Orthogonal polynomials and second-order pseudo-spectral linear differential equations.

Francisco Marcellán [a,*], Ridha Sfaxi [b]

[a] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Spain.*

[b] *Faculté des Sciences de Gabès, Département de Mathématiques, Cité Erriadh 6072 Gabès, Tunisie*

Abstract

In this paper we deal with monic orthogonal polynomial sequences which satisfy the second-order pseudo-spectral linear differential equation:

$$\phi_2(x)y''(x) + \phi_1(x)y'(x) = \chi(x,n)y(x),\ n \in \mathbb{N},$$

where ϕ_i, $i = 1, 2$ are polynomials with ϕ_2 monic, and the degrees of the polynomials $\chi(.,n)$ are uniformly bounded. These polynomial sequences are semiclassical of class either $s = 0$ or 1. They are, up to a linear change of variable, the classical polynomials (Hermite, Laguerre, Bessel, and Jacobi) and symmetric semiclassical polynomials of class one. For them, We deduce the three-term recurrence relations, the structure relations, and the second-order linear differential equations that these polynomial sequences satisfy.

Key words: Orthogonal polynomials, three-term recurrence relations, semiclassical polynomials, first structure relations, second-order linear differential equations.
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1 Introduction and preliminary results

Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ its algebraic dual space . We denote by $\langle \mathfrak{L}, f\rangle$ the action of $\mathfrak{L} \in \mathbb{P}'$ on $f \in \mathbb{P}$.

* Corresponding author.
Email addresses: pacomarc@ing.uc3m.es (Francisco Marcellán), ridhasfaxi@yahoo.fr (Ridha Sfaxi).

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Let us introduce some useful operations in $\mathbb{P}'$ which will be used in the sequel. For any linear functional $\mathfrak{L}$, any polynomial q, and complex numbers b, c and a a nonzero complex number, let $D\mathfrak{L} = \mathfrak{L}'$, $q\mathfrak{L}$, $\tau_b\mathfrak{L}$, $h_a\mathfrak{L}$, and $(x-c)^{-1}\mathfrak{L}$ be the linear functionals defined by duality

$$\begin{aligned}
\langle \mathfrak{L}', f\rangle &:= -\langle \mathfrak{L}, f'\rangle,\\
\langle q\mathfrak{L}, f\rangle &:= \langle \mathfrak{L}, qf\rangle,\\
\langle \tau_{-b}\mathfrak{L},\ f\rangle &:= \langle \mathfrak{L},\ \tau_b f\rangle = \langle \mathfrak{L},\ f(x-b)\rangle,\\
\langle h_a\mathfrak{L},\ f\rangle &:= \langle \mathfrak{L},\ h_a f\rangle = \langle \mathfrak{L},\ f(ax)\rangle,\\
\langle (x-c)^{-1}\mathfrak{L},\ f\rangle &:= \langle \mathfrak{L},\ \theta_c f\rangle = \left\langle \mathfrak{L},\ \frac{f(x)-f(c)}{x-c}\right\rangle, \quad f\in\mathbb{P}.
\end{aligned}$$

A linear functional $\mathfrak{L}$ is said to be quasi-definite (regular) if there exists a sequence of monic polynomials $\{B_n\}_{n\in\mathbb{N}}$ with $\deg B_n = n,\ n\in\mathbb{N}$, such that

$$\langle \mathfrak{L}, B_nB_m\rangle = r_n\delta_{n,m},\ n,m\in\mathbb{N}, \quad r_n\neq 0, \tag{1.1}$$

where $\delta_{n,m}$ is the Kronecker symbol.
In such a case, $\{B_n\}_{n\in\mathbb{N}}$ is said to be the monic orthogonal polynomial sequence (MOPS) with respect to $\mathfrak{L}$. The orthogonality of the sequence $\{B_n\}_{n\in\mathbb{N}}$ can be characterized by a three-term recurrence relation (TTRR) [5] :

$$\begin{cases} B_{n+2}(x) &= (x-\beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n\in\mathbb{N},\\ B_1(x) &= x-\beta_0, \quad B_0(x) = 1, \end{cases} \tag{1.2}$$

where β_n and γ_{n+1} are complex numbers and $\gamma_{n+1}\neq 0$ for every $n\in\mathbb{N}$.
Notice that if $\mathfrak{L}$ is a quasi-definite linear functional, then the shifted linear functional $\tilde{\mathfrak{L}} = (h_{a^{-1}}\circ\tau_{-b})\mathfrak{L}$, where a, b are complex numbers and $a\neq 0$, is also quasi-definite. Furthermore, if $\{B_n\}_{n\in\mathbb{N}}$ is the MOPS with respect to $\mathfrak{L}$, then the shifted sequence $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ where $\tilde{B}_n(x) = a^{-n}B_n(ax+b),\ n\in\mathbb{N}$, is the MOPS with respect to $\tilde{\mathfrak{L}}$ (see [5], [9]).
A linear functional $\mathfrak{L}$ is said to be semiclassical if it is quasi-definite and there exists an admissible pair of polynomials (ϕ,ψ), i.e., ϕ is monic, $\deg\phi = t$, $\deg\psi = p\geq 1$, with $(1/p!)\psi^{(p)}(0)\notin\mathbb{N}^*$ if $p = t-1$, such that

$$D(\phi\mathfrak{L}) + \psi\mathfrak{L} = 0. \tag{1.3}$$

The pair (ϕ,ψ) is not unique [10], because (1.3) can be simplified if there exists a zero ξ of ϕ satisfying

$$\phi'(\xi) + \psi(\xi) = 0 \quad \text{and} \quad \langle \mathfrak{L}, \theta_\xi^2(\phi) + \theta_\xi(\psi)\rangle = 0. \tag{1.4}$$

The division by $x-\xi$, yields

$$\Big(\theta_\xi(\phi)\mathfrak{L}\Big)' + \Big(\theta_\xi^2(\phi) + \theta_\xi(\psi)\Big)\mathfrak{L} = 0. \tag{1.5}$$

The minimum value of $\max(t-2,\ p-1)$, among all possible admissible pairs (ϕ,ψ), is said to be the class of $\mathfrak{L}$. The pair (ϕ,ψ) associated with the class s, $s\in\mathbb{N}$, is unique. Moreover, when $\mathfrak{L}$ is semiclassical of class s, its corresponding MOPS is said to be semiclassical of class s.
If $s=0$, then $\mathfrak{L}$ is said to be classical (Hermite, Laguerre, Bessel, and Jacobi). For more details, see [8] and [9].
When $\mathfrak{L}$ is semiclassical of class s and satisfies (1.3), $\tilde{\mathfrak{L}}=(h_{a^{-1}}\circ\tau_{-b})\mathfrak{L}$ is also semiclassical of class s and

$$(\tilde{\phi}\tilde{\mathfrak{L}})'+\tilde{\psi}\tilde{\mathfrak{L}}=\ 0, \tag{1.6}$$

holds with $\tilde{\phi}(x)=a^{-t}\phi(ax+b)$ and $\tilde{\psi}(x)=a^{1-t}\psi(ax+b)$.
The semiclassical orthogonal polynomial sequences are characterized by the following equivalent relations.
1. First structure relation [9].

$$\phi(x)B_n'(x)=\frac{C_n(x)-C_0(x)}{2}B_n(x)-\gamma_nD_n(x)B_{n-1}(x),\ n\in\mathbb{N}, \tag{1.7}$$

where $\deg C_n\leq s+1$, $\deg D_n\leq s$, and

$$C_{n+1}(x)+C_n(x)=2(x-\beta_n)D_n(x),\ n\in\mathbb{N}, \tag{1.8}$$
$$\gamma_{n+1}D_{n+1}(x)+\phi(x)=\gamma_nD_{n-1}(x)+(x-\beta_n)^2D_n(x)-(x-\beta_n)C_n(x), \tag{1.9}$$

for every $n\in\mathbb{N}$, where

$$C_0(x)=-\phi'(x)-\psi(x), \tag{1.10}$$
$$D_0(x)=-(\mathfrak{L}\theta_0\phi)'(x)-(\mathfrak{L}\theta_0\psi)(x). \tag{1.11}$$

Here $(\mathfrak{L}\theta_0f)(x)=\langle\mathfrak{L}_y,\ \theta_x(f)(y)\rangle=\langle\mathfrak{L}_y,\ \frac{f(x)-f(y)}{x-y}\rangle$, $f\in\mathbb{P}$, with the convention $\gamma_0=1$ and $D_{-1}(x)=B_{-1}(x)=0$.
In the sequel, we will use the following relations:

$$\frac{C_n(x)-C_0(x)}{2}=\eta_nC_0(x)+(-1)^{n-1}\sum_{\nu=0}^{n-1}(-1)^{\nu}(x-\beta_\nu)D_\nu(x),\ n\in\mathbb{N}, \tag{1.12}$$

where $\eta_n=\dfrac{(-1)^n-1}{2}$, $n\in\mathbb{N}$, and

$$\frac{C_n^2(x)-C_0^2(x)}{4}-\gamma_nD_{n-1}(x)D_n(x)=\phi(x)\sum_{\nu=0}^{n-1}D_\nu(x),\ n\in\mathbb{N}. \tag{1.13}$$

2. Second-order linear differential equation of Maroni type [9].

$$\phi(x)B_n''(x)-\psi(x)B_n'(x)=M(x,n)B_n(x)-\gamma_nD_n'(x)B_{n-1}(x),\ n\in\mathbb{N}, \tag{1.14}$$

with

$$M(x,n)=\frac{C_n'(x)-C_0'(x)}{2}+\sum_{\nu=0}^{n-1}D_\nu(x),$$

and where $C_n(x)$ and $D_n(x)$ are given by (1.8) – (1.11).

3. Second-order linear differential equation of Laguerre-Perron type [2] (holonomic equation).

$$J(x,n)B_n''(x)+K(x,n)B_n'(x)+L(x,n)B_n(x)=0,\ n\in\mathbb{N}, \tag{1.15}$$

where

$$\begin{aligned} J(x,n)\ &=\phi(x)D_n(x),\\ K(x,n)\ &=\Big(\phi'(x)+C_0(x)\Big)D_n(x)-\phi(x)D_n'(x),\\ L(x,n)\ &=\Big(\frac{C_n(x)-C_0(x)}{2}\Big)D_n'(x)-\Big(\sum_{\nu=0}^{n-1}D_\nu(x)+\frac{C_n'(x)-C_0'(x)}{2}\Big)D_n(x). \end{aligned}$$

Obviously, the first structure relation (1.7) can be simplified. The triplet $(\phi,\{C_n\}_{n\in\mathbb{N}},\{D_n\}_{n\in\mathbb{N}})$ is not unique. Indeed, for each $\chi\in\mathbb{P}$, $\chi\neq 0$, the sequence $\{B_n\}_{n\in\mathbb{N}}$ also fulfils where

$$\hat{\phi}(x)=\chi(x)\phi(x),\ \hat{C}_n(x)=\chi(x)C_n(x),\ \hat{D}_n(x)=\chi(x)D_n(x).$$

So, (1.7) can be simplified if and only if there exists a zero ξ of ϕ such that $C_0(\xi)=D_0(\xi)=0$, i.e., the two conditions given in (1.4). Taking into account (1.8) – (1.10), we get $C_n(\xi)=D_n(\xi)=0,\ n\in\mathbb{N}$. Thus, dividing by $x-\xi$ in (1.7), we obtain

$$\theta_\xi(\phi)(x)B_n'(x)=\frac{\theta_\xi(C_n)(x)-\theta_\xi(C_0)(x)}{2}B_n(x)-\gamma_n\theta_\xi(D_n)(x)B_{n-1}(x),$$

where $\deg\theta_\xi(C_n)\le s$ and $\deg\theta_\xi(D_n)\le s-1$, for every $n\in\mathbb{N}$.

Furthermore, the following result will be needed.

Lemma 1 *Let $\{B_n\}_{n\ge 0}$ be a semiclassical polynomial sequence satisfying* (1.7) *and let $s=\max\Big(\deg\phi-2,\deg\psi-1\Big)$. Then,*

- (i) *If there exists a pair $(k,l)\in\mathbb{N}^2$ such that $\deg D_n\le l$ for every $n\ge k$, then $s\le l$.*
- (ii) *The set $\{n\in\mathbb{N}\mid \deg D_n=s\}$ is infinite.*

Proof. Assume the existence of a pair $(k,l)\in\mathbb{N}^2$ such that $\deg D_n\le l$, $n\ge k$. Then, if we take $n=k+1$, (resp. $n=k+2$) in (1.9), we get

$$\begin{aligned} \gamma_{k+2}D_{k+2}(x)+\phi(x)&=\gamma_{k+1}D_k(x)+(x-\beta_{k+1})^2D_{k+1}(x)-(x-\beta_{k+1})C_{k+1}(x),\\ \gamma_{k+3}D_{k+3}(x)+\phi(x)&=\gamma_{k+2}D_{k+1}(x)+(x-\beta_{k+2})^2D_{k+2}(x)-(x-\beta_{k+2})C_{k+2}(x). \end{aligned}$$

Subtracting the two previous expressions, we deduce

$$(x-\beta_{k+2})C_{k+2}(x)-(x-\beta_{k+1})C_{k+1}(x)=\gamma_{k+2}D_{k+1}(x)+(x-\beta_{k+2})^2D_{k+2}(x)-$$
$$\gamma_{k+3}D_{k+3}(x)-\gamma_{k+1}D_k(x)-(x-\beta_{k+1})^2D_{k+1}(x)+\gamma_{k+2}D_{k+2}(x). \tag{1.16}$$

The analysis of the degrees of both sides in (1.16) yields

$$\deg\Big((x-\beta_{k+2})C_{k+2}(x)-(x-\beta_{k+1})C_{k+1}(x)\Big)\le l+2. \tag{1.17}$$

Now, from (1.2) we can write (1.16) as follows:

$$(x-\beta_{k+2})C_{k+2}(x)-(x-\beta_{k+1})C_{k+1}(x)=x\Big(C_{k+2}(x)-C_{k+1}(x)\Big)+$$
$$\beta_{k+1}C_{k+1}(x)-\beta_{k+2}C_{k+2}(x). \tag{1.18}$$

Inserting (1.8) into (1.9) with $n=k+1$, we have

$$\phi(x)=\frac{C_{k+2}(x)-C_{k+1}(x)}{2}(x-\beta_{k+1})-\gamma_{k+2}D_{k+2}(x)+\gamma_{k+1}D_k(x), \tag{1.19}$$

Notice that $\deg\phi\le l+2$. Otherwise, if $\deg\phi\ge l+3$, then from (1.19) and the assumption $\deg D_{k+1}\le l$, we get

$$\deg\Big(C_{k+2}(x)-C_{k+1}(x)\Big)=\deg(\phi)-1\ge l+2. \tag{1.20}$$

On the other hand, rewriting (1.8) with $n=k+1$ in two ways:

$$C_{k+1}(x)=(x-\beta_{k+1})D_{k+1}(x)-\frac{C_{k+2}(x)-C_{k+1}(x)}{2},$$
$$C_{k+2}(x)=(x-\beta_{k+1})D_{k+1}(x)+\frac{C_{k+2}(x)-C_{k+1}(x)}{2},$$

and using (1.20) as well as the assumption $\deg D_{k+1}\le l$, we obtain

$$\deg C_{k+2}=\deg C_{k+1}=\deg\phi-1. \tag{1.21}$$

From (1.18), (1.20), and (1.21), we get

$$\deg\Big((x-\beta_{k+2})C_{k+2}(x)-(x-\beta_{k+1})C_{k+1}(x)\Big)=\deg\phi,$$

and together with (1.17), we get $\deg\phi\le l+2$ holds. This is a contradiction.
Next, since $\deg\phi\le l+2$ and $\deg D_n\le l$, $n\ge k$, then taking into account (1.9) $\deg C_n\le l+1$, $n\ge k+1$, follows.
From (1.7), with $n=k+1$, we have

$$C_0(x)B_{k+1}(x)=-2\phi(x)B'_{k+1}(x)+C_{k+1}(x)B_{k+1}(x)-2\gamma_{k+1}D_{k+1}B_k(x).$$

Since $\deg\phi\le l+2$, $\deg D_{k+1}\le l$ and $\deg C_{k+1}\le l+1$, it follows that $\deg C_0B_{k+1}\le k+l+2$, and so $\deg C_0\le l+1$. Since $\psi(x)=-C_0(x)-\phi'(x)$,

then $\deg\psi \le l+1$. Thus, $\max\Big(\deg\phi-2, \deg\psi-1\Big) \le l$.
Hence, (i) holds.

The set $\{n\in\mathbb{N} \mid \deg D_n = s\}$ is infinite. Otherwise, there exists $k\in\mathbb{N}$ such that $\deg D_n \le s-1,\ n\ge k$, since $\deg D_n \le s,\ n\in\mathbb{N}$. But, according to (i) this implies $s \le s-1$. This is a contradiction.
Hence, (ii) holds. ∎

The aim of this manuscript is to solve the following problem raised by S. Belmehdi (see [2])

Classify all MOPS $\{B_n\}_{n\ge0}$ satisfying the second-order pseudo-spectral linear differential equation

$$\phi_2(x)y''(x)+\phi_1(x)y'(x)=\chi(x,n)y(x),\ n\in\mathbb{N}, \tag{1.22}$$

where ϕ_i, $i=1,2$ are polynomials with ϕ_2 monic, and the degrees of the polynomials $\chi(.,n)$ are uniformly bounded.

The differential equation (1.22) and the more general (1.15) having an MOPS as solution are very important in the theory of orthogonal polynomials. Such equations are used as a basic tool in the electrostatic interpretation of the zeros of the polynomials $B_n(x)$ (see [4],[7] and the references therein).
When $\max\Big(\deg\phi_2-2, \deg\phi_1-1\Big)=0$, then the polynomial solutions of (1.22) are the classical orthogonal polynomials. This result has been obtained in 1889 by E. J. Routh [11] (see also [3]). However, when $\max\Big(\deg\phi_2-2, \deg\phi_1-1\Big)\ge 1$, the classification is not completed and only some particular cases have beeen studied in the literature. As sake of an example, We can mention the generalized Gegenbauer sequence (GG) analyzed in [2] by S. Belmehdi. In addition, the same author shows that (1.15) is pseudo-spectral if and only if $D_n(x)$ is the product of a function of n and a polynomial in x, i.e., $D_n(x)=\varrho_n D(x),\ n\ge0$. In the present paper, we obtain all MOPS which are solutions of (1.22).

The structure of this paper is as follows. In Section 2, we prove (see Theorem 3) that the MOPS which are solutions of (1.22) are semiclassical of class either $s=0$ or 1. They also satisfy a pseudo-spectral second-order linear differential equation of Laguerre-Perron type, with $D_n(x)=\varrho_n(x-d)^s$, $n\ge0$. In Section 3, we establish the system that satisfy the corresponding elements $C_n(x),\ D_n(x)$ in the pseudo-spectral cases. Finally, we describe all the resulting cases.

2 Main Results.

Let $\{B_n\}_{n\in\mathbb{N}}$ be an MOPS with respect to a linear functional $\mathfrak{L}$) satisfying the TTRR (1.2) as well as the second-order pseudo-spectral linear differential equation

$$\phi_2(x)B_n''(x) + \phi_1(x)B_n'(x) = \chi(x,n)B_n(x), \; n \geq 0, \tag{2.1}$$

where $\phi_\nu, \; \nu = 1,2$ and $\chi(.,n)$ are polynomials, ϕ_2 is monic, the coefficients of $\chi(\;.,n)$ depend on n, and there exists $\sigma \in \mathbb{N}$ such that

$$\deg \chi(\;.,n) \leq \sigma, \; \text{for every } n \in \mathbb{N}. \tag{2.2}$$

Setting

$$\kappa = \max(\sigma,\mu), \tag{2.3}$$

where $\mu = \max\Big(\deg\phi_2 - 2, \deg\phi_1 - 1\Big)$, then

Proposition 2 *The MOPS $\{B_n\}_{n\in\mathbb{N}}$ is semiclassical of class less than or equal to κ. Furthermore, the corresponding quasi-definite linear functional $\mathfrak{L}$ satisfies*

$$D(\phi_2\mathfrak{L}) + \psi_2\mathfrak{L} = 0, \tag{2.4}$$

where

$$\begin{aligned} \psi_2(x) = \; & \frac{1}{2}\Big(A_0(x)B_1(x) - \phi_1(x) - A_1(x)\Big), \quad \deg\psi_2 \geq 1, \qquad \textit{and} \\ A_i(x) = \; & \sum_{\nu=0}^{\sigma+i} \frac{\langle \mathfrak{L}, \chi(x,\nu)B_iB_\nu\rangle}{\langle \mathfrak{L}, B_\nu^2\rangle} B_\nu(x), \; i = 0, \; 1. \end{aligned}$$

Proof. Using (1.1), we get

$$\langle \mathfrak{L}, (\phi_2 B_n'' + \phi_1 B_n')B_i\rangle = 0, \; n \geq \sigma + 1 + i, \quad i = 0, \; 1,$$

or, equivalently,

$$\langle (\phi_2 B_i \mathfrak{L})'' - (\phi_1 B_i \mathfrak{L})', B_n\rangle = 0, \; n \geq \sigma + 1 + i, \quad i = 0, \; 1.$$

Then, for $i \in \{0,1\}$ there exists a polynomial A_i with $\deg A_i \leq \sigma + i$ such that

$$(\phi_2 B_i \mathfrak{L})'' - (\phi_1 B_i \mathfrak{L})' = A_i \mathfrak{L}, \tag{2.5}$$

where $\quad A_i(x) = \sum_{\nu=0}^{\sigma+i} \langle \mathfrak{L}, B_\nu^2\rangle^{-1} \langle \mathfrak{L}, \chi(x,\nu)B_iB_\nu\rangle B_\nu(x), \; i = 0, \; 1.$

In particular, for $i = 1$, (2.5) becomes

$$B_1\Big((\phi_2\mathfrak{L})'' - (\phi_1\mathfrak{L})'\Big) + 2(\phi_2\mathfrak{L})' - \phi_1\mathfrak{L} = A_1\mathfrak{L},$$

and by inserting (2.5) with $i = 0$, we obtain $B_1 A_0 \mathfrak{L} + 2(\phi_2 \mathfrak{L})' - \phi_1 \mathfrak{L} = A_1 \mathfrak{L}$. This yields (2.4), where $\psi_2(x) = \frac{1}{2}\Big(A_0(x)B_1(x) - \phi_1(x) - A_1(x)\Big)$.
Notice that $\deg \psi_2 \leq \max(\sigma + 1, \deg \phi_1)$, since $\deg A_i \leq \sigma + i,\ i = 0,\ 1$. Besides, $\deg \psi_2 \geq 1$, taking into account that ϕ_2 is a non-zero polynomial and $\mathfrak{L}$ is quasi-definite we get $\deg \psi_2 \geq 1$. Using (2.3) we deduce that $\max\Big(\deg \phi_2 - 2, \deg \psi_2 - 1\Big) \leq \kappa$. Thus, $\{B_n\}_{n\geq 0}$ is semiclassical of class less than or equal to κ. ■

In the sequel, we will assume that s is the class of the semiclassical MOPS $\{B_n\}_{n\in\mathbb{N}}$, orthogonal with respect to the linear functional $\mathfrak{L}$ such that

$$D(\phi \mathfrak{L}) + \psi \mathfrak{L} = 0 \tag{2.6}$$

holds, where (ϕ, ψ) is a pair of polynomials with ϕ monic , $\deg \psi \geq 1$, and $s = \max\Big(\deg \phi - 2, \deg \psi - 1\Big)$.
From Proposition 2 we deduce that

$$s \leq \kappa. \tag{2.7}$$

Of course, the equation (2.4) can be simplified and then we can assume that there exists a monic polynomial $\Xi(x)$ such that

$$\phi_2(x) = \Xi(x)\phi(x), \quad \psi_2(x) = \Xi(x)\psi(x) - \Xi'(x)\phi(x). \tag{2.8}$$

Now, we can give the main result of this section.

Theorem 3 *The sequence of monic orthogonal polynomials which are solutions of a second-order pseudo-spectral linear differential equation is semiclassical of class at most one and satisfies a second-order pseudo-spectral linear differential equations of Laguerre-Perron type.*

First, we need the three following technical lemmas.

Lemma 4 *Let $\{B_n\}_{n\in\mathbb{N}}$ be defined as above and satisfying* (1.7). *Then, there exist a monic polynomial $D(x)$, with $\deg D = s$, and an integer number k with $k \leq \kappa + s + 1$, such that*

$$D_n(x) = \varrho_n D(x),\ n \geq k, \tag{2.9}$$

$$\Big(\phi_1(x) + \Xi(x)\psi(x)\Big)D(x) + \Xi(x)\phi(x)D'(x) = 0. \tag{2.10}$$

Proof. Multiplying both sides of (1.14) by $\Xi(x)$ and using (2.8),

$$\phi_2(x)B_n''(x) - \Xi(x)\psi(x)B_n'(x) = \Xi(x)M(x,n)B_n(x) - \gamma_n \Xi(x)D_n'(x)B_{n-1}(x).$$

The elimination of $\phi_2(x)B''_n(x)$ between (2.1) and the previous equation yields

$$\Big(\phi_1(x)+\Xi(x)\psi(x)\Big)B'_n(x) = \Big(\chi(x,n)-M(x,n)\Xi(x)\Big)B_n(x)+ \gamma_n\Xi(x)D'_n(x)B_{n-1}(x),\ n\geq 0. \qquad (2.11)$$

Multiplying both sides of (2.11) by $\phi(x)$ and inserting (1.7),

$$\Omega_2(x,n)B_n(x) = \Omega_1(x,n)B_{n-1}(x),\ n\geq 0, \qquad (2.12)$$

with

$$\Omega_1(x,n) = \gamma_n\Big[\Big(\phi_1(x)+\psi(x)\Xi(x)\Big)D_n(x)+\Xi(x)\phi(x)D'_n(x)\Big],$$
$$\Omega_2(x,n) = \frac{C_n(x)-C_0(x)}{2}\Big[\phi_1(x)+\psi(x)\Xi(x)\Big]-\phi(x)\Big[\chi(x,n)-M(x,n)\Xi(x)\Big],$$

and

$$\deg\Omega_i(x,n)\leq\sigma+s+i,\quad i=1,\ 2,\quad \text{for every } n\in\mathbb{N}.$$

But, as a consequence of the TTRR $B_n(x)$ and $B_{n-1}(x)$ are coprime polynomials, for each integer $n\geq 1$. Thus from (2.12) there exists an integer k with $0\leq k\leq\kappa+s+1$ such that

$$\Omega_1(x,n)=\Omega_2(x,n)=0,\ n\geq k.$$

In particular, we have

$$\Big(\phi_1(x)+\Xi(x)\psi(x)\Big)D_n(x)+\Xi(x)\phi(x)D'_n(x)=0,\ n\geq k, \qquad (2.13)$$

and, as a consequence, $D_n(x)=\varrho_n D(x),\ n\geq k$.
Hence (2.9) holds.

According to (2.9), (2.13) becomes

$$\varrho_n\Big(\Big(\phi_1(x)+\Xi(x)\psi(x)\Big)D(x)+\Xi(x)\phi(x)D'(x)\Big)=0,\ n\geq k.$$

But, from Lemma 1, (*ii*), with $D_n(x)=\varrho_n D(x),\ n\geq k$, we deduce that $\deg D=s$ and the set $\{n\in\mathbb{N}\mid n\geq k \text{ and } \varrho_n\neq 0\}$ is infinite. Hence, (2.10) follows immediately. ■

Lemma 5 *If* $\deg D\geq 1$, *then for any zero* d *of* $D(x)$, *we have*

(i) $C_n(d)=(-1)^{n-k}C_k(d),\ n\geq k.$
(ii) $C_n(d)\neq 0,\ n\geq k.$
(iii) $\beta_n=\dfrac{\phi(d)}{C_k(d)}(-1)^{n-k}+d,\ n\geq k+1.$

Proof. Assume that d is a zero of $D(x)$. Then, (1.8) with $D_n(x) = \varrho_n D(x)$, $n \geq k$ and $x = d$, yields

$$C_{n+1}(d) + C_n(d) = 0, \ n \geq k.$$

Hence, (i) follows as a straightforward consequence.

From (i) and (1.9) we get

$$\phi(d) = (-1)^{n-k+1}(d - \beta_n)C_k(d), \ n \geq k+1. \tag{2.14}$$

Notice that $C_k(d) \neq 0$. Otherwise, $C_n(d) = 0, \ n \geq k$, and $\phi(d) = 0$. Then, from (1.13) with $x = d$ and $n \geq k$, we obtain $C_0(d) = 0$.
Let us consider the following recurrence property. For each integer $i, \ 0 \leq i \leq k$, we have

$$D_{n-i}(d) = 0 \quad \text{and} \quad C_{n-i}(d) = 0, \ n \geq k.$$

For $i = 0$, it is easy to check that the recurrence property is true. Assume that it holds until order $i, \ i \leq k-1$, and let us show that it remains valid for order $i+1$.
From (1.9) with $n = k - i$ and $x = d$, as well as the recurrence hypothesis, we get $\gamma_{k-i} D_{k-i-1}(d) = 0$. Since $\gamma_n \neq 0, \ n \geq 1$, it follows that $D_{k-i-1}(d) = 0$.
Moreover, (1.13) with $n = k - i - 1$ and $x = d$, leads to $C_{k-i-1}(d) = 0$. Then, $D_{n-i-1}(d) = 0$ and $C_{n-i-1}(d) = 0, \ n \geq k$. Thus, the recurrence property holds.
Therefore, $\phi(d) = C_n(d) = D_n(d) = 0, \ n \geq 0$. As a consequence, we can divide by $x - d$ in (1.7) and in an analog way in (2.6). This is a contradiction. Hence, (ii) holds.

The statement (iii) follows in a straightforward way from (2.14) and (ii).■

Lemma 6

$$D(x) = (x-d)^s, \quad \textit{where} \quad s \in \{0, 1\}.$$

Proof. If $d_i, \ i \in \{1, 2\}$ are two distinct zeros of the polynomial $D(x)$, then from Lemma 5, (iii), we have

$$\beta_n = \frac{\phi(d_i)}{C_k(d_i)}(-1)^{n-k} + d_i, \ n \geq k+1, \qquad i = 1, \ 2.$$

Handling in the previous relation we get

$$\left(\frac{\phi(d_1)}{C_k(d_1)} - \frac{\phi(d_2)}{C_k(d_2)}\right)(-1)^{n-k} + d_1 - d_2 = 0, \ n \geq k+1.$$

This means that $\frac{\phi(d_1)}{C_k(d_1)} - \frac{\phi(d_2)}{C_k(d_2)} = 0$ and $d_1 = d_2$. But, $d_1 \neq d_2$ by hypothesis. Thus, $D(x) = (x-d)^s$.

Notice that $s \in \{0,1\}$. Otherwise, if $s \geq 2$, then $D(d) = D'(d) = 0$. Differentiating both sides of (1.8) with $D_n(x) = \varrho_n D(x)$, $n \geq k$, and evaluating at $x = d$, we get

$$C'_{n+1}(d) + C'_n(d) = 0,\ n \geq k,$$

or, equivalently,

$$C'_n(d) = (-1)^{n-k} C'_k(d),\ n \geq k. \tag{2.15}$$

In the same way, differentiating both sides of (1.9) with $D_n(x) = \varrho_n D(x)$, $n \geq k$, and evaluating at $x = d$ we get

$$0 = -\phi'(d) - (d - \beta_n) C'_n(d) - C_n(d),\ n \geq k+1.$$

Then, by using (2.15) and Lemma 5, (iii), it follows that

$$(-1)^{n-k} C_k(d) + \phi'(d) - \frac{\phi(d) C'_k(d)}{C_k(d)} = 0,\ n \geq k+1.$$

This means that $\phi'(d) - \frac{\phi(d)C'_k(d)}{C_k(d)} = 0$ and $C_k(d) = 0$ hold. Thus, we get a contradiction and, as a consequence, our statement follows. ∎

Now, we prove the main result of this section.

Proof of Theorem 3. According to Lemma 6, we will analyze two cases.

• $s = 0$. Then $\deg D_n \leq 0,\ n \in \mathbb{N}$, and $D_n(x) = \varrho_n \in \mathbb{C},\ n \in \mathbb{N}$. From (1.14), the sequence $\{B_n\}_{n\geq 0}$ satisfies

$$\phi(x) B''_n(x) - \psi(x) B'_n(x) = \lambda_n B_n(x),\ n \in \mathbb{N},$$

where $\deg \phi \leq 2$, $\deg \psi = 1$, and

$$\lambda_n = M(x,n) = \frac{C'_n(x) - C'_0(x)}{2} + \sum_{\nu=0}^{n-1} \varrho_\nu = n\Big((n-1)\frac{\phi''(0)}{2} - \psi'(0)\Big). \tag{2.16}$$

• $s = 1$. Then, $\deg D_n \leq 1,\ n \in \mathbb{N}$, and

$$D_n(x) = \varrho_n (x - d) + \upsilon_n,\ n \in \mathbb{N}, \tag{2.17}$$

where $\upsilon_n \in \mathbb{C},\ n \in \mathbb{N}$, and $\upsilon_n = 0,\ n \geq k$.
Let us prove that $\upsilon_n = 0,\ 0 \leq n < k$.
If we multiply both sides of (2.1) by $x - d$ and we use (2.8) and (2.10), then we get

$$\Xi(x)\Big((x-d)\Big(\phi(x)B''_n(x) - \psi(x)B'_n(x)\Big) - \phi(x)B'_n(x)\Big) = (x-d)\chi(x,n)B_n(x),$$

for every $n \in \mathbb{N}$.
Then, (1.14) and the fact that $D'_n(x) = \varrho_n,\ n \in \mathbb{N}$, yield

$$\Xi(x)\phi(x)B'_n(x) = \frac{\widetilde{C}_n(x) - \widetilde{C}_0(x)}{2} B_n(x) - \gamma_n \widetilde{D}_n(x) B_{n-1}(x),\ n \in \mathbb{N}, \tag{2.18}$$

where

$$\widetilde{C}_0(x) = -\Xi(x)\Big(\phi'(x)+\psi(x)\Big),$$
$$\frac{\widetilde{C}_n(x)-\widetilde{C}_0(x)}{2} = (x-d)\Big(\Xi(x)M(x,n)-\chi(x,n)\Big),$$
$$\widetilde{D}_n(x) = \varrho_n(x-d)\Xi(x),\ n\in\mathbb{N}.$$

According to (1.7), we can divide by $\Xi(x)$ in the structure relation (2.18). In this case, $\Xi(x)$ divides $\widetilde{C}_n(x)$, $n\in\mathbb{N}$, and we can write $\widetilde{C}_n(x) = \Xi(x)\widehat{C}_n(x)$, $n\in\mathbb{N}$. After division $\Xi(x)$ in (2.18), we obtain

$$\phi(x)B_n'(x) = \frac{\widehat{C}_n(x)-\widehat{C}_0(x)}{2}B_n(x) - \gamma_n\varrho_n(x-d)B_{n-1}(x),\ n\in\mathbb{N}, \tag{2.19}$$

with $\widehat{C}_0(x) = C_0(x)$.
Cancelling $\phi(x)B_n'(x)$ from (1.7) and (2.19), and taking into account (2.17) we get

$$\Big(\frac{C_n(x)-\widehat{C}_n(x)}{2}\Big)B_n(x) = -\gamma_n\upsilon_n B_{n-1}(x),\ n\in\mathbb{N}.$$

Thus $\upsilon_n = 0,\ n\in\mathbb{N}$, and $\widehat{C}_n(x) = C_n(x),\ n\in\mathbb{N}$, since $\gamma_n\neq 0,\ n\geq 1$, $\deg B_n = n,\ n\geq 0$, and $B_{-1}(x)=0$.
Finally, since $D_n(x) = \varrho_n(x-d),\ n\in\mathbb{N}$, $\{B_n\}_{n\in\mathbb{N}}$ satisfies a second-order pseudo-spectral linear differential equation of Laguerre-Perron type [2]. ■

3 Orthogonal polynomials as solutions of second-order pseudo-spectral differential equation of Laguerre-Perron type.

Let $\{B_n\}_{n\in\mathbb{N}}$ be a semiclassical MOPS of class s which is a solution of a second-order pseudo-spectral linear differential equation of Laguerre-Perron type with $D_n(x) = \varrho_n D(x),\ n\in\mathbb{N}$. Then (1.15) becomes

$$J(x)B_n''(x) + K(x)B_n'(x) + L(x,n)B_n(x) = 0,\ n\in\mathbb{N}, \tag{3.1}$$

where

$$J(x) = \phi(x)D(x),$$
$$K(x) = \Big[\phi'(x)+C_0(x)\Big]D(x) - \phi(x)D'(x),$$
$$L(x,n) = \Big[\frac{C_n(x)-C_0(x)}{2}\Big]D'(x) - \Big[D(x)\sum_{\nu=0}^{n-1}\varrho_\nu + \frac{C_n'(x)-C_0'(x)}{2}\Big]D(x),\ n\in\mathbb{N}.$$

The first structure relation (1.7) reads

$$\phi(x)B_n'(x) = \frac{C_n(x)-C_0(x)}{2}B_n(x) - \gamma_n\varrho_n D(x)B_{n-1}(x),\ n\in\mathbb{N}, \tag{3.2}$$

where

$$\begin{cases} C_{n+1}(x) + C_n(x) = 2\varrho_n(x-\beta_n)D(x), \\ \gamma_{n+1}\varrho_{n+1}D(x) = -\phi(x) + \gamma_n\varrho_{n-1}D(x) + \\ \qquad\qquad \varrho_n(x-\beta_n)^2 D(x) - (x-\beta_n)C_n(x), \\ C_0(x) = -\phi'(x) - \psi(x),\ D_0(x) = -(\mathfrak{L}\theta_0\psi)(x) - (\mathfrak{L}\theta_0\phi)'(x), \end{cases} \tag{3.3}$$

and the convention, $\varrho_{-1} = 0$.
In addition, from (1.12) and (1.13),

$$\frac{C_n(x) - C_0(x)}{2} = \eta_n C_0(x) + (a_n x - b_n)D(x),\ n \in \mathbb{N}, \tag{3.4}$$

with

$$a_n = (-1)^{n-1}\sum_{\nu=0}^{n-1}(-1)^\nu \varrho_\nu \quad , \quad b_n = (-1)^{n-1}\sum_{\nu=0}^{n-1}(-1)^\nu \varrho_\nu\beta_\nu, \tag{3.5}$$

$$\frac{C_n^2(x) - C_0^2(x)}{4} - \gamma_n\varrho_n\varrho_{n-1}D^2(x) = \phi(x)D(x)\sum_{\nu=0}^{n-1}\varrho_\nu,\ n \in \mathbb{N}. \tag{3.6}$$

3.1 The associated system in the pseudo-spectral case.

From (3.3),

$$\phi(x) = \Big(\varrho_n(x-\beta_n)^2 - \varrho_{n+1}\gamma_{n+1} + \gamma_n\varrho_{n-1}\Big)D(x) - (x-\beta_n)C_n(x),\ n \in \mathbb{N}. \tag{3.7}$$

. For $n = 0$, (3.7) reads

$$\phi(x) = \Big(\varrho_0(x-\beta_0)^2 - \varrho_1\gamma_1\Big)D(x) - (x-\beta_0)C_0(x). \tag{3.8}$$

Cancelling $\phi(x)$ in (3.7) and (3.8), from (3.4) we obtain

$$E(x,n)D(x) = F(x,n)C_0(x),\ n \geq 1, \tag{3.9}$$

where

$$E(x,n) = \Sigma_{n,2}x^2 + \Sigma_{n,1}x + \Sigma_{n,0},\ n \geq 1, \tag{3.10}$$

with

$$\begin{aligned} \Sigma_{n,0} &= \frac{1}{2}\Big[-\varrho_n\beta_n^2 + \varrho_0\beta_0^2 + \varrho_{n+1}\gamma_{n+1} - \gamma_n\varrho_{n-1} - \varrho_1\gamma_1 + 2b_n\beta_n\Big], \\ \Sigma_{n,1} &= -\beta_0\varrho_0 + (\varrho_n - a_n)\beta_n - b_n, \\ \Sigma_{n,2} &= \frac{\varrho_0 - \varrho_n}{2} + a_n, \end{aligned}$$

and

$$F(x,n) = -\eta_n x + \frac{(-1)^n\beta_n - \beta_0}{2}. \tag{3.11}$$

For $n=1$, (3.9) reads

$$C_0^*(x)D(x) = D^*(x)C_0(x), \tag{3.12}$$

where $C_0^*(x) := E(x,1)$ and $D^*(x) := F(x,1)$.
Multiplying both sides of (3.9) by $D^*(x)$ and using (3.12), we get

$$D^*(x)E(x,n) = C_0^*(x)F(x,n),\ n \geq 1. \tag{3.13}$$

Thus, the identification of the coefficients yields the following system

$$\begin{cases} \Sigma_{n,2} = \dfrac{\varrho_1 - 3\varrho_0}{2}\eta_n, \quad n \geq 1, \\ \Sigma_{n,1} = \ \eta_n \mathbf{R} + \dfrac{(-1)^n\beta_n - \beta_0}{2}\Sigma_{1,2}, \quad n \geq 1, \\ \Sigma_{n,0} = \ -\eta_n \mathbf{S} - \dfrac{(-1)^n\beta_n - \beta_0}{2}\mathbf{R}, \quad n \geq 1, \\ \Big[- (\beta_1 + \beta_0)\eta_n \ \ + (-1)^n\beta_n - \beta_0 \Big]\mathbf{S} = \ 0, \quad n \geq 1, \end{cases}$$

where

$$\mathbf{R} = -\frac{\beta_1 + \beta_0}{2}\Sigma_{1,2} - \Sigma_{1,1},$$
$$\mathbf{S} = -\frac{\beta_1 + \beta_0}{2}\mathbf{R} + \Sigma_{1,0}.$$

Using (3.5) and (3.10), the first equation of the previous system reads

$$\varrho_{n+1} + \varrho_n = 2\varrho_n + \varrho_1 - \varrho_0,\ n \in \mathbb{N}, \quad \text{i.e.,}\ \varrho_n = (\varrho_1 - \varrho_0)n + \varrho_0,\ n \in \mathbb{N}.$$

Then, from (3.5) we get

$$a_n = \frac{\varrho_1 - \varrho_0}{2}n + \frac{\varrho_1 - 3\varrho_0}{2}\eta_n,\ n \in \mathbb{N}. \tag{3.14}$$

As a consequence, the previous system is equivalent to the following one

$$\varrho_n = (\varrho_1 - \varrho_0)n + \varrho_0,\ n \in \mathbb{N} \tag{3.15}$$

$$b_n = -\frac{\varrho_1 + \varrho_0}{4}\beta_0 + \Big[\frac{\varrho_1 - \varrho_0}{2}n + \frac{\varrho_1 + \varrho_0}{4}\Big]\beta_n - \mathbf{R}\eta_n,\ n \geq 1 \tag{3.16}$$

$$\varrho_{n+1}\gamma_{n+1} - \gamma_n\varrho_{n-1} = \Big[\frac{\varrho_0 - \varrho_1}{2}\beta_n - \mathbf{R} + \frac{\varrho_1 + \varrho_0}{2}\beta_0\Big]\beta_n - 2\mathbf{S}\eta_n + \gamma_1\varrho_1 + \beta_0(\mathbf{R} - \beta_0\varrho_0),\ n \geq 1 \tag{3.17}$$

$$\Big[- (\beta_1 + \beta_0)\eta_n + (-1)^n\beta_n - \beta_0 \Big]\mathbf{S} = \ 0,\ n \geq 1. \tag{3.18}$$

where

$$\mathbf{R} = \frac{\varrho_1 + 5\varrho_0}{4}\beta_0 + \frac{\varrho_0 - 3\varrho_1}{4}\beta_1,$$
$$\mathbf{S} = \frac{1}{2}\Big[\varrho_2\gamma_2 - (\varrho_0 + \varrho_1)\Big(\gamma_1 + (\frac{\beta_0 - \beta_1}{2})^2\Big)\Big].$$

3.2 Classification and resolution of the system.

First, we give the following result:

Lemma 7 *Let* $\{B_n\}_{n\in\mathbb{N}}$ *be an MOPS which is a solution of* (3.1). *Then it is semiclassical of class at most one and satisfies the following first structure relation*:

$$\phi^*(x)B'_n(x) = \frac{C^*_n(x) - C^*_0(x)}{2}B_n(x) - \gamma_n\varrho_n D^*(x)B_{n-1}(x),\ n \in \mathbb{N}, \tag{3.19}$$

where

$$\begin{aligned}
\phi^*(x) &= \Big[\varrho_0(x-\beta_0)^2 - \varrho_1\gamma_1\Big]D^*(x) - (x-\beta_0)C^*_0(x),\\
\frac{C^*_n(x) - C^*_0(x)}{2} &= \eta_n C^*_0(x) + (a_n x - b_n)D^*(x),\ n \in \mathbb{N},\\
C^*_0(x) &= \Big[\frac{3\varrho_0 - \varrho_1}{2}x - \mathbf{R}\Big]D^*(x) + \mathbf{S},\\
D^*(x) &= x - \frac{\beta_1 + \beta_0}{2}.
\end{aligned} \tag{3.20}$$

Proof. If we multiply both sides of (3.4), resp. (3.8), by $D^*(x)$ and taking into account (3.12), we obtain

$$\frac{C_n(x) - C_0(x)}{2}D^*(x) = \frac{C^*_n(x) - C^*_0(x)}{2}D(x),\ n \geq 1, \tag{3.21}$$
$$\phi(x)D^*(x) = \phi^*(x)D(x). \tag{3.22}$$

Multiplying both sides of (3.2) by $D^*(x)$ and using (3.21) − (3.22), then (3.19) follows.
Notice that the quasi-definite linear functional $\mathfrak{L}$ satisfies

$$D(\phi^*\mathfrak{L}) + \psi^*\mathfrak{L} = 0, \tag{3.23}$$

where

$$\begin{aligned}
\phi^*(x) &= \Big(\varrho_0(x-\beta_0)^2 - \varrho_1\gamma_1\Big)D^*(x) - (x-\beta_0)C^*_0(x),\\
\psi^*(x) &= -\phi^{*\prime}(x) - C^*_0(x) = -\varrho_1 B_2(x).
\end{aligned}$$

Since $\max\Big(\deg\phi^* - 2, \deg\psi^* - 1\Big) = 1$, then $\{B_n\}_{n\in\mathbb{N}}$ is semiclassical of class at most one. ■

As a consequence, $\{B_n\}_{n\in\mathbb{N}}$ satisfies

$$J^*(x)B_n''(x) + K^*(x)B_n'(x) + L^*(x,n)B_n(x) = 0,\ n \in \mathbb{N}, \tag{3.24}$$

where

$$\begin{aligned} J^*(x) &= \phi^*(x)D^*(x), \\ K^*(x) &= \Big(\phi^{*\prime}(x) + C_0^*(x)\Big)D^*(x) - \phi^*(x), \\ L^*(x,n) &= \frac{C_n^*(x) - C_0^*(x)}{2} - \Big(D^*(x)\sum_{\nu=0}^{n-1}\varrho_\nu + \frac{C_n^{*\prime}(x) - C_0^{*\prime}(x)}{2}\Big)D^*(x). \end{aligned}$$

Notice that from Lemma 7

$$\phi^*\Big(\frac{\beta_0+\beta_1}{2}\Big) = \frac{\beta_0-\beta_1}{2}C_0^*\Big(\frac{\beta_0+\beta_1}{2}\Big).$$

Since,

$$C_0^*\Big(\frac{\beta_0+\beta_1}{2}\Big) = \mathbf{S}, \tag{3.25}$$

then,

$$\phi^*\Big(\frac{\beta_0+\beta_1}{2}\Big) = \frac{\beta_0-\beta_1}{2}\mathbf{S}. \tag{3.26}$$

Proposition 8 *Let $\{B_n\}_{n\in\mathbb{N}}$ be an MOPS satisfying the TTRR* (1.2). *Let assume that it is a solution of* (3.1) *and let* $\mathbf{S} = \frac{1}{2}\Big[\varrho_2\gamma_2 - (\varrho_0+\varrho_1)\Big(\gamma_1 + (\frac{\beta_0-\beta_1}{2})^2\Big)\Big]$.

(i) *If $\mathbf{S} = 0$, then $\{B_n\}_{n\in\mathbb{N}}$ is a classical MOPS.*
(ii) *If $\mathbf{S} \neq 0$, then $\{B_n\}_{n\in\mathbb{N}}$ is a semiclassical MOPS of class one.*

Proof. According to Lemma 7, the structure relation (3.19) can be simplified since the class of $\{B_n\}_{n\in\mathbb{N}}$ is less than or equal to one. Clearly, the simplification in (3.19) is possible, only by dividing by $D^*(x) = x - \frac{\beta_0+\beta_1}{2}$. This requires that $\mathbf{S} = 0$. In this case, $C_0^*\Big(\frac{\beta_0+\beta_1}{2}\Big) = \phi^*\Big(\frac{\beta_0+\beta_1}{2}\Big) = 0$ and $C_n^*\Big(\frac{\beta_0+\beta_1}{2}\Big) = 0$, $n \in \mathbb{N}$, taking into account (3.25) – (3.26) and (3.20), respectively. These are the conditions that we need in order to divide by $D^*(x)$. ■

Now, we can give the description of all MOPS $\{B_n\}_{n\in\mathbb{N}}$ solutions of a second-order pseudo-spectral linear differential equation of Laguerre-Perron type.

I. $\mathbf{S} = 0$, i.e., $\varrho_2\gamma_2 = (\varrho_0+\varrho_1)\Big[\gamma_1 + (\frac{\beta_0-\beta_1}{2})^2\Big]$.

In this case,

$$\begin{aligned} C_0^*(x) =& \ D^*(x)\hat{C}_0(x), \\ \phi^*(x) =& \ \hat{\phi}(x)D^*(x), \\ \frac{C_n^*(x)-C_0^*(x)}{2} =& \ \frac{\hat{C}_n(x)-\hat{C}_0(x)}{2}D^*(x), \ n\geq 1, \end{aligned}$$

where

$$\begin{aligned} \hat{\phi}(x) =& \ \frac{\varrho_1-\varrho_0}{2}(x-\beta_0)^2+\frac{(\beta_0-\beta_1)(3\varrho_1-\varrho_0)}{4}(x-\beta_0)-\gamma_1\varrho_1, \\ \frac{\hat{C}_n(x)-\hat{C}_0(x)}{2} =& \ \Big(\frac{3\varrho_0-\varrho_1}{2}\eta_n+a_n\Big)x-\mathbf{R}\eta_n-b_n, \ n\geq 1, \\ \hat{C}_0(x) =& \ \frac{3\varrho_0-\varrho_1}{2}x-\mathbf{R}. \end{aligned}$$

Division by $D^*(x)$ in (3.19) leads to

$$\hat{\phi}(x)B'_{n+1}(x)=\frac{\hat{C}_{n+1}(x)-\hat{C}_0(x)}{2}B_{n+1}(x)-\gamma_{n+1}\varrho_{n+1}B_n(x), \ n\in\mathbb{N}. \tag{3.27}$$

Thus, $\{B_n\}_{n\in\mathbb{N}}$ is a classical MOPS and the corresponding linear functional $\mathfrak{L}$ satisfies

$$D(\hat{\phi}\mathfrak{L})+\hat{\psi}\mathfrak{L}=0, \tag{3.28}$$

where

$$\begin{aligned} \hat{\phi}(x) =& \ \frac{\varrho_1-\varrho_0}{2}(x-\beta_0)^2+\frac{(\beta_0-\beta_1)(3\varrho_1-\varrho_0)}{4}(x-\beta_0)-\gamma_1\varrho_1, \\ \hat{\psi}(x) =& \ -\hat{\phi}'(x)-\hat{C}_0(x)=-\frac{\varrho_0+\varrho_1}{2}B_1(x). \end{aligned}$$

Notice that, if $\varrho_1\neq\varrho_0$, then

$$\hat{\phi}(x)= \ \frac{\varrho_1-\varrho_0}{2}\Big[(x-d)^2-\frac{1}{4}\mu\Big],$$

where

$$\begin{aligned} d =& \ \beta_0-\frac{(\beta_0-\beta_1)(3\varrho_1-\varrho_0)}{4(\varrho_1-\varrho_0)}, \\ \mu =& \ \frac{(\beta_0-\beta_1)^2(3\varrho_1-\varrho_0)^2}{4(\varrho_1-\varrho_0)^2}+\frac{8\varrho_1\gamma_1}{\varrho_1-\varrho_0}. \end{aligned}$$

The second-order linear differential equation satisfied by B_n is reduced to the following one

$$\hat{J}(x)B''_n(x)+\hat{K}(x)B'_n(x)+\hat{L}(x,n)B_n(x)=0, \ n\in\mathbb{N}, \tag{3.29}$$

where

$$\hat{J}(x) = \hat{\phi}(x),$$
$$\hat{K}(x) = -\hat{\psi}(x),$$
$$\hat{L}(x,n) = -n(n-1)\frac{(\varrho_1 - \varrho_0)}{2} - \frac{(\varrho_1 + \varrho_0)}{2}n.$$

According to the number of zeros of the polynomial $\hat{\phi}(x)$ we recover, by using a suitable shifting, the well-known four canonical cases of classical orthogonal polynomials [8],[9].

Table 1

($\mathbf{I}_1$) Hermite:

$\hat{\phi}(x) = 1, \quad \hat{\psi}(x) = 2x,$

$\beta_n = 0, \quad \gamma_{n+1} = \dfrac{n+1}{2},\ n \in \mathbb{N},$

$\hat{C}_n(x) = -2x,\ n \in \mathbb{N},$

$\varrho_n = -2,\ a_n = 2\eta_n,\ b_n = 0,\ n \in \mathbb{N}, \quad \mathbf{R} = \mathbf{S} = 0,$

$B_n'(x) = nB_{n-1}(x),\ n \in \mathbb{N},$

$B_n''(x) - 2xB_n'(x) = -2nB_n(x),\ n \in \mathbb{N}.$

Table 2

($\mathbf{I}_2$) Laguerre:

$\hat{\phi}(x) = x, \quad \hat{\psi}(x) = x - \alpha - 1,$

$\beta_n = 2n + \alpha + 1, \quad \gamma_{n+1} = (n+1)(n+\alpha+1),\ n \in \mathbb{N}$, with $\alpha \neq -n,\ n \geq 1,$

$\hat{C}_n(x) = -x + 2n + \alpha,\ n \in \mathbb{N},$

$\varrho_n = -1,\ a_n = \eta_n,\ b_n = -n + \alpha\eta_n,\ n \in \mathbb{N},\ \mathbf{R} = -\alpha, \quad \mathbf{S} = 0,$

$xB_n'(x) = nB_n(x) + n(n+\alpha)B_{n-1}(x),\ n \in \mathbb{N},$

$xB_n''(x) - (x - \alpha - 1)B_n'(x) = -nB_n(x),\ n \in \mathbb{N}.$

Table 3

($\mathbf{I}_3$) Bessel:

$\hat{\phi}(x) = x^2, \quad \hat{\psi}(x) = -2(\alpha x + 1),$

$\beta_0 = -\frac{1}{\alpha},\ \beta_{n+1} = \frac{1-\alpha}{(n+\alpha)(n+\alpha+1)},\ n \in \mathbb{N},$

$\gamma_1 = -\frac{1}{\alpha^2(2\alpha+1)},\ \gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)},\ n \geq 1,$ with $\alpha \neq -\frac{n}{2},\ n \in \mathbb{N},$

$\hat{C}_0(x) = 2(\alpha-1)x + 2, \qquad \hat{C}_n(x) = 2(n+\alpha-1)x + \frac{2(\alpha-1)}{(n+\alpha-1)},\ n \geq 1,$

$\varrho_n = 2n + 2\alpha - 1,\ a_n = n - 2(\alpha+1)\eta_n,\ b_n = \frac{n}{n+\alpha-1} + 2\eta_n,\ n \in \mathbb{N},\ \mathbf{R} = -2,\ \mathbf{S} = 0,$

$x^2 B_n'(x) = n(x - \frac{1}{n+\alpha-1})B_n(x) - (2n+2\alpha-1)\gamma_n B_{n-1}(x),\ n \in \mathbb{N},$

$x^2 B_n''(x) + 2(\alpha x + 1)B_n'(x) = n(n+2\alpha-1)B_n(x),\ n \in \mathbb{N}.$

Table 4

($\mathbf{I}_4$) Jacobi:

$\hat{\phi}(x) = x^2 - 1,\ \hat{\psi}(x) = -(\alpha+\beta+2)x + \alpha - \beta,$

$\beta_0 = \frac{\alpha-\beta}{\alpha+\beta+2},\ \beta_{n+1} = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)},\ n \in \mathbb{N},$

$\gamma_1 = \frac{4(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)},\ \gamma_{n+1} = \frac{4(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)},\ n \geq 1,$

with $\alpha, \beta \neq -n,\ \alpha + \beta \neq -n-1,\ n \geq 1,$

$\hat{C}_0(x) = (\alpha+\beta)x + \beta - \alpha, \qquad \hat{C}_n(x) = (2n+\alpha+\beta)x + \frac{\beta^2-\alpha^2}{2n+\alpha+\beta},\ n \geq 1,$

$\varrho_n = 2n + \alpha + \beta + 1,\ a_n = n - (\alpha+\beta)\eta_n,\ b_n = (\beta-\alpha)(\eta_n + \frac{n}{2n+\alpha+\beta}),\ n \in \mathbb{N},$

$\mathbf{R} = \alpha - \beta,\ \mathbf{S} = 0,$

$(x^2-1)B_n'(x) = n(x + \frac{\alpha-\beta}{2n+\alpha+\beta})B_n(x) - (2n+2\alpha+\beta+1)\gamma_n B_{n-1}(x),\ n \in \mathbb{N},$

$(x^2-1)B_n''(x) + \big((\alpha+\beta+2)x - (\alpha-\beta)\big)B_n'(x) = n(n+\alpha+\beta+1)B_n(x),\ n \in \mathbb{N}.$

II. $\mathbf{S} \neq 0$, i.e., $\varrho_2\gamma_2 \neq (\varrho_0 + \varrho_1)\Big[\gamma_1 + (\dfrac{\beta_0 - \beta_1}{2})^2\Big]$.

In this case, $\{B_n\}_{n\in\mathbb{N}}$ is a semiclassical MOPS of class one.

Lemma 9 *When* $\mathbf{S} \neq 0$, *the system* (3.15) − (3.18) *becomes*

$$\beta_n = \beta_0,\ n \in \mathbb{N}, \tag{3.30}$$

$$\varrho_{n+1}\gamma_{n+1} - \varrho_{n-1}\gamma_n = -2\eta_n\mathbf{S} + \gamma_1\varrho_1,\ n \geq 1, \tag{3.31}$$

where

$$\varrho_n = (\varrho_1 - \varrho_0)n + \varrho_0,\ n \in \mathbb{N},$$
$$\mathbf{S} = \frac{1}{2}\Big[\varrho_2\gamma_2 - (\varrho_0 + \varrho_1)\gamma_1\Big], \qquad \mathbf{R} = \frac{3\varrho_0 - \varrho_1}{2}\beta_0.$$

Proof. From (3.18) and the assumption $\mathbf{S} \neq 0$, we obtain

$$\beta_n = \frac{\beta_0 - \beta_1}{2}(-1)^n + \frac{\beta_0 + \beta_1}{2},\ n \in \mathbb{N}. \tag{3.32}$$

It is easy to see that

$$\sum_{\nu=0}^{n-1}(-1)^\nu = -\eta_n,\ n \in \mathbb{N}, \qquad \sum_{\nu=0}^{n-1}\nu(-1)^\nu = \frac{(2n-1)(-1)^n+1}{4},\ n \in \mathbb{N}.$$

From (3.5), (3.15), and (3.32), we get

$$b_n = (-1)^{n-1}\Big[\frac{(\varrho_1-\varrho_0)(\beta_0-\beta_1)}{4}n(n-1) + \frac{\beta_0-\beta_1}{2}\varrho_0 n - \frac{(\beta_0+\beta_1)(\varrho_1-\varrho_0)}{8}\Big] +$$

$$\frac{(\beta_0+\beta_1)(\varrho_1-\varrho_0)}{8}(2n-1) - \frac{\beta_0+\beta_1}{2}\varrho_0\eta_n,\ n \geq 0. \tag{3.33}$$

On the other hand, from (3.16) and (3.32), we get

$$b_n = -\frac{\varrho_1+\varrho_0}{4}\beta_0 + \Big[\frac{\varrho_1-\varrho_0}{2}n + \frac{\varrho_1+\varrho_0}{4}\Big]\Big[\frac{\beta_0-\beta_1}{2}(-1)^n + \frac{\beta_0+\beta_1}{2}\Big] - \mathbf{R}\eta_n,\ n \in \mathbb{N},$$

and by identification with (3.33), after some easy computation it follows that $(\beta_0 - \beta_1)\varrho_i = 0$, $i = 0, 1$. This implies, $\beta_1 = \beta_0$. Otherwise, according to (3.15) we get $\varrho_1 = \varrho_0 = 0$, and $\varrho_n = 0$, $n \in \mathbb{N}$. Therefore, $\mathbf{S} = 0$ which yields a contradiction.
Since $\beta_1 = \beta_0$, then by (3.32) we deduce (3.30).

Finally, (3.31) follows from (3.17) and (3.30). ■

By a suitable shifting, we can take $\beta_0 = 0$. Thus, $\beta_n = 0,\ n \in \mathbb{N}$, and then $\{B_n\}_{n\in\mathbb{N}}$ will be a symmetric polynomial sequence, i.e., if $B_n(-x) = (-1)^n B_n(x),\ n \in \mathbb{N}$.
In this case, the polynomials ϕ^* and ψ^* can be written as follows:

$$\phi^*(x) = x\Big[\frac{\varrho_1-\varrho_0}{2}x^2 - \varrho_1\gamma_1 - \mathbf{S}\Big], \tag{3.34}$$

$$\psi^*(x) = -\varrho_1(x^2 - \gamma_1). \tag{3.35}$$

In [1], (see also [6]), a description of all symmetric semiclassical polynomial sequences of class one is done. There are three canonical choices for ϕ^*

$$\phi^*(x) = x, \quad \phi^*(x) = x(x^2-1), \quad \phi^*(x) = x^3.$$

Table 5

($\mathbf{II}_1$) The generalized Hermite:

$\phi^*(x) = x, \quad \psi^*(x) = 2x^2 - (1+2\mu),$

$\beta_n = 0, \quad \gamma_{n+1} = \frac{1}{2}\big(n+1+\mu\big(1+(-1)^n\big)\big),\ n \in \mathbb{N},$

with $\mu \neq 0$ and $\mu \neq -\frac{2n+1}{2}$, $n \in \mathbb{N}$,

$C_n^*(x) = -2\big(x^2 - (-1)^n\mu\big),\ n \in \mathbb{N}, \quad D^*(x) = x,$

$\varrho_n = -2,\ a_n = 2\eta_n,\ b_n = 0,\ n \in \mathbb{N}, \quad \mathbf{R} = 0,\ \mathbf{S} = 2\mu,$

$$xB_n'(x) = 2\mu\eta_n B_n(x) + 2\gamma_n x B_{n-1}(x),\ n \in \mathbb{N},$$

$$x^2 B_n''(x) + 2x(-x^2+\mu)B_n'(x) = -2(nx^2 + \mu\eta_n)B_n(x),\ n \in \mathbb{N}.$$

Table 6

($\mathbf{II}_2$) The symmetric Generalized Gegenbauer:

$\phi^*(x) = x(x^2-1), \quad \psi^*(x) = -2(\alpha+\beta+2)x^2 + 2(\beta+1),$

$\beta_n = 0,\ n \in \mathbb{N},$

$\gamma_{n+1} = \frac{\big[2n+2+(2\beta+1)(1+(-1)^n)\big]\big[2n+2+4\alpha+(2\beta+1)(1+(-1)^n)\big]}{16(n+\alpha+\beta+1)(n+\alpha+\beta+2)},\ n \in \mathbb{N},$

with $\alpha \neq -n$, $\alpha+\beta \neq -n$, $n \geq 1$ and $\beta \neq -(1/2)$,

$C_n^*(x) = (2n+2\alpha+2\beta+1)x^2 - (2\beta+1)(-1)^n,\ n \in \mathbb{N}, \quad D^*(x) = x,$

$\varrho_n = 2(n+\alpha+\beta+1),\ a_n = n - (2\alpha+2\beta+1)\eta_n,\ b_n = 0,\ n \in \mathbb{N},$

$\mathbf{R} = 0, \quad \mathbf{S} = -(1+2\beta),$

$$x(x^2-1)B_n'(x) = \big(nx^2 - (2\beta+1)\eta_n\big)B_n(x) - 2(n+\alpha+\beta)\gamma_n x B_{n-1}(x),\ n \in \mathbb{N},$$

$$x^2(x^2-1)B_n''(x) + x\big((2\alpha+2\beta+3)x^2 - 2\beta - 1\big)B_n'(x) = \big(n(2\alpha+2\beta+2+n)x^2 + (2\beta+1)\eta_n\big)B_n(x),\ n \in \mathbb{N}.$$

Table 7

($\mathbf{II}_3$) The symmetric Generalized Bessel:

$\phi^*(x) = x^3, \quad \psi^*(x) = -2(\nu+1)x^2 - \frac{1}{2},$

$\beta_n = 0, \quad \gamma_{n+1} = \frac{1-2\nu-(-1)^n(2n+2\nu+1)}{16(n+\nu)(n+\nu+1)},\ n \in \mathbb{N}, \quad$ with $\nu \neq -n,\ n \geq 0,$

$C_n^*(x) = (2n+2\nu-1)x^2 + \frac{(-1)^n}{2},\ n \geq 0, \quad D^*(x) = x,$

$\varrho_n = 2(n+\nu),\ a_n = n-1-2\nu,\ b_n = 0,\ n \in \mathbb{N},\ \mathbf{R} = 0,\ \mathbf{S} = (1/2),$

$$x^3 B_n'(x) = \big(nx^2 + \frac{1+(-1)^n}{4}\big)B_n(x) - 2(n+\nu)\gamma_n x B_{n-1}(x),\ n \in \mathbb{N},$$

$$x^4 B_n''(x) + x\big((2\nu+1)x^2 + \frac{1}{2}\big)B_n'(x) = \big(n(n+2\nu)x^2 - \frac{1}{2}\eta_n\big)B_n(x),\ n \in \mathbb{N}.$$

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