recursive way we get

$$Y_n(z) = \sum_{k=0}^{n} (-1)^k B_k^{(n)}(\lambda) V_{n-k}(z),$$

where $B_k^{(n)}(\lambda) = \prod_{j=1}^{k} D_{n-j}(\lambda)$ and $B_0^{(n)}(\lambda) = 1$. Moreover, by using the same argument as in Proposition 2 we have $B_k^{(n)}(\lambda) < c\frac{1}{n2^k}$ for every $n \geq 1$ and $1 \leq k \leq n$. Thus

$$|Y_n(z)| \leq \sum_{k=0}^{n} B_k^{(n)}(\lambda)\, |V_{n-k}(z)|\,.$$

On the other hand, from (7), we have that $\{V_n(z)\}_{n=0}^{\infty}$ is uniformly bounded on compact subsets of $\mathbf{C}$. Thus, for a fixed compact set $K \subset \mathbf{C}$ there exists a constant C, depending only on K, such that when $z \in K$

$$|V_n(z)| < C, \quad n \geq 1.$$

Thus, the sequence $\{Y_n(z)\}_{n=0}^{\infty}$ is uniformly bounded on $K \subset \mathbf{C}$. As a conclusion

$$Y_n(z) = V_n(z) - O(n^{-2}), \quad z \in K,$$

and using (7) we obtain the result.

(ii) Since we have uniform convergence in (18), taking derivatives and using a very well known result concerning the derivative of the Bessel functions (see [17, formula (1.71.5)]) we obtain (19). □

Now we give the inner strong asymptotics of $Q_n^{(\alpha,\beta)}$ on $(-1,1)$.

Proposition 7 *For $\theta \in [\epsilon, \pi-\epsilon]$ and $\epsilon > 0$*

$$Q_n^{(\alpha,\beta)}(\cos\theta) = \pi^{-1/2} n^{-1/2} \times\left[\left(\sin\frac{\theta}{2}\right)^{-\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta+1/2}\cos(k_1\theta+\gamma) + O(n^{-1})\right], \tag{21}$$

$$Q_n'^{(\alpha,\beta)}(\cos\theta) = \pi^{-1/2}\frac{(n+\alpha+\beta)(n-1)^{-1/2}}{2} \times\left[\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\cos(k_1\theta+\gamma_1) + O(n^{-1})\right], \tag{22}$$

where $k_1 = n + (\alpha+\beta-1)/2$, $\gamma = -(\alpha-1/2)\pi/2$, and $\gamma_1 = -(\alpha+1/2)\pi/2$.

Proof From Proposition 3(i) the sequence $\{n^{1/2}Q_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is uniformly bounded on compact subsets of $(-1,1)$. Multiplication by $n^{1/2}$ in (15) yields

$$n^{1/2}Q_n^{(\alpha,\beta)}(x)=n^{1/2}P_n^{(\alpha-1,\beta-1)}(x)-d_{n-1}(\lambda)\sqrt{\frac{n}{n-1}}(n-1)^{1/2}Q_{n-1}^{(\alpha,\beta)}(x).$$

Since

$$d_{n-1}(\lambda)\sqrt{\frac{n}{n-1}}=O\left(\frac{1}{n^2}\right),$$

we have

$$n^{1/2}Q_n^{(\alpha,\beta)}(x)=n^{1/2}P_n^{(\alpha-1,\beta-1)}(x)+O\left(n^{-2}\right).$$

Using (8), the relation (20) follows.

Concerning (21), it can be obtained in a similar way by using [17, formula (4.21.7)] and Proposition 3(ii). □

Now, we can give the best possible estimate for the Sobolev norms of the Jacobi-Sobolev polynomials.

Proposition 8 *For $\alpha\geq\beta\geq 1/2$ and $1\leq p\leq\infty$*

$$\|Q_n^{(\alpha,\beta)}\|_{W^{1,p}}\sim\begin{cases} n^{1/2} & \text{if } 4(\alpha+1)/(2\alpha+1)>p,\\ n^{1/2}(\log n)^{1/p} & \text{if } 4(\alpha+1)/(2\alpha+1)=p,\\ n^{\alpha+1-\frac{2\alpha+2}{p}} & \text{if } 4(\alpha+1)/(2\alpha+1)<p.\end{cases}\tag{23}$$

Notice that for $p=\infty$ we get the statement of Proposition 5. Thus in the proof we will assume $1\leq p<\infty$.

Proof In order to prove the upper bound of (22) it is necessary to prove

$$\|Q_n^{(\alpha,\beta)}\|_{W^{1,p}}\leq cn\|P_n^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})}.\tag{24}$$

Using (17) and the Minkowski's inequality we have

$$\|Q_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})}\leq c\sum_{k=0}^{n}b_k^{(n)}(\lambda)\|P_{n-k}^{(\alpha-1,\beta-1)}\|_{L^p(\nu_{\alpha-1,\beta-1})}.$$

For $\alpha,\beta>-1$ and $k=0,1,\dots,n$, from (9) it yields

$$(n-k)^{1/2}\|P_{n-k}^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})}\leq cn^{1/2}\|P_n^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})}.$$

Thus

$$\|P_{n-k}^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})}\leq c\sqrt{\frac{n}{n-k}}\|P_n^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})},\quad 0\leq k\leq n-1.$$

From the above inequality and Proposition 2, we get

$$\sum_{k=0}^{n}b_k^{(n)}(\lambda)\|P_{n-k}^{(\alpha-1,\beta-1)}\|_{L^p(\nu_{\alpha-1,\beta-1})}$$

$$\leq cb_n^{(n)}(\lambda) + \sum_{k=0}^{n-1} b_k^{(n)}(\lambda)\|P_{n-k}^{(\alpha-1,\beta-1)}\|_{L^p(\nu_{\alpha-1,\beta-1})}$$

$$\leq c\|P_n^{(\alpha-,\beta-1)}\|_{L^p(\nu_{\alpha-1,\beta-1})} \sum_{k=0}^{n-1} \frac{1}{2^k} \leq c_1\|P_n^{(\alpha-1,\beta-1)}\|_{L^p(\nu_{\alpha-1,\beta-1})}.$$

Thus

$$\|Q_n^{(\alpha,\beta)}\|_{L^p(\mu_{\alpha,\beta})} \leq c\|P_n^{(\alpha-1,\beta-1)}\|_{L^p(\nu_{\alpha-1,\beta-1})} \leq cn\|P_n^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})}. \tag{25}$$

In the same way as above we get

$$\|Q'^{(\alpha,\beta)}_n\|_{L^p(\nu_{\alpha,\beta})} \leq cn \sum_{k=0}^{n} b_k^{(n)}(\lambda)\|P_{n-k-1}^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})} \leq cn\|P_n^{(\alpha,\beta)}\|_{L^p(\nu_{\alpha,\beta})}. \tag{26}$$

Thus from (24) and (25) we get (23).

Notice that the upper estimate in (22) can also be proved using the bounds for Jacobi-Sobolev polynomials given in Corollary 2.

In order to prove the lower bound in (22) we will need the following

Proposition 9 *For* $\alpha > 0$ *and* $1 \leq p < \infty$

$$\|Q'^{(\alpha,\beta)}_n\|_{L^p(\nu_{\alpha,\beta})} \geq c \begin{cases} n^{1/2} & \text{if } 4(\alpha+1)/(2\alpha+1) > p, \\ n^{1/2}(\log n)^{1/p} & \text{if } 4(\alpha+1)/(2\alpha+1) = p, \\ n^{\alpha+1-\frac{2\alpha+2}{p}} & \text{if } 4(\alpha+1)/(2\alpha+1) < p. \end{cases} \tag{27}$$

Proof We will use a technique similar to [17, Theorem 7.34]. According to (19)

$$\begin{aligned}
&\int_0^{\pi/2} \theta^{2\alpha+1}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta \\
&\quad > \int_0^{\omega/n} \theta^{2\alpha+1}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta \\
&\quad \geq cn^{-2\alpha-2}\int_0^{\omega} t^{2\alpha+1}\left|Q'^{(\alpha,\beta)}_n\left(\cos\frac{t}{n}\right)\right|^p dt \\
&\quad \cong cn^{p(\alpha+1)-2\alpha-2}\int_0^{\omega} t^{2\alpha+1}|t^{-\alpha}J_\alpha(t)|^p dt \\
&\quad = cn^{p(\alpha+1)-2\alpha-2}\int_0^{\omega} t^{2\alpha+1-p\alpha}|J_\alpha(t)|^p dt.
\end{aligned}$$

On the other hand, from [16, Lemma 2.1], for $\gamma > -1 - p\alpha$ and $1 \leq p < \infty$ we have

$$\int_0^{\omega} t^{\gamma}|J_\alpha(t)|^p dt \sim \begin{cases} c & \text{if } \gamma < p/2-1, \\ c\log\omega & \text{if } \gamma = p/2-1. \end{cases}$$

Thus, for $4(\alpha+1)/(2\alpha+1) \leq p$ and taking ω large enough, (26) follows.

Finally, from (21) we obtain

$$\int_0^{\pi/2} \theta^{2\alpha+1}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta > \int_{\pi/4}^{\pi/2} \theta^{2\alpha+1}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta \sim n^{p/2}. \qquad \square$$

For $\alpha > 0$ and $1 \le p < \infty$, using (26) we get

$$\|Q_n^{(\alpha,\beta)}\|_{W^{1,p}} \ge c \begin{cases} n^{1/2} & \text{if } 4(\alpha+1)/(2\alpha+1) > p, \\ n^{1/2}(\log n)^{1/p} & \text{if } 4(\alpha+1)/(2\alpha+1) = p, \\ n^{\alpha+1-\frac{2\alpha+2}{p}} & \text{if } 4(\alpha+1)/(2\alpha+1) < p. \end{cases} \tag{28}$$

Thus, from (23) and (27) the statement follows. $\square$

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