QUADRATIC DECOMPOSITION OF A LAGUERRE-HAHN POLYNOMIAL SEQUENCE I

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Abstract

Given two sequences of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ such that $B_{2n}(x) = P_n(x^2), n \geq 0$, we show that the Laguerre-Hahn character of one of them remains valid for the other. Then we give relations between their classes and the coefficients of their structure relations. As an application, with an appropriate choice of the sequence $\{P_n\}_{n\geq 0}$, we obtain a new nonsymmetric semi-classical sequence of polynomials $\{B_n\}_{n\geq 0}$ of class $s = 1$.

1 Introduction

A sequence of monic polynomials $\{B_n\}_{n\geq 0}$, $\deg B_n = n,\ n \geq 0$, is said to be a Laguerre-Hahn sequence of class s if it is orthogonal with respect to a linear functional w satisfying

$$(\Phi w)' + \Psi w + B(x^{-1}w^2) = 0$$

where

$$\prod_{c\in Z_\Phi} \left(|\Psi(c) + \Phi'(c)| + |B(c)| + \left|\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\right\rangle\right|\right) \neq 0.$$

Here $\Phi, \Psi,$ and B are polynomials, Z_Φ denotes the set of zeros of Φ, and $s = \max(\deg\Phi - 2, \deg B - 2, \deg\Psi - 1)$. When $B = 0$, the sequence $\{B_n\}_{n\geq 0}$ is

said to be semi-classical (see [13]. When $\Phi = 0$, w is said to be a second degree linear functional.
Laguerre-Hahn orthogonal polynomials were introduced in [11] in connection with continued fractions and the Riccati equations that the corresponding Stieltjes functions satisfy. Among the Laguerre-Hahn sequences of orthogonal polynomials only those which are of class $s = 0$ and those of class $s = 1$ which are symmetric are completely described in the literature (see [1] and [5]).
In general the problem of determining in an explicit way the Laguerre-Hahn polynomial sequences becomes a very difficult task when the class is greater than or equal to one. This is a consequence of the fact that for such sequences the coefficients of the three term recurrence relation satisfy a very complex non linear system. In this paper, we present a constructive process for some special cases of sequences of Laguerre-Hahn polynomials in such a way that we give the answer to the following question.

Consider two sequences of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ such that $B_{2n}(x) = P_n(x^2), n \geq 0$. Notice that this relation holds when the sequence $\{B_n\}_{n\geq 0}$ is symmetric and furthermore $B_{2n+1}(x) = xP_n^*(x^2), n \geq 0$, where P_n^* denotes the nth kernel polynomial associated with $\{P_n\}_{n\geq 0}$. These polynomials have been introduced in [8].
Let assume that either $\{P_n\}_{n\geq 0}$ or $\{B_n\}_{n\geq 0}$ is a Laguerre-Hahn sequence. What can be said about the Laguerre-Hahn character of the other one?.

This question was partially answered by many authors. For instance in [2] the authors proved that, in a symmetric case, if $\{P_n\}_{n\geq 0}$ is semi-classical then the sequence $\{B_n\}_{n\geq 0}$ is also semi-classical. Some extension for general quadratic transforms in the variable have been analyzed in [12]. The first author and coworkers (see [6] and [7]) proved that for some particular symmetric semi-classical sequences of polynomials $\{B_n\}_{n\geq 0}$ the corresponding sequences $\{P_n\}_{n\geq 0}$ are semi-classical of class one as well as some generalizations of this result. More precisely, there is shown that a symmetric sequence $\{B_n\}_{n\geq 0}$ is semi-classical if and only if $\{P_n\}_{n\geq 0}$ is semi-classical. Finally D. Beghdadi [4] showed that if w, the linear functional such that $\{B_n\}_{n\geq 0}$ is the corresponding sequence of orthogonal polynomials, is a second degree linear form then σw, the linear functional corresponding to $\{P_n\}_{n\geq 0}$, is also a second degree linear functional.

The answer we will give to the previous question constitutes a generalization of all these results.

In section 2, we introduce some notations and preliminary results to be used in the sequel. In section 3, we show that if one of the two monic orthogonal polynomial sequences $\{P_n\}_{n\geq 0}$ or $\{B_n\}_{n\geq 0}$ is a Laguerre-Hahn sequence, then the other one is also a Laguerre-Hahn sequence. In this case we prove that

$s \leq 2s'+3$ where s' and s are, respectively, the classes of $\{P_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$. In section 4 we give the expressions of the structure relation coefficients of $\{B_n\}_{n\geq 0}$ in terms of those of $\{P_n\}_{n\geq 0}$. From this structure relation a fourth order differential equation with polynomial coefficients that such polynomials satisfy can be deduced using the techniques introduced in [9]. Finally, in section 5, as an example, we study the class of the linear functional w when $\{P_n\}_{n\geq 0}$ is a particular Laguerre-Hahn sequence of monic polynomials of class zero. We focus our attention on the associated Laguerre polynomials of the first kind (see [3] and [5]) and a singular case (see [1] and [5]), respectively. Thus we obtain a new symmetric sequence of monic Laguerre-Hahn polynomials $\{B_n\}_{n\geq 0}$ of class$s = 2$ as well as a new non symmetric Laguerre-Hahn polynomial sequence of class $s = 2$. The recurrence relation coefficients for such sequences are explicitely given.

2 Notations and preliminary results

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and $\mathcal{P}'$ be its algebraic dual space. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ and $S(u)(z) = -\sum_{n\geq 0} \frac{u_n}{z^{n+1}}$ the formal Stieltjes function of u where $u_n = \langle u, x^n\rangle$, $n \geq 0$, are the moments of u. Let introduce the following operations on $\mathcal{P}'$.

(i) The left multiplication of a linear functional by a polynomial

(2.1) $$\langle gu, f\rangle = \langle u, gf\rangle, f, g \in \mathcal{P}, \ u \in \mathcal{P}'.$$

(ii) The right multiplication of a linear functional by a polynomial

(2.2) $$(uf)(x) = \left\langle u, \frac{xf(x) - \xi f(\xi)}{x - \xi} \right\rangle, \ f \in \mathcal{P}, \ u \in \mathcal{P}'.$$

(iii) The product of two linear functionals

(2.3) $$\langle vu, f\rangle = \langle u, vf\rangle, f \in \mathcal{P}, u,v \in \mathcal{P}'.$$

(iv) The dilation of a linear functional

(2.4) $$\langle h_a u, f\rangle = \langle u, h_a f\rangle, a \in \mathbb{C} - \{0\}, f \in \mathcal{P}, u \in \mathcal{P}',$$

where

(2.5) $$(h_a f)(x) = f(ax).$$

(v) The shift of a linear functional

(2.6) $$\langle \tau_{-b} u, f\rangle = \langle u, \tau_b f\rangle, b \in \mathbb{C}, f \in \mathcal{P}, u \in \mathcal{P}',$$

where

$$(\tau_b f)(x) = f(x-b). \tag{2.7}$$

(vi) The even part of a linear functional

$$\langle \sigma(u), f \rangle = \langle u, \sigma f \rangle \,, \tag{2.8}$$

where

$$(\sigma f)(x) = f(x^2). \tag{2.9}$$

(vii) The division of a linear functional by a polynomial of first degree

$$\left\langle (x-c)^{-1}u, f \right\rangle = \langle u, \theta_c f \rangle \,, c \in \mathbb{C}, f \in \mathcal{P}, u \in \mathcal{P}', \tag{2.10}$$

where

$$(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}. \tag{2.11}$$

(viii). The derivative Du of a linear functional u

$$\langle Du, f \rangle = -\langle u, f' \rangle \,. \tag{2.12}$$

Definition 2.1 *([8]) A sequence of polynomials $\{B_n\}_{n\geq 0}$ is said to be a monic orthogonal polynomial sequence (MOPS) with respect to a linear functional w if*
i) $\deg B_n = n$ *and the leading coefficient of $B_n(x)$ is equal to 1*
ii) $\langle w, B_n B_m \rangle = r_n \delta_{n,m}, n, m \geq 0, r_n \neq 0, n \geq 0.$

It is well known (see [8]) that a sequence of monic orthogonal polynomials satisfies a three-term recurrence relation

$$\begin{gathered} B_0(x) = 1, B_1(x) = x - \beta_0, \\ B_{n+2}(x) = (x - \beta_{n+1})\, B_{n+1}(x) - \gamma_{n+1} B_n(x), n \geq 0 \end{gathered} \tag{2.13}$$

with

$$(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times (\mathbb{C} - \{0\}), n \geq 0.$$

In such conditions w is said to be regular or quasi-definite [8]. In the sequel we consider regular linear functionals w with $w_0 = 1$.

Notice that the orthogonality is preserved by a shifting on the variable. Indeed the shifted sequence $\left\{\widehat{B}_n\right\}_{n\geq 0}$ defined by $\widehat{B}_n(x) = a^{-n} B_n(ax+b)$, $n \geq 0$, $a \neq 0$, satisfies the recurrence relation ([8], [13])

$$(2.14)\qquad \begin{array}{c}\widehat{B}_0(x)=1, \widehat{B}_1(x)=x-\frac{\beta_0-b}{a},\\ \widehat{B}_{n+2}(x)=(x-\frac{\beta_{n+1}-b}{a})\widehat{B}_{n+1}(x)-\frac{\gamma_{n+1}}{a^2}\widehat{B}_n(x), n\geq 0.\end{array}$$

Such a sequence is orthogonal with respect to the linear functional $\widehat{w} = (h_{a^{-1}}\circ\tau_{-b})w$.

The sequence $\{B_n^{(1)}\}_{n\geq 0}$ of associated polynomials of first kind for the sequence $\{B_n\}_{n\geq 0}$ is defined by

$$(2.15)\qquad B_n^{(1)}(x)=\left\langle w, \frac{B_{n+1}(x)-B_{n+1}(\xi)}{x-\xi}\right\rangle, n\geq 0.$$

It satisfies the shifted recurrence relation (see [8], [13])

$$(2.16)\qquad \begin{array}{c}B_0^{(1)}(x)=1, B_1^{(1)}(x)=x-\beta_1\\ B_{n+2}^{(1)}(x)=(x-\beta_{n+2})B_{n+1}^{(1)}(x)-\gamma_{n+2}B_n^{(1)}(x), n\geq 0.\end{array}$$

A linear functional w is said to be symmetric if and only if $w_{2n+1}=0$, $n\geq 0$. Equivalently, in (2.13) we get $\beta_n=0$, $n\geq 0$.

If the sequence $\{B_n\}_{n\geq 0}$ satisfies

$$(2.17)\qquad \begin{array}{c}B_0(x)=1,\ B_1(x)=x-\beta_0,\\ B_{n+2}(x)=(x-(-1)^{n+1}\beta_0)B_{n+1}(x)-\gamma_{n+1}B_n(x),\ n\geq 0,\end{array}$$

with $\gamma_{n+1}\in\mathbb{C}-\{0\}$, $n\geq 0$, then there exist two MOPS $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ such that (see [15])

$$(2.18)\qquad \begin{array}{c}B_{2n}(x)=P_n(x^2),\ \ n\geq 0,\\ B_{2n+1}(x)=(x-\beta_0)R_n(x^2),\ \ n\geq 0.\end{array}$$

Conversely, if $\{P_n\}_{n\geq 0}$ is a MOPS and $\{B_n\}_{n\geq 0}$ is a sequence of monic polynomials such that $B_{2n}(x)=P_n(x^2)$, $n\geq 0$, then $\{B_n\}_{n\geq 0}$ is a MOPS if and only if there exists a MOPS $\{R_n\}_{n\geq 0}$ such that $B_{2n+1}(x)=(x-\beta_0)R_n(x^2)$, $n\geq 0$. The sequence $\{P_n\}_{n\geq 0}$ satisfies the following recurrence relation

$$(2.19)\qquad \begin{array}{c}P_0(x)=1,\ P_1(x)=x-\beta_0^P\\ P_{n+2}(x)=(x-\beta_{n+1}^P)\ P_{n+1}(x)-\gamma_{n+1}^P P_n(x), n\geq 0,\end{array}$$

where

$$(2.20)\qquad \left\{\begin{array}{c}\beta_0^P=\gamma_1+\beta_0^2,\\ \beta_{n+1}^P=\gamma_{2n+2}+\gamma_{2n+3}+\beta_0^2, n\geq 0,\\ \gamma_{n+1}^P=\gamma_{2n+1}\gamma_{2n+2},\ n\geq 0.\end{array}\right.$$

On the other hand, $\{R_n\}_{n\geq 0}$ satisfies

$$R_0(x)=1, R_1(x)=x-\beta_0^R,\qquad (2.21)$$
$$R_{n+2}(x)=(x-\beta_{n+1}^R)R_{n+1}(x)-\gamma_{n+1}^R R_n(x), n\geq 0,$$

where

$$\begin{cases} \beta_0^R=\gamma_1+\gamma_2+\beta_0^2, \\ \beta_{n+1}^R=\gamma_{2n+3}+\gamma_{2n+4}+\beta_0^2, n\geq 0, \\ \gamma_{n+1}^R=\gamma_{2n+2}\gamma_{2n+3}, n\geq 0. \end{cases}\qquad (2.22)$$

Equations (2.20) and (2.22) are obtained from [13, Proposition 4.4] and [14].

Denoting by w, u, and v the linear functionals associated with $\{B_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, and $\{R_n\}_{n\geq 0}$, respectively, we get

$$u=\sigma w,\qquad (2.23)$$

$$\beta_0 u=\sigma(xw),\qquad (2.24)$$

$$v=\gamma_1^{-1}(x-\beta_0^2)u.\qquad (2.25)$$

The regularity of v and (2.25) mean that (see [8], [9], [10], [13])

$$P_n(\beta_0^2)\neq 0, n\geq 0.\qquad (2.26)$$

Conversely, if $\beta_0\in\mathbb{C}$ and u is a linear functional such that the corresponding MOPS $\{P_n\}_{n\geq 0}$ satisfies (2.19) and (2.26) then the linear functional v defined in (2.25) where, $\gamma_1=\beta_0^P-\beta_0^2$, is regular ([8], [9], [10], [13]). Denote by $\{R_n\}_{n\geq 0}$ the MOPS corresponding to v. Then the sequence $\{B_n\}_{n\geq 0}$ defined by (2.18) satisfies (2.17) with (2.20) and (2.22). Furthermore, its corresponding linear functional w satisfies (2.23) and (2.24).

Definition 2.2 *Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional w. $\{B_n\}_{n\geq 0}$ is said to be a Laguerre-Hahn orthogonal polynomial sequence (respectively, w is said to be a Laguerre-Hahn linear functional) of class s if the following conditions hold.*
There exist Φ, a nonzero monic polynomial of degree t, Ψ, a polynomial of degree p, and B, a polynomial of degree r, such that

$$(\Phi w)'+\Psi w+B(x^{-1}w^2)=0\qquad (2.27)$$

as well as

$$\prod_{c\in Z_\Phi}\left(|\Psi(c)+\Phi'(c)|+|B(c)|+\left|\left\langle w,\theta_c\Psi+\theta_c^2\Phi+w\theta_0\theta_c B\right\rangle\right|\right)\neq 0 \tag{2.28}$$

where Z_Φ denotes the set of zeros of Φ. The class s of $\{B_n\}_{n\geq 0}$ is given by $s=\max(p-1,t-2,r-2)$.

In terms of the Stieltjes function, $S(w)$, (2.27) reads (see [9] and [11])

$$A(z)S'(w)(z)=B(z)S^2(w)(z)+C(z)S(w)(z)+D(z) \tag{2.29}$$

with

$$\begin{aligned} A(z)&=\Phi(z),\\ C(z)&=-\Phi'(z)-\Psi(z),\\ D(z)&=-(w\theta_0\Phi)'(z)-(w\theta_0\Psi)(z)-(w^2\theta_0^2B)(z). \end{aligned} \tag{2.30}$$

The condition (2.28) is equivalent to the fact that A,B,C, and D are coprime polynomials and the class s is given by $s=\max(\deg A-2,\deg B-2,\deg C-1)$. If w is a Laguerre-Hahn linear functional of class s fulfilling (2.27), then $\widehat{w}=(h_{a^{-1}}\circ\tau_{-b})w$ is also a Laguerre-Hahn linear functional of the same class and satisfies [4]

$$(\widehat{\Phi}\widehat{w})'+\widehat{\Psi}\widehat{w}+\widehat{B}(x^{-1}\widehat{w}^2)=0 \tag{2.31},$$

where

$$\widehat{\Phi}(z)=a^{-t}\Phi(az+b),\ \widehat{B}(z)=a^{-t}B(az+b),\ \widehat{\Psi}(z)=a^{1-t}\Psi(az+b). \tag{2.32}$$

A Laguerre-Hahn orthogonal polynomial sequence $\{B_n\}_{n\geq 0}$ satisfies the following structure relation (see [1] and [9]). For every $n\geq 0$

$$\Phi(x)B_{n+1}^{'}(x)-B(x)B_n^{(1)}(x)=\frac{(C_{n+1}(x)-C_0(x))}{2}B_{n+1}(x)-\gamma_{n+1}D_{n+1}(x)B_n(x),$$

where

$$C_0(x)=C(x), \tag{2.33}$$

$$E_0(x)=B(x), \tag{2.34}$$

$$D_0(x)=D(x), \tag{2.35}$$

$$E_{n+1}(x)=\gamma_{n+1}D_n(x), n\geq 0, \tag{2.36}$$

$$C_{n+1}(x) = -C_n(x) + 2(x-\beta_n)D_n(x),\ n \geq 0, \tag{2.37}$$

$$\gamma_{n+1}D_{n+1}(x) = -\Phi(x)+E_n(x)+(x-\beta_n)^2 D_n(x)-(x-\beta_n)C_n(x), n \geq 0. \tag{2.38}$$

Remark. If c is a common zero of Φ, B, C, and D then it is also a common zero of E_n, C_n, and $D_n,\ n \geq 0$. Thus in both sides of (2.33) we can divide by $x - c$.

In the sequel we will assume that $\{B_n\}_{n\geq 0}, \{P_n\}_{n\geq 0}, \{R_n\}_{n\geq 0}, u, w$, and v satisfy (2.17)-(2.26).

3 The case when either $\{P_n\}_{n\geq 0}$ or $\{B_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS.

First of all we will prove the two following lemmas

Lemma 3.1 *Using the notations introduced in section 2*

$$\sigma(f(x^2)u) = f(x)\sigma u, \tag{3.1}$$

$$\sigma u' = 2(\sigma(xu))', \tag{3.2}$$

$$\sigma u^2 = (\sigma u)^2 + x^{-1}(\sigma(xu))^2, \tag{3.3}$$

$$\sigma(x^{-1}u^2) = 2x^{-1}\sigma(xu)\sigma u. \tag{3.4}$$

Proof.
For (3.1) and (3.2) see ([13, formula (1.10)]).
For (3.3) we have

$$\begin{aligned}
\left\langle (\sigma u)^2 + x^{-1}(\sigma(xu))^2, x^n \right\rangle &= \sum_{k=0}^{n} (\sigma u)_k (\sigma u)_{n-k} + \sum_{k=0}^{n-1} (\sigma(xu))_k (\sigma(xu))_{n-1-k} \\
&= \sum_{k=0}^{n} u_{2k} u_{2n-2k} + \sum_{k=0}^{n-1} u_{2k+1} u_{2n-2k-1} \\
&= \sum_{k=0}^{2n} u_k u_{2n-k} \\
&= \left\langle \sigma u^2, x^n \right\rangle, n \geq 1,
\end{aligned}$$

and $\left\langle (\sigma u)^2 + x^{-1}(\sigma(xu))^2, 1 \right\rangle = 1 = \left\langle \sigma u^2, 1 \right\rangle$.
So, the desired formula follows.
Finally, for (3.4) we have

$$
\begin{aligned}
\left\langle\ \sigma(x^{-1}u^2), x^n\right\rangle &= \left\langle x^{-1}u^2, x^{2n}\right\rangle \\
&= \left\langle u^2, x^{2n-1}\right\rangle \\
&= \sum_{k=0}^{2n-1} u_k u_{2n-1-k} \\
&= \sum_{k=0}^{n-1} u_{2k+1}u_{2n-2k-2} + \sum_{k=0}^{n-1} u_{2k}u_{2n-2k-1}, n \geq 1.
\end{aligned}
$$

The change of indices $k \to n-k-1$ in the second sum gives

$$
\left\langle\ \sigma(x^{-1}u^2), x^n\right\rangle = 2\sum_{k=0}^{n-1} u_{2k+1}u_{2n-2k-2}, n \geq 1.
$$

On the other hand,

$$
\begin{aligned}
\left\langle 2x^{-1}\sigma(xu)\sigma u, x^n\right\rangle &= 2\left\langle \sigma(xu)\sigma u, x^{n-1}\right\rangle \\
&= 2\sum_{k=0}^{n-1}(\sigma(xu))_k(\sigma u)_{n-1-k} \\
&= 2\sum_{k=0}^{n-1} u_{2k+1}u_{2n-2k-2}, n \geq 1.
\end{aligned}
$$

Hence

$$
\left\langle\ \sigma(x^{-1}u^2), x^n\right\rangle = \left\langle 2x^{-1}\sigma(xu)\sigma u, x^n\right\rangle, n \geq 1.
$$

Notice that this equality is also true for $n = 0$ Indeed,
$\left\langle\ \sigma(x^{-1}u^2), 1\right\rangle = 0 = \left\langle 2x^{-1}\sigma(xu)\sigma u, 1\right\rangle$.

Then (3.4) follows. □

Lemma 3.2 *Let $\widetilde{u}$ be a linear functional and let Φ, Ψ, and B be three polynomials. Let us split these polynomials as follows*
(3.5)
$\Phi(x) = \Phi^e(x^2) + x\Phi^o(x^2), \Psi(x) = \Psi^e(x^2) + x\Psi^o(x^2), B(x) = B^e(x^2) + xB^o(x^2).$

Then we get
(3.6)

$$
\begin{aligned}
&\sigma((\Phi\widetilde{u})' + \Psi\widetilde{u} + B(x^{-1}\widetilde{u}^2)) = (2x\Phi^o(x)\sigma\widetilde{u})' + \Psi^e(x)\sigma\widetilde{u} + xB^o(x)(x^{-1}(\sigma\widetilde{u})^2) \\
&+2(\Phi^e(x)\sigma(x\widetilde{u}))' + \Psi^o(x)\sigma(x\widetilde{u}) + B^o(x)(x^{-1}(\sigma(x\widetilde{u}))^2) + 2B^e(x)(x^{-1}(\sigma\widetilde{u})\sigma(x\widetilde{u})).
\end{aligned}
$$

Proof. From the linearity of the operator σ we have

$$
\sigma((\Phi\widetilde{u})' + \Psi\widetilde{u} + B(x^{-1}\widetilde{u}^2)) = \sigma((\Phi\widetilde{u})') + \sigma(\Psi\widetilde{u}) + \sigma(B(x^{-1}\widetilde{u}^2))). \tag{3.7}
$$

On the other hand, taking into account (3.2)

$$\begin{aligned}\sigma((\Phi\widetilde{u})') &= 2(\sigma(x\Phi\widetilde{u}))' \\ &= 2(\sigma[(x\Phi^e(x^2)+x^2\Phi^o(x^2))\widetilde{u}])' \text{ (by (3.5))} \\ &= 2(\Phi^e(x)\sigma(x\widetilde{u}))' + 2(x\Phi^o(x)\sigma\widetilde{u})' \text{ (by (3.1)).}\end{aligned}$$

From (3.1) and (3.5)

$$\begin{aligned}\sigma(\Psi\widetilde{u}) &= \sigma((\Psi^e(x^2)+x\Psi^o(x^2))\widetilde{u}) \\ &= \Psi^e(x)\sigma\widetilde{u} + \Psi^o(x)\sigma(x\widetilde{u}).\end{aligned}$$

From (3.5) we have

$$\begin{aligned}\sigma(B(x^{-1}\widetilde{u}^2)) &= \sigma(B^e(x^2)(x^{-1}\widetilde{u}^2) + B^o(x^2)\widetilde{u}^2) \\ &= B^e(x)\sigma(x^{-1}\widetilde{u}^2) + B^o(x)\sigma\widetilde{u}^2 \text{ (by (3.1))} \\ &= 2B^e(x)(x^{-1}\sigma(x\widetilde{u})\sigma\widetilde{u}) + \\ &\quad B^o(x)(\sigma\widetilde{u})^2 + B^o(x)(x^{-1}(\sigma(x\widetilde{u}))^2) \text{ by (3.3) and (3.4)} \\ &= 2B^e(x)(x^{-1}\sigma(x\widetilde{u})\sigma\widetilde{u}) + xB^o(x)(x^{-1}(\sigma\widetilde{u})^2) + \\ &\quad B^o(x)(x^{-1}(\sigma(x\widetilde{u}))^2).\end{aligned}$$

So, the statement follows. □

Proposition 3.3 *If $\{P_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS with respect to a linear functional u of class s' then $\{B_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS with respect to the linear functional w of class $s \leq 2s'+3$. Furthermore, if u satisfies*

$$(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2) = 0 \tag{3.8}$$

then, w satisfies (2.27) *with*

$$\Phi(x) = (x+\beta_0)\Phi^P(x^2), \tag{3.9}$$

$$B(x) = 2xB^P(x^2), \tag{3.10}$$

and

$$\Psi(x) = 2x(x+\beta_0)\Psi^P(x^2) - 2\Phi^P(x^2). \tag{3.11}$$

Proof. Put $\widetilde{w} = (\Phi w)' + \Psi w + B(x^{-1}w^2)$. To prove that $\widetilde{w} = 0$ it is enough to prove that $\sigma\widetilde{w} = 0$ and $\sigma(x\widetilde{w}) = 0$.

The even and odd parts of the polynomials $\Phi, \Psi,$ and B in (3.9), (3.10), and (3.11) are
$\Phi^e(x) = \beta_0\Phi^P(x), \Phi^o(x) = \Phi^P(x),$
$\Psi^e(x) = 2x\Psi^P(x) - 2\Phi^P(x), \Psi^o(x) = 2\beta_0\Psi^P(x),$
$B^e(x) = 0, B^o(x) = 2B^P(x).$

Then, from Lemma 3.2 we get

$$\begin{aligned} \sigma\widetilde{w} &= (2x\Phi^P(x)\sigma w)' + 2(x\Psi^P(x) - \Phi^P(x))\sigma w + 2xB^P(x)(x^{-1}(\sigma w)^2) + \\ &\quad + (2\beta_0\Phi^P(x)\sigma(xw))' + 2\beta_0\Psi^P(x)\sigma(xw) + 2B^P(x)(x^{-1}(\sigma(xw))^2). \end{aligned}$$

Taking into account (2.23) and (2.24) we obtain

$$\sigma\widetilde{w} = 2(x+\beta_0^2)[(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2)] = 0 \text{ (by (3.8))}.$$

Similarly,

$$\begin{aligned} \sigma(x\widetilde{w}) &= \sigma(x(\Phi w)' + x\Psi w + xB(x^{-1}w^2)) \\ &= \sigma((x\Phi w)' + (x\Psi - \Phi)w + xB(x^{-1}w^2)) \\ &= 4\beta_0 x[(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2)] \\ &= 0. \end{aligned}$$

So, $\widetilde{w} = 0$. Hence w satisfies (2.27) with (3.9), (3.10), and (3.11), which implies that $\{B_n\}_{n\geq 0}$ is a Laguerre-Hahn polynomial sequence.

Next we will prove that $s \leq 2s'+3$. If $\deg\Psi = p$, $\deg\Phi = t$, $\deg B = r$, $\deg\Psi^P = p'$, $\deg\Phi^P = t'$ and $\deg B^P = r'$, then from (3.9), (3.10) and (3.11) we get $t = 2t'+1$, $r = 2r'+1$, and $p \leq \max(2p'+2, 2t')$. As a consequence:

If either $t' = s'+2$ or $r' = r+2$ and $p' \leq s'+1$ then $t = 2s'+5$ or $r = 2s'+5$ and $p \leq 2s'+4$. Therefore $s \leq 2s'+3$.

If $t' \leq s'+1$, $r' \leq s'+1$ and $p' = s'+1$ then $t \leq 2s'+5$, $r \leq 2s'+5$, and $p = 2s'+4$. Therefore $s \leq 2s'+3$. □

Remark. If we take $\beta_0 = 0$, then we recover the following result stated in [1]:

Corollary 3.4 *Let w be a symmetric Laguerre-Hahn linear functional of class s satisfying* (2.27) *and* (2.28). *If s is even, then Φ and B are even and Ψ is odd. If s is odd, then Φ and B are odd and Ψ is even.*

Proposition 3.5 *If $\{B_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS then $\{P_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS. Furthermore, if w satisfies* (2.27) *and we split these polynomials according to their even and odd parts as in* (3.5) *then u satisfies:*

$$(\Phi_1^P u)' + \Psi_1^P u + B_1^P(x^{-1}u^2) = 0 \qquad (3.12)$$

where

$$\begin{cases} \Phi_1^P(x) = 2(x\Phi^o(x) + \beta_0\Phi^e(x)), \\ \Psi_1^P(x) = \Psi^e(x) + \beta_0\Psi^o(x), \\ B_1^P(x) = (x+\beta_0^2)B^o(x) + 2\beta_0 B^e(x), \end{cases} \qquad (3.13)$$

and

$$(\Phi_2^P u)' + \Psi_2^P u + B_2^P(x^{-1}u^2) = 0 \qquad (3.14)$$

where

$$(3.15)\qquad \begin{cases} \Phi_2^P(x) = 2(x\Phi^e(x) + \beta_0 x\Phi^o(x)), \\ \Psi_2^P(x) = x\Psi^o(x) - \Phi^e(x) + \beta_0(\Psi^e(x) - \Phi^o(x)), \\ B_2^P(x) = (x+\beta_0^2)B^e(x) + 2\beta_0 xB^o(x)\ . \end{cases}$$

Proof.
Applying Lemma 3.2 to the functional equation (2.27) and using (2.23) and (2.24) we get (3.12) with (3.13).
On the other hand, multiplication of (2.27) by x gives

$$(3.16)\qquad (x\Phi(x)w)' + (x\Psi(x) - \Phi(x))w + xB(x)(x^{-1}w^2) = 0.$$

Applying Lemma 3.2 to (3.16) and using (2.22) and (2.23) we get (3.14) with (3.15).

Notice that from the functional equation (3.12) we can not conclude that u is a Laguerre-Hahn linear functional since we have not proved that at least one of the polynomials Φ_1^P, Ψ_1^P, and B_1^P is not zero, which is not always true. For example if w is symmetric ($\beta_0 = 0$) of even class s, then, according to the Corollary 3.4, we have $\Phi_1^P = \Psi_1^P = B_1^P = 0$. This is the reason of the functional equation (3.14).
Now assume that $\Phi_1^P = \Psi_1^P = B_1^P = 0$ and $\Phi_2^P = \Psi_2^P = B_2^P = 0$. Then, from (3.13) and (3.15), we get

$$\begin{cases} x\Phi^o(x) + \beta_0\Phi^e(x) = 0, \\ x\Phi^e(x) + \beta_0 x\Phi^o(x) = 0, \end{cases}$$

and, as a consequence,

$$\begin{cases} (x-\beta_0^2)\Phi^e(x) = 0, \\ (x^2-\beta_0^2)\Phi^o(x) = 0. \end{cases}$$

Hence

$$\Phi^e(x) = \Phi^o(x) = 0$$

and therefore $\Phi(x) = 0$.
Similarly, using (3.13) and (3.15), we prove that $\Psi = B = 0$, a contradiction. Consequently, at least one of the polynomials Φ_1^P, Ψ_1^P, B_1^P, Φ_2^P, Ψ_2^P, and B_2^P is not zero. Therefore u is a Laguerre-Hahn linear functional. □

4 The structure relation

Let assume that $\{P_n\}_{n\geq 0}$ is a Laguerre-Hahn polynomial sequence. According to Proposition 3.3, $\{B_n\}_{n\geq 0}$ is also a Laguerre-Hahn polynomial sequence. Then it satisfies a structure relation. Our aim is to express C_n and D_n, $n \geq 0$, the polynomial coefficients of the structure relation for the sequence $\{B_n\}_{n\geq 0}$, in terms of those of $\{P_n\}_{n\geq 0}$ which we will denote C_n^P and D_n^P, $n \geq 0$. We get the following result:

Proposition 4.1 *If $\{P_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS of class s' satisfying the structure relation*

(4.1)

$$\Phi^P(x)P'_{n+1}(x)-B^P(x)P_n^{(1)}(x) = \frac{C_{n+1}^P(x)-C_0^P(x)}{2}P_{n+1}(x)-\gamma_{n+1}^P D_{n+1}^P(x)P_n(x),$$

for every $n \geq 0$, then the Laguerre-Hahn MOPS $\{B_n\}_{n\geq 0}$ of class s satisfies (2.33) *where for every $n \geq s'+2$,*

$$C_{2n}(x) = \Phi^P(x^2) + 2x(x+\beta_0)(C_n^P(x^2) + 2\frac{\gamma_n^P D_n^P(x^2)}{\gamma_{2n-1}}), \tag{4.2}$$

(4.3)

$$C_{2n+1}(x) = -\Phi^P(x^2) - 2x(x+\beta_0)C_n^P(x^2) + 4x(x+\beta_0)(x^2-\beta_0^2 - \frac{\gamma_n^P}{\gamma_{2n-1}})D_n^P(x^2),$$

$$D_{2n}(x) = 2x(x+\beta_0)^2 D_n^P(x^2), \tag{4.4}$$

(4.5)

$$D_{2n+1}(x) = x(C_{n+1}^P(x^2) - C_n^P(x^2) + 2\frac{\gamma_{n+1}^P}{\gamma_{2n+1}}D_{n+1}^P(x^2) + 2(x^2-\beta_0^2-\frac{\gamma_n^P}{\gamma_{2n-1}})D_n^P(x^2)).$$

To prove this proposition we need the following lemma.

Lemma 4.2 *The associated polynomials of the first kind for the sequence of monic polynomials $\{B_n\}_{n\geq 0}$ are given by*

$$B_{2n+1}^{(1)}(\zeta) = (\zeta+\beta_0)P_n^{(1)}(\zeta^2),\ n \geq 0, \tag{4.6}$$

$$B_{2n+2}^{(1)}(\zeta) = P_{n+1}^{(1)}(\zeta^2) + \gamma_{2n+3}P_n^{(1)}(\zeta^2),\ n \geq 0. \tag{4.7}$$

Proof.

Using (2.14) one has

$$\begin{aligned}
B_{2n+1}^{(1)}(\zeta) &= \left\langle w, \tfrac{B_{2n+2}(x) - B_{2n+2}(\zeta)}{x-\zeta}\right\rangle \text{ (w acts on the variable x)} \\
&= \left\langle w, \tfrac{P_{n+1}(x^2) - P_{n+1}(\zeta^2)}{x-\zeta}\right\rangle \text{ (from (2.18))} \\
&= \left\langle w, (x+\zeta)\tfrac{P_{n+1}(x^2) - P_{n+1}(\zeta^2)}{x^2-\zeta^2}\right\rangle \\
&= \left\langle \sigma((x+\zeta)w), \tfrac{P_{n+1}(x) - P_{n+1}(\zeta^2)}{x-\zeta^2}\right\rangle \\
&= (\beta_0+\zeta)\left\langle u, \tfrac{P_{n+1}(x) - P_{n+1}(\zeta^2)}{x-\zeta^2}\right\rangle \text{ (from (2.23) and (2.24))} \\
&= (\beta_0+\zeta)P_n^{(1)}(\zeta^2).
\end{aligned}$$

Hence, one get (4.6).

From (2.15) and (2.17)

$B^{(1)}_{2n+3}(\zeta) = (\zeta + \beta_0)B^{(1)}_{2n+2}(\zeta) - \gamma_{2n+3}B^{(1)}_{2n+1}(\zeta),$

which implies

$$\begin{aligned} B^{(1)}_{2n+2}(\zeta) &= \tfrac{B^{(1)}_{2n+3}(\zeta)+\gamma_{2n+3}B^{(1)}_{2n+1}(\zeta)}{\zeta+\beta_0} \\ &= P^{(1)}_{n+1}(\zeta^2) + \gamma_{2n+3}P^{(1)}_{n}(\zeta^2)(\text{ according to (4.6) }). \end{aligned}$$

Therefore (4.7) follows. □

Next, we will give the proof of Proposition 4.1.

Proof. Differentiation of (2.18) gives

$$B'_{2n+2}(x) = 2xP'_{n+1}(x^2),\ n \geq 0. \tag{4.8}$$

The change of variable $x \to x^2$ in (4.1) yields, for $n \geq 0$,

$$\Phi^P(x^2)P'_{n+1}(x^2) - B^P(x^2)P^{(1)}_n(x^2) = \tfrac{C^P_{n+1}(x^2)-C^P_0(x^2)}{2}P_{n+1}(x^2) - \gamma^P_{n+1}D^P_{n+1}(x^2)P_n(x^2).$$

Multiplying both sides of the above expression by $2x(x+\beta_0)$ and taking into account (2.18), (3.9), (3.10), (4.6), and (4.8) we get

$$\begin{aligned} \Phi(x)B'_{2n+2}(x) - B(x)B^{(1)}_{2n+1}(x) = x(x+\beta_0)(C^P_{n+1}(x^2) - C^P_0(x^2))B_{2n+2}(x) - \\ 2\gamma^P_{n+1}x(x+\beta_0)D^P_{n+1}(x^2)B_{2n}(x),\ n \geq 0. \end{aligned} \tag{4.9}$$

The change of indices $n \to 2n$ in (2.17) gives

$$B_{2n}(x) = \frac{1}{\gamma_{2n+1}}(-B_{2n+2}(x) + (x+\beta_0)B_{2n+1}(x)),\ n \geq 0. \tag{4.10}$$

Substitution of (4.10) in (4.9) gives for every $n \geq 0$

$$\begin{aligned} \Phi(x)B'_{2n+2}(x) - B(x)B^{(1)}_{2n+1}(x) = -2x(x+\beta_0)^2\tfrac{\gamma^P_{n+1}D^P_{n+1}(x^2)}{\gamma_{2n+1}}B_{2n+1}(x) + \\ x(x+\beta_0)(C^P_{n+1}(x^2) - C^P_0(x^2) + \tfrac{2\gamma^P_{n+1}D^P_{n+1}(x^2)}{\gamma_{2n+1}})B_{2n+2}(x). \end{aligned} \tag{4.11}$$

The identification with (2.33), where $n \to 2n+1$, leads to

$$M(x,n)B_{2n+2}(x) = N(x,n)B_{2n+1}(x),$$

where for every $n \geq 0$,

$$M(x,n) = 2x(x+\beta_0)(\tfrac{C_{n+1}^P(x^2)-C_0^P(x^2)}{2} + \tfrac{\gamma_{n+1}^P D_{n+1}^P(x^2)}{\gamma_{2n+1}}) - \tfrac{C_{2n+2}(x)-C_0(x)}{2},$$

$$N(x,n) = 2x(x+\beta_0)^2 \tfrac{\gamma_{n+1}^P D_{n+1}^P(x^2)}{\gamma_{2n+1}} - \gamma_{2n+2}D_{2n+2}(x).$$

Taking into account that B_{2n+1} and B_{2n+2} have no common zeros, then B_{2n+2} divides $N(x,n)$, which is a polynomial of degree at most $2s'+3$.

As a consequence, $M(x,n) = N(x,n) = 0$, $n \geq s'+1$. Therefore

(4.12) $$D_{2n+2}(x) = 2\frac{\gamma_{n+1}^P}{\gamma_{2n+1}\gamma_{2n+2}}x(x+\beta_0)^2 D_{n+1}^P(x^2),\ n \geq s'+1,$$

and

(4.13)

$$C_{2n+2}(x) = C_0(x) + 4x(x+\beta_0)(\frac{C_{n+1}^P(x^2)-C_0^P(x^2)}{2} + \frac{\gamma_{n+1}^P D_{n+1}^P(x^2)}{\gamma_{2n+1}}),\ n \geq s'+1.$$

Taking into account (2.20) in (4.12), where $n \to n-1$, we get (4.4).

From (4.13), where $n \to n-1$, (2.30), (2.34), and (3.11) we get (4.2).

Substituting (4.2) and (4.4) in (2.38), where $n \to 2n$, and taking into account $\beta_{2n} = \beta_0$, (4.3) follows.

Finally substitution of (4.2), where $n \to n+1$, and (4.3) in (2.38), where $n \to 2n+1$, gives (4.5). □

5 Examples

5.1 A symmetric Laguerre-Hahn sequence of orthogonal polynomials of class $s = 2$

Let us consider $\beta_0 = 0$ and let $\{P_n\}_{n\geq 0}$ be the Laguerre-Hahn polynomial sequence satisfying the recurrence (2.19) with

(5.1) $$\beta_0^P = \alpha - 1 + \lambda, \quad \beta_{n+1}^P = 2n + \alpha + 1, \quad n \geq 0$$

$$(5.2) \qquad \gamma_1^P = \rho, \quad \gamma_{n+1}^P = n(n+\alpha), \quad n \geq 1,$$

where $\lambda \in \mathbb{C}$, $\alpha + n \neq 0, \quad n \geq 1$, and $\rho \in \mathbb{C} - \{0\}$.
This sequence has been studied in [1] and [5] in the framework of the Laguerre-Hahn analogues of class 0 of the Laguerre polynomial sequence since its corresponding linear functional u satisfies (3.8) with

$$(5.3) \qquad \Phi^P(x) = x,$$

$$(5.4) \qquad \Psi^P(x) = -x + \alpha - 1,$$

$$(5.5) \qquad B^P(x) = x^2 + [2(1-\alpha) - \lambda]x + \alpha(\alpha - 1 + \lambda) - \rho.$$

Using (5.1), (5.2), and (2.18) we can show by an easy recurrence that

$$(5.6) \qquad P_n(0) = (-1)^n((\alpha)_n(\alpha + \lambda - 1 - \frac{\rho}{\alpha}) + (n-1)!\rho)/\alpha, \quad n \geq 2.$$

Here, for every complex number α, $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1), n \geq 1$, will denote the Pochhammer symbol.
Hence, under some conditions on the parameters ρ , λ, and α, we can assume that $\{P_n\}_{n\geq 0}$ verifies

$$P_n(0) \neq 0, \quad n \geq 0.$$

Therefore, according to Proposition 3.3, w is a Laguerre-Hahn linear functional of class $s \leq 3$ and satisfies (2.27) with $\Phi(x) = x^3$, $\Psi(x) = -2x^2(x^2 - \alpha + 2)$,

$$B(x) = 2x\{x^4 + [2(1-\alpha) - \lambda]x^2 + \alpha(\alpha - 1 + \lambda) - \rho\}.$$

To determine the class s of w we use (2.28). One has $\Phi(0) = 0$, $(\Phi' + \Psi)(0) = 0$, $B(0) = 0$, and $\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0$.
As a consequence, we can divide by x in (2.27) and we get

$$(5.7) \qquad \Phi(x) = x^2,$$

$$(5.8) \qquad \Psi(x) = -2x^3 + (2\alpha - 4)x,$$

$$(5.9) \qquad B(x) = 2\{x^4 + [2(1-\alpha) - \lambda]x^2 + \alpha(\alpha - 1 + \lambda) - \rho\}.$$

So w is a Laguerre-Hahn linear functional of class $s \leq 2$. Notice that when $\rho \neq \alpha(\alpha-1+\lambda)$ a new symmetric Laguerre-Hahn form of class $s = 2$ is obtained.

Concerning the expression of γ_n, the change of indices $n \to 2n$ in (2.17) yields

$$B_{2n+2}(x) = xB_{2n+1}(x) - \gamma_{2n+1}B_{2n}(x), \quad n \geq 0. \tag{5.10}$$

Taking into account (2.18),

$$P_{n+1}(x^2) = x^2R_n(x^2) - \gamma_{2n+1}P_n(x^2), \quad n \geq 0. \tag{5.11}$$

Thus for $x = 0$ (5.11) becomes

$$P_{n+1}(0) = -\gamma_{2n+1}P_n(0)$$

or, equivalently,

$$\gamma_{2n+1} = -\frac{P_{n+1}(0)}{P_n(0)}, \quad n \geq 0. \tag{5.12}$$

From (2.20), one has

$$\gamma_{2n+2} = -\frac{\gamma_{n+1}^P P_n(0)}{P_{n+1}(0)}, \quad n \geq 0. \tag{5.13}$$

From (5.2), (5.6), (5.12), and (5.13) we get

$$\gamma_1 = \alpha - 1 + \lambda \tag{5.14}$$

$$\gamma_{2n+1} = \frac{(\alpha)_{n+1}(\alpha + \lambda - 1 - \frac{\rho}{\alpha}) + \rho n!}{(\alpha)_n(\alpha + \lambda - 1 - \frac{\rho}{\alpha}) + \rho(n-1)!}, \quad n \geq 1, \tag{5.15}$$

$$\gamma_{2n+2} = \frac{(\alpha)_{n+1}n(\alpha + \lambda - 1 - \frac{\rho}{\alpha}) + \rho(n+\alpha)n!}{(\alpha)_{n+1}(\alpha + \lambda - 1 - \frac{\rho}{\alpha}) + \rho n!}, \quad n \geq 0. \tag{5.16}$$

5.2 A non symmetric Laguerre-Hahn sequence of orthogonal polynomials of class $s = 2$

Let us consider a complex number $\beta_0 \neq 0$ and let $\{P_n\}_{n\geq 0}$ be the Laguerre-Hahn polynomial sequence satisfying the recurrence (2.17) with

$$\beta_0^P = \alpha + 3, \quad \beta_{n+1}^P = 2n + \alpha + 5, \quad n \geq 0 \tag{5.17}$$

$$\gamma_1^P = \rho, \quad \gamma_{n+1}^P = (n+2)(n+\alpha+2), \quad n \geq 1, \tag{5.18}$$

where $\alpha + n + 2 \neq 0, \quad n \geq 1$.

This sequence has been studied in [3] from the point of view of the analytic properties of general associated Laguerre and Hermite polynomials as well as in [1] and [5] in the framework of the Laguerre-Hahn analogues of class 0 of the Laguerre polynomial sequence since its corresponding linear functional u satisfies (3.8) with

$$\Phi^P(x) = x, \tag{5.19}$$

$$\Psi^P(x) = x - \alpha - 3, \tag{5.20}$$

$$B^P(x) = -\alpha - 1. \tag{5.21}$$

We will assume that

$$P_n(\beta_0^2) \neq 0, n \geq 0,$$

so that the linear functional w is regular. Notice that an explicit expression of P_n in terms of hypergeometric functions is given in [3, formula (2.8)].

Therefore, according to Proposition 3.3, w is a Laguerre-Hahn linear functional of class $s \leq 3$ and satisfies (2.27) with

$$\Phi(x) = (x + \beta_0)x^2, \tag{5.22}$$

$$\Psi(x) = 2x(x + \beta_0)(x^2 - \alpha - 3) - 2x^2, \tag{5.23}$$

$$B(x) = -2(\alpha + 1)x. \tag{5.24}$$

To determine the class s of w we use (2.28). One has $\Phi(0) = 0$, $(\Phi' + \Psi)(0) = 0$, $B(0) = 0$, and $\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B\rangle = 0$.

As a consequence, we can divide by x in (2.27) and we get

$$\Phi(x) = (x + \beta_0)x^2, \tag{5.25}$$

$$\Psi(x) = 2(x + \beta_0)(x^2 - \alpha - 3) - 2x, \tag{5.26}$$

$$B(x) = -2(\alpha + 1). \tag{5.27}$$

So w is a Laguerre-Hahn linear functional of class $s = 2$.

Concerning the expression of γ_n, the change of indices $n \to 2n$ in (2.17) yields

$$B_{2n+2}(x) = (x + \beta_0)B_{2n+1}(x) - \gamma_{2n+1}B_{2n}(x), \quad n \geq 0. \tag{5.28}$$

Taking into account (2.18),

$$P_{n+1}(x^2) = (x^2 - {\beta_0}^2)R_n(x^2) - \gamma_{2n+1}P_n(x^2), \quad n \geq 0. \tag{5.29}$$

Thus, for $x = \beta_0^2$ (5.29) becomes

$$P_{n+1}(\beta_0^2) = -\gamma_{2n+1}P_n(\beta_0^2)$$

or, equivalently,

$$\gamma_{2n+1} = -\frac{P_{n+1}(\beta_0^2)}{P_n(\beta_0^2)}, \quad n \geq 0. \tag{5.30}$$

From (2.20), one has

$$\gamma_{2n+2} = \frac{(n+2)(n+\alpha+2)}{\gamma_{2n+1}}, \quad n \geq 0. \tag{5.31}$$

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