A COHEN TYPE INEQUALITY FOR FOURIER EXPANSIONS OF ORTHOGONAL POLYNOMIALS WITH A NON-DISCRETE JACOBI-SOBOLEV INNER PRODUCT

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Abstract. Let $\{Q_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ denote the sequence of polynomials orthogonal with respect to the non-discrete Sobolev inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)d\mu_{\alpha,\beta}(x) + \lambda\int_{-1}^{1} f'(x)g'(x)d\mu_{\alpha+1,\beta}(x)$$

where $\lambda > 0$ and $d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$ with $\alpha > -1$, $\beta > -1$. In this paper we prove a Cohen type inequality for the Fourier expansion in terms of the orthogonal polynomials $\{Q_n^{(\alpha,\beta)}(x)\}_n$. Necessary conditions for the norm convergence of such a Fourier expansion are given. Finally, the failure of a.e. convergence of the Fourier expansion of a function in terms of the orthogonal polynomials associated with the above Sobolev inner product is proved.

1. Introduction

Let $d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$ with $\alpha,\beta > -1$, be the Jacobi measure supported on the interval $[-1,1]$. We say that $f \in L^p(d\mu_{\alpha,\beta})$ if f is measurable on $[-1,1]$ and $\|f\|_{L^p(d\mu_{\alpha,\beta})}< \infty$, where

$$\|f\|_{L^p(d\mu_{\alpha,\beta})}= \begin{cases} \left(\int_{-1}^{1}|f(x)|^p d\mu_{\alpha,\beta}(x)\right)^{\frac{1}{p}} & \text{if } 1\leq p<\infty,\\ \underset{-1<x<1}{ess\ sup}\ |f(x)| & \text{if } p=\infty. \end{cases}$$

Let us introduce the Sobolev-type spaces (see, for instance, [2, Chapter III] in a more general framework)

$$S_p^{\alpha,\beta} = \{f \ :\ \|f\|^p_{S_p^{\alpha,\beta}}= \|f\|^p_{L^p(d\mu_{\alpha,\beta})}+\lambda\|f'\|^p_{L^p(d\mu_{\alpha+1,\beta})}< \infty\}, \quad 1\leq p<\infty,$$

$$S_\infty^{\alpha,\beta} = \{f \ :\ \|f\|_{S_\infty^{\alpha,\beta}}= \max\{||f||_{L^\infty(d\mu_{\alpha,\beta})}, ||f'||_{L^\infty(d\mu_{\alpha+1,\beta})}\} < \infty\},$$

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where $\lambda > 0$, as well as the linear space $[S_p^{\alpha,\beta}]$ of all bounded linear operators $T : S_p^{\alpha,\beta} \to S_p^{\alpha,\beta}$, with the usual operator norm

$$||T||_{[S_p^{\alpha,\beta}]} = \sup_{0 \neq f \in S_p^{\alpha,\beta}} \frac{||T(f)||_{S_p^{\alpha,\beta}}}{||f||_{S_p^{\alpha,\beta}}}.$$

Let f and g in $S_2^{\alpha,\beta}$. Let consider the Sobolev-type inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)d\mu_{\alpha,\beta}(x) + \lambda \int_{-1}^{1} f'(x)g'(x)d\mu_{\alpha+1,\beta}(x) \tag{1}$$

where $\lambda > 0$. Let $\{Q_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ denote the sequence of polynomials orthogonal with respect to (1), normalized by the condition that $Q_n^{(\alpha,\beta)}$ has the same leading coefficient as the classical Jacobi polynomial

$P_n^{(\alpha,\beta-1)}(x) = \frac{1}{2^n}\binom{2n+\alpha+\beta-1}{n}x^n + lower \quad degree \quad terms.$

We call them the Jacobi-Sobolev orthogonal polynomials.

The measures $\mu_{\alpha,\beta}$ and $\mu_{\alpha+1,\beta}$ constitute a particular case of the so-called coherent pairs of measures studied in [16]. In [11] (see also [17]) the authors established the asymptotics of the zeros of such Jacobi-Sobolev polynomials.

The aim of our contribution is to obtain a lower bound for the norm of the partial sums of the Fourier expansion in terms of Jacobi-Sobolev polynomials, the well-known Cohen type inequality in the framework of Approximation Theory. A Cohen type inequality has been established in other contexts, e.g., on compact groups or for classical orthogonal expansions. See [3], [4], [5], [7], [9], [12] and references therein.

Throughout the manuscript positive constants are denoted by $c,\ c_1, ...$ and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and $u_n \sim v_n$ means $c_1 u_n \leq v_n \leq c_2 u_n$ for sufficiently large n, where c_1 and c_2 are positive real numbers.

The structure of the manuscript is as follows. In Section 2 we introduce the basic background about Jacobi polynomials to be used in the manuscript. In particular, we focus our attention in some estimates and the strong asymptotics on [-1,1] for such polynomials as well as the Mehler-Heine formula. In Section 3 we analyze the polynomials orthogonal with respect to the inner product (1). Their representation in terms of Jacobi polynomials yields estimates, inner strong asymptotics, and a Mehler-Heine type formula. Some estimates of the weighted p Sobolev norm of these polynomials will be needed in the sequel and we show them in Proposition 9. In section 4, a Cohen-type inequality, associated with the Fourier expansions in terms of the Jacobi-Sobolev orthogonal polynomials, is deduced. In Section 5 we focus our attention in the norm convergence of the above Fourier expansions. Finally, Section 6 is devoted to the analysis of the divergence a.e. of such expansions.

2. Jacobi polynomials

For $\alpha, \beta > -1$, we denote by $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ the sequence of Jacobi polynomials which are orthogonal on $[-1,1]$ with respect to the measure $d\mu_{\alpha,\beta}$. They are normalized in such a way that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. We denote the n-th monic Jacobi polynomial by

$$\hat{P}_n^{(\alpha,\beta)}(x) = (h_n^{\alpha,\beta})^{-1} P_n^{(\alpha,\beta)}(x), \tag{2}$$

where (see [1, formula (22.3.1)])

$$h_n^{\alpha,\beta} = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n}.$$

Now we list some basic properties of Jacobi polynomials which will be used in the sequel. The following integral formula for Jacobi polynomials holds (see (2) and [1, formula (22.2.1)])

$$\begin{aligned} \int_{-1}^{1} [\hat{P}_n^{(\alpha,\beta)}(x)]^2 d\mu_{\alpha,\beta}(x) \\ = 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)}. \end{aligned} \tag{3}$$

They satisfy a connection formula (see [1, formula (22.7.19)], [11, formula (2.5)])

$$\hat{P}_n^{(\alpha,\beta-1)}(x) = \hat{P}_n^{(\alpha,\beta)}(x) + a_{n-1}(\alpha,\beta)\hat{P}_{n-1}^{(\alpha,\beta)}(x), \tag{4}$$

where

$$a_n(\alpha,\beta) = \frac{2(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 0, \tag{5}$$

as well as the following relation for the derivatives (see [23, formula (4.21.7)])

$$\frac{d}{dx} P_n^{(\alpha,\beta-1)}(x) = \frac{n+\alpha+\beta}{2} P_{n-1}^{(\alpha+1,\beta)}(x). \tag{6}$$

The following estimate for $P_n^{(\alpha,\beta)}$ holds (see [23, formula (7.32.6)], [6])

$$|P_n^{(\alpha,\beta)}(x)| \leq c n^{-1/2}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}, \tag{7}$$

where $x \in (-1,1)$ and $\alpha, \beta \geq -1/2$.

The formula of Mehler-Heine for Jacobi orthogonal polynomials is (see [23, Theorem 8.1.1])

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = (z/2)^{-\alpha} J_\alpha(z), \tag{8}$$

where α, β are real numbers, and $J_\alpha(z)$ is the Bessel function. This formula holds locally uniformly, that is, on every compact subset of the complex plane.

The inner strong asymptotics of $P_n^{(\alpha,\beta)}$, for $\theta \in [\epsilon, \pi-\epsilon]$ and $\epsilon > 0$, reads as (see [23, Theorem 8.21.8])

$$\begin{aligned} P_n^{(\alpha,\beta)}(cos\theta) &= \pi^{-1/2} n^{-1/2} \\ &\times \left[\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2} \left(\cos\frac{\theta}{2}\right)^{-\beta-1/2} \cos(k\theta+\gamma) + O(n^{-1}) \right], \end{aligned} \tag{9}$$

where $k = n + (\alpha+\beta+1)/2$, $\gamma = -(\alpha+1/2)\pi/2$.

For $\alpha, \beta, \mu > -1$ and $1 \leq q \leq \infty$ (see [23, p. 391. Exercise 91] as well as [12, (2.2)])

$$\left(\int_0^1 (1-x)^\mu |P_n^{(\alpha,\beta)}(x)|^p dx\right)^{1/p} \sim \begin{cases} n^{-1/2} & \text{if } 2\mu > p\alpha - 2 + p/2, \\ n^{-1/2}(\log\ n)^{1/p} & \text{if } 2\mu = p\alpha - 2 + p/2, \\ n^{\alpha - \frac{2\mu+2}{p}} & \text{if } 2\mu < p\alpha - 2 + p/2. \end{cases} \tag{10}$$

3. Asymptotics of Jacobi-Sobolev orthogonal polynomials

Let us denote by $\hat{Q}_n^{(\alpha,\beta)}$ the monic Jacobi-Sobolev polynomial of degree n, i.e. $\hat{Q}_n^{(\alpha,\beta)}(x) = (h_n^{\alpha,\beta-1})^{-1} Q_n^{(\alpha,\beta)}(x)$. From (4) and [11, formula (2.7)] (see also [17] and [10] in a more general framework) we have the following relation between the Jacobi-Sobolev and Jacobi monic orthogonal polynomials:

Proposition 1. *For $\alpha, \beta > -1$*

$$\hat{P}_n^{(\alpha,\beta)}(x) + a_{n-1}(\alpha,\beta)\hat{P}_{n-1}^{(\alpha,\beta)}(x) = \hat{Q}_n^{(\alpha,\beta)}(x) + \hat{d}_{n-1}(\lambda)\hat{Q}_{n-1}^{(\alpha,\beta)}(x), \qquad n \geq 1,$$

where $a_{n-1}(\alpha,\beta)$ is given in (5) *and*

$$\hat{d}_n(\lambda) = a_n(\alpha,\beta)\frac{\|\hat{P}_n^{(\alpha,\beta)}\|^2_{L^2(d\mu_{\alpha,\beta})}}{\|\hat{Q}_n^{(\alpha,\beta)}\|^2_{S_2^\alpha}}, \quad n \geq 0. \tag{11}$$

Proposition 2.

$$||\hat{Q}_n^{(\alpha,\beta)}||^2_{S_2^{\alpha,\beta}} \cong \lambda n^2 ||\hat{P}_{n-1}^{(\alpha+1,\beta)}||^2_{L^2(d\mu_{\alpha+1,\beta})}. \tag{12}$$

In particular, for $\hat{d}_n(\lambda)$ defined in (11) *we obtain*

$$\hat{d}_n(\lambda) \cong \frac{1}{4\lambda n^2}.$$

Proof. We apply the same argument as in the proof of Theorem 2 in [13]. Using the extremal property

$$||\hat{P}_n^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})} = \inf\{||P||^2_{L^2(d\mu_{\alpha,\beta})} : \deg P = n,\ P\ monic\}$$

we get

$$\begin{aligned}(13)\quad ||\hat{Q}_n^{(\alpha,\beta)}||^2_{S_2^{\alpha,\beta}} &= ||\hat{Q}_n^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})} + \lambda ||\hat{Q}'^{(\alpha,\beta)}_n||^2_{L^2(d\mu_{\alpha+1,\beta})} \\ &\geq ||\hat{P}_n^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})} + \lambda n^2 ||\hat{P}_{n-1}^{(\alpha+1,\beta)}||^2_{L^2(d\mu_{\alpha+1,\beta})}.\end{aligned}$$

On the other hand, from the extremal property of $||\hat{Q}_n^{(\alpha,\beta)}||^2_{S_2^{\alpha,\beta}}$, (4), and (6) we have

$$\begin{aligned}(14)\quad ||\hat{Q}_n^{(\alpha,\beta)}||^2_{S_2^{\alpha,\beta}} &\leq ||\hat{P}_n^{(\alpha,\beta)} + a_{n-1}(\alpha,\beta)\hat{P}_{n-1}^{(\alpha,\beta)}||^2_{S_2^{\alpha,\beta}} \\ &= ||\hat{P}_n^{(\alpha,\beta)} + a_{n-1}(\alpha,\beta)\hat{P}_{n-1}^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})} + \lambda n^2 ||\hat{P}_{n-1}^{(\alpha+1,\beta)}||^2_{L^2(d\mu_{\alpha+1,\beta})} \\ &\leq ||\hat{P}_n^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})} + (a_{n-1}(\alpha,\beta))^2 ||\hat{P}_{n-1}^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})} \\ &\quad + \lambda n^2 ||\hat{P}_{n-1}^{(\alpha+1,\beta)}||^2_{L^2(d\mu_{\alpha+1,\beta})}.\end{aligned}$$

Since by (3) and (5) we have $||\hat{P}_n^{(\alpha,\beta)}||_{L^2(d\mu_{\alpha,\beta})} \cong ||\hat{P}_{n-1}^{(\alpha+1,\beta)}||_{L^2(d\mu_{\alpha+1,\beta})}$ and $a_n(\alpha,\beta) \cong 1/2$, then (13) and (14) yield (12). □

As a straightforward consequence of Proposition 1 and Proposition 2, using (2) we deduce

Corollary 1. *For $\alpha, \beta > -1$*

$$\frac{n+\alpha+\beta}{2n+\alpha+\beta} P_n^{(\alpha,\beta)}(x) + \frac{n+\alpha}{2n+\alpha+\beta} P_{n-1}^{(\alpha,\beta)}(x) = Q_n^{(\alpha,\beta)}(x) + d_{n-1}(\lambda) Q_{n-1}^{(\alpha,\beta)}(x), \tag{15}$$

where $n \geq 1$ and

$$d_n(\lambda) = \hat{d}_n(\lambda) \frac{h_n^{\alpha,\beta-1}}{h_{n-1}^{\alpha,\beta-1}} \cong \frac{1}{2\lambda n^2}. \tag{16}$$

Corollary 2. *For $\alpha > -1$ and $\beta > 0$*

$$P_n^{(\alpha,\beta-1)}(x) = Q_n^{(\alpha,\beta)}(x) + d_{n-1}(\lambda)Q_{n-1}^{(\alpha,\beta)}(x), \qquad n \geq 1, \tag{17}$$

and for $\alpha, \beta > -1$

$$\frac{n+\alpha+\beta}{2}P_{n-1}^{(\alpha+1,\beta)}(x) = \left(Q_n^{(\alpha,\beta)}(x)\right)' + d_{n-1}(\lambda)\left(Q_{n-1}^{(\alpha,\beta)}(x)\right)', \qquad n \geq 1. \tag{18}$$

Proof. The first statement follows from Proposition 1 and (4). The second one follows by taking derivatives in (17) and using (6). □

Using (17) in a recursive way the representation of the polynomials $Q_n^{(\alpha,\beta)}$ in terms of the elements of the sequence $\{P_n^{(\alpha,\beta-1)}(x)\}_{n=0}^{\infty}$ becomes

$$Q_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}(-1)^k b_k^{(n)}(\lambda)P_{n-k}^{(\alpha,\beta-1)}(x), \tag{19}$$

where $b_k^{(n)}(\lambda) = \prod_{j=1}^{k} d_{n-j}(\lambda)$ and $b_0^{(n)}(\lambda) = 1$.

Proposition 3. *There exists a constant $c > 1$ such that the coefficients $b_k^{(n)}(\lambda)$ in (18) satisfy $b_k^{(n)}(\lambda) < c\frac{1}{n2^k}$ for all $n \geq 1$ and $1 \leq k \leq n$.*

Proof. From (16) we have $\lim_n 2(n+1)d_n(\lambda) = 0$. Thus, there exist $n_0 \in \mathbf{N}$ and a constant $c > 1$ such that $2(n+1)d_n(\lambda) < 1$ for all $n \geq n_0$ and $2(n+1)d_n(\lambda) < c$ for $n = 1, ..., n_0 - 1$. Therefore, for $1 \leq k \leq n - n_0$,

$$b_k^{(n)}(\lambda) = \prod_{j=1}^{k} d_{n-j}(\lambda) < \frac{1}{n2^k},$$

and, for $n - n_0 \leq k \leq n$,

$$\begin{aligned} b_k^{(n)}(\lambda) &= \prod_{j=1}^{n-n_0} d_{n-j}(\lambda) \prod_{j=n-n_0+1}^{k} d_{n-j}(\lambda) \\ &\leq \frac{1}{n2^{n-n_0}}\left(\frac{c}{2}\right)^{k-n+n_0} = c^{k-n+n_0}\frac{1}{n2^k} \leq c^{n_0}\frac{1}{n2^k}. \end{aligned}$$

□

Proposition 4. a) *For the polynomials $Q_n^{(\alpha,\beta)}$ we obtain*

$$|Q_n^{(\alpha,\beta)}(x)| \leq cn^{-1/2}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2+1/4}$$

for $x \in (-1,1)$, $\alpha \geq -1/2$, and $\beta \geq 1/2$.

b) *For the polynomials* ${Q'}_n^{(\alpha,\beta)}$ *we have the estimate*

$$|{Q'}_n^{(\alpha,\beta)}(x)| \le cn^{1/2}(1-x)^{-\alpha/2-3/4}(1+x)^{-\beta/2-1/4}$$

for $x \in (-1,1)$, $\alpha > -1$, *and* $\beta \ge -1/2$.

Proof. *a*) Using (19) we have

$$|Q_n^{(\alpha,\beta)}(\cos\theta)| \le \sum_{k=0}^{n} b_k^{(n)}(\lambda)|P_{n-k}^{(\alpha,\beta-1)}(\cos\theta)|.$$

From (7), it is straightforward to prove that, for $\alpha, \beta \ge -1/2$ and $k = 0, 1, ..., n-1$,

$$|P_{n-k}^{(\alpha,\beta)}(\cos\theta)| \le c\sqrt{\frac{n}{n-k}}\, n^{-1/2}\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}.$$

Thus, according to Proposition 3,

$$\begin{aligned}|Q_n^{(\alpha,\beta)}(\cos\theta)| &\le \sum_{k=0}^{n} b_k^{(n)}(\lambda)|P_{n-k}^{(\alpha,\beta-1)}(\cos\theta)|\\ &\le cb_n^{(n)}(\lambda) + cn^{-1/2}\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta+1/2}\sum_{k=0}^{n-1}\frac{1}{2^k}\\ &\le cn^{-1/2}\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta+1/2}.\end{aligned}$$

On the other hand, from (18), the proof of the case *b*) can be done in a similar way. □

Proposition 5. *Let* $\alpha, \beta > -1$. *Then*

$$|Q_n^{(\alpha,\beta)}(x)| \le \begin{cases} cn^{\alpha} & \textit{for } x \in [0,1],\ \alpha \ge -1/2,\\ cn^{\beta-1} & \textit{for } x \in [-1,0],\ \beta \ge 1/2,\\ cn^{-1/2} & \textit{for } x \in [-1,1],\ \alpha \le -1/2,\ \beta \le 1/2,\end{cases}$$

and

$$|{Q'}_n^{(\alpha,\beta)}(x)| \le \begin{cases} cn^{\alpha+2} & \textit{for } x \in [0,1],\ \alpha > -1,\\ cn^{\beta+1} & \textit{for } x \in [-1,0],\ \beta \ge -1/2.\end{cases}$$

Proof. Taking into account that the Jacobi polynomials satisfy (see [23, paragraph below Theorem 7.32.1])

$$|P_n^{(\alpha,\beta)}(x)| \le \begin{cases} cn^{\alpha} & \text{for } x \in [0,1],\ \alpha \ge -1/2,\\ cn^{\beta} & \text{for } x \in [-1,0],\ \beta \ge -1/2,\\ cn^{-1/2} & \text{for } x \in [-1,1],\ \alpha \le -1/2,\ \beta \le -1/2,\end{cases}$$

for $n \geq 1$, thus, for $0 \leq j \leq n-1$,

$$|P_{n-j}^{(\alpha,\beta)}(x)| \leq \begin{cases} c\left(\frac{n-j}{n}\right)^{\alpha} n^{\alpha} & \text{for } x \in [0,1],\ \alpha \geq -1/2, \\ c\left(\frac{n-j}{n}\right)^{\beta} n^{\beta} & \text{for } x \in [-1,0],\ \beta \geq -1/2, \\ c\left(\frac{n-j}{n}\right)^{-1/2} n^{-1/2} & \text{for } x \in [-1,1],\ \alpha \leq -1/2,\ \beta \leq -1/2. \end{cases}$$

As a consequence, the statement follows from the latter estimates and arguments similar to those we used in the proof of Proposition 4. □

Corollary 3. *For $\alpha \geq -1/2$ and $\beta \geq 1/2$*

$$|Q_n^{(\alpha,\beta)}(\cos\theta)| \leq cA(n,\alpha,\beta-1,\theta),$$

and for $\alpha > -1$ and $\beta \geq -1/2$

$$|Q'^{(\alpha,\beta)}_n(\cos\theta)| \leq cA(n,\alpha+1,\beta,\theta),$$

where

$$A(n,\alpha,\beta,\theta) = \begin{cases} n^{-1/2}(\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}) & \text{if } c/n \leq \theta \leq \pi - c/n, \\ n^{\alpha} & \text{if } 0 \leq \theta \leq c/n, \\ n^{\beta} & \text{if } \pi - c/n \leq \theta \leq \pi. \end{cases}$$

Proof. The inequality

$$n^{\alpha} \leq cn^{-1/2}\theta^{-\alpha-1/2}$$

holds for $\theta \in (0, c/n]$, as well as

$$n^{\beta} \leq cn^{-1/2}(\pi-\theta)^{-\beta-1/2}$$

holds for $\theta \in [\pi - c/n, \pi)$. Therefore, the statement follows from Proposition 4 and Proposition 5. □

Next we show that the Jacobi-Sobolev polynomial $Q_n^{(\alpha,\beta)}(x)$ attains its maximum in $[-1,1]$ at the end-points. To be more precise,

Proposition 6. *a) For $\alpha \geq -1/2$, $\beta \geq 1/2$, and $q = \max\{\alpha, \beta-1\}$*

$$\max_{-1 \leq x \leq 1} |Q_n^{(\alpha,\beta)}(x)| = |Q_n^{(\alpha,\beta)}(a)| \sim n^q,$$

where $a = 1$ if $q = \alpha$ and $a = -1$ if $q = \beta - 1$.

b) For $\alpha > -1$, $\beta \geq -1/2$ and $q = \max\{\alpha+1, \beta\}$

$$\max_{-1\leq x\leq 1} |{Q'}_n^{(\alpha,\beta)}(x)| = |{Q'}_n^{(\alpha,\beta)}(b)| \sim n^{q+1},$$

where $b = 1$ if $q = \alpha + 1$ and $b = -1$ if $q = \beta$.

Proof. *a)* We shall prove only the case $q = \alpha$. If $q = \beta - 1$ the the proof can be done in a similar way. From (16), (17), and Proposition 5

$$Q_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta-1)}(x) - d_{n-1}(\lambda)Q_{n-1}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta-1)}(x) - O(n^{\alpha-2}).$$

Now, from [23, Theorem 7.32.1] and Proposition 5, the result follows.

Taking into account (6), the case *b)* can be proved in a similar way. □

Next, we deduce a Mehler-Heine type formula for $Q_n^{(\alpha,\beta)}$ and $(Q_n^{(\alpha,\beta)})'$.

Proposition 7. *Let $\alpha, \beta > -1$. Uniformly on compact subsets of $\boldsymbol{C}$*

a)

$$\lim_{n\to\infty} n^{-\alpha} Q_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = (z/2)^{-\alpha} J_\alpha(z), \tag{20}$$

b)

$$\lim_{n\to\infty} n^{-\alpha-2} {Q'}_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = (z/2)^{-(\alpha+1)} J_{\alpha+1}(z). \tag{21}$$

Proof. *a)* Multiplying in (15) by $(n+1)^{-\alpha}$ we obtain

$$V_n(z) = Y_n(z) + D_{n-1}(\lambda)Y_{n-1}(z),$$

where $V_n(z) = (n+1)^{-\alpha}\left[\frac{n+\alpha+\beta}{2n+\alpha+\beta}P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) + \frac{n+\alpha}{2n+\alpha+\beta}P_{n-1}^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right)\right]$, $Y_n(z) = (n+1)^{-\alpha}Q_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right)$ and $D_{n-1}(\lambda) = d_{n-1}(\lambda)\left(\frac{n}{n+1}\right)^{\alpha} \cong \frac{c}{n^2}$ according to (16).

Using above relation in a recursive way we obtain

$$Y_n(z) = \sum_{k=0}^{n}(-1)^k B_k^{(n)}(\lambda)V_{n-k}(z),$$

where $B_k^{(n)}(\lambda) = \prod_{j=1}^{k} D_{n-j}(\lambda)$ and $B_0^{(n)}(\lambda) = 1$. Moreover, by using the same argument as in Proposition 3 we have $B_k^{(n)}(\lambda) < c\frac{1}{n2^k}$ for every $n \geq 1$ and $1 \leq k \leq n$. Thus

$$|Y_n(z)| \leq \sum_{k=0}^{n} B_k^{(n)}(\lambda)\,|V_{n-k}(z)|\,.$$

On the other hand, from (8), we have that $\{V_n(z)\}_{n=0}^{\infty}$ is uniformly bounded on compact subsets of $\mathbf{C}$. Thus, for a fixed compact set $K \subset \mathbf{C}$ there exists a constant C, depending only on K, such that when $z \in K$

$$|V_n(z)| < C, \quad n \geq 1.$$

Thus, the sequence $\{Y_n(z)\}_{n=0}^{\infty}$ is uniformly bounded on $K \subset \mathbf{C}$. As a conclusion

$$Y_n(z) = V_n(z) + O(n^{-2}), \qquad z \in K,$$

and using (8) we obtain the result.

b) Since we have uniform convergence in (20), taking derivatives and using some properties of Bessel functions we obtain (21). □

Now we give the inner strong asymptotics of $Q_n^{(\alpha,\beta)}$ on $(-1, 1)$.

Proposition 8. *Let* $\theta \in [\epsilon, \pi - \epsilon]$ *and* $\epsilon > 0$. *For* $\alpha \geq -1/2$, $\beta \geq 1/2$

$$\text{(22)} \quad Q_n^{(\alpha,\beta)}(\cos\theta) = \pi^{-1/2} n^{-1/2} \times \left[\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2} \left(\cos\frac{\theta}{2}\right)^{-\beta+1/2} \cos(k_1\theta + \gamma) + O(n^{-1}) \right],$$

and for $\alpha > -1$, $\beta \geq -1/2$

$$\text{(23)} \quad Q'^{(\alpha,\beta)}_n(\cos\theta) = \pi^{-1/2} \frac{(n+\alpha+\beta+1)(n-1)^{-1/2}}{2} \times \left[\left(\sin\frac{\theta}{2}\right)^{-\alpha-3/2} \left(\cos\frac{\theta}{2}\right)^{-\beta-1/2} \cos(k_1\theta + \gamma_1) + O(n^{-1}) \right],$$

where $k_1 = n + (\alpha+\beta)/2$, $\gamma = -(\alpha + 1/2)\pi/2$, *and* $\gamma_1 = -(\alpha + 3/2)\pi/2$.

Proof. From Proposition 4 (*a*) the sequence $\{n^{1/2} Q_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$ is uniformly bounded on compact subsets of $(-1, 1)$. Multiplication by $n^{1/2}$ in (17) yields

$$n^{1/2} Q_n^{(\alpha,\beta)}(x) = n^{1/2} P_n^{(\alpha,\beta-1)}(x) - d_{n-1}(\lambda) \sqrt{\frac{n}{n-1}} (n-1)^{1/2} Q_{n-1}^{(\alpha,\beta)}(x).$$

Since

$$d_{n-1}(\lambda) \sqrt{\frac{n}{n-1}} = O\left(\frac{1}{n^2}\right),$$

we have

$$n^{1/2} Q_n^{(\alpha,\beta)}(x) = n^{1/2} P_n^{(\alpha,\beta-1)}(x) + O\left(n^{-2}\right).$$

Now (22) follows from (9).

Concerning (23), it can be obtained in a similar way by using (18) and Proposition 4 (b). $\square$

Next we obtain an estimate for the Sobolev norms of the Jacobi-Sobolev polynomials.

Proposition 9. *For* $\alpha > -1/2$, $\alpha+1 \geq \beta \geq -1/2$, *and* $1 \leq p \leq \infty$

$$||Q_n^{(\alpha,\beta)}||_{S_p^{\alpha,\beta}} \sim \begin{cases} n^{1/2} & \text{if } 4(\alpha+2)/(2\alpha+3) > p, \\ n^{1/2}\log\ n & \text{if } 4(\alpha+2)/(2\alpha+3) = p, \\ n^{\alpha+2-\frac{2\alpha+4}{p}} & \text{if } 4(\alpha+2)/(2\alpha+3) < p. \end{cases} \tag{24}$$

Notice that if $p=\infty$ then we have Proposition 6(b). Thus in the proof we will assume $1 \leq p < \infty$.

Proof. In order to establish the upper bound in (22) it is enough to prove that

$$||Q_n^{(\alpha,\beta)}||_{S_p^{\alpha,\beta}} \leq cn||P_n^{(\alpha+1,\beta)}||_{L^p(d\mu_{\alpha+1,\beta})}. \tag{25}$$

Using (15) in a recurrence way and then Minkowski's inequality we obtain

$$||Q_n^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})} \leq c\sum_{k=0}^{n} b_k^{(n)}(\lambda)||P_{n-k}^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})} + c\sum_{k=0}^{n} b_k^{(n)}(\lambda)||P_{n-k-1}^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})}.$$

On the other hand, for $\alpha,\beta > -1$ and $k = 0,1,...,n$, (10) implies

$$(n-k)^{1/2}||P_{n-k}^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})} \leq cn^{1/2}||P_n^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})}.$$

Thus

$$||P_{n-k}^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})} \leq \sqrt{\frac{n}{n-k}}||P_n^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})}, \quad 0 \leq k \leq n-1.$$

On the other hand, from Proposition 3,

$$\sum_{k=0}^{n} b_k^{(n)}(\lambda)||P_{n-k}^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})} \leq cb_n^{(n)}(\lambda) + \sum_{k=0}^{n-1} b_k^{(n)}(\lambda)||P_{n-k}^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})}$$
$$\leq c||P_{n-1}^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})}\sum_{i=0}^{n-1}\frac{1}{2^k} \leq c||P_n^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})}.$$

Thus

$$||Q_n^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})} \leq c||P_n^{(\alpha,\beta)}||_{L^p(d\mu_{\alpha,\beta})} \leq cn||P_n^{(\alpha+1,\beta)}||_{L^p(d\mu_{\alpha+1,\beta})}. \tag{26}$$

In the same way as above we conclude that

$$(27)\quad ||Q'^{(\alpha,\beta)}_n||_{L^p(d\mu_{\alpha+1,\beta})} \le cn\sum_{k=0}^{n} b_k^{(n)}(\lambda)||P^{(\alpha+1,\beta)}_{n-k-1}||_{L^p(d\mu_{\alpha+1,\beta})} \le cn||P^{(\alpha+1,\beta)}_n||_{L^p(d\mu_{\alpha+1,\beta})}.$$

Thus (25) follows from (26) and (27).

In order to prove the lower bound in relation (24) we shall need the following:

Proposition 10. *For $\alpha > -1$ and $1 \le p < \infty$*

$$(28)\qquad ||Q'^{(\alpha,\beta)}_n||_{L^p(d\mu_{\alpha+1,\beta})} \ge c\begin{cases} n^{1/2} & \textit{if } 4(\alpha+2)/(2\alpha+3) > p, \\ n^{1/2}\log\, n & \textit{if } 4(\alpha+2)/(2\alpha+3) = p, \\ n^{\alpha+2-\frac{2\alpha+4}{p}} & \textit{if } 4(\alpha+2)/(2\alpha+3) < p. \end{cases}$$

Proof. We shall use a technique similar to [23, Theorem 7.34]. According to (18)

$$\int_0^{\pi/2}\theta^{2\alpha+3}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta > \int_0^{\omega/n}\theta^{2\alpha+3}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta$$
$$\ge cn^{-2\alpha-4}\int_0^{\omega} t^{2\alpha+3}|Q'^{(\alpha,\beta)}_n(\cos\frac{t}{n})|^p dt \cong cn^{p(\alpha+2)-2\alpha-4}$$
$$\times\int_0^{\omega} t^{2\alpha+3}|t^{-(\alpha+1)}J_{\alpha+1}(t)|^p dt = cn^{p(\alpha+2)-2\alpha-4}$$
$$\times\int_0^{\omega} t^{2\alpha+3-p(\alpha+1)}|J_{\alpha+1}(t)|^p dt.$$

On the other hand, Stempak's lemma (see [22, Lemma 2.1]), for $\gamma > -1-p\alpha$ and $1 \le p < \infty$ implies

$$\int_0^{\omega} t^{\gamma}|J_{\alpha+1}(t)|^p dt \sim \begin{cases} c & \text{if } \gamma < p/2-1, \\ c\log\omega & \text{if } \gamma = p/2-1. \end{cases}$$

Thus, for $4(\alpha+2)/(2\alpha+3) \le p$ and ω large enough, (28) follows.

Finally, from (23) we obtain

$$\int_0^{\pi/2}\theta^{2\alpha+3}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta > \int_{\pi/4}^{\pi/2}\theta^{2\alpha+3}|Q'^{(\alpha,\beta)}_n(\cos\theta)|^p d\theta \sim n^{p/2}.$$

□

For the proof of Proposition 9, from (28), for $\alpha > -1$ and $1 \le p < \infty$ we get

$$||Q_n^{(\alpha,\beta)}||_{S_p^{\alpha,\beta}} \ge c \begin{cases} n^{1/2} & \text{if } 4(\alpha+2)/(2\alpha+3) > p, \\ n^{1/2} \log\ n & \text{if } 4(\alpha+2)/(2\alpha+3) = p, \\ n^{\alpha+2-\frac{2\alpha+4}{p}} & \text{if } 4(\alpha+2)/(2\alpha+3) < p. \end{cases} \tag{29}$$

Thus, using (25) and (29), the statement follows. □

4. A Cohen type inequality for Jacobi-Sobolev expansions

For $f \in S_1^{\alpha,\beta}$, its Fourier expansion in terms of Jacobi-Sobolev polynomials is

$$\sum_{k=0}^{\infty} \hat{f}(k) Q_k^{(\alpha,\beta)}(x), \tag{30}$$

where

$$\hat{f}(k) = (\|Q_k^{(\alpha,\beta)}\|^2_{S_2^{\alpha,\beta}})^{-1} \langle f, Q_k^{(\alpha,\beta)} \rangle, \qquad k = 0,\ 1,\ \dots .$$

The Cesàro means of order δ of the expansion (30) is defined by (see [24, p. 76-77])

$$\sigma_n^{\delta} f(x) = \sum_{k=0}^{n} \frac{C_{n-k}^{\delta}}{C_n^{\delta}} \hat{f}(k) Q_k^{(\alpha,\beta)}(x),$$

where $C_k^{\delta} = \binom{k+\delta}{k}$.

For a function $f \in S_p^{\alpha,\beta}$ and a fixed sequence $\{c_{k,n}\}_{k=0}^{n}$, $n \in \mathbf{N} \cup \{\mathbf{0}\}$, of real numbers with $c_{n-1,n} = o(n^2 c_{n,n})$, we define the operators $T_n^{\alpha,\beta}$ by

$$T_n^{\alpha,\beta}(f) = \sum_{k=0}^{n} c_{k,n} \hat{f}(k) Q_k^{(\alpha,\beta)}.$$

Let $q_0 = (4\alpha+8)/(2\alpha+3)$ and let p_0 be the conjugate of q_0. Now we can state our main result.

Theorem 1. *For $\alpha > -1/2$ and $\alpha + 1 \ge \beta \ge -1/2$*

$$\|T_n^{\alpha,\beta}\|_{[S_p^{\alpha}]} \ge c|c_{n,n}| \begin{cases} n^{\frac{2\alpha+4}{p} - \frac{2\alpha+5}{2}} & \text{if } 1 \le p < p_0, \\ (\log\ n)^{\frac{2\alpha+3}{4\alpha+8}} & \text{if } p = p_0,\ p = q_0, \\ n^{\frac{2\alpha+3}{2} - \frac{2\alpha+4}{p}} & \text{if } q_0 < p \le \infty. \end{cases}$$

Corollary 4. *Let $\alpha, \beta,\ p_0,\ q_0$, and p be as in Theorem 1. For $c_{k,n} = 1$, $k = 0, \dots, n$, and for p outside the interval (p_0, q_0)*

$$\|\sigma_n^0\|_{[S_p^{\alpha,\beta}]} \to \infty, \qquad n \to \infty.$$

For $c_{k,n} = \frac{C^\delta_{n-k}}{C^\delta_n}$, $0 \le k \le n$, Theorem 1 yields.

Corollary 5. *For $\alpha > -1/2$ and $\alpha + 1 \ge \beta \ge -1/2$*

$$\begin{cases} 0 < \delta < \frac{2\alpha+4}{p} - \frac{2\alpha+5}{2} & \text{if } 1 \le p < p_0, \\ 0 < \delta < \frac{2\alpha+3}{2} - \frac{2\alpha+4}{p} & \text{if } q_0 < p \le \infty, \end{cases}$$

and $p \notin [p_0, q_0]$

$$\|\sigma^\delta_n\|_{[S^{\alpha,\beta}_p]} \to \infty, \qquad n \to \infty.$$

We shall use as test functions (see [12, formula (2.8)] and [1, formula (22.7.19)])

$$(31) \quad g^{\alpha,\beta-1,j}_n(x) = (1-x^2)^j P^{(\alpha+j,\beta-1+j)}_n(x)$$
$$= \sum_{m=0}^{2j} c_{m,j}(\alpha, \beta-1, n) P^{(\alpha,\beta-1)}_{n+m}(x) = \sum_{m=0}^{2j} c_{m,j}(\alpha, \beta-1, n)$$
$$\times \left(A_{n+m}(\alpha,\beta) P^{(\alpha,\beta)}_{n+m}(x) + B_{n+m}(\alpha,\beta) P^{(\alpha,\beta)}_{n+m-1}(x) \right),$$

where $j \in \mathbf{N} \backslash \{\mathbf{1}\}$, and

$$c_{0,j}(\alpha,\beta,n) = \frac{4^j \Gamma(n+\alpha+j+1)\Gamma(n+\beta+j+1)\Gamma(2n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(2n+\alpha+\beta+2j+2)},$$

$$c_{1,j}(\alpha,\beta,n)$$
$$= -\frac{4^j A_1^{-j-1}(n+1)\Gamma(n+\alpha+j+1)\Gamma(n+\beta+j+1)\Gamma(2n+\alpha+\beta+3)}{(2n+\alpha+\beta+j+2)\Gamma(n+\alpha+1)\Gamma(n+\beta+2)\Gamma(2n+\alpha+\beta+2j+2)}$$
$$+ \frac{4^j A_1^{-j-1}(n+1)\Gamma(n+\alpha+j+1)\Gamma(n+\beta+j+2)\Gamma(2n+\alpha+\beta+4)}{(2n+\alpha+\beta+j+3)\Gamma(n+\alpha+2)\Gamma(n+\beta+2)\Gamma(2n+\alpha+\beta+2j+3)},$$

$$c_{2j,j}(\alpha,\beta,n) = \frac{(-4)^j \Gamma(n+2j+1)\Gamma(2n+2j+\alpha+\beta+1)}{\Gamma(n+1)\Gamma(2n+4j+\alpha+\beta+1)},$$

$$A_n(\alpha,\beta) = \frac{n+\alpha+\beta}{2n+\alpha+\beta}, \quad B_n(\alpha,\beta) = \frac{n+\alpha}{2n+\alpha+\beta}.$$

Applying the operator $T^{\alpha,\beta}_n$ to $g^{\alpha,\beta-1,j}_n$, for some $j > \alpha + 5/2 - 2(\alpha+2)/p$, we get

$$(32) \qquad T^{\alpha,\beta}_n(g^{\alpha,\beta-1,j}_n) = \sum_{k=0}^{n} c_{k,n} (g^{\alpha,\beta-1,j}_n)^\wedge(k) Q^{(\alpha,\beta)}_k,$$

where

$$(g^{\alpha,\beta-1,j}_n)^\wedge(k) = (\|Q^{(\alpha,\beta)}_k\|^2_{S^{\alpha,\beta}_2})^{-1} \langle g^{\alpha,\beta-1,j}_n, Q^{(\alpha,\beta)}_k \rangle, \qquad k = 0,\ 1,\ \dots,\ n,$$

and, using (3) and (12), we deduce

$$\|Q_n^{(\alpha,\beta)}\|^2_{S_2^{\alpha,\beta}} \cong \lambda 2^{\alpha+\beta} n.$$

Taking into account (31), for $0 \leq k \leq n-2$,

$$\int_{-1}^{1} g_n^{\alpha,\beta-1,j}(x) Q_k^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) = 0.$$

If $k = n-1$, then we get

$$\begin{aligned}\int_{-1}^{1} & g_n^{\alpha,\beta-1,j}(x) Q_{n-1}^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) \\ &= c_{0,j}(\alpha,\beta,n) A_{n-1}(\alpha,\beta) B_n(\alpha,\beta) \\ &\qquad \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x) P_{n-1}^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) \cong 2^{\alpha+\beta+2j-2} n^{-1}.\end{aligned}$$

If $k = n$, then

$$\begin{aligned}\int_{-1}^{1} & g_n^{\alpha,\beta-1,j}(x) Q_n^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) \\ &= c_{0,j}(\alpha,\beta,n)(A_n(\alpha,\beta))^2 \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) \\ &+ c_{0,j}(\alpha,\beta,n)(B_n(\alpha,\beta))^2 \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x) P_{n-1}^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) \\ &- c_{0,j}(\alpha,\beta,n) A_{n-1}(\alpha,\beta) B_n(\alpha,\beta) b_1^{(n)}(\lambda) \int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(x) P_{n-1}^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) \\ &+ c_{1,j}(\alpha,\beta,n) A_n(\alpha,\beta) B_{n+1}(\alpha,\beta) \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x) \\ &\cong 2^{\alpha+\beta+2j-1} n^{-1}.\end{aligned}$$

On the other hand, for $0 \leq k \leq n-1$,

$$\int_{-1}^{1} \left(g_n^{\alpha,\beta-1,j}(x)\right)' \left(Q_k^{(\alpha,\beta)}(x)\right)' d\mu_{\alpha+1,\beta}(x) = 0,$$

and, for $k = n$,

$$\begin{aligned}\int_{-1}^{1} & \left(g_n^{\alpha,\beta-1,j}(x)\right)' \left(Q_n^{(\alpha,\beta)}(x)\right)' d\mu_{\alpha+1,\beta}(x) \\ &= \left(\frac{n+\alpha+\beta}{2}\right)^2 c_{0,j}(\alpha,\beta-1,n) \int_{-1}^{1} P_{n-1}^{(\alpha+1,\beta)}(x) P_{n-1}^{(\alpha+1,\beta)}(x) d\mu_{\alpha+1,\beta}(x) \\ &\cong 2^{\alpha+\beta+2j-1} n.\end{aligned}$$

Thus

$$\begin{cases} \langle g_n^{\alpha,\beta-1,j}, Q_k^{(\alpha,\beta)}\rangle = 0, & \text{if } 0 \le k \le n-2, \\ \langle g_n^{\alpha,\beta-1,j}, Q_{n-1}^{(\alpha,\beta)}\rangle \cong 2^{\alpha+\beta+2j-2} n^{-1}, & \\ \langle g_n^{\alpha,\beta-1,j}, Q_n^{(\alpha,\beta)}\rangle \cong 2^{\alpha+\beta+2j-1} n. & \end{cases}$$

As a conclusion,

$$\begin{cases} (g_n^{\alpha,\beta-1,j})\,\hat{}\,(k) = 0 & \text{if } 0 \le k \le n-2, \\ (g_n^{\alpha,\beta-1,j})\,\hat{}\,(n-1) \cong \frac{2^{2j-2}}{\lambda n^2}, & \\ (g_n^{\alpha,\beta-1,j})\,\hat{}\,(n) \cong \frac{2^{2j-2}}{\lambda}. & \end{cases} \tag{33}$$

Now we shall estimate

$$\|g_n^{\alpha,\beta-1,j}\|^p_{S_p^{\alpha,\beta}} = \|g_n^{\alpha,\beta-1,j}\|^p_{L^p(d\mu_{\alpha,\beta})} + \lambda \|(g_n^{\alpha,\beta-1,j})'\|^p_{L^p(d\mu_{\alpha+1,\beta})}. \tag{34}$$

From [12, formula (3.1)]

$$\|g_n^{\alpha,\beta-1,j}\|^p_{L^p(d\mu_{\alpha,\beta})} \le c n^{-p/2}, \tag{35}$$

for $j > \alpha + 1/2 - (2\alpha+2)/p \ge \beta - 1/2 - (2\beta+2)/p$.

On the other hand, from (6), (31), and [23, formula (4.5.4)]

$$\begin{aligned} \left(g_n^{\alpha,\beta-1,j}(x)\right)' &= \left((1-x^2)^j P_n^{(\alpha+j,\beta-1+j)}(x)\right)' \\ &= -2j(1-x^2)^{j-1} x P_n^{(\alpha+j,\beta-1+j)}(x) \\ &\quad + \frac{n+\alpha+\beta+2j}{2}(1-x^2)^j P_n^{(\alpha+1+j,\beta+j)}(x) \\ &= \frac{4j(n+\alpha+j)}{2n+\alpha+\beta+2j}(1-x^2)^{j-1} P_n^{(\alpha-1+j,\beta-1+j)}(x) \\ &\quad - \frac{4j(n+1)}{2n+\alpha+\beta+2j}(1-x^2)^{j-1} P_{n+1}^{(\alpha-1+j,\beta-1+j)}(x) \\ &\quad - 2j(1-x^2)^{j-1} P_n^{(\alpha+j,\beta-1+j)}(x) \\ &\quad + \frac{n+\alpha+\beta+2j}{2}(1-x^2)^j P_n^{(\alpha+1+j,\beta+j)}(x). \end{aligned}$$

From (10), for $j > \max\{\alpha + 3/2 - (2\alpha+4)/p,\ \beta + 3/2 - (2\beta+2)/p\}$

$$||(1-x^2)^{j-1} P_n^{(\alpha-1+j,\beta-1+j)}||_{L^p(d\mu_{\alpha+1,\beta})} \sim n^{-1/2},$$

for $\alpha + 1 \ge \beta$ and $j > \alpha + 5/2 - (2\alpha+4)/p$

$$||(1-x^2)^{j-1} P_n^{(\alpha+j,\beta-1+j)}||_{L^p(d\mu_{\alpha+1,\beta})} \sim n^{-1/2},$$

and for $\alpha+1 \geq \beta$ and $j > \alpha + 3/2 - (2\alpha+4)/p$

$$||(1-x^2)^j P_n^{(\alpha+1+j,\beta+j)}||_{L^p(d\mu_{\alpha+1,\beta})} \sim n^{-1/2}.$$

Thus, for $\alpha+1 \geq \beta$ and $j > \alpha + 5/2 - (2\alpha+4)/p$

$$||(g_n^{\alpha,\beta-1,j})'||_{L^p(d\mu_{\alpha+1,\beta})} \leq cn^{1/2}. \tag{36}$$

By using (35) and (36), we find from (34) that

$$||g_n^{\alpha,\beta-1,j}||_{S_p^{\alpha,\beta}} \leq cn^{1/2}, \tag{37}$$

for $\alpha+1 \geq \beta$ and $j > \alpha + 5/2 - (2\alpha+4)/p$.

Now we can prove our main result.

Proof of Theorem 1. By duality, it is enough to assume that $q_0 \leq p \leq \infty$. From (32), (33), and (37)

$$\begin{aligned}\|T_n^{\alpha,\beta}\|_{S_p^{\alpha,\beta}} &\geq [\|g_n^{(\alpha,\beta-1,j)}\|_{S_p^{\alpha,\beta}}]^{-1}\|T_n^{\alpha,\beta}(g_n^{(\alpha,\beta-1,j)})\|_{S_p^{\alpha,\beta}} \\ &\geq cn^{-1/2}|c_{n,n}(g_n^{\alpha,\beta-1,j})\hat{}\,(n)|||Q_n^{(\alpha,\beta)}||_{S_p^{\alpha,\beta}} \\ &- cn^{-1/2}|c_{n-1,n}(g_n^{\alpha,\beta-1,j})\hat{}\,(n-1)|||Q_{n-1}^{(\alpha,\beta)}||_{S_p^{\alpha,\beta}} \\ &\sim cn^{-1/2}|c_1 c_{n,n}|||Q_n^{(\alpha,\beta)}||_{S_p^{\alpha,\beta}}\left(1-\left|\frac{c_2 c_{n-1,n}}{c_1 n^2 c_{n,n}}\right|\right).\end{aligned}$$

Now from Proposition 9 the statement of the theorem follows. □

5. Necessary conditions for the norm convergence

The problem of the convergence in the norm of partial sums of the Fourier expansions in terms of Jacobi polynomials has been discussed by many authors. See, for instance, [18], [19], [20], and the references therein.

Let $q_n^{(\alpha,\beta)}$ be the Jacobi-Sobolev orthonormal polynomials i.e.

$$q_n^{(\alpha,\beta)}(x) = (\|Q_n^{(\alpha,\beta)}\|_{S_2^{\alpha,\beta}})^{-1} Q_n^{(\alpha,\beta)}(x).$$

For $f \in S_1^{\alpha,\beta}$ the Fourier expansion in terms of Jacobi-Sobolev orthonormal polynomials is

$$\sum_{k=0}^{\infty} \hat{f}(k) q_k^{(\alpha,\beta)}(x), \tag{38}$$

where

$$\hat{f}(k) = \langle f, q_k^{(\alpha,\beta)} \rangle, \qquad k = 0,\ 1,\ \ldots\ .$$

Let $S_n f$ be the n-th partial sum of the expansion (38)

$$S_n(f,x) = \sum_{k=0}^{n} \hat{f}(k) q_k^{(\alpha,\beta)}(x).$$

Theorem 2. *Let $\alpha > -1/2$, $\alpha+1 \geq \beta \geq -1/2$, and $1 < p < \infty$. If there exists a constant $c > 0$ such that*

$$\|S_n f\|_{S_p^{\alpha,\beta}} \leq c\|f\|_{S_p^{\alpha,\beta}} \tag{39}$$

for every $f \in S_p^{\alpha,\beta}$ then $p \in (p_0, q_0)$.

Proof. For the proof, we apply the same argument as in [19]. Assume that (39) holds. Then

$$\|\langle f, q_n^{(\alpha,\beta)}\rangle q_n^{(\alpha,\beta)}(x)\|_{S_p^{\alpha,\beta}} = \|S_n f - S_{n-1} f\|_{S_p^{\alpha,\beta}} \leq 2c\|f\|_{S_p^{\alpha,\beta}}.$$

Therefore

$$\|q_n^{(\alpha,\beta)}(x)\|_{S_p^{\alpha,\beta}} \|q_n^{(\alpha,\beta)}(x)\|_{S_q^{\alpha}} < \infty, \tag{40}$$

where p is the conjugate of q.

On the other hand, from (24) we obtain the Sobolev norms of Jacobi-Sobolev orthonormal polynomials.

$$||q_n^{(\alpha,\beta)}||_{S_p^{\alpha,\beta}} \sim \begin{cases} c & \text{if } p < q_0, \\ \log\ n & \text{if } p = q_0, \\ n^{\alpha+3/2-\frac{2\alpha+4}{p}} & \text{if } p > q_0, \end{cases} \tag{41}$$

for $\alpha > -1/2$, $\alpha+1 \geq \beta \geq -1/2$, and $1 \leq p \leq \infty$. Now, from (41) it follows that the inequality (40) holds if and only if $p \in (p_0, q_0)$.

The proof of Theorem 2 is complete. □

6. Divergence almost everywhere

For $\lambda = 0$ and $\alpha = \beta = 0$, Pollard [21] showed that for each $p < 4/3$ there exists a function $f \in L^p(dx)$ such that its Fourier expansion (36) diverges a.e. on $[-1,1]$. Later on, Meaney [14] extended the result to $p = 4/3$. Furthermore, he proved that this is a special case of a divergence result for the Fourier expansion in terms of Jacobi polynomials. The failure of a.e. convergence of the Fourier expansions associated with systems of orthogonal polynomials on $[-1,1]$ and Bessel systems has been discussed in [8], [22].

If the sequence $\{S_n(f)\}_{n\geq 0}$ is uniformly bounded on a set, say E, of positive measure in $[-1,1]$, then

$$||\hat{f}(n)q_n^{(\alpha,\beta)}(x)||_{S_\infty^{\alpha,\beta},E} < c, \quad n \in \mathbf{N}, \quad x \in E.$$

Therefore,

$$|\hat{f}(n){q'}_n^{(\alpha,\beta)}(x)| < c, \quad n \in \mathbf{N},$$

a.e. on E. From Egorov's Theorem it follows that there is a subset $E_1 \subset E$ of positive measure such that

$$|\hat{f}(n){q'}_n^{(\alpha,\beta)}(x)| < c$$

uniformly for $x \in E_1$. On the other hand, from (23)

$$|\hat{f}(n)\left(\cos(k_1\theta + \gamma_1) + O(n^{-1})\right)| < c$$

uniformly for $\cos\theta \in E_1$. Using the Cantor-Lebesgue Theorem, as described in [15, Subsection 1.5] (see also [24, p.316]), we obtain

$$|\hat{f}(n)| < c. \tag{42}$$

Theorem 3. *Let $\alpha > -1/2$ and $\alpha + 1 \geq \beta \geq -1/2$. There is an $f \in S_p^{\alpha,\beta}$, $1 \leq p \leq p_0$, whose Fourier expansion* (38) *diverges almost everywhere on* $[-1,1]$ *in the norm of $S_\infty^{\alpha,\beta}$.*

Proof. Consider the linear functionals

$$T_n(f) = \hat{f}(n) = \langle f, q_n^{(\alpha,\beta)}\rangle$$

on $S_p^{\alpha,\beta}$, $1 \leq p \leq p_0$. By using [2, Theorem 3.8] we have

$$||T_n|| = \|q_n^{(\alpha,\beta)}\|_{S_p^{\alpha,\beta}}, \;\; q_0 \leq p \leq \infty.$$

Thus, from (41)

$$\sup_n ||T_n|| = \infty.$$

As a consequence of the Banach-Steinhaus theorem there exists $f \in S_p^{\alpha,\beta}$, $1 \leq p \leq p_0$, such that

$$\sup_n |T_n(f)| = \infty.$$

Since this result contradicts (42) then for this f the Fourier series diverges almost everywhere on $[-1,1]$ in the norm of $S_{\infty}^{\alpha,\beta}$. □

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