Analytic properties of Laguerre-type orthogonal polynomials

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Abstract

In this paper we consider sequences of monic polynomials orthogonal with respect to an inner product

$$\langle p,q\rangle = \int_0^\infty pqx^\alpha e^{-x}dx + Mp(a)q(a),$$

where $M \in \mathbb{R}_+$, and $a \in \mathbb{R}_-$.

We focus our attention in the representation of these polynomials in terms of the standard Laguerre polynomials as well as hypergeometric functions. The lowering and raising operators associated with these polynomials are obtained. The distribution of their zeros is analyzed in terms of their dependence of M. Finally, some outer asymptotic properties of such orthogonal polynomials are discussed.

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1 Introduction

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle p,q\rangle_\alpha = \int_0^\infty pqx^\alpha e^{-x}dx,\ \alpha > -1,\ p,q \in \mathbb{P}. \tag{1}$$

The expression of these polynomials as an ${}_1F_1$ hypergeometric function is very well known in the literature (see [2], [9], [13], [18], [19], among others) and they constitute a family of classical orthogonal polynomials (see [14] and [18]). The connection between these two facts follows from a characterization of such orthogonal polynomials as eigenfunctions of a second order linear differential

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operator with polynomial coefficients. Nevertheless, we are interested in other structural properties of Laguerre polynomials which will be useful in the sequel (see [4] and [14]).

Proposition 1 *Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials. Then the following statements hold.*

1. *Three term recurrence relation. For every $n \in \mathbb{N}$,*
$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + \beta_n L_n^\alpha(x) + \gamma_n L_{n-1}^\alpha(x), \quad n \geq 1 \tag{2}$$
with $L_0^\alpha(x) = 1, L_1^\alpha(x) = x- \ (\alpha+1), \beta_n = 2n+\alpha+1,$ and $\gamma_n = n(n+\alpha), n \geq 1.$

2. *Structure relation. For every $n \in \mathbb{N}$,*
$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x). \tag{3}$$

3. *For every $n \in \mathbb{N}$,*
$$\|L_n^\alpha\|_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{4}$$

4. *Hahn condition. For every $n \in \mathbb{N}$,*
$$(L_n^\alpha)'(x) = nL_{n-1}^{\alpha+1}(x). \tag{5}$$

5. *Lowering operator. For every $n \in \mathbb{N}$,*
$$x(L_n^\alpha(x))' - nL_n^\alpha(x) = n(n+\alpha)L_{n-1}^\alpha(x). \tag{6}$$

6. *For every $n \in \mathbb{N}$, $L_n^\alpha(x)$ is an eigenfunction of the differential operator*
$$xD^2 + (\alpha+1-x)D \tag{7}$$
with $-n$ as eigenvalue.

   If we denote by $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$ as the sequence of Laguerre polynomials with leading coefficients $\frac{(-1)^n}{n!}$, i.e.
$$\widehat{L}_n^\alpha(x) = \frac{(-1)^n}{n!}L_n^\alpha(x), \tag{8}$$
then we have the following asymptotic formulas

7. *Outer strong asymptotics (Perron's asymptotics formula on $\mathbb{C}-\mathbb{R}_+$). Let $\alpha \in \mathbb{R}$. Then*
$$\widehat{L}_n^\alpha(x) = \frac{1}{2}\pi^{-1/2}e^{x/2}(-x)^{-\alpha/2-1/4}n^{\alpha/2-1/4}e^{2(-nx)^{1/2}}\left\{\sum_{k=0}^{p-1} C_k(x)n^{-k/2} + \mathcal{O}(n^{-p/2})\right\}. \tag{9}$$
Here $C_k(x)$ is independent of n. This relation holds for x in the complex plane with a cut along the positive real semiaxis. The bound for the remainder holds uniformly in every closed domain with no points in common with $x \geq 0$ (see [19], Theorem 8.22.3).

8. *Mehler-Heine type formula. Fixed j, with $j \in N \cup \{0\}$ and J_α the Bessel function of the first kind, then*
$$\lim_{n\to\infty} \frac{\widehat{L}_n^\alpha(x/(n+j))}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}), \tag{10}$$
uniformly over compact subsets of $\mathbb{C}$ (see [19], Theorem 8.1.3).

On the other hand, if

$$K_n(x,y)=\sum_{k=0}^{n}\frac{L_k^\alpha(x)L_k^\alpha(y)}{\|L_k^\alpha\|_\alpha^2} \tag{11}$$

denotes the n-th kernel polynomial associated with the Laguerre orthogonal polynomials, then, according to the Christoffel-Darboux formula, for every $n\in\mathbb{N}$ we get

$$K_n(x,y)=\frac{L_{n+1}^\alpha(x)L_n^\alpha(y)-L_{n+1}^\alpha(y)L_n^\alpha(x)}{x-y}\frac{1}{\|L_n^\alpha\|_\alpha^2}. \tag{12}$$

Notice that if $deg\ f\leq n$, then the n-th kernel polynomial satisfies the so-called "reproducing property"

$$\int K_n\left(x,y\right)f\left(x\right)x^\alpha e^{-x}dx=f\left(y\right) \tag{13}$$

The kernel polynomials associated with Laguerre polynomials will play a key role in order to prove some of the basic results of the manuscript.

Let us introduce the following inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients

$$\langle p,q\rangle=\int_0^\infty pqx^\alpha e^{-x}dx+Mp(a)q(a), \tag{14}$$

where $M\in\mathbb{R}_+$, and $a\in\mathbb{R}_-$. We will denote by $\{\widetilde{L}_n^\alpha(x)\}_{n\geq0}$ the sequence of monic polynomials orthogonal with respect to the above inner product.

The case $a=0$ has been extensively studied in the literature, mainly in connection with spectral problems for higher order linear differential operators.A systemic approach was done in [11] where a representation of these polynomials in terms of Laguerre polynomials is given. From it, the author deduces a hypergeometric representation as well as a second order linear differential equation that they satisfy. When α is a positive integer number, they are eigenfunctions of a linear differential operator of order $2\alpha+4$ with polynomial coefficients which are independent of the degree of the polynomial (see [12]). These polynomials are called either Laguerre-type polynomials (see, for instance, [5] and [17]) or Laguerre- Krall polynomials ([8]). The distribution of their zeros, their dependence on the mass M as well as their interlacing properties have been analyzed in [6]. An electrostatic interpretation of these zeros as equilibrium points with respect to a logarithmic potential under the action of an external field has been done in [10] and [7]. Finally, outer relative asymptotics and Mehler-Heine formulas for these polynomials have been obtained in [1] and [16].

Nevertheless, when a is a negative number, i.e. the mass point is located outside the support of the measure, the study of their analytic properties has not attracted the interest of researchers, up to in the general framework of semi-classical functionals [15]. Indeed, one the main goals of this manuscript is to consider some algebraic and analytic properties of such polynomials and to present a comparison with those of Laguerre polynomials.

The structure of the manuscript is as follows. In Section 2, the connection formula between Laguerre-type and Laguerre polynomials is given. As a consequence, a representation of these polynomials as hypergeometric functions appears in a natural way. In Section 3 the coefficients of the corresponding three term recurrence relation are deduced. Notice that this result is related to the Uvarov spectral transformation of the Laguerre weight and thus the connection between the entries of their corresponding Jacobi matrices follows in a straightforward way. In Section 4 we obtain the lowering and raising operators associated with the Laguerre type polynomials. Thus the corresponding holonomic equation follows in a simple way. Section 5 is focused on the behavior of the zeros of these polynomials in terms of the mass M. Finally, in Section 6 we analyze the outer relative asymptotics as well as the Mehler-Heine formula for these polynomials.

2 The Connection Formula

We can expand $\widetilde{L}_n^\alpha(x)$ in terms of the classical Laguerre polynomials

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) + \sum_{k=0}^{n-1} a_{n,k} L_k^\alpha(x), \tag{15}$$

where

$$a_{n,k} = \frac{\left\langle L_k^\alpha(x), \widetilde{L}_n^\alpha(x) \right\rangle_\alpha}{\|L_k^\alpha\|_\alpha^2}, \quad 0 \leq k \leq n-1. \tag{16}$$

Thus, (15) becomes

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - M\widetilde{L}_n^\alpha(a) \sum_{k=0}^{n-1} \frac{L_k^\alpha(a) L_k^\alpha(x)}{\|L_k^\alpha\|_\alpha^2},$$

and taking into account (11) we get

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - M\widetilde{L}_n^\alpha(a) K_{n-1}(x,a). \tag{17}$$

In order to find $\widetilde{L}_n^\alpha(a)$, we evaluate (17) in $x = a$. Thus

$$\widetilde{L}_n^\alpha(a) = \frac{L_n^\alpha(a)}{1 + MK_{n-1}(a,a)}, \quad n \geq 1, \tag{18}$$

and replacing this value in (17), we have

Proposition 2 *Let $\{\widetilde{L}_n^\alpha\}_{n\geq 0}$ be the sequence of monic Laguerre-type orthogonal polynomials. Then*

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - \frac{ML_n^\alpha(a)}{1 + MK_{n-1}(a,a)} K_{n-1}(x,a), \quad n \geq 1. \tag{19}$$

On the other hand, from (12)

$$(x-a)K_{n-1}(x,a) = \frac{1}{\left\|L_{n-1}^\alpha\right\|_\alpha^2} \left(L_n^\alpha(x) L_{n-1}^\alpha(a) - L_{n-1}^\alpha(x) L_n^\alpha(a) \right),$$

and (19) becomes

$$(x-a)\widetilde{L}_n^\alpha(x) = L_{n+1}^\alpha(x) + A_n L_n^\alpha(x) + B_n L_{n-1}^\alpha(x), \quad n \geq 1,$$

where

$$\begin{aligned} A_n &= \beta_n - a - \frac{ML_n^\alpha(a) L_{n-1}^\alpha(a)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2 (1 + MK_{n-1}(a,a))}, \\ B_n &= \gamma_n + \frac{M\left(L_n^\alpha(a)\right)^2}{\left\|L_{n-1}^\alpha\right\|_\alpha^2 (1 + MK_{n-1}(a,a))} = \gamma_n \left(\frac{1 + MK_n(a,a)}{1 + MK_{n-1}(a,a)} \right). \end{aligned}$$

Introducing the notation

$$a_n = \frac{L_{n+1}^\alpha(a)}{L_n^\alpha(a)}, \qquad b_n = \frac{1 + MK_n(a,a)}{1 + MK_{n-1}(a,a)}, \qquad n \geq 1, \tag{20}$$

we get

$$
\begin{aligned}
A_n &= \beta_n - a - M\frac{\gamma_n}{\|L_n^\alpha\|_\alpha^2}\frac{L_{n-1}^\alpha(a)}{L_n^\alpha(a)}\frac{(L_n^\alpha(a))^2}{(1+MK_{n-1}(a,a))} \\
&= -a_n - \gamma_n\frac{b_n}{a_{n-1}} = -a_n - \frac{B_n}{a_{n-1}} \qquad (21)
\end{aligned}
$$

and

$$
B_n = \gamma_n b_n. \qquad (22)
$$

Notice that from (20)

$$
B_n = \frac{\left\langle (x-a)\,\widetilde{L}_n^\alpha(x), L_{n-1}^\alpha(x)\right\rangle_\alpha}{\left\|L_{n-1}^\alpha\right\|_\alpha^2} = \frac{\left\|\widetilde{L}_n^\alpha\right\|^2}{\left\|L_{n-1}^\alpha\right\|_\alpha^2} = \gamma_n\frac{\left\|\widetilde{L}_n^\alpha\right\|^2}{\|L_n^\alpha\|_\alpha^2}.
$$

This yields an expression for the ratio of the energy of polynomials $\widetilde{L}_n^\alpha$ and L_n^α with respect to the norms associated with their corresponding inner products.

Proposition 3 *Let* $\left\|\widetilde{L}_n^\alpha\right\|^2$ *be the norm of Laguerre-type monic polynomials with respect to (14). Then*

$$
\frac{\left\|\widetilde{L}_n^\alpha\right\|^2}{\|L_n^\alpha\|_\alpha^2} = \frac{1+MK_n(a,a)}{1+MK_{n-1}(a,a)}, \quad n\geq 1.
$$

Proof. Taking in account

$$
\left\|\widetilde{L}_n^\alpha\right\|^2 = \left\langle \widetilde{L}_n^\alpha(x), x^n\right\rangle = \left\langle \widetilde{L}_n^\alpha(x), L_n^\alpha(x)\right\rangle_\alpha + M\widetilde{L}_n^\alpha(a)L_n^\alpha(a)
$$

and using (18), we get

$$
\left\|\widetilde{L}_n^\alpha\right\|^2 = \|L_n^\alpha\|_\alpha^2 + \frac{M\left(L_n^\alpha(a)\right)^2}{1+MK_{n-1}(a,a)} = \|L_n^\alpha\|_\alpha^2\frac{1+MK_n(a,a)}{1+MK_{n-1}(a,a)}.
$$

■

Remark. From the expressions (21) and (22) we observe that two basic parameters (a_n, b_n) are needed in the connection formula.

As a conclusion,

Theorem 1 *Let* $\{\widetilde{L}_n^\alpha\}_{n\geq 0}$ *be the sequence of monic Laguerre-type polynomials orthogonal with respect to (14). Then*

$$
(x-a)\widetilde{L}_n^\alpha(x) = L_{n+1}^\alpha(x) + A_nL_n^\alpha(x) + B_nL_{n-1}^\alpha(x), \quad n\geq 1, \qquad (23)
$$

where

$$
\begin{aligned}
A_n &= (2n+1+\alpha-a) - \frac{ML_n^\alpha(a)L_{n-1}^\alpha(a)}{\left\|L_{n-1}^\alpha\right\|_\alpha^2(1+MK_{n-1}(a,a))} = -a_n - \gamma_n\frac{b_n}{a_{n-1}}, \qquad (24) \\
B_n &= n(n+\alpha)\frac{1+MK_n(a,a)}{1+MK_{n-1}(a,a)} = \gamma_n b_n, \qquad (25)
\end{aligned}
$$

with

$$
a_n = \frac{L_{n+1}^\alpha(a)}{L_n^\alpha(a)}, \qquad b_n = \frac{1+MK_n(a,a)}{1+MK_{n-1}(a,a)}, \qquad n\geq 1.
$$

Next we will focus our attention in the representation of this new family of orthogonal polynomials as hypergeometric functions. From (23) and the hypergeometric representation of the classical Laguerre polynomials

$$L_n^{\alpha}(x) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \, {}_1F_1(-n;\alpha+1;x),$$

we can write

$$\begin{aligned}(x-a)\,\tilde{L}_n^{\alpha}(x) &= (-1)^n (\alpha+1)_n \sum_{k=0}^{\infty} \left(\frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!} \right) \\ &\times \left[\frac{-(\alpha+n+1)(-n-1)}{(-n-1+k)} - \left(a_n + \gamma_n \frac{b_n}{a_{n-1}} \right) - \frac{\gamma_n b_n}{(\alpha+n)} \frac{(-n+k)}{(-n)} \right].\end{aligned}$$

Now let's write the expression in brackets as a rational function in the variable k. A careful computation of the elements of the sum inside these brackets, we obtain

$$\left[\frac{-(\alpha+n+1)(-n-1)}{(-n-1+k)} - \left(a_n + \gamma_n \frac{b_n}{a_{n-1}} \right) - \frac{\gamma_n b_n}{(\alpha+n)} \frac{(-n+k)}{(-n)} \right] = b_n \frac{(k-e_1)(k-e_2)}{(k-(n+1))}$$

where

$$\begin{aligned}e_1 &= \frac{-1}{2b_n a_{n-1}} \left(d_1 + \sqrt{d_1^2 - 4d_0 b_n a_{n-1}} \right), \\ e_2 &= \frac{-1}{2b_n a_{n-1}} \left(d_1 - \sqrt{d_1^2 - 4d_0 b_n a_{n-1}} \right), \\ d_0 &= (n+1)\left(\gamma_n b_n + a_{n-1}\left(a_n + 1 + \alpha + n(1+b_n)\right)\right), \\ d_1 &= -\gamma_n b_n - a_{n-1}\left(b_n(2n+1) + a_n\right).\end{aligned}$$

Thus

$$(x-a)\,\tilde{L}_n^{\alpha}(x) = \sum_{k=0}^{\infty} \left(\frac{(-1)^n (\alpha+1)_n (-n)_k}{(\alpha+1)_k} \frac{x^k}{k!} \right) \times \left[b_n \frac{(k-e_1)(k-e_2)}{(k-(n+1))} \right]. \tag{26}$$

Notice that although the sum is up to infinity, this is a terminating hypergeometric series, because the Pochhammer symbol $(-n)_k$ becomes zero if $k > n+1$. Now, using some properties of the Pochhammer symbol, we can write

$$\frac{(k-e_1)(k-e_2)}{(k-(n+1))} = \left(\frac{e_1 e_2}{(n+1)} \right) \frac{(1-e_1)_k (1-e_2)_k (-n-1)_k}{(-e_1)_k (-e_2)_k (-n)_k}$$

and, therefore,

$$\begin{aligned}(x-a)\,\tilde{L}_n^{\alpha}(x) &= (-1)^n b_n \left(\frac{e_1 e_2}{(n+1)} \right) (\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!} \times \frac{(1-e_1)_k (1-e_2)_k (-n-1)_k}{(-e_1)_k (-e_2)_k (-n)_k} \\ &= (-1)^n b_n \left(\frac{e_1 e_2}{(n+1)} \right) (\alpha+1)_n \sum_{k=0}^{\infty} \frac{(1-e_1)_k (1-e_2)_k (-n-1)_k}{(-e_1)_k (-e_2)_k (\alpha+1)_k} \frac{x^k}{k!} \\ &= C_{n,\alpha}\, {}_3F_3(1-e_1, 1-e_2, -n-1; -e_0, -e_1, \alpha+1; x).\end{aligned}$$

Finally, the hypergeometric representation is

$$\tilde{L}_n^{\alpha}(x) = \left(\frac{C_{n,\alpha}}{x-a} \right) {}_3F_3(1-e_1, 1-e_2, -n-1; -e_0, -e_1, \alpha+1; x). \tag{27}$$

3 The Three Term Recurrence Relation

Taking into account

$$x\widetilde{L}_n^{\alpha}(x) = \widetilde{L}_{n+1}^{\alpha}(x) + \widetilde{\beta}_n \widetilde{L}_n^{\alpha}(x) + \widetilde{\gamma}_n \widetilde{L}_{n-1}^{\alpha}(x), \quad n \geq 1, \tag{28}$$

we get

$$\widetilde{\beta}_n = \frac{\left\langle x\widetilde{L}_n^{\alpha}(x), \widetilde{L}_n^{\alpha}(x) \right\rangle}{\left\| \widetilde{L}_n^{\alpha} \right\|^2} = a + \frac{\left\langle (x-a)\,\widetilde{L}_n^{\alpha}(x), \widetilde{L}_n^{\alpha}(x) \right\rangle_{\alpha}}{\left\| \widetilde{L}_n^{\alpha} \right\|^2}, \quad n \geq 1.$$

But, from the connection formula, the previous expression becomes

$$\begin{aligned}
\widetilde{\beta}_n &= a + \frac{\left\langle L_{n+1}^{\alpha}(x) - \left(a_n + \gamma_n \frac{b_n}{a_{n-1}}\right) L_n^{\alpha}(x) + \gamma_n b_n L_{n-1}^{\alpha}(x), \widetilde{L}_n^{\alpha}(x) \right\rangle_{\alpha}}{\left\| \widetilde{L}_n^{\alpha} \right\|^2} \\
&= \beta_n + a_n \left(1 - \frac{1}{b_n}\right) - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right), \quad n \geq 1.
\end{aligned}$$

On the other hand

$$\widetilde{\gamma}_n = \frac{\left\langle x\widetilde{L}_n^{\alpha}(x), \widetilde{L}_{n-1}^{\alpha}(x) \right\rangle}{\left\| \widetilde{L}_{n-1}^{\alpha} \right\|^2} = \frac{\left\| \widetilde{L}_n^{\alpha} \right\|^2}{\left\| \widetilde{L}_{n-1}^{\alpha} \right\|^2} = \frac{b_n}{b_{n-1}}\gamma_n, \quad n \geq 1.$$

Thus, as a conclusion

Proposition 4 *The coefficients of the three term recurrence relation for the sequence of monic orthogonal polynomials* $\{\widetilde{L}_n^{\alpha}(x)\}_{n\geq 0}$ *are*

$$\widetilde{\beta}_n = \beta_n + a_n \left(1 - \frac{1}{b_n}\right) - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right), \quad n \geq 1, \tag{29}$$

$$\widetilde{\gamma}_n = \frac{b_n}{b_{n-1}}\gamma_n, \quad n \geq 1. \tag{30}$$

Remark. An alternative way to find $\left(\widetilde{\beta}_n, \widetilde{\gamma}_n\right)$ is based on the analysis of the **UL** factorization of the Jacobi matrix associated with such orthogonal polynomials. This is essentially the well known matrix interpretation of the Uvarov transform (see [3] and [20] for such a matrix approach).

Now we are going to analyze how these parameters behave. Indeed,

$$\begin{aligned}
\frac{\widetilde{\beta}_n}{\beta_n} &= 1 + \frac{a_n}{\beta_n}\left(1 - \frac{1}{b_n}\right) - \frac{a_{n-1}}{\beta_n}\left(1 - \frac{1}{b_{n-1}}\right) \\
&= 1 + \frac{1}{\beta_n}\frac{L_{n+1}^{\alpha}(a)}{L_n^{\alpha}(a)}\left(M\frac{\left(L_n^{\alpha}(a)\right)^2 \diagup \|L_n^{\alpha}\|_{\alpha}^2}{1 + MK_n(a,a)}\right) - \frac{1}{\beta_n}\frac{L_n^{\alpha}(a)}{L_{n-1}^{\alpha}(a)}\left(M\frac{\left(L_{n-1}^{\alpha}(a)\right)^2 \diagup \|L_{n-1}^{\alpha}\|_{\alpha}^2}{1 + MK_{n-1}(a,a)}\right), n \geq 1.
\end{aligned}$$

But

$$\frac{ML_{n+1}^{\alpha}(a)\,L_{n}^{\alpha}(a)}{\|L_{n}^{\alpha}\|_{\alpha}^{2}+M\,(n+1)!n!\left(\widehat{L}_{n}^{\alpha+1}(a)\right)^{2}\left(1-\dfrac{\widehat{L}_{n+1}^{\alpha+1}(a)}{\widehat{L}_{n}^{\alpha+1}(a)}\dfrac{\widehat{L}_{n-1}^{\alpha+1}(a)}{\widehat{L}_{n}^{\alpha+1}(a)}\right)}$$

$$=\frac{\left(1-\dfrac{\widehat{L}_{n+1}^{\alpha+1}(a)}{\widehat{L}_{n}^{\alpha+1}(a)}\right)\left(1-\dfrac{\widehat{L}_{n-1}^{\alpha+1}(a)}{\widehat{L}_{n}^{\alpha+1}(a)}\right)}{\dfrac{\|L_{n}^{\alpha}\|_{\alpha}^{2}}{M\,(n+1)!n!\left(\widehat{L}_{n}^{\alpha+1}(a)\right)^{2}}+\left(1-\dfrac{\widehat{L}_{n+1}^{\alpha+1}(a)}{\widehat{L}_{n}^{\alpha+1}(a)}\dfrac{\widehat{L}_{n-1}^{\alpha+1}(a)}{\widehat{L}_{n}^{\alpha+1}(a)}\right)}. \tag{31}$$

According to Perron's formula

$$\frac{\widehat{L}_{n}^{\alpha+1}(a)}{\widehat{L}_{n-1}^{\alpha+1}(a)}=1+\frac{\sqrt{|a|}}{\sqrt{n}}+\mathcal{O}\left(\frac{1}{n}\right)$$

and the expression (31) becomes

$$\begin{aligned}\frac{\left(-\dfrac{\sqrt{|a|}}{\sqrt{n+1}}+\mathcal{O}\left(\frac{1}{n}\right)\right)\left(\dfrac{\sqrt{|a|}}{\sqrt{n}}+\mathcal{O}\left(\frac{1}{n}\right)\right)}{1-\left(1+\dfrac{\sqrt{|a|}}{\sqrt{n+1}}\right)\left(1-\dfrac{\sqrt{|a|}}{\sqrt{n}}\right)} &= \frac{\dfrac{-\sqrt{|a|}}{\sqrt{n}\sqrt{n+1}}+\mathcal{O}\left(n^{-3/2}\right)}{\dfrac{\sqrt{|a|}}{\sqrt{n}}-\dfrac{\sqrt{|a|}}{\sqrt{n+1}}+\mathcal{O}\left(\frac{1}{n}\right)}\\ &= -2\sqrt{|a|}n^{1/2}+\mathcal{O}\left(n^{-1/2}\right).\end{aligned}$$

Thus

$$\frac{\widetilde{\beta}_{n}}{\beta_{n}}=1-\frac{\sqrt{|a|}}{2}n^{-3/2}+\mathcal{O}\left(n^{-5/2}\right).$$

On the other hand, from (29) and (20)

$$\frac{\widetilde{\gamma}_{n}}{\gamma_{n}}=\frac{b_{n}}{b_{n-1}},\quad n\geq 1.$$

again we take in account

$$\begin{aligned}\frac{1+MK_{n}(a,a)}{1+MK_{n-1}(a,a)} &= 1+\frac{M\left(\widehat{L}_{n}^{\alpha}(a)\right)^{2}\diagup\|L_{n}^{\alpha}\|_{\alpha}^{2}}{1+MK_{n-1}(a,a)}\\ &\sim 1+\frac{\dfrac{|a|}{n}+\mathcal{O}\left(n^{-3/2}\right)}{(n+\alpha)\left(\sqrt{|a|}\diagup 2n^{3/2}\right)}=1+2\sqrt{|a|}n^{-1/2}+\mathcal{O}\left(n^{-1}\right).\end{aligned}$$

As a conclusion, we have

Proposition 5

$$\begin{aligned}\frac{\widetilde{\beta}_{n}}{\beta_{n}} &= 1-\frac{\sqrt{|a|}}{2}n^{-3/2}+\mathcal{O}(n^{-5/2}),\quad n\geq 1,\\ \frac{\widetilde{\gamma}_{n}}{\gamma_{n}} &= 1+2\sqrt{|a|}n^{-1/2}+\mathcal{O}(n^{-1}),\quad n\geq 1.\end{aligned}$$

4 Lowering and raising operators

From the connection formula (23) we get

$$(x-a)\widetilde{L}_n^{\alpha}(x) = (x + A_n - \beta_n)\, L_n^{\alpha}(x) + (B_n - \gamma_n)\, L_{n-1}^{\alpha}(x), \quad n \geq 1. \tag{32}$$

Notice that for $x = a$

$$(a + A_n - \beta_n)\, L_n^{\alpha}(a) = (\gamma_n - B_n)\, L_{n-1}^{\alpha}(a).$$

Thus

$$a + A_n - \beta_n = (\gamma_n - B_n)\, \frac{1}{a_{n-1}}$$

and (32) becomes

$$(x-a)\widetilde{L}_n^{\alpha}(x) = \left(x - a - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right)\right) L_n^{\alpha}(x) + \gamma_n\, (b_n - 1)\, L_{n-1}^{\alpha}(x). \tag{33}$$

On the other hand, introducing the shift $n \to n-1$, the above expression becomes

$$(x-a)\widetilde{L}_{n-1}^{\alpha}(x) = \left(x - a - a_{n-2}\left(1 - \frac{1}{b_{n-2}}\right)\right) L_{n-1}^{\alpha}(x) + \gamma_{n-1}\, (b_{n-1} - 1)\, L_{n-2}^{\alpha}(x)$$

and, using the three term recurrence relation, we get

$$\begin{aligned} & (x-a)\widetilde{L}_{n-1}^{\alpha}(x) \\ = & \left(x - a - a_{n-2}\left(1 - \frac{1}{b_{n-2}}\right) + (b_{n-1} - 1)\,(x - \beta_{n-1})\right) L_{n-1}^{\alpha}(x) - (b_{n-1} - 1)\, L_n^{\alpha}(x). \end{aligned} \tag{34}$$

Thus, taking $x = a$, we obtain

$$\left(-a_{n-2}\left(1 - \frac{1}{b_{n-2}}\right) + (b_{n-1} - 1)\,(a - \beta_{n-1})\right) = (b_{n-1} - 1)\, a_{n-1}.$$

Replacing it in (34) we get

$$(x-a)\widetilde{L}_{n-1}^{\alpha}(x) = (b_{n-1}\,(x-a) + a_{n-1}\,(b_{n-1} - 1))\, L_{n-1}^{\alpha}(x) + (1 - b_{n-1})\, L_n^{\alpha}(x). \tag{35}$$

As a conclusion, from (33) and (35) deduce the representation of Laguerre polynomials in terms of Laguerre-type polynomials which will very useful in the sequel.

$$L_n(x) = \frac{\left[(x-a) + a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right)\right] \widetilde{L}_n^{\alpha}(x) - \frac{a_{n-1}^2}{b_{n-1}}\left(1 - \frac{1}{b_{n-1}}\right) \widetilde{L}_{n-1}^{\alpha}(x)}{(x-a)}, \tag{36}$$

$$L_{n-1}(x) = \frac{\left(1 - \frac{1}{b_{n-1}}\right) \widetilde{L}_n^{\alpha}(x) + \frac{1}{b_{n-1}}\left[(x-a) - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right)\right] \widetilde{L}_{n-1}^{\alpha}(x)}{(x-a)}. \tag{37}$$

Next, taking derivatives in (33) and multiplying by x

$$\begin{aligned} x\widetilde{L}_n^{\alpha}(x) + x\,(x-a)\, D\widetilde{L}_n^{\alpha}(x) &= xL_n^{\alpha}(x) + \left(x - a - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right)\right) xDL_n^{\alpha}(x) \\ &\quad + \gamma_n (b_n - 1) xDL_{n-1}^{\alpha}(x), \end{aligned}$$

but, according to the lowering operator for Laguerre polynomials (6) and the three term recurrence relation that they satisfy, the above expression becomes

$$\begin{aligned}&x\widetilde{L}_n^{\alpha}(x)+x\,(x-a)\,D\widetilde{L}_n^{\alpha}(x)\\=\;&\left[(n+1)\,x-na-na_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-n\,(n+\alpha)\,(b_n-1)\right]L_n^{\alpha}(x)\\&+\gamma_n\left[b_nx-a-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-(n+\alpha)\,(b_n-1)\right]L_{n-1}^{\alpha}(x).\end{aligned}$$

Multiplying both hand sides by $(x-a)$ and using (36) and (37) we get

$$\begin{aligned}&x\,(x-a)\,\widetilde{L}_n^{\alpha}(x)+x\,(x-a)^2\,D\widetilde{L}_n^{\alpha}(x)\\=\;&\left[(n+1)\,x-na-na_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-n\,(n+\alpha)\,(b_n-1)\right](x-a)\,L_n^{\alpha}(x)\\&+\gamma_n\left[b_nx-a-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-(n+\alpha)\,(b_n-1)\right](x-a)\,L_{n-1}^{\alpha}\,(x)\,.\end{aligned}$$

Introducing the parameter

$$c_n=a+a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-(n+\alpha)\,(b_n-1)$$

the above expression reads as

$$\begin{aligned}&x\,(x-a)\,\widetilde{L}_n^{\alpha}(x)+x\,(x-a)^2\,D\widetilde{L}_n^{\alpha}(x)\\=\;&[(n+1)\,x-nc_n]\left\{\left[(x-a)+a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right]\widetilde{L}_n^{\alpha}(x)-\frac{a_{n-1}^2}{b_{n-1}}\left(1-\frac{1}{b_{n-1}}\right)\widetilde{L}_{n-1}^{\alpha}(x)\right\}\\&+\gamma_n\,[b_nx-c_n]\left\{\left(1-\frac{1}{b_{n-1}}\right)\widetilde{L}_n^{\alpha}(x)+\frac{1}{b_{n-1}}\left[(x-a)-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right]\widetilde{L}_{n-1}^{\alpha}(x)\right\}.\end{aligned}$$

As a conclusion,

$$\begin{aligned}&\left[x\,(x-a)^2\,D+x\,(x-a)-[(n+1)\,x-nc_n]\left(x-a+a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right)\right.\\&\left.-\,\gamma_n\,[b_nx-c_n]\left(1-\frac{1}{b_{n-1}}\right)\right]\widetilde{L}_n^{\alpha}(x)\\=\;&\left[-\,[(n+1)\,x-nc_n]\,\frac{a_{n-1}^2}{b_{n-1}}\left(1-\frac{1}{b_{n-1}}\right)\right.\\&\left.+\frac{\gamma_n}{b_{n-1}}\,[b_nx-c_n]\left(x-a-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right)\right]\widetilde{L}_{n-1}^{\alpha}(x).\end{aligned}$$

Thus, we get the expression for the lowering operator $\mathcal{L}_n$

$$\mathcal{L}_n\widetilde{L}_n^{\alpha}(x)=u_n\,(x)\,\widetilde{L}_{n-1}^{\alpha}(x)$$

where

$$\mathcal{L}_n=x\,(x-a)^2\,D-nx^2+D_nx+E_n$$

and

$$D_n = \frac{1}{b_{n-1}}\left(a_{n-1} + \gamma_n b_n\right) + \gamma_n - a_{n-1} - 2\left(\gamma_n b_n - an\right),$$
$$E_n = c_n\left(\left(n a_{n-1} + \gamma_n\right)\left(1 - \frac{1}{b_{n-1}}\right) - an\right).$$

Notice that $u_n(x)$ is a quadratic polynomial

$$u_n\left(x\right) = F_n x^2 + G_n x + H_n,$$

with

$$F_n = \gamma_n \frac{b_n}{b_{n-1}},$$
$$G_n = \left(\frac{1}{b_{n-1}} - 1\right)\frac{a_{n-1}^2}{b_{n-1}}\left(n+1\right) - \frac{\gamma_n}{b_{n-1}}\left(c_n + b_n\left(a - \left(\frac{1}{b_{n-1}} - 1\right)a_{n-1}\right)\right),$$
$$H_n = \gamma_n c_n \frac{a - ((1/b_{n-1}) - 1)a_{n-1}}{b_{n-1}} - n c_n\left(\frac{1}{b_{n-1}} - 1\right)\frac{a_{n-1}^2}{b_{n-1}}.$$

Taking into account the three term recurrence relation (28)

$$\widetilde{\gamma}_n \mathcal{L}_n \widetilde{L}_n^{\alpha}(x) = u_n\left(x\right)\left[\left(x - \widetilde{\beta}_n\right)\widetilde{L}_n^{\alpha}(x) - \widetilde{L}_{n+1}^{\alpha}(x)\right].$$

Thus, we get the raising operator

$$\mathcal{R}_n = -\widetilde{\gamma}_n \mathcal{L}_n + \left(x - \widetilde{\beta}_n\right)u_n\left(x\right)$$

i.e

$$\mathcal{R}_n \widetilde{L}_n^{\alpha}(x) = u_n\left(x\right)\widetilde{L}_{n+1}^{\alpha}(x).$$

As a consequence,

$$\frac{1}{u_n\left(x\right)}\mathcal{R}_n \widetilde{L}_n^{\alpha}(x) = \widetilde{L}_{n+1}^{\alpha}(x).$$

Applying the lowering operator in the above expression,

$$\mathcal{L}_{n+1}\left[\frac{1}{u_n\left(x\right)}\mathcal{R}_n\right]\left(\widetilde{L}_n^{\alpha}(x)\right) = u_{n+1}\left(x\right)\widetilde{L}_n^{\alpha}(x)$$

and thus, the linear differential equation (holonomic equation) satisfied by the Laguerre-type polynomials follows after a tedious computation. When $a = 0$ such equation has been obtained in [11].

5 The Zeros

We will give a new expression of $\widetilde{L}_n^{\alpha}(x)$ in terms of the generalized moments using the Gram-Schmidt orthonormalization process for the family of polynomials $\{(x-a)^k\}_{k=0}^n$. Indeed, if $\left\langle (x-a)^k, (x-a)^j \right\rangle_{\alpha} = \zeta_{k+j}$ and denoting

$$\Omega_n\left(d\mu\right) = \begin{vmatrix} \zeta_0 & \zeta_1 & \cdots & \zeta_n \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n-1} & \zeta_n & \cdots & \zeta_{2n-1} \\ \zeta_n & \zeta_{n+1} & \cdots & \zeta_{2n} \end{vmatrix}, \quad n \geq 1,$$

where $d\mu(x) = x^{\alpha} e^{-x} dx$, then we get

$$L_n^\alpha(x) = \frac{1}{\Omega_{n-1}(d\mu)} \begin{vmatrix} \zeta_0 & \zeta_1 & \cdots & \zeta_n \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n-1} & \zeta_n & \cdots & \zeta_{2n-1} \\ 1 & x-a & \cdots & (x-a)^n \end{vmatrix}, \quad n \geq 1, \tag{38}$$

If $d\widetilde{\mu} = d\mu + M\delta(x-a)$, then

$$\widetilde{L}_n^\alpha(x) = \frac{1}{\Omega_{n-1}(d\widetilde{\mu})} \begin{vmatrix} \zeta_0 + M & \zeta_1 & \cdots & \zeta_n \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n-1} & \zeta_n & \cdots & \zeta_{2n-1} \\ 1 & x-a & \cdots & (x-a)^n \end{vmatrix}, \quad n \geq 1.$$

But

$$\Omega_{n-1}(d\widetilde{\mu}) = \begin{vmatrix} \zeta_0 + M & \zeta_1 & \cdots & \zeta_{n-1} \\ \zeta_1 & \zeta_2 & \cdots & \zeta_n \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n-1} & \zeta_n & \cdots & \zeta_{2n-2} \end{vmatrix} = \Omega_{n-1}(d\mu) + M\Omega_{n-2}\left((x-a)^2 d\mu\right).$$

As a consequence,

$$\widetilde{L}_n^\alpha(x) = \frac{\Omega_{n-1}(d\mu) L_n^\alpha(x) + M(x-a)\,\Omega_{n-2}\left((x-a)^2 d\mu\right) R_{n-1}^\alpha(x)}{\Omega_{n-1}(d\mu) + M\Omega_{n-2}\left((x-a)^2 d\mu\right)}, \quad n \geq 1,$$

where $R_{n-1}^\alpha(x)$ is the polynomial of degree $n-1$ orthogonal with respect to the inner product associated with the measure $(x-a)^2 d\mu$. Taking limit when M tends to infinity,

$$\lim_{M\to\infty} \widetilde{L}_n^\alpha(x) = (x-a)\, R_{n-1}^\alpha(x).$$

From this expression we can see that when $M \to \infty$, the mass point a attracts one of the zeros of $\widetilde{L}_n^\alpha(x)$, and each zero of $R_{n-1}^\alpha(x)$ attracts one of the remainder $n-1$ zeros of $\widetilde{L}_n^\alpha(x)$.

5.1 Interlacing

Let $\{x_{n,k}\} \equiv x_{n,1} < x_{n,2} < ... < x_{n,n}$ be the zeros of $L_n^\alpha(x)$ and $\{\widetilde{x}_{n,k}\} \equiv \widetilde{x}_{n,1} < \widetilde{x}_{n,2} < ... < \widetilde{x}_{n,n}$ the zeros of $\widetilde{L}_n^\alpha(x)$, with $1 \leq k \leq n$. Notice thah they are simple and real numbers greater than a. Assuming that $a < 0$,we will prove that the zeros $\{\widetilde{x}_{n,k}\}_{k=1}^n$ interlace with $\{x_{n,k}\}_{k=1}^n$.

We know from (*32*) that

$$(x-a)\widetilde{L}_n^\alpha(x) = (A_n + (x - (2n+\alpha+1)))\, L_n^\alpha(x) + (B_n - n(n+\alpha))\, L_{n-1}^\alpha(x), \quad n \geq 1.$$

Evaluating this expression in $x_{n,k}$ we have

$$(x_{n,k} - a)\widetilde{L}_n^\alpha(x_{n,k}) = (B_n - n(n+\alpha))\, L_{n-1}^\alpha(x_{n,k}). \tag{39}$$

Since

$$B_n = n(n+\alpha) + \frac{M\left(L_n^\alpha(a)\right)^2}{\|L_{n-1}\|_\alpha^2 \left(1 + MK_{n-1}(a,a)\right)}$$

(39) becomes

$$(x_{n,k}-a)\widetilde{L}_n^\alpha(x_{n,k}) = \frac{M\left(L_n^\alpha(a)\right)^2}{\|L_{n-1}\|_\alpha^2\left(1+MK_{n-1}(a,a)\right)}L_{n-1}^\alpha(x_{n,k}). \tag{40}$$

Taking into account $a<0$, $x_{n,k}-a$ is always positive, so $sign\left(\widetilde{L}_n^\alpha(x_{n,k})\right) = sign\left(L_{n-1}^\alpha(x_{n,k})\right)$, and since the zeros of $L_{n-1}^\alpha(x)$ interlace with the zeros of $L_n^\alpha(x)$ then the positive zeros of $\widetilde{L}_n^\alpha(x)$ interlace in similar way with the zeros of $L_n^\alpha(x)$, so we have

$$\widetilde{x}_{n,1} < x_{n1} < \widetilde{x}_{n,2} < x_{n2} < \ldots < \widetilde{x}_{n,n} < x_{n,n}.$$

5.2 Numerical experiments

Next we will show some numerical experiments dealing with the least zero of Laguerre-type polynomials. We are interested to analyze when such a zero is negative as well as how converges to the point a. To obtain these results we have implemented a Matlab program using the classical Chebyshev algorithm, since these polynomials satisfy a RRTT and, therefore, this technique can be used.

In the first two tables, for $M=0$ obviously we recover the least zero and the second zero of the classical Laguerre polynomials (in bold).

The following table shows this effect for the first and second zeros of Laguerre-type polynomial of degree $n=18$ and $\alpha=0$, for some choices of the mass M

$\tilde{x}_{18,k}$	$M=0$	$M=5.0\cdot10^{-12}$	$M=5.0\cdot10^{-8}$	$M=5.0\cdot10^{-4}$	$M=5.0\cdot10^{-2}$
$k=1$	**0.078214**	-9.999962	-9.999999	-9.999999	-9.99999
$k=2$	**0.412713**	0.091897	0.092041	0.092138	0.09146

as well as when the masspoint is located at $a=-1$.

$\tilde{x}_{18,k}$	$M=0$	$M=5.0\cdot10^{-12}$	$M=5.0\cdot10^{-8}$	$M=5.0\cdot10^{-4}$	$M=5.0\cdot10^{-2}$
$k=1$	**0.078214**	0.077724	0.078693	-0.982153	-0.999814
$k=2$	**0.412713**	0.410213	0.416097	0.130937	0.131273

In the next table we give the least zero for polynomials of degree $n=7, \alpha=0$, as well as we pint out the fact that there exists $M=M_0$ such that this zero is negative. In this particular example, with the mass point located at $a=-10$, this value is roughly $1,0\cdot10^{-9} < M_0 < 2\cdot10^{-9}$

$\tilde{x}_{7,k}$:	$M=0$	$M=1.0\cdot10^{-9}$	$M=2.0\cdot10^{-9}$	$M=5.0\cdot10^{-9}$	$M=5.0\cdot10^{-5}$
$k=1$	**0.193044**	0.048634	**−0.775950**	-3.598918	-9.998880

and with the mass point located at $a=-1$, we need larger values of M_0 to get the least zero as a negative real number. Now the estimate is $1.0\cdot10^{-3} < M_0 < 2\cdot10^{-3}$

$\tilde{x}_{7,k}$:	$M=0$	$M=5.0\cdot10^{-9}$	$M=1.0\cdot10^{-3}$	$M=2.0\cdot10^{-3}$	$M=5.0\cdot10^{-2}$
$k=1$	**0.193044**	0.193043	0.059013	**−0.094368**	-0.915571

Another interesting question is to analyze, for a fixed value M, the behavior of zeros of Laguerre-type polynomials in terms of the parameter α. Notice that, for a fixed value of α we can lose its negative zero. Again we show the behavior of the first two zeros to give more information about their relative spacing.

For instance, let us show the first two zeros of the Laguerre-type polynomials of degree $n=6$, when $M=5.0\cdot10^{-8}$ and the mass point is located at $a=-10$

$\tilde{x}_{6,k}$:	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 3$	$\alpha = 10$
$k = 1$	-3.498898	-1.606946	-0.173020	**1.271640**	**4.890738**
$k = 2$	0.333321	0.592795	1.031807	3.044173	8.143534

and again, the first two zeros when $M = 5.0 \cdot 10^{-3}$ and $a = -1$

$\tilde{x}_{6,k}$:	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 3$	$\alpha = 10$
$k = 1$	-0.145632	-0.146124	-0.083058	**0.835712**	**4.890712**
$k = 2$	0.714756	0.982986	1.270833	2.677161	8.143506

Finally, another interesting numerical approximation is to consider a different choice of the fixed parameters. For instance, for a fixed M we would find values of the degree n for which $\tilde{L}_n^\alpha(x)$ has a negative zero. Locating the mass point at $a = -10$, we have

$M = 5.0 \cdot 10^{-8}$	$n \geq n_0$
$\alpha = -0.99$	$n \geq 5$
$\alpha = 0$	$n \geq 6$
$\alpha = 1$	$n \geq 6$
$\alpha = 2.5$	$n \geq 7$
$\alpha = 5$	$n \geq 10$
$\alpha = 8$	$n \geq 14$

$M = 5.0 \cdot 10^{-6}$	$n \geq n_0$
$\alpha = -0.99$	$n \geq 4$
$\alpha = 0$	$n \geq 4$
$\alpha = 1$	$n \geq 5$
$\alpha = 2.5$	$n \geq 5$
$\alpha = 5$	$n \geq 7$
$\alpha = 8$	$n \geq 11$

$M = 5.0 \cdot 10^{-4}$	$n \geq n_0$
$\alpha = -0.99$	$n \geq 3$
$\alpha = 0$	$n \geq 3$
$\alpha = 1$	$n \geq 3$
$\alpha = 2.5$	$n \geq 4$
$\alpha = 5$	$n \geq 5$
$\alpha = 8$	$n \geq 8$

6 Outer Asymptotics

In this section, we will obtain some results concerning the asymptotic behavior of Laguerre-type orthogonal polynomials in the exterior of the positive real semi axis. According to the Perron's formula (9), taking into account (19)

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - \frac{M L_n^\alpha(a)}{1 + M K_{n-1}(a,a)} K_{n-1}(x,a),$$

using the standard normalization for Laguerre polynomials (8), and dividing by $\widehat{L}_n^\alpha(x)$ in both hand sides of the above expression, we obtain

$$\frac{\widehat{\widetilde{L}}_n^\alpha(x)}{\widehat{L}_n^\alpha(x)} = 1 - \frac{M \widehat{L}_n^\alpha(a)}{1 + M K_{n-1}(a,a)} \frac{K_{n-1}(x,a)}{\widehat{L}_n^\alpha(x)}.$$

Using the Christoffel-Darboux formula, this expression becomes

$$\frac{\widehat{\widetilde{L}}_n^\alpha(x)}{\widehat{L}_n^\alpha(x)} = 1 + \frac{M \left[1 - \frac{\widehat{L}_n^{\alpha+1}(a)}{\widehat{L}_{n-1}^{\alpha+1}(a)}\right]^2 \left(\frac{\widehat{L}_{n-1}^\alpha(a)}{\widehat{L}_n^\alpha(a)} - \frac{\widehat{L}_{n-1}^\alpha(x)}{\widehat{L}_n^\alpha(x)}\right) \frac{1}{x-a}}{\frac{\left\|L_{n-1}^\alpha\right\|_\alpha^2}{n!\,(n-1)!\left(\widehat{L}_{n-1}^{\alpha+1}(a)\right)^2} + M \frac{\widehat{L}_{n-2}^{\alpha+1}(a)}{\widehat{L}_{n-1}^{\alpha+1}(a)} \left[\frac{\widehat{L}_{n-1}^{\alpha+1}(a)}{\widehat{L}_{n-2}^{\alpha+1}(a)} - \frac{\widehat{L}_n^{\alpha+1}(a)}{\widehat{L}_{n-1}^{\alpha+1}(a)}\right]}. \tag{41}$$

But from the Perron's formula

$$\frac{\widehat{L}_n^\alpha(x)}{\widehat{L}_{n-1}^\alpha(x)} = 1 + \frac{\sqrt{-x}}{\sqrt{n}} + \mathcal{O}\left(n^{-1}\right).$$

Thus, in (41) we get

$$\begin{aligned}
\frac{\widehat{\widetilde{L}}_n^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)} &\sim 1+\frac{M\left(\frac{\sqrt{|a|}}{\sqrt{n}}\right)^2\left(1-\frac{\sqrt{|a|}}{\sqrt{n}}-1+\frac{\sqrt{-x}}{\sqrt{n}}\right)\frac{1}{x-a}}{M\left(-1-\frac{\sqrt{|a|}}{\sqrt{n}}+1+\frac{\sqrt{|a|}}{\sqrt{n-1}}\right)} \\
&\sim 1+2\frac{\sqrt{|a|}\left(\sqrt{(-x)}-\sqrt{|a|}\right)}{x-a}=1-2\sqrt{|a|}\frac{\sqrt{-x}-\sqrt{|a|}}{-x+a} \\
&= 1-2\sqrt{|a|}\frac{1}{\sqrt{(-x)}+\sqrt{|a|}}=\frac{\sqrt{-x}-\sqrt{|a|}}{\sqrt{-x}+\sqrt{|a|}}
\end{aligned}$$

and the convergence is locally uniformly on $\mathbb{C}-\mathbb{R}_+$. Notice that, according to the Hurwitz theorem, the above result shows that the point $x=a$ attracts one zero of $\widetilde{L}_n^{\alpha}(x)$ for n large enough.

6.1 Mehler-Heine formula

Concerning the Mehler-Heine formula, notice that from (41)

$$\frac{\widehat{\widetilde{L}}_n^{\alpha}(x/n)}{\widehat{L}_n^{\alpha}(x/n)}=1+\frac{M\left[1-\frac{\widehat{L}_n^{\alpha+1}(a)}{\widehat{L}_{n-1}^{\alpha+1}(a)}\right]^2\left(\frac{\widehat{L}_{n-1}^{\alpha}(a)}{\widehat{L}_n^{\alpha}(a)}-\frac{\widehat{L}_{n-1}^{\alpha}(x/n)}{\widehat{L}_n^{\alpha}(x/n)}\right)\frac{1}{\frac{x}{n}-a}}{\frac{\|L_n^{\alpha}\|_{\alpha}^2}{n!\,(n-1)!\left(\widehat{L}_{n-1}^{\alpha+1}(a)\right)^2}+M\frac{\widehat{L}_{n-2}^{\alpha+1}(a)}{\widehat{L}_{n-1}^{\alpha+1}(a)}\left[\frac{\widehat{L}_{n-1}^{\alpha+1}(a)}{\widehat{L}_{n-2}^{\alpha+1}(a)}-\frac{\widehat{L}_n^{\alpha+1}(a)}{\widehat{L}_{n-1}^{\alpha+1}(a)}\right]}.$$

Proceeding as above

$$\begin{aligned}
\frac{\widehat{\widetilde{L}}_n^{\alpha}(x/n)}{n^{\alpha}} &\sim \frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}}-\frac{M\frac{|a|}{n}\frac{1}{a}\widehat{L}_n^{\alpha}(a)}{M\sqrt{|a|}\left(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right)}\left(\frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}}-\left(1+\frac{\sqrt{|a|}}{\sqrt{n}}\right)\frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}}\right) \\
&\sim \frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}}-2\frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}}=-x^{-\alpha/2}J_{\alpha}\left(2\sqrt{x}\right).
\end{aligned}$$

Notice that the addition of a mass point changes the sign in the Mehler-Heine formula that the standard Laguerre polynomials satisfy. Furthermore, the Mehler-Heine formula for the Laguerre-Sobolev type orthogonal polynomials is the same independently if the point a is located outside the support of the measure or in its boundary. For this last situation, see Theorem 2.b and Theorem 3 in [1] and Proposition 2 in [16].

References

[1] R. Alvarez Nodarse and J. J. Moreno-Balcázar, *Asymptotic properties of generalized Laguerre orthogonal polynomials*, Indag. Math. N. S. **15** (2004), 151–165.

[2] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, Vol **71**, Cambridge University Press, Cambridge, UK. (1999).

[3] M. I. Bueno and F. Marcellán, *Darboux transformation and perturbation of linear functionals*, Linear Algebra Appl. **384** (2004), 215–242.

[4] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York. (1978).

[5] T. S. Chihara, *Orthogonal polynomials and measures with end point masses*, Rocky Mountain J. Math. **15** (1985), 705–719.

[6] D. K. Dimitrov, F. Marcellán, and F. R. Rafaeli, *Monotonicity of zeros of Laguerre-Sobolev-type orthogonal polynomials* , J. Math. Anal. Appl. **368** (2010), 80-89.

[7] H. Dueñas and F. Marcellán, *Laguerre-Type orthogonal polynomials. Electrostatic interpretation*, Int. J. Pure and Appl. Math. **38** (2007), 345-358.

[8] F. A. Grunbaum, L. Haine and E. Horozov, *Some functions that generalize the Krall-Laguerre polynomials*, J. Comput. Appl. Math. **106** (1999), 271–297.

[9] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in one variable*, Encyclopedia of Mathematics and its Applications Vol **98**, Cambridge University Press, Cambridge, UK. (2005).

[10] M. E. H. Ismail, *More on electrostatic models for zeros of orthogonal polynomials*, Numer. Funct. Anal. Optimiz. **21** (2000), 191–204.

[11] R. Koekoek, *Generalizations of classical Laguerre polynomials and some q-analogues.* Doctoral Dissertation, Technical University Delft.(1990)

[12] J. Koekoek and R. Koekoek, *Differential equation for Koornwinder's generalized Laguerre polynomials.* Proc. Amer. Math. Soc. **112** (1991), 1045–1054.

[13] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publications, New York. (1972).

[14] F. Marcellán, A. Branquinho, and J. C. Petronilho, *Classical Orthogonal Polynomials: A Functional Approach*, Acta Appl. Math. **34** (1994), 283–303.

[15] F. Marcellán and P. Maroni, *Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique* Annal. Mat. Pura ed Appl. **CLXII** (1992), 1–22.

[16] F. Marcellán and J. J. Moreno-Balcázar, *Asymptotics and Zeros of Sobolev Orthogonal Polynomials on Unbounded Supports* Acta Appl. Math. **94** (2006), 163–192.

[17] F. Marcellán and A. Ronveaux, *Differential equations for classical type orthogonal polynomials*, Canad. Math. Bull. **32** (1989), 404-411.

[18] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics: An Unified Approach*, Birkhauser Verlag, Basel. (1988).

[19] G. Szegő, *Orthogonal Polynomials*, 4^{th} ed., Amer. Math. Soc. Colloq. Publ. Series, vol **23**, Amer. Math. Soc., Providence, RI. (1975).

[20] G. J. Yoon, *Darboux transforms and orthogonal polynomials*, Bull. Korean Math. Soc. **39** (2002), 359–376.