# LAURENT POLYNOMIAL PERTURBATIONS OF LINEAR FUNCTIONALS. AN INVERSE PROBLEM. *

K. CASTILLO †, L. GARZA ‡, AND F. MARCELLÁN §

**Abstract.** Given a linear functional $\mathcal{L}$ in the linear space $\mathbb{P}$ of polynomials with complex coefficients we analyze those linear funcionals $\tilde{\mathcal{L}}$ such that for a fixed $\alpha \in \mathbb{C}$

$$\left\langle \tilde{\mathcal{L}}, (z + z^{-1} - (\alpha + \bar{\alpha}))p \right\rangle = \langle \mathcal{L}, p \rangle,$$

for every $p \in \mathbb{P}$.
We obtain the relation between the corresponding Carathéodory functions in such a way that a linear spectral transform appears.
If $\mathcal{L}$ is a positive definite linear functional, the necessary and sufficient conditions in order for $\tilde{\mathcal{L}}$ to be a quasi-definite linear functional are given. The relation between the corresponding sequences of monic orthogonal polynomials is presented.

**Key words.** Orthogonal polynomials, linear functionals, Laurent polynomials, linear spectral transformations.

**AMS subject classifications.** 42C05

## 1. Introduction.

Let $\mathbf{T} = \{c_{k-l}\}_{k,l\geqslant 0}$ be an Hermitian, Toeplitz matrix. On the linear space $\Lambda$ of the Laurent polynomials ($\Lambda = span\{z^n\}_{n\in\mathbb{Z}}$) with complex coefficients, we can introduce a linear functional $\mathcal{L} : \Lambda \to \mathbb{C}$ such that

$$\langle \mathcal{L}, z^n \rangle = c_n, \quad n \geqslant 0.$$

The complex number $c_n$ is said to be the *nth* moment associated with $\mathcal{L}$. From the Hermitian character of $\mathbf{T}$ we have

$$c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \bar{c}_{-n}, \quad n \in \mathbb{Z}.$$

Then, a bilinear functional associated with $\mathcal{L}$ in the linear space $\mathbb{P}$ of polynomials with complex coefficients can be defined as follows (see [7], [11])

$$\langle p(z), q(z) \rangle_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\bar{q}(z^{-1}) \right\rangle \tag{1.1}$$

where $p, q \in \mathbb{P}$.

$\mathcal{L}$ is said to be quasi-definite if the principal leading submatrices of $\mathbf{T}$ are non-singular. In this case, there exists a unique sequence of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ such that

$$\langle \Phi_n, \Phi_m \rangle_{\mathcal{L}} = k_n \delta_{n,m}, \tag{1.2}$$

where $k_n = \|\Phi_n\|^2 \neq 0$ for every $n \geqslant 0$. It is said to be the monic orthogonal polynomial sequence associated with $\mathcal{L}$.

These polynomials satisfy the following recurrence relations due to G. Szegő (see [7], [10], [16], [19])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \geqslant 0, \tag{1.3}$$

$$\Phi_{n+1}(z) = (1 - |\Phi_{n+1}(0)|^2)z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \quad n \geqslant 0, \tag{1.4}$$

*Dedicated to professor Richard Varga with occasion of his $80^{th}$ birthday.

†Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911, Leganés, Spain.(kcastill@math.uc3m.es).

‡Universidad Autónoma de Tamaulipas, Carretera Sendero Nacional Km. 3, AP2005, Matamoros, México. (lgaona@uat.edu.mx).

§Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911, Leganés, Spain.(pacomarc@ing.uc3m.es).

where $\Phi_n^*(z) = z^n\bar{\Phi}_n(z^{-1})$ is the so-called reversed polynomial associated with $\Phi_n(z)$ (see [16]), and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$, with $|\Phi_n(0)| \neq 1$ for every $n \geqslant 1$, are called reflection (or Verblunsky) parameters. Moreover, we have

$$\frac{k_{n+1}}{k_n} = 1 - |\Phi_{n+1}(0)|^2, \quad n \geqslant 0. \tag{1.5}$$

On the other hand, if the determinants of the leading principal submatrices of $\mathbf{T}$ are positive, then the linear functional is said to be positive definite and it has the following integral representation

$$\langle \mathcal{L}, p(z)\rangle = \int_{\mathbb{T}} p(z)d\sigma(z), \quad p \in \Lambda, \tag{1.6}$$

where $\sigma$ is a nontrivial probability measure supported on the unit circle (see [7], [10], [11], [16]), assuming $c_0 = 1$. We will maintain this assumption throughout the remaining of the manuscript.

The measure $\sigma$ can be decomposed as the sum of an absolutely continuous measure with respect to the Lebesgue measure $\frac{d\theta}{2\pi}$ and a singular measure. Thus, if $\omega = \sigma'$, then

$$d\sigma(\theta) = \omega(\theta)\frac{d\theta}{2\pi} + d\sigma_s(\theta). \tag{1.7}$$

In the positive definite case, there exists a unique sequence of orthonormal polynomials $\{\varphi_n\}_{n\geqslant 0}$ such that

$$\langle \varphi_n, \varphi_m\rangle_{\mathcal{L}} = \delta_{n,m}. \tag{1.8}$$

Notice that $\varphi_n(z) = \Phi_n(z)/\|\Phi_n\|$. Moreover, we have $|\Phi_n(0)| < 1$ for every $n \geqslant 1$.

The $n$-th reproducing kernel $K_n(z,y)$ associated with $\{\Phi_n\}_{n\geqslant 0}$, is defined by

$$K_n(z,y) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(z)}{k_j} = \frac{\overline{\Phi_{n+1}^*(y)}\Phi_{n+1}^*(z) - \overline{\Phi_{n+1}(y)}\Phi_{n+1}(z)}{k_{n+1}(1-\bar{y}z)}.$$

Furthermore,

$$\Phi_n^*(z) = k_n K_n(z,0).$$

In terms of the moments $\{c_n\}_{n\geqslant 0}$, an analytic function in a neighborhood of $z = 0$

$$F(z) = 1 + 2\sum_{n=1}^{\infty} c_{-n}z^n \tag{1.9}$$

can be introduced. If $\mathcal{L}$ is a positive definite linear functional, then $F(z)$ is analytic in $|z| < 1$ and $\mathfrak{Re}\,(F(z)) > 0$ therein. In such a case $F(z)$ is said to be a Carathéodory function and it can be represented as a Riesz-Herglotz transform of the nontrivial probability measure $\sigma$ introduced in (1.6) (see [7], [11], [16])

$$F(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\sigma(w).$$

As a convention, if $\{c_k\}_{k\in\mathbb{Z}}$ is the sequence of moments associated with a quasi-definite functional $\mathcal{L}$, then the function given in (1.9) is said to be the Carathéodory function associated with $\mathcal{L}$. $F(z)$ can be interpreted as a functional "mirror" of the sequence $\{c_n\}_{n\geqslant 0}$. The Carathéodory functions for some perturbations of a measure $\sigma$ (or its associated linear functional) have been studied in [13] for the following three canonical cases (*i*) $d\tilde{\sigma} = |z-\alpha|^2 d\sigma$, $\alpha \in \mathbb{C}$, $|z| = 1$ (Christoffel transformation).

(*ii*) $d\tilde{\sigma} = d\sigma + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$, $\alpha, \boldsymbol{m} \in \mathbb{C}$, $\alpha \neq 0$. (Uvarov transformation).

(*iii*) $d\tilde{\sigma} = \frac{1}{|z-\alpha|^2} d\sigma + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$, $|z| = 1$, $\boldsymbol{m} \in \mathbb{C}$, and $|\alpha| \in \mathbb{R} \setminus \{0, 1\}$. (Geronimus transformation).

These three examples of canonical spectral transforms (see [3], [8], [9], [12], [14], among others) are the analogues on the unit circle of the canonical spectral transforms on the real line considered by several authors (see [1], [17], [21] and [22]). Moreover, if we denote these transformations by $\mathcal{F}_C(\alpha)$, $\mathcal{F}_U(\alpha, \boldsymbol{m})$, and $\mathcal{F}_G(\alpha)$, respectively, then we have

Proposition 1.1.

(*i*) $\mathcal{F}_G(\alpha, \boldsymbol{m}) \circ \mathcal{F}_C(\alpha) = \mathcal{F}_U(\alpha, \boldsymbol{m})$.

(*ii*) $\mathcal{F}_C(\alpha) \circ \mathcal{F}_G(\alpha, \boldsymbol{m}) = \mathcal{I}$ *(Identity transformation).*

Notice that in these three cases, the corresponding Carathéodory functions are related by

$$\tilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)},$$

where $A, B$, and $D \neq 0$ are polynomials in the variable $z$. They constitute examples of the so-called *linear spectral transformations*. Other examples of spectral transformations have been analyzed in [13].

Furthermore, in [5], [6] we have studied a perturbation $\mathcal{L}_R$ of $\mathcal{L}$ defined by

$$\langle \mathcal{L}_R, q \rangle = \left\langle \mathcal{L}, \frac{1}{2}(z - \alpha + z^{-1} - \bar{\alpha})q \right\rangle, \quad q \in \Lambda, \tag{1.10}$$

where $\alpha \in \mathbb{C}$. Here the relation between the associated Carathéodory functions is

$$F_R(z) = \frac{[z^2 - (\alpha + \bar{\alpha})z + 1]F(z) + z^2 + (c_1 - c_{-1})z - 1}{z}, \tag{1.11}$$

i.e. this is a linear spectral transform that is not one of the canonical linear spectral transformations above mentioned. Indeed, the Christoffel transformation is a particular case of this transformation when $|\mathfrak{Re}[\alpha]| > 1$.

On the other hand, assuming that $\mathcal{L}$ is a quasi-definite linear functional, necessary and sufficient conditions for the quasi-definiteness of $\mathcal{L}_R$ are obtained in [2] and [18]. This transformation is denoted by $\mathcal{F}_R(\alpha)$.

It is natural to analyze the existence of the inverse transformation, i.e. if there exists a linear functional $\mathcal{L}_{R^{-1}}$ such that

$$\left\langle \mathcal{L}_{R^{-1}}, [z + z^{-1} - (\alpha + \bar{\alpha})]p(z) \right\rangle = \langle \mathcal{L}, p(z) \rangle, \quad p \in \Lambda, \tag{1.12}$$

as well as if the quasi-definite character of the linear functional is preserved by such a transformation. This is one of the goals of our contribution. Notice that the transformation (1.12) does not define a unique linear functional $\mathcal{L}_{R^{-1}}$. As we will show in Section 3, uniqueness depends on a free parameter.

The structure of the manuscript is as follows. In section 2, we assume the linear functional $\mathcal{L}_{R^{-1}}$ is quasi-definite and we obtain the relation between the corresponding Carathéodory functions. In Section 3, we analyze the conditions on the nontrivial probability measure $\sigma$ such that $\mathcal{L}_{R^{-1}}$ is a quasi-definite linear functional and we obtain the necessary conditions for $\mathcal{L}_{R^{-1}}$ to be quasi-definite, an expression for the corresponding sequence of monic orthogonal polynomials, and a recursive algorithm to compute its family of Verblunsky parameters. Finally, in Section 4, several examples of this transformation for three illustrative cases of nontrivial measures are analyzed.

**2. Carathéodory functions.** Assuming that $\mathcal{L}_{R^{-1}}$ is a quasi-definite linear functional, we will denote its associated Carathéodory function by $F_{R^{-1}}(z)$. First we will study the relation between $F(z)$ and $F_{R^{-1}}(z)$.

Proposition 2.1. *$F_{R^{-1}}(z)$, the Carathéodory function associated to $\mathcal{L}_{R^{-1}}$, is a linear spectral transformation of $F(z)$ given by*

$$F_{R^{-1}}(z) = \frac{zF(z)}{z^2-(\alpha+\bar{\alpha})z+1} + \boldsymbol{m}_1\frac{z+b}{z-b} + \boldsymbol{m}_2\frac{z+\bar{b}}{z-\bar{b}}, \tag{2.1}$$

*where $b, \bar{b}$ are the zeros of $z^2-(\alpha+\bar{\alpha})z+1$, with $|b|=1$, and*

$$\boldsymbol{m}_1 = -\frac{1}{2}\left(\tilde{c}_0 + \frac{\mathfrak{Im}(\tilde{c}_1)}{\mathfrak{Im}(b)}\right), \quad \boldsymbol{m}_2 = -\frac{1}{2}\left(\tilde{c}_0 - \frac{\mathfrak{Im}(\tilde{c}_1)}{\mathfrak{Im}(b)}\right).$$

*Proof.* From (1.12), we get

$$c_{-k} = \tilde{c}_{-(k+1)} + \tilde{c}_{-(k-1)} - (\alpha+\bar{\alpha})\tilde{c}_{-k}, \tag{2.2}$$

Multiplying (2.2) by $z^k$, $k=1,2,\ldots$, and replacing in (1.9), we get

$$\begin{aligned}
\sum_{k=1}^{\infty} c_{-k}z^k &= \sum_{k=1}^{\infty}\tilde{c}_{-(k+1)}z^k + \sum_{k=1}^{\infty}\tilde{c}_{-(k-1)}z^k - (\alpha+\bar{\alpha})\sum_{k=1}^{\infty}\tilde{c}_{-k}z^k,\\
\frac{F(z)-1}{2} &= z^{-1}\left(\frac{F_{R^{-1}}(z)-\tilde{c}_0}{2} - \tilde{c}_{-1}z\right) + z\left(\frac{F_{R^{-1}}(z)-\tilde{c}_0}{2} + \tilde{c}_0\right)\\
&\quad - (\alpha+\bar{\alpha})\left(\frac{F_{R^{-1}}(z)-\tilde{c}_0}{2}\right),\\
F(z)-1 &= [z+z^{-1}-(\alpha+\bar{\alpha})]F_{R^{-1}}(z) + \tilde{c}_0[z-z^{-1}+(\alpha+\bar{\alpha})] - 2\tilde{c}_{-1}.
\end{aligned}$$

Therefore

$$F_{R^{-1}}(z) = \frac{F(z) + [z^{-1}-z-(\alpha+\bar{\alpha})]\tilde{c}_0 + 2\tilde{c}_{-1} - 1}{z+z^{-1}-(\alpha+\bar{\alpha})}. \tag{2.3}$$

Notice that, from (2.2), $1+(\alpha+\bar{\alpha})\tilde{c}_0 = \tilde{c}_1 + \tilde{c}_{-1}$, and thus

$$F_{R^{-1}}(z) = \frac{zF(z) - \tilde{c}_0z^2 + (\tilde{c}_{-1}-\tilde{c}_1)z + \tilde{c}_0}{z^2-(\alpha+\bar{\alpha})z+1}, \tag{2.4}$$

which is equivalent to (2.1). □

On the other hand, from (2.1)

$$\begin{aligned}
F_{R^{-1}}(z) &= \left(\frac{\frac{b}{b-\bar{b}}}{z-b} - \frac{\frac{\bar{b}}{b-\bar{b}}}{z-\bar{b}}\right)F(z) - \boldsymbol{m}_1(1+\bar{b}z)\sum_{k=0}^{\infty}\frac{z^k}{b^k} - \boldsymbol{m}_2(1+bz)\sum_{k=0}^{\infty}b^kz^k,\\
&= \left(\sum_{k=1}^{\infty}\frac{b^k-\bar{b}^k}{b-\bar{b}}z^k\right)F(z) - \boldsymbol{m}_1\left(1+2\sum_{k=1}^{\infty}\bar{b}^kz^k\right) - \boldsymbol{m}_2\left(1+2\sum_{k=1}^{\infty}b^kz^k\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\tilde{c}_0 + 2\sum_{k=1}^{\infty}\tilde{c}_{-k}z^k &= \left(\sum_{k=1}^{\infty}\frac{b^k-\bar{b}^k}{b-\bar{b}}z^k\right)\left(1+2\sum_{k=1}^{\infty}c_{-k}z^k\right) - \boldsymbol{m}_1\left(1+2\sum_{k=1}^{\infty}\bar{b}^kz^k\right)\\
&\quad - \boldsymbol{m}_2\left(1+2\sum_{k=1}^{\infty}b^kz^k\right).
\end{aligned}$$

Therefore, comparing coefficients of $z^n$ on both sides of the last expression, we have for $n \geqslant 2$

$$\tilde{c}_{-n} = \sum_{k=1}^{n-1} \frac{b^k - \bar{b}^k}{b - \bar{b}} c_{-(n-k)} + \frac{1}{2} \frac{b^n - \bar{b}^n}{b - \bar{b}} - \boldsymbol{m}_1 \bar{b}^n - \boldsymbol{m}_2 b^n. \tag{2.5}$$

Comparing the independent terms and the coefficients on $z$ we can deduce (2.2) for $n = 0$ and $n = 1$.

Furthermore, denoting this transformation by $\mathcal{F}_{R^{-1}}$, we obtain

Proposition 2.2.

($i$) $\mathcal{F}_R(\alpha) \circ \mathcal{F}_{R^{-1}}(\alpha) = \mathcal{I}$,

($ii$) $\mathcal{F}_{R^{-1}}(\alpha) \circ \mathcal{F}_R(\alpha) = \mathcal{F}_U(b, \hat{\boldsymbol{m}}_1) \circ \mathcal{F}_U(\bar{b}, \hat{\boldsymbol{m}}_2)$.

*Proof.*

($i$) It is evident from the definition of $\mathcal{F}_R$ and $\mathcal{F}_{R^{-1}}$.

($ii$) Denoting $H(z) = \mathcal{F}_{R^{-1}}(\alpha) \circ \mathcal{F}_R(\alpha)[F(z)]$,

$$\begin{aligned}
H(z) &= \frac{zF_R(z) - \tilde{c}_0 z^2 + (\tilde{c}_{-1} - \tilde{c}_1)z + \tilde{c}_0}{z^2 - (\alpha + \bar{\alpha})z + 1}, \\
&= \frac{z\left[\frac{[z^2 - (\alpha+\bar{\alpha})z + 1]F(z) + z^2 + (c_1 - c_{-1})z - 1}{z}\right] - \tilde{c}_0 z^2 + (\tilde{c}_{-1} - \tilde{c}_1)z + \tilde{c}_0}{z^2 - (\alpha + \bar{\alpha})z + 1}, \\
&= F(z) + \frac{(1 - \tilde{c}_0)z^2 + (c_1 - c_{-1} + \tilde{c}_{-1} - \tilde{c}_1)z + \tilde{c}_0 - 1}{z^2 - (\alpha + \bar{\alpha})z + 1} \\
&= F(z) + \hat{\boldsymbol{m}}_1 \frac{z + b}{z - b} + \hat{\boldsymbol{m}}_2 \frac{z + \bar{b}}{z - \bar{b}},
\end{aligned}$$

with $\hat{\boldsymbol{m}}_1 = \tilde{\boldsymbol{m}}_1 + \boldsymbol{m}_1$, $\hat{\boldsymbol{m}}_2 = \tilde{\boldsymbol{m}}_2 + \boldsymbol{m}_2$, and

$$\tilde{\boldsymbol{m}}_1 = \frac{1}{2}\left(1 + \frac{\Im\mathfrak{m}(c_1)}{\Im\mathfrak{m}(b)}\right), \quad \tilde{\boldsymbol{m}}_2 = \frac{1}{2}\left(1 - \frac{\Im\mathfrak{m}(c_1)}{\Im\mathfrak{m}(b)}\right).$$

□

Remark 2.3. Notice that if $a = -2\Re\mathfrak{e}[\alpha]$ and using (2.2) then we obtain

$$\mathbf{T} = \mathbf{Z}\widetilde{\mathbf{T}} + a\widetilde{\mathbf{T}} + \widetilde{\mathbf{T}}\mathbf{Z}^t,$$

where $\mathbf{Z}$ is the shift matrix with ones on the first upper-diagonal and zeros on the remaining entries, and $\mathbf{M}^t$ denotes the transpose of the matrix $\mathbf{M}$. Furthermore, notice that Hermitian Toeplitz matrices can be characterized as $\mathbf{T} = \mathbf{T}^*$ together with $\mathbf{Z}\mathbf{T}\mathbf{Z}^t = \mathbf{T}$, and therefore

$$\mathbf{T}\mathbf{Z}^t = \widetilde{\mathbf{T}}\mathbf{B},$$

where $\mathbf{B} = \mathbf{I} + a\mathbf{Z}^t + (\mathbf{Z}^t)^2$ is an infinite lower triangular matrix with ones in the main diagonal, with the following structure

$$\mathbf{B} = \left(\begin{array}{c|c|c} \mathbf{A} & 0 & \cdots \\ \hline \mathbf{A}^t & \mathbf{A} & \ddots \\ \hline 0 & \ddots & \ddots \end{array}\right), \tag{2.6}$$

where $\mathbf{A} = \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$. On the other hand, is not difficult to show that

$$\mathbf{B}^{-1} = \left(\begin{array}{c|c|c|c} \mathbf{A}_1 & 0 & 0 & \dots \\ \hline \mathbf{A}_2 & \mathbf{A}_1 & 0 & \ddots \\ \hline \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right), \tag{2.7}$$

where $\mathbf{A}_1 = \mathbf{A}^{-1}$, $\mathbf{A}_k = (-1)^{k-1}\mathbf{A}^{-1}\mathbf{M}^{k-1}$, $k \geqslant 2$, and $\mathbf{M} = \mathbf{A}^{-1}\mathbf{A}^t = \left(\begin{smallmatrix} 1 & a \\ -a & 1-a^2 \end{smallmatrix}\right)$. In other words, $\mathbf{B}^{-1}$ is a lower triangular block matrix, with Toeplitz structure. Finally,

$$\mathbf{TS} = \widetilde{\mathbf{T}},$$

where $\mathbf{S}$ is given by

$$\mathbf{S} = \mathbf{Z}^t\mathbf{B}^{-1} = \left(\begin{array}{c|c|c|c} \mathbf{Z}_1^t & 0 & 0 & \dots \\ \hline \mathbf{Z}_1 & \mathbf{Z}_1^t & 0 & \ddots \\ \hline 0 & \mathbf{Z}_1 & \mathbf{Z}_1^t & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right)\left(\begin{array}{c|c|c|c} \mathbf{A}_1 & 0 & 0 & \dots \\ \hline \mathbf{A}_2 & \mathbf{A}_1 & 0 & \ddots \\ \hline \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right), \tag{2.8}$$

with $\mathbf{Z}_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, i.e. $\mathbf{S}$ is also a lower triangular block matrix with Toeplitz structure.

## 3. Quasi-definiteness of $\mathcal{L}_{R^{-1}}$.

Let us consider a linear functional $\mathcal{L}_{R^{-1}}$, such that

$$\left\langle \mathcal{L}_{R^{-1}}, [z + z^{-1} - (\alpha + \bar{\alpha})]p \right\rangle = \langle \mathcal{L}, p \rangle, \tag{3.1}$$

where $\mathcal{L}$ is a positive definite Hermitian linear functional on the linear space of Laurent polynomials. Notice that we will assume that $\mathcal{L}_{R^{-1}}$ is also Hermitian.

For all values of $\alpha$ such that $|\Re\mathrm{e}(\alpha)| > 1$, the Laurent polynomial $z + z^{-1} - (\alpha + \bar{\alpha})$ can be represented as a polynomial of the form $c|z-\beta|^2$, $c \in \mathbb{R}, \beta \in \mathbb{C}$, i.e $\mathcal{L}_{R^{-1}}$ is a Geronimus transformation, studied in [4], [15]. For this reason, we are only interested in those values of $\alpha$ such that $0 < |\Re\mathrm{e}(\alpha)| < 1$. However, in this case the zeros $b$ and $\bar{b}$ of $z^2 - (\alpha + \bar{\alpha})z + 1$ are complex conjugates and, furthermore, $|b| = 1$.

We will denote by $\sigma$ and $\tilde{\sigma}$ the measures associated with $\mathcal{L}$ and $\mathcal{L}_{R^{-1}}$, respectively, i.e.

$$d\tilde{\sigma} = \frac{d\sigma}{2\Re\mathrm{e}(z-\alpha)} + \boldsymbol{m}_1\delta(z-b) + \boldsymbol{m}_2\delta(z-\bar{b}), \quad \boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathbb{R}. \tag{3.2}$$

Here $\sigma$ is a nontrivial probability measure supported on $\mathbb{T}$, which can be decomposed as in (1.7).

Thus, if $\sigma_s = 0$, then the integral

$$\tilde{c}_n = \int_0^{2\pi} \frac{e^{in\theta}\omega(\theta)}{z + z^{-1} - (\alpha + \bar{\alpha})}\frac{d\theta}{2\pi} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^n\omega(z)}{z^2 - (\alpha + \bar{\alpha})z + 1}dz, \tag{3.3}$$

has singularities in $z = b$ and $z = \bar{b}$. These singularities can be removed if we consider

$$\tilde{c}_n = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^n\omega(\theta)}{z^2-(\alpha+\bar{\alpha})z+1}dz, \quad z=e^{i\theta}, \tag{3.4}$$

$$= \frac{1}{2\pi i(b-\bar{b})}\left(\int_{\mathbb{T}}\frac{z^n\omega(z)}{z-b}dz - \int_{\mathbb{T}}\frac{z^n\omega(z)}{z-\bar{b}}dz\right), \tag{3.5}$$

$$= \frac{1}{2\pi i(b-\bar{b})}\left(\int_{\mathbb{T}}\frac{z^n[\omega(z)-\omega(b)]}{z-b}dz - \int_{\mathbb{T}}\frac{z^n[\omega(z)-\omega(\bar{b})]}{z-\bar{b}}dz\right. \tag{3.6}$$

$$\left. + \frac{1}{2}b^n\omega(b) - \frac{1}{2}\bar{b}^n\omega(\bar{b})\right), \tag{3.7}$$

assuming that $\omega(z)$ satisfies a Lipschitz condition of order $\tau$ $(0 < \tau \leqslant 1)$ on $\mathbb{T}$ (see [20]). Notice than this is also valid if $\sigma_s \neq 0$, as long as $\sigma_s$ has a finite number of mass points different from $b$ and $\bar{b}$.

Now, assume $\mathcal{L}_{R^{-1}}$ is quasi-definite and let $\{\Psi_n\}_{n\geqslant 0}$ be the its corresponding sequence of monic orthogonal polynomials. Next, we will state the relation between $\{\Psi_n\}_{n\geqslant 0}$ and $\{\Phi_n\}_{n\geqslant 0}$.

Proposition 3.1. *Let $\mathcal{L}$ be a positive definite linear functional. If $\mathcal{L}_{R^{-1}}$ given as in* (1.12) *is a quasi-definite linear functional, then $\Psi_n(z)$, the nth monic polynomial orthogonal with respect to $\mathcal{L}_{R^{-1}}$, is*

$$\Psi_n(z) = \left(z + \frac{\tilde{k}_n}{k_{n-1}}\right)\Phi_{n-1}(z) + \left(\Phi_n(0) - \frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right)\Phi^*_{n-1}(z). \tag{3.8}$$

*Conversely, if $\{\Psi_n\}_{n\geqslant 0}$ is given by* (3.8) *and assuming $|\Psi_n(0)| \neq 1$, $n \geqslant 1$, then $\{\Psi_n\}_{n\geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_{R^{-1}}$.*

*Proof*. Let

$$\Psi_n(z) = \Phi_n(z) + \sum_{m=0}^{n-1}\lambda_{n,m}\Phi_m(z). \tag{3.9}$$

Multiplying the above expression by $\overline{\Phi_m(z)}$ and applying $\mathcal{L}$, for $0 \leqslant m \leqslant n-1$ we get

$$\left\langle \mathcal{L}, \Psi_n(z)\overline{\Phi_m(z)}\right\rangle = \lambda_{n,m}k_m, \text{or equivalently,} \tag{3.10}$$

$$\left\langle \mathcal{L}_{R^{-1}}, [z+z^{-1}-(\alpha+\bar{\alpha})]\Psi_n(z)\overline{\Phi_m(z)}\right\rangle = \lambda_{n,m}k_m. \tag{3.11}$$

Thus

$$\lambda_{n,m} = \frac{1}{k_m}\left\langle \mathcal{L}_{R^{-1}}, [z+z^{-1}-(\alpha+\bar{\alpha})]\Psi_n(z)\overline{\Phi_m(z)}\right\rangle, \quad 0 \leqslant m \leqslant n-1. \tag{3.12}$$

If $m = n-1$,

$$\begin{aligned}
\lambda_{n,n-1} &= \frac{1}{k_{n-1}}\left[\left\langle \mathcal{L}_{R^{-1}}, z\Psi_n(z)\overline{\Phi_{n-1}(z)}\right\rangle + \left\langle \mathcal{L}_{R^{-1}}, z^{-1}\Psi_n(z)\overline{\Phi_{n-1}(z)}\right\rangle\right],\\
&= \frac{1}{k_{n-1}}\left[\left\langle \mathcal{L}_{R^{-1}}, [\Psi_{n+1}(z) - \Psi_{n+1}(0)\Psi^*_n(z)]\overline{\Phi_{n-1}(z)}\right\rangle + \tilde{k}_n\right],\\
&= \frac{\tilde{k}_n}{k_{n-1}}(1 - \Psi_{n+1}(0)\overline{\Phi_{n-1}(0)}).
\end{aligned}$$

On the other hand, for $0 \leqslant m \leqslant n-2$,

$$\begin{aligned}\lambda_{n,m} &= \frac{1}{k_m}\left\langle \mathcal{L}_{R^{-1}}, z\Psi_n(z)\overline{\Phi_m(z)}\right\rangle,\\ &= \frac{1}{k_m}\left\langle \mathcal{L}_{R^{-1}}, [\Psi_{n+1}(z)-\Psi_{n+1}(0)\Psi_n^*(z)]\overline{\Phi_m(z)}\right\rangle,\\ &= -\frac{\tilde{k}_n}{k_m}\Psi_{n+1}(0)\overline{\Phi_m(0)}.\end{aligned}$$

Substituting these values in (3.9), we obtain

$$\Psi_n(z) = \Phi_n(z) + \frac{\tilde{k}_n}{k_{n-1}}\Phi_{n-1}(z) - \tilde{k}_n\Psi_{n+1}(0)\sum_{m=0}^{n-1}\frac{\overline{\Phi_m(0)}\Phi_m(z)}{k_m}, \tag{3.13}$$

$$= \Phi_n(z) + \frac{\tilde{k}_n}{k_{n-1}}\Phi_{n-1}(z) - \tilde{k}_n\Psi_{n+1}(0)K_{n-1}(z,0), \tag{3.14}$$

$$= \Phi_n(z) + \frac{\tilde{k}_n}{k_{n-1}}\Phi_{n-1}(z) - \frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\Phi_{n-1}^*(z). \tag{3.15}$$

Using the recurrence relation, we get

$$\Psi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z) + \frac{\tilde{k}_n}{k_{n-1}}\Phi_{n-1}(z) - \frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\Phi_{n-1}^*(z), \tag{3.16}$$

$$= \left(z + \frac{\tilde{k}_n}{k_{n-1}}\right)\Phi_{n-1}(z) + \left(\Phi_n(0) - \frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right)\Phi_{n-1}^*(z), \tag{3.17}$$

which proves the first statement of the proposition.

Notice that evaluating (3.15) at $z = 0$, we get

$$\Psi_n(0) = \frac{\tilde{k}_n}{k_{n-1}}\Phi_{n-1}(0) + \Phi_n(0) - \frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0), \tag{3.18}$$

and thus (3.17) becomes

$$\Psi_n(z) = \left(z + \frac{\tilde{k}_n}{k_{n-1}}\right)\Phi_{n-1}(z) + \left(\Psi_n(0) - \frac{\tilde{k}_n}{k_{n-1}}\Phi_{n-1}(0)\right)\Phi_{n-1}^*(z). \tag{3.19}$$

On the other hand, if we denote $\nu_n = \tilde{k}_{n+1}/k_n$ and $l_n = \Psi_{n+1}(0) - \nu_n\Phi_n(0)$, and considering the reversed polynomial of $\Psi_{n+1}(z)$, then we obtain the following linear transfer equation

$$\begin{bmatrix}\Psi_{n+1}(z)\\ \Psi_{n+1}^*(z)\end{bmatrix} = \begin{bmatrix} z+\nu_n & l_n\\ \overline{l_n}z & \nu_n z + 1\end{bmatrix}\begin{bmatrix}\Phi_n(z)\\ \Phi_n^*(z)\end{bmatrix}.$$

Notice that the determinant of the above transfer matrix is

$$\begin{aligned}(z+\nu_n)(\nu_n z+1) - |l_n|^2 z &= \nu_n z^2 + (\nu_n^2 + 1 - |l_n|^2)z + \nu_n,\\ &= \nu_n(z^2+1) + [\nu_n^2(1-|\Phi_n(0)|^2) + 1 - |\Psi_{n+1}(0)|^2]z\\ &+ \nu_n[\Psi_{n+1}(0)\overline{\Phi_n(0)} + \overline{\Psi_{n+1}(0)}\Phi_n(0)]z,\\ &= \nu_n(z^2 - (\alpha+\bar{\alpha})z + 1),\end{aligned}$$

where the last equality will become clear looking at (3.22) in the following Proposition. Furthermore, we get

$$\Phi_n(z) = \frac{(\nu_n z+1)\Psi_{n+1}(z) - l_n\Psi^*_{n+1}(z)}{\nu_n[z^2-(\alpha+\bar{\alpha})z+1]},$$
$$\Phi^*_n(z) = \frac{(z+\nu_n)\Psi^*_{n+1}(z) - \bar{l}_n z\Psi_{n+1}(z)}{\nu_n[z^2-(\alpha+\bar{\alpha})z+1]},$$

and thus we obtain the following alternative expression that relates both sequences of polynomials

$$[z^2-(\alpha+\bar{\alpha})z+1]\Phi_n(z) = \Psi_{n+2}(z) + \nu_n^{-1}[\Psi_{n+1}(z) - \Phi_{n+1}(0)\Psi^*_{n+1}(z)]. \tag{3.20}$$

Now we prove that the sequence of monic polynomials $\{\Psi_n\}_{n\geqslant 0}$ given in (3.8) is orthogonal with respect to $\mathcal{L}_{R^{-1}}$. Notice that $\Psi_{n+1}(z) - \Psi_{n+1}(0)\Psi^*_n(z)$ is a polynomial of degree $n+1$ that vanishes in $z=0$ and thus $\Psi_{n+1}(z)-\Psi_{n+1}(0)\Psi^*_n(z) = zp(z)$ where $p(z)$ is a polynomial of degree $n$. Then,

$$\begin{aligned}
zp(z) &= (z+\nu_n)\Phi_n(z) + l_n\Phi^*_n(z) - \Psi_{n+1}(0)[\overline{l_{n-1}}z\Phi_{n-1}(z) + (\nu_{n-1}z+1)\Phi^*_{n-1}(z)],\\
&= (z+\nu_n)[z\Phi_{n-1}(z) + \Phi_n(0)\Phi^*_{n-1}(z)] + l_n[\Phi^*_{n-1}(z) + \overline{\Phi_n(0)}z\Phi_{n-1}(z)]\\
&- \Psi_{n+1}(0)[\overline{l_{n-1}}z\Phi_{n-1}(z) + (\nu_{n-1}z+1)\Phi^*_{n-1}(z)],\\
&= z\Phi_{n-1}(z)[z+\nu_n + l_n\overline{\Phi_n(0)} - \Psi_{n+1}(0)\overline{l_{n-1}}]\\
&+ \Phi^*_{n-1}(z)[(z+\nu_n)\Phi_n(0) + l_n - \Psi_{n+1}(0)(\nu_{n-1}z+1)],\\
&= z(z+\nu_{n-1})\Phi_{n-1}(z) + zl_{n-1}\Phi^*_{n-1}(z),\\
&= z\Psi_n(z),
\end{aligned}$$

where the fourth equality follows from (1.5) and (3.18). This is, $\{\Psi_n\}_{n\geqslant 0}$ satisfies a recurrence relation like (1.3) and therefore it is an orthogonal sequence with respect to some linear functional $\tilde{\mathcal{L}}$. We will prove that $\tilde{\mathcal{L}} = \mathcal{L}_{R^{-1}}$. For $0\leqslant k\leqslant n-1$, consider

$$\begin{aligned}
\left\langle \tilde{\mathcal{L}}, [z+z^{-1}-(\alpha+\bar{\alpha})]\Phi_n(z)\bar{z}^k\right\rangle &= \left\langle \tilde{\mathcal{L}}, [z^2-(\alpha+\bar{\alpha})z+1]\Phi_n(z)\bar{z}^{k+1}\right\rangle\\
&= \left\langle \tilde{\mathcal{L}}, \Psi_{n+2}(z)\bar{z}^{k+1}\right\rangle\\
&+ \nu_n^{-1}\left\langle \tilde{\mathcal{L}}, [\Psi_{n+1}(z) - \Phi_{n+1}(0)\Psi^*_{n+1}(z)]\bar{z}^{k+1}\right\rangle\\
&= 0.
\end{aligned}$$

On the other hand, for $k=n$ we get

$$\left\langle \tilde{\mathcal{L}}, \Psi_{n+2}(z) + \nu_n^{-1}[\Psi_{n+1}(z) - \Phi_{n+1}(0)\Psi^*_{n+1}(z)]\bar{z}^{k+1}\right\rangle = \nu_n^{-1}\tilde{k}_{n+1} = k_n$$

Thus, $\{\Phi_n\}_{n\geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to $[z+z^{-1}-(\alpha+\bar{\alpha})]\tilde{\mathcal{L}}$. But then $[z+z^{-1}-(\alpha+\bar{\alpha})]\tilde{\mathcal{L}} = \mathcal{L}$ and, therefore, $\tilde{\mathcal{L}} = \mathcal{L}_{R^{-1}}$. □

Proposition 3.2. *Let $\mathcal{L}$ be a positive definite linear functional and $\sigma$ its associated measure. If $\mathcal{L}_{R^{-1}}$ is a quasi-definite linear functional then*

(*i*) $[\Im\mathfrak{m}(\tilde{c}_1)]^2 \neq (1-[\Re\mathfrak{e}(\alpha)]^2)\tilde{c}_0^2 - \Re\mathfrak{e}(\alpha)\tilde{c}_0 - \frac{1}{4}$,

(*ii*) $\left(1-|\Phi_n(0)|^2\right)\nu_n^2 + A_{n+1}\nu_n + 1 - |\Psi_{n+1}(0)|^2 = 0$, *for* $n\geqslant 1$, *where* $A_n = \overline{\Psi_n(0)}\Phi_{n-1}(0) + \Psi_n(0)\overline{\Phi_{n-1}(0)} + \alpha + \bar{\alpha}$.

*Proof*. From (2.2), for $k = 0$ and assuming $c_0 = 1$, we have

$$\Re\mathfrak{e}(\tilde{c}_1) = \frac{1}{2} + \Re\mathfrak{e}(\alpha)\tilde{c}_0. \tag{3.21}$$

In addition, for $\mathcal{L}_{R^{-1}}$ to be a quasi-definite functional we need

$$\det \widetilde{\mathbf{T}}_2 = \begin{vmatrix} \tilde{c}_0 & \tilde{c}_1 \\ \tilde{c}_{-1} & \tilde{c}_0 \end{vmatrix} = \tilde{c}_0^2 - [\Re\mathfrak{e}(\tilde{c}_1)]^2 - [\Im\mathfrak{m}(\tilde{c}_1)]^2 \neq 0,$$

where $\widetilde{\mathbf{T}}$ is the Toeplitz matrix associated with $\mathcal{L}_{R^{-1}}$ and $\widetilde{\mathbf{T}}_n$ is its corresponding $n \times n$ leading principal submatrix. Therefore, for the choice of $\alpha$, we get

$$[\Im\mathfrak{m}(\tilde{c}_1)]^2 \neq \tilde{c}_0^2 - \left[\frac{1}{2} + \Re\mathfrak{e}(\alpha)\tilde{c}_0\right]^2,$$

which is $(i)$. Thus, $\tilde{c}_0$ and $\Im\mathfrak{m}[\tilde{c}_1]$ are free parameters, while $\Re\mathfrak{e}[\tilde{c}_1]$ is determined by $\tilde{c}_0$ and the choice of $\alpha$.

Furthermore, we have

$$\begin{aligned} k_n &= \langle \Phi_n(z), \Phi_n(z)\rangle_{\mathcal{L}} = \langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}}, \\ &= \left\langle [z + z^{-1} - (\alpha + \bar{\alpha})]\Psi_n(z), \Phi_n(z)\right\rangle_{\mathcal{L}_{R^{-1}}}, \\ &= \langle z\Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}_{R^{-1}}} + \langle \Psi_n(z), z\Phi_n(z)\rangle_{\mathcal{L}_{R^{-1}}} - (\alpha + \bar{\alpha})\langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}_{R^{-1}}}, \\ &= -[\Psi_{n+1}(0)\overline{\Phi_n(0)} + \alpha + \bar{\alpha}]\tilde{k}_n + \langle \Psi_n(z), z\Phi_n(z)\rangle_{\mathcal{L}_{R^{-1}}}. \end{aligned}$$

On the other hand, from (3.15)

$$\begin{aligned} \langle \Psi_n(z), z\Phi_n(z)\rangle_{\mathcal{L}_{R^{-1}}} &= \left\langle \Psi_n(z), z\Psi_n(z) - \frac{\tilde{k}_n}{k_{n-1}} z\Phi_{n-1}(z) + \frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0) z\Phi^*_{n-1}(z)\right\rangle_{\mathcal{L}_{R^{-1}}}, \\ &= -\overline{\Psi_{n+1}(0)}\Psi_n(0)\tilde{k}_n - \frac{\tilde{k}_n}{k_{n-1}}\tilde{k}_n + \frac{\tilde{k}_n}{k_{n-1}}\overline{\Psi_{n+1}(0)}\Phi_{n-1}(0)\tilde{k}_n, \end{aligned}$$

and from (3.18),

$$\begin{aligned} \langle \Psi_n(z), z\Phi_n(z)\rangle_{\mathcal{L}_{R^{-1}}} &= -\overline{\Psi_{n+1}(0)}\Psi_n(0)\tilde{k}_n - \frac{\tilde{k}_n}{k_{n-1}}\tilde{k}_n \\ &\quad + \left(\Psi_n(0) - \Phi_n(0) + \frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right)\overline{\Psi_{n+1}(0)}\tilde{k}_n, \\ &= -\overline{\Psi_{n+1}(0)}\Phi_n(0)\tilde{k}_n - \frac{\tilde{k}_n}{k_{n-1}}\tilde{k}_n + \frac{\tilde{k}_n}{k_{n-1}}|\Psi_{n+1}(0)|^2\tilde{k}_n. \end{aligned}$$

Thus, if $A_{n+1} = \overline{\Psi_{n+1}(0)}\Phi_n(0) + \Psi_{n+1}(0)\overline{\Phi_n(0)} + \alpha + \bar{\alpha}$,

$$\begin{aligned} k_n &= -A_{n+1}\tilde{k}_n + \left(|\Psi_{n+1}(0)|^2 - 1\right)\frac{\tilde{k}_n}{k_{n-1}}\tilde{k}_n, \\ 1 - |\Phi_n(0)|^2 &= -A_{n+1}\frac{\tilde{k}_n}{k_{n-1}} + \left|\frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right|^2 - \left(\frac{\tilde{k}_n}{k_{n-1}}\right)^2. \end{aligned}$$

Since, from (3.18), $\frac{\tilde{k}_n}{k_{n-1}}\Psi_{n+1}(0) = \frac{\tilde{k}_n}{k_{n-1}}\Phi_{n-1}(0) + \Phi_n(0) - \Psi_n(0)$, we obtain

$$1 - |\Psi_n(0)|^2 = -A_n \frac{\tilde{k}_n}{k_{n-1}} - \left(1 - |\Phi_{n-1}(0)|^2\right)\left(\frac{\tilde{k}_n}{k_{n-1}}\right)^2.$$

Therefore,

$$\left(1 - |\Phi_{n-1}(0)|^2\right)\left(\frac{\tilde{k}_n}{k_{n-1}}\right)^2 + A_n \frac{\tilde{k}_n}{k_{n-1}} + 1 - |\Psi_n(0)|^2 = 0, \tag{3.22}$$

which is (*ii*). □

Now, from (3.18),

$$\Psi_{n+1}(0) = \Phi_{n-1}(0) + [\Phi_n(0) - \Psi_n(0)]\frac{k_{n-1}}{\tilde{k}_n}, \tag{3.23}$$

$$= \frac{[\Phi_n(0) - \Psi_n(0)]\prod_{k=1}^{n-1}(1 - |\Phi_k(0)|^2)}{\prod_{k=1}^{n}(1 - |\Psi_k(0)|^2)\tilde{c}_0} + \Phi_{n-1}(0), \quad n \geqslant 1. \tag{3.24}$$

Thus, we can built an algorithm to compute recursively the sequence $\{\Psi_{n+1}(0)\}_{n\geqslant 1}$, starting from $\Psi_1(0) = -\frac{\tilde{c}_1}{\tilde{c}_0}$. Namely,

**Input:** $\alpha$, $\tilde{c}_0$, $\{\Phi_n(0)\}_{n\geqslant 1}$.
Compute $\mathfrak{Re}(\tilde{c}_1) = \frac{1}{2} + \mathfrak{Re}(\alpha)\tilde{c}_0$.
**IF (Initial condition)** (*i*) in Proposition 3.2 holds, then
$\Psi_1(0) = -\frac{\tilde{c}_1}{\tilde{c}_0}$
**FOR** $n = 1, 2, \ldots$

$$\Psi_{n+1}(0) = \frac{[\Phi_n(0) - \Psi_n(0)]\prod_{k=1}^{n-1}(1 - |\Phi_k(0)|^2)}{\prod_{k=1}^{n}(1 - |\Psi_k(0)|^2)\tilde{c}_0} + \Phi_{n-1}(0)$$

**IF** $|\Psi_{n+1}(0)| = 1$, break
**END** (FOR)

## 4. Examples.

**4.1. A Christoffel case.** Let $d\sigma = |z-1|^2 \frac{d\theta}{2\pi}$. Is well known (see [16]) that the family of Verblunsky parameters associated with $\sigma$ is

$$\Phi_n(0) = \frac{1}{n+1}, \quad n \geqslant 1.$$

Now, let us consider the perturbation

$$d\tilde{\sigma} = \frac{|z-1|^2}{z + z^{-1} - (\alpha + \bar{\alpha})}\frac{d\theta}{2\pi}, \quad |z| = 1, \tag{4.1}$$

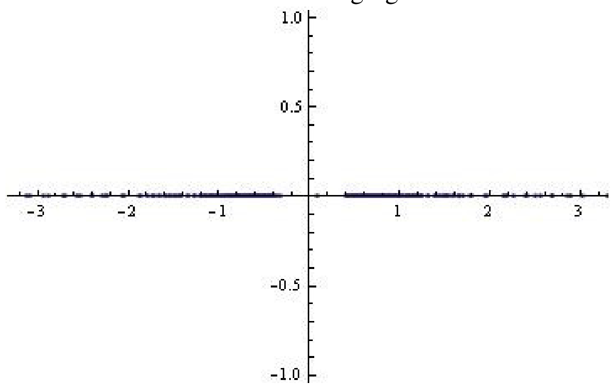
where $\mathfrak{Re}[\alpha] = 0.6$. Notice that $b = 0.6 + 0.8i$. Then, according to (3.4)

$$\tilde{c}_0 = \frac{1}{1.6i}\left[\int_0^{2\pi} \frac{|e^{i\theta} - 1|^2 - 0.8}{1 - (0.6 + 0.8i)e^{-i\theta}}\frac{d\theta}{2\pi} - \int_0^{2\pi} \frac{|e^{i\theta} - 1|^2 - 0.8}{1 - (0.6 - 0.8i)e^{-i\theta}}\frac{d\theta}{2\pi}\right],$$
$$= \frac{1}{1.6i}(0.6 - 0.8i - (0.6 + 0.8i)) = -1,$$

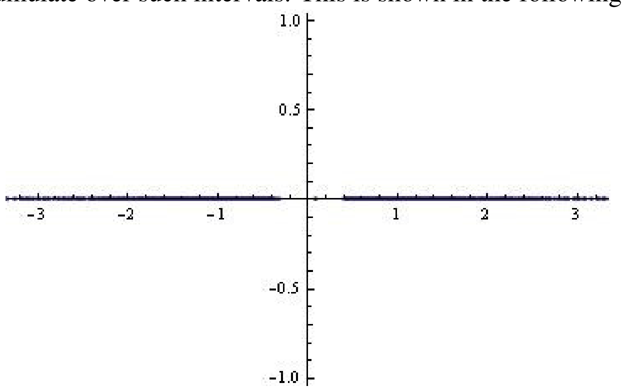
and

$$\tilde{c}_1 = \frac{1}{1.6i}\left[\int_0^{2\pi} \frac{(|e^{i\theta} - 1|^2 - 0.8)e^{i\theta}}{1 - (0.6 + 0.8i)e^{-i\theta}}\frac{d\theta}{2\pi} - \int_0^{2\pi} \frac{(|e^{i\theta} - 1|^2 - 0.8)e^{i\theta}}{1 - (0.6 - 0.8i)e^{-i\theta}}\frac{d\theta}{2\pi}\right.$$
$$\left. + \frac{1}{2}(0.6 + 0.8i)(0.8) - \frac{1}{2}(0.6 - 0.8i)(0.8)\right],$$
$$= 0.4.$$

Observe that $(i)$ in Proposition 3.2 holds. Applying the algorithm, the first 500 Verblunsky parameters are shown in the following figure.




Notice that all of the new Verblunsky parameters are real. They are distributed at both sides of the origin, in nearly symmetric intervals. If we calculate for $n = 2000$, then the values accumulate over such intervals. This is shown in the following figure.




**4.2. The case of constant Verblunsky parameters.** We consider linear functionals such that the corresponding measures are supported on an arc of the unit circle which doesn't contain $b_1$ and $b_2$.

Such a situation appears (see [7], [16]) when $\Phi_n(0) = a$, $n \geqslant 1$, with $0 < |a| < 1$. Here the measure $\sigma$ associated with $\{\Phi_n(0)\}_{n\geqslant 1}$ is supported on the arc $\Delta_\nu = \{e^{i\theta} : \nu \leqslant \theta \leqslant 2\pi - \nu\}$, with

$$\cos(\nu/2) := \sqrt{1 - |a|^2},$$

but it can have a mass point located on $\mathbb{T}$. The orthogonality measure $\sigma$ is given by

$$d\sigma = \frac{\sqrt{\sin(\frac{\theta+\nu}{2})\sin(\frac{\theta-\nu}{2})}}{2\pi\sin(\frac{\theta-\tau}{2})}d\theta + \boldsymbol{m}_\tau\delta(z - e^{i\tau}), \tag{4.2}$$

where $e^{i\tau} = \frac{1-a}{1-\bar{a}}$ and

$$\boldsymbol{m}_\tau = \begin{cases} \frac{2|a|^2-a-\bar{a}}{|1-a|} & \text{if} \quad |1-2a| > 1, \\ 0 & \text{if} \quad |1-2a| \leqslant 1. \end{cases}$$

Moreover, the orthonormal polynomials associated with $\sigma$ are given by

$$\varphi_n(z) = \frac{1}{(1-|a|^2)^{n/2}}\left((z+a)\frac{z_1^n - z_2^n}{z_1 - z_2} - z(1-|a|^2)\frac{z_1^{n-1} - z_2^{n-1}}{z_1 - z_2}\right), \quad n \in \mathbb{N}, \tag{4.3}$$

with

$$z_1 = \frac{z+1+\sqrt{(z-e^{i\nu})(z-e^{-i\nu})}}{2},$$
$$z_2 = \frac{z+1-\sqrt{(z-e^{i\nu})(z-e^{-i\nu})}}{2}.$$

Consider a perturbation of (4.2) given by

$$d\tilde{\sigma} = \frac{d\sigma}{z+z^{-1}-(\alpha+\bar{\alpha})},$$

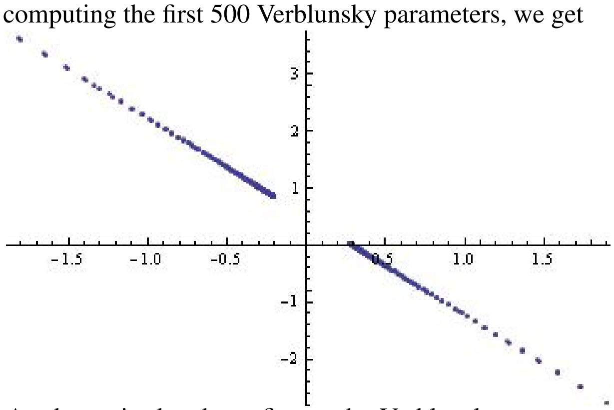
with $\Re\mathfrak{e}[\alpha] = 0.8$ and $a = 0.5i$. Notice that in this case, $b = 0.8 + 0.6i$ and thus $b \notin \Delta_\nu$. Then,

$$\tilde{c}_0 = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{\sqrt{\sin(\frac{1}{2}\theta+\frac{1}{6}\pi)\sin(\frac{1}{2}\theta-\frac{1}{6}\pi)}}{2(\cos\theta-0.8)\pi\sin(\frac{\theta}{2})} = -0.458\,76,$$
$$\tilde{c}_1 = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{(\cos\theta+i\sin\theta)\sqrt{\sin(\frac{1}{2}\theta+\frac{1}{6}\pi)\sin(\frac{1}{2}\theta-\frac{1}{6}\pi)}}{2(\cos\theta-0.8)\pi\sin(\frac{\theta}{2})} = 0.132\,99,$$

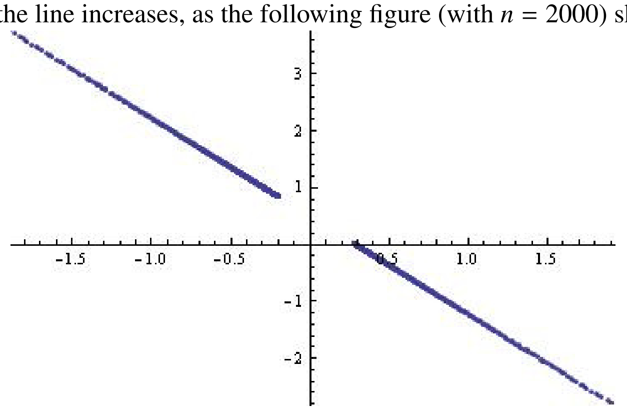
and $(i)$ holds. In such a situation, the algorithm becomes

$$\Psi_{n+1}(0) = \frac{[a-\Psi_n(0)](1-|a|^2)^{n-1}}{\prod_{k=1}^{n}(1-|\Psi_k(0)|^2)\tilde{c}_0} + a, \quad n \geqslant 1, \tag{4.4}$$

and computing the first 500 Verblunsky parameters, we get



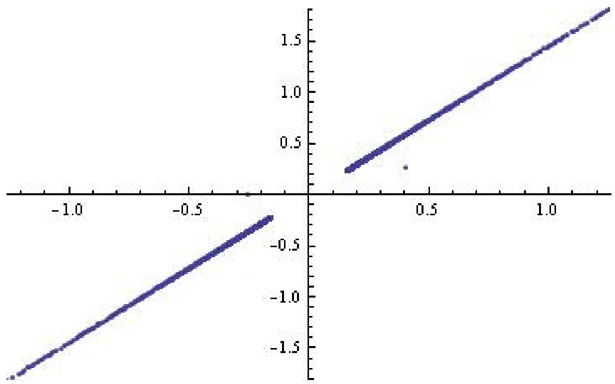

As shown in the above figure, the Verblunsky parameters associated with the modified measure have the same argument with respect to a certain point (the value of $a$). This is, they are located on a straight line, at both sides of $a$. When $n$ increases, the density of the points on the line increases, as the following figure (with $n = 2000$) shows



**4.3. A Bernstein-Szegő case.** Consider the Bernstein-Szegő, $d\sigma = \frac{1-|\beta|}{|e^{i\theta}-\beta|^2}\frac{d\theta}{2\pi}$, $0 < |\beta| < 1$. It is well known ([16]) that in this case the Verblunsky parameters are given by $\Phi_1(0) = -\beta$, and $\Phi_n(0) = 0$, $n \geqslant 2$. Consider the measure $d\tilde{\sigma}$ defined by

$$d\tilde{\sigma} = \frac{1}{[z + z^{-1} - (\alpha + \bar{\alpha})]|e^{i\theta} - \beta|^2}\frac{d\theta}{2\pi}.$$

Setting $\Re\mathfrak{e}[\alpha] = 0.8$, $\beta = 0.5$ and computing the first moments using (3.4), we get $\tilde{c}_0 = -0.9308$ and $\tilde{c}_1 = -0.2395$. Thus, the algorithm gives, for the first 2000 Verblunsky parameters




Notice that the behavior of the Verblunsky parameters is similar to the previous example.

**Acknowledgements.** We thank the anonymous referees for their valuable comments and suggestions which have improved the presentation of the manuscript. The work of the second author has been supported by a grant of Universidad Autónoma de Tamaulipas. The work of the third author has been supported by Dirección General de Investigación, Ministerio de Educación y Ciencia of Spain, grant MTM06-13000-C03-02. Second and third authors have been supported by project CCG07-UC3M/ESP-3339 with the financial support of Comunidad de Madrid-Universidad Carlos III de Madrid.

## REFERENCES

[1] M. I. Bueno and F. Marcellán, *Darboux transformations and perturbation of linear functionals*, Linear Alg. and Appl., 384 (2004), pp. 215–242.

[2] M. J. Cantero, *Polinomios ortogonales sobre la circunferencia unidad. Modificaciones de los parámetros de Schur*, PhD thesis, Universidad de Zaragoza, 1997. In Spanish.

[3] L. Daruis, J. Hernández, and F. Marcellán, *Spectral transformations for Hermitian Toeplitz matrices*, J. Comput. Appl. Math., 202 (2007), pp. 155–176.

[4] L. Garza, J. Hernández, and F. Marcellán, *Orthogonal polynomials and measures on the unit circle. The Geronimus transformations*, J. Comput. Appl. Math., (2009). To appear.

[5] L. Garza and F. Marcellán, *Linear spectral transformations and Laurent polynomials*, Mediterr. J. Math., (2009). To appear.

[6] ———, *Verblunsky parameters and linear spectral transformations*, Meth. Appl. Anal, (2009). To appear.

[7] Ya. L. Geronimus, *Polynomials orthogonal on a circle and their applications*, Amer. Math. Soc. Transl. Series, Amer. Math. Soc. Providence, RI., 1 (1962), pp. 1–78.

[8] E. Godoy and F. Marcellán, *An analogue of the Christoffel formula for polynomial modification of a measure on the unit circle*, Boll. Un. Mat. Ital., 5-A (1991), pp. 1–12.

[9] ———, *Orthogonal polynomials and rational modifications of measures*, Canad. J. Math., 45 (1993), pp. 930–943.

[10] U. Grenander and G. Szegő, *Toeplitz Forms and their Applications*, University of California Press, Berkeley 1958, Chelsea, New York, $2^{nd}$ edition, 1984.

[11] W. B. Jones, O. Njåstad, and W. J. Thron, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc., 21 (1989), pp. 113–152.

[12] X. Li and F. Marcellán, *Representation of Orthogonal Polynomials for modified measures*, Comm. Anal. Theory of Cont. Fract., 7 (1999), pp. 9–22.

[13] F. Marcellán, *Polinomios ortogonales no estándar. Aplicaciones en Análisis Numérico y Teoría de Aproximación*, Rev. Acad. Colomb. Ciencias Exactas, Físicas y Naturales, 30 (2006), pp. 563–579. In Spanish.

[14] F. Marcellán and J. Hernández, *Christoffel transforms and Hermitian linear functionals* , Mediterr. J. Math., 2 (2005), pp. 451–458.

[15] ———, *Geronimus spectral transforms and measures on the complex plane* , J. Comput. Appl. Math., 217 (2008), pp. 441–456.

[16] B. Simon, *Orthogonal Polynomials on the Unit Circle*, Amer. Math. Soc. Coll. Publ. Series, vol. 54, Amer. Math. Soc. Providence, RI, 2005.

[17] V. Spiridonov, L. Vinet, and A. Zhedanov, *Spectral transformations, self-similar reductions and orthogonal polynomials*, J. Phys. A: Math. Gen., 30 (1997), pp. 7621–7637.

[18] C. Suárez, *Polinomios ortogonales relativos a modificaciones de funcionales regulares y hermitianos*, PhD thesis, Universidad of Vigo, 1993. In Spanish.

[19] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Series. vol 23, Amer. Math. Soc., Providence, RI, $4^{th}$ edition, 1975.

[20] L. Volkovyski, G. Lunts, and I. Aramanovich, *Problems in the Theory of Functions of a Complex Variable*, Mir Publishers, Moscow, 1972.

[21] G. J. Yoon, *Darboux transforms and orthogonal polynomials*, Bull. Korean Math. Soc., 39 (2002), pp. 359–376.

[22] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math., 85 (1997), pp. 67–83.