On semiclassical linear functionals of class $s = 2$. Classification and Integral representations

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Abstract

In this paper we obtain all the semiclassical linear functionals of class two taking into account the irreducible expression of the corresponding Pearson equation. We focus our attention in their integral representations. Thus, some linear functionals very well known in the literature, associated with perturbations of classical linear functionals, appear as well as new linear functionals which have not been studied as far as we know.

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1. INTRODUCTION

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. The elements of $\mathcal{P}'$ will be called either linear functionals or linear forms. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \geq 0$, the moments of u. For any linear functional u and any polynomial h let $Du = u'$, hu, δ_c, and $(x-c)^{-1}u$ be the linear functionals defined by $\langle u', f\rangle := -\langle u, f'\rangle$, $\langle hu, f\rangle := \langle u, hf\rangle$, $\langle \delta_c, f\rangle := f(c)$, and $\big\langle (x-c)^{-1}u, f\big\rangle := \langle u, \theta_c f\rangle$ where $\big(\theta_c f\big)(x) = \dfrac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $c, d \in \mathbb{C}, c \neq d$, and $u \in \mathcal{P}'$, we have [12]

$$(x-c)^{-1}((x-c)u) = u - (u)_0\delta_c, \tag{1.1}$$

$$(x-c)((x-c)^{-1}u) = u, \tag{1.2}$$

$$(x-c)^{-2}((x-c)^2u) = u - (u)_0\delta_c + ((u)_1 - c(u)_0)\delta_c', \tag{1.3}$$

$$((x-d)(x-c))^{-1}((x-d)(x-c)u) = u + \frac{1}{c-d}((u)_0(d\delta_c - c\delta_d) - (u)_1(\delta_c - \delta_d)). \tag{1.4}$$

In the sequel, we will denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ as well as $\mathbb{R}_+^* = \mathbb{R}_+\backslash\{0\}$, and $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.

The linear functional u is said to be regular (quasi-definite) if there exists a sequence $\{P_n\}_{n\geq 0}$ of polynomials with $\deg P_n = n, n \geq 0$, such that

$$\langle u, P_nP_m\rangle = r_n\delta_{n,m}\ , \quad n, m \geq 0\ , \quad r_n \neq 0\ , \quad n \geq 0\ .$$

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We can always assume that each P_n is monic i.e. $P_n(x) = x^n + lower \quad degree \quad terms$. Then the sequence $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to u (MOPS in short). It is a very well known fact that the sequence $\{P_n\}_{n\geq 0}$ satisfies a three term recurrence relation $\big($see, for instance, the monograph by T. S. Chihara [6]$\big)$

$$\begin{array}{l} P_{n+2}(x) = (x - \xi_{n+1})P_{n+1}(x) - \rho_{n+1}P_n(x) \; , \quad n \geq 0 \; , \\ P_1(x) = x - \xi_0 \; , \qquad P_0(x) = 1 \; , \end{array} \tag{1.5}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times \mathbb{C}^* \; , \quad n \geq 0$. By convention we set $\rho_0 = (u)_0 = 1$.

A linear functional u is said to be symmetric if $(u)_{2n+1} = 0, n \geq 0$. The conditions $(u)_{2n+1} = 0, n \geq 0$, are equivalent to the fact that the corresponding MOPS, $\{P_n\}_{n\geq 0}$, satisfies the recurrence relation (1.5) with $\xi_n = 0, n \geq 0$ (see [6]).

A regular linear functional u is said to be positive definite if $\langle u, f\rangle > 0$ for all $f \in \mathcal{P}$ such that $f(x) \geq 0$, for every $x \in \mathbb{R}$ and $f \neq 0$. Equivalently, its MOPS satisfies (1.5) with $\xi_n \in \mathbb{R}$ and $\rho_n \in \mathbb{R}_+^*$ for all n (see [14]).

From a structural point of view, a very important family of regular linear functionals has been exhaustively analyzed in the literature during the last two decades. Let us recall that a linear functional $\tilde{u}$ is called semiclassical when it is regular and there exist two polynomials $\tilde{\Phi}$, a monic polynomial, and $\tilde{\Psi}, \deg \tilde{\Psi} \geq 1$, such that

$$\Big(\tilde{\Phi}\tilde{u}\Big)' + \tilde{\Psi}\tilde{u} = 0 \; . \tag{1.6}$$

The class of the semiclassical linear functional $\tilde{u}$ is a nonnegative number $\tilde{s} = \max\big(\deg \tilde{\Psi} - 1, \deg \tilde{\Phi} - 2\big)$ if and only if the following condition is satisfied

$$\prod_c \Big(|\tilde{\Phi}'(c) + \tilde{\Psi}(c)| + \Big|\Big\langle \tilde{u}, \theta_c\tilde{\Psi} + \theta_c^2\tilde{\Phi}\Big\rangle\Big| \Big) > 0 \; , \tag{1.7}$$

where c belongs to the set of zeros of $\tilde{\Phi}$. Notice that this condition means that the equation (1.6) is irreducible (see [14]).

The corresponding MOPS $\{P_n\}_{n\geq 0}$ is said to be semiclassical of class $\tilde{s}$. When $\tilde{s} = 0$, $\tilde{u}$ is a classical linear functional (Hermite, Laguerre, Jacobi, and Bessel).

Semiclassical linear functionals associated with weight functions were considered first by J. Shohat [17] in the framework of the existence of sequences of orthogonal polynomials satisfying second order linear differential equations with polynomial coefficients (holonomic equations). Later on, P. Maroni and coworkers have extensively studied such a kind of linear functionals with a special emphasis on their structure properties. For instance, [14] constitutes a relevant survey on the subject. Many examples appearing in the literature, mainly related to spectral perturbations of classical linear functionals, are semiclassical but a constructive theory of such linear functionals remains open. On the other hand, complex path integral representations for semiclassical linear functionals in a more general framework have been studied in [11] and [12].

Indeed, the classification of semiclassical linear functionals according to some criteria of optimal information from the so-called Pearson equation, i.e., a first order linear differential equation satisfied by the linear functional, plays a central role in the constructive theory of such linear functionals. In [4], S. Belmehdi makes use of this approach to provide a full description of all semiclassical linear functionals of class $s = 1$. See also [1] for the classification of symmetric semiclassical linear functionals of such a class.

It is reasonable to deal with the description of the semiclassical linear functionals of class $s = 2$. Some particular examples of these functionals are known. Indeed, in [7], A. M. Delgado and F. Marcellán have deduced all the symmetric semiclassical linear functionals of this class . The aim of our contribution is to fulfil all the semiclassical linear functionals of class $s = 2$.

The structure of the manuscript is as follows. In Section 2, the irreducible canonical Pearson equations associated with semiclassical linear functionals of class $s = 2$ are obtained. Thus, we deduce the fourteen irreducible canonical cases. In Section 3, the integral representation of such linear functionals is given, with a special emphasis in the positive definite case.

2. Irreducible canonical functional equations

In the sequel, we will assume that the linear functional $\tilde{u}$ is semiclassical of class $\tilde{s} = 2$ and satisfies (1.6).

The semiclassical character of a linear functional is kept by shifting. Indeed, the shifted linear functional $u = (h_{a^{-1}} \circ \tau_{-b})\tilde{u}$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$, satisfies

$$(\Phi u)' + \Psi u = 0 \tag{2.1}$$

with

$$\Phi(x) = a^{-t}\tilde{\Phi}(ax + b) \ , \quad \Psi(x) = a^{1-t}\tilde{\Psi}(ax + b) \ , \quad t = deg\tilde{\Phi} \ . \tag{2.2}$$

The linear functionals $\tau_{-b}\tilde{u}$ $\big($ translation of $\tilde{u}$ $\big)$ and $h_a\tilde{u}$ $\big($ dilation of $\tilde{u}$ $\big)$ are defined by

$$\langle \tau_b\tilde{u}, f\rangle := \langle \tilde{u}, \tau_{-b}f\rangle := \langle \tilde{u}, f(x + b)\rangle \ , \quad \langle h_a\tilde{u}, f\rangle := \langle \tilde{u}, h_af\rangle := \langle \tilde{u}, f(ax)\rangle \ , \quad f \in \mathcal{P} \ . \tag{2.3}$$

The sequence $\{\hat{P}_n(x) = a^{-n}P_n(ax + b)\}_{n\geq 0}$ is orthogonal with respect to u and fulfils (1.5) with

$$\hat{\xi}_n = \frac{\xi_n - b}{a} \ , \quad \hat{\rho}_{n+1} = \frac{\rho_{n+1}}{a^2} \ , \quad n \geq 0 \ . \tag{2.4}$$

As we can see, a displacement $h_a \circ \tau_b$ does not modify the nature of a semiclassical linear functional. We shall apply this process to the distributional equation satisfied by a semiclassical linear functional of class $s = 2$. $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ are arbitrary complex numbers . A convenient choice of a and b, according to the expression of $\tilde{\Phi}$, allows us to re-locate the zeros of $\tilde{\Phi}$ in the complex plane. In this way, (2.1) can be reduced either to some situations which appear in the literature or lead to new linear functionals not yet studied. Thus we get canonical distributional equations of class two in a simple way, that becomes a pattern for the family of equations which can be reduced using a shifting.

We distinguish two situations:
A. deg $\tilde{\Phi} = 4$ and $1 \leq$ deg $\tilde{\Psi} \leq 3$,
B. deg $\tilde{\Psi} = 3$ and $0 \leq$ deg $\tilde{\Phi} \leq 3$.
Case A. Set
$\tilde{\Phi}(x) = \prod_{i=1}^{4}(x - \alpha_i)$,
$\tilde{\Psi} = \tilde{a}_3x^3 + \tilde{a}_2x^2 + \tilde{a}_1x + \tilde{a}_0$, $|\tilde{a}_3| + |\tilde{a}_2| + |\tilde{a}_1| \neq 0$.
Then u satisfies

$$\left(\prod_{i=1}^{4}\left(x - \frac{\alpha_i - b}{a}\right)u\right)' + \left(\tilde{a}_3x^3 + \frac{a^{-1}}{2}\tilde{\Psi}''(b)x^2 + a^{-2}\tilde{\Psi}'(b)x + a^{-3}\tilde{\Psi}(b)\right)u = 0. \tag{2.5}$$

We will discuss the following cases.
$\mathbf{A}_1$. $\tilde{\Phi}$ has four simple zeros.
We choose a and b such that $\alpha_1 - b = a$, $\alpha_2 - b = -a$, $\alpha_3 - b = ac$, and $\alpha_4 - b = ad$, with $c \neq \pm 1$, $d \neq \pm 1$ and $d \neq c$, in such a way that (2.5) reduces to

$$\big((x^2 - 1)(x - c)(x - d)u\big)' + (a_3x^3 + a_2x^2 + a_1x + a_0)u = 0. \tag{2.6}$$

The rational fraction $\dfrac{-\Phi'-\Psi}{\Phi}$ has four simple poles $1, -1, c$, and d. We denote by $\alpha = \dfrac{-\Phi'(1)-\Psi(1)}{\Phi'(1)}, \beta = \dfrac{-\Phi'(-1)-\Psi(-1)}{\Phi'(-1)}, \gamma = \dfrac{-\Phi'(c)-\Psi(c)}{\Phi'(c)}$ and $\rho = \dfrac{-\Phi'(d)-\Psi(d)}{\Phi'(d)}$ its residues. Then, after some straightforward calculations, we obtain

$$\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = M \begin{pmatrix} \alpha+1 \\ \beta+1 \\ \gamma+1 \\ \rho+1 \end{pmatrix} \text{ where } M = \begin{pmatrix} -1 & -1 & -1 & -1 \\ c+d+1 & c+d-1 & d & c \\ -c-d-cd & c+d-cd & 1 & 1 \\ cd & -cd & -d & -c \end{pmatrix}.$$

Notice that $\det M = 2(d-c)(c^2-1)(d^2-1) \neq 0$. This means that this change of parameters is bijective. (We have the same kind of relation between the new parameters and the old ones in the other cases we will study in the sequel)

Now, changing the parameters in (2.6), according to the condition of irreducibility (1.7) we get

$$\begin{cases} \big((x^2-1)(x-c)(x-d)u\big)' + \big(-(\alpha+\beta+\gamma+\rho+4)x^3 \\ \quad +(d(\alpha+\beta+\gamma+3)+c(\alpha+\beta+\rho+3)+\alpha-\beta)x^2 + (-cd(\alpha+\beta+2) \\ \quad\quad +(\beta-\alpha)(c+d)+\gamma+\rho+2)x - cd(\beta-\alpha) - c(\rho+1) - d(\gamma+1)\big)u = 0, \\ \big\{|\beta| + |-(\alpha+\beta+\gamma+\rho+3)(u)_2 + (d(\alpha+\beta+\gamma+2)+c(\alpha+\beta+\rho+2) \\ \quad -2\beta-\gamma-\rho-2)(u)_1 - cd(\alpha+\beta+1) + c(2\beta+\rho+1) + d(2\beta+\gamma+1) - 2\beta|\big\} \\ \times\big\{|\alpha| + |-(\alpha+\beta+\gamma+\rho+3)(u)_2 + (c(\alpha+\beta+\rho+2)+d(\alpha+\beta+\gamma+2)+ \\ \quad 2\alpha+\gamma+\rho+2)(u)_1 - cd(\alpha+\beta+1) - c(2\alpha+\rho+1) - d(2\alpha+\gamma+1) - 2\alpha|\big\} \\ \times\big\{|\gamma| + |-(\alpha+\beta+\gamma+\rho+3)(u)_2 + (d(\alpha+\beta+\gamma+2) \\ \quad +\alpha-\beta-c\gamma)(u)_1 + d(\beta-\alpha) + c\gamma(d-c) + \gamma+\rho+1|\big\} \\ \times\big\{|\rho| + |-(\alpha+\beta+\gamma+\rho+3)(u)_2 + (c(\alpha+\beta+\rho+2) \\ \quad -d\rho+\alpha-\beta)(u)_1 + c(\beta-\alpha) + d\rho(c-d) + \gamma+\rho+1|\big\} \neq 0. \end{cases} \tag{2.7}$$

This is the irreducible distributional equation of class two associated with the case when Φ in (2.1) has four simple zeros.
We will proceed in a similar way in the cases listed in below.

A$_2$. $\tilde{\Phi}$ has three different zeros and one of them is a double zero.
We choose a and b such that $\alpha_1 - b = \alpha_2 - b = 0$, $\alpha_3 - b = a$, and $\alpha_4 - b = ac$, $c \neq 0$ and $c \neq 1$, then (2.5) can be written as

$$\big(x^2(x-1)(x-c)u\big)' + (a_3x^3 + a_2x^2 + a_1x + a_0)u = 0. \tag{2.8}$$

By an appropriate choice of the coefficients a_3, a_2, a_1, and a_0, (2.8) becomes

$$\begin{cases} \big(x^2(x-1)(x-c)u\big)' + \big(-(\alpha+2\beta+\rho+5)x^3 + (c(\alpha+2\beta+4) \\ \quad\quad +2\beta+\gamma+\rho+4)x^2 - (c(2\beta+\gamma+3)+\gamma)x + c\gamma\big)u = 0, \\ \big\{|\gamma| + |-(\alpha+2\beta+\rho+4)(u)_2 + (c(\alpha+2\beta+3) \\ \quad\quad +2\beta+\gamma+\rho+3)(u)_1 - c(2\beta+\gamma+2) - \gamma|\big\} \\ \times\big\{|\alpha| + |-(\alpha+2\beta+\rho+4)(u)_2 + (c(\alpha+2\beta+3)+\gamma-\alpha)(u)_1 + c(\alpha-\gamma) - \alpha|\big\} \\ \times\big\{|\rho| + |-(\alpha+2\beta+\rho+4)(u)_2 + (\rho(1-c)+2\beta+\gamma+3)(u)_1 + c\rho(1-c) - \gamma|\big\} \neq 0. \end{cases} \tag{2.9}$$

This is the corresponding irreducible distributional equation of class two, when Φ in (2.1) has two simple zeros and another one of multiplicity two.

A$_3$. $\tilde{\Phi}$ has two different zeros. This means that either $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$, or $\alpha_1 = \alpha_2 = \alpha_3 \neq \alpha_4$.

We have to consider two subcases.

$\mathbf{A}_{3.1}$. If we choose a and b such that $\alpha_1 - b = \alpha_2 - b = a$ and $\alpha_3 - b = \alpha_4 - b = -a$, then (2.5) becomes

$$\big((x^2-1)^2u\big)' + (a_3x^3 + a_2x^2 + a_1x + a_0)u = 0. \tag{2.10}$$

From a suitable choice of the coefficients a_3, a_2, a_1, and a_0, (2.10) yields

$$\begin{cases} \big((x^2-1)^2u\big)' + \big(-(\alpha+\beta+4)x^3 + (\beta+\gamma+\rho-\alpha)x^2 \\ \qquad\qquad +(\alpha+\beta+2\gamma-2\rho+4)x + \alpha+\gamma+\rho-\beta\big)u = 0, \\ \big\{|\gamma| + |-(\alpha+\beta+3)(u)_2 + (\gamma+\rho-2\alpha-2)(u)_1 + \beta+3\gamma-\alpha-\rho+1|\big\} \\ \times\big\{|\rho| + |-(\alpha+\beta+3)(u)_2 + (2\beta+\gamma+\rho+2)(u)_1 + \alpha+\gamma-\beta-3\rho+1|\big\} \neq 0. \end{cases} \tag{2.11}$$

This is an irreducible distributional equation of class two, when Φ in (2.1) has two different double zeros.

$\mathbf{A}_{3.2}$. Choosing a and b such that $b = \alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_4 - b = a$, then (2.5) becomes

$$\big(x^3(x-1)u\big)' + (a_3x^3 + a_2x^2 + a_1x + a_0)u = 0. \tag{2.12}$$

An appropriate choice of the coefficients a_3, a_2, a_1, and a_0, reduces (2.12) to

$$\begin{cases} \big(x^3(x-1)u\big)' + \big(-(\alpha+\beta+2)x^3 + (\beta+\gamma+1)x^2 + (2\lambda-\gamma)x - 2\lambda\big)u = 0, \\ \big\{|\lambda| + |-(\alpha+\beta+1)(u)_2 + (\beta+\gamma)(u)_1 + 2\lambda-\gamma|\big\} \\ \times\big\{|\alpha| + |-(\alpha+\beta+1)(u)_2 - (\alpha-\gamma)(u)_1 + 2\lambda-\alpha|\big\} \neq 0. \end{cases} \tag{2.13}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) has two different zeros: one is simple and the other one has multiplicity three.

$\mathbf{A}_4$. $\tilde{\Phi}$ has one zero, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$.
If we choose b such that $\alpha_1 - b = 0$, then the equation (2.5) becomes

$$(x^4u)' + \left(\tilde{a}_3x^3 + \frac{a^{-1}}{2}\tilde{\Psi}''(\alpha_1)x^2 + a^{-2}\tilde{\Psi}'(\alpha_1)x + a^{-3}\tilde{\Psi}(\alpha_1)\right)u = 0. \tag{2.14}$$

We can consider three subcases.
$\mathbf{A}_{4.1}$. $\tilde{\Psi}(\alpha_1) \neq 0$.
Let a be such that $a^{-3}\tilde{\Psi}(\alpha_1) = 3$. Then (2.14) reduces to

$$(x^4u)' + (a_3x^3 + a_2x^2 + a_1x + 3)u = 0. \tag{2.15}$$

By a skilful choice of the coefficients a_3, a_2, and a_1, (2.15) can be written

$$(x^4u)' + \big(-(\alpha+4)x^3 + \beta x^2 - 2\gamma x + 3\big)u = 0, \tag{2.16}$$

since $\Phi'(0) + \Psi(0) = 3 \neq 0$, then this is the irreducible distributional equation of class two, when Φ in (2.1) has a zero of multiplicity four and Ψ doesn't vanish at this zero.

$\mathbf{A}_{4.2}$. $\tilde{\Psi}(\alpha_1) = 0$ and $\tilde{\Psi}'(\alpha_1) \neq 0$.
If we choose a such that $a^{-2}\tilde{\Psi}'(\alpha_1) = -4$, then we obtain

$$(x^4u)' + (a_3x^3 + a_2x^2 - 4x)u = 0. \tag{2.17}$$

By an appropriate choice of the coefficients a_3 and a_2, (2.17) reads as

$$\begin{cases} (x^4u)' + \big(-(4\alpha-1)x^3 + 2\beta x^2 - 4x\big)u = 0, \\ (2\alpha-1)(u)_2 - \beta(u)_1 + 2 \neq 0. \end{cases} \tag{2.18}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) has a zero of multiplicity four and this zero is a simple zero of Ψ.

$\mathbf{A}_{4.3}$. $\tilde{\Psi}(\alpha_1) = \tilde{\Psi}'(\alpha_1) = 0$ and $\tilde{\Psi}''(\alpha_1) \neq 0$.
Let a be such that $a^{-1}\tilde{\Psi}''(\alpha_1) = -4$, then we have

$$(x^4 u)' + (a_3 x^3 - 2x^2)u = 0. \tag{2.19}$$

By a suitable choice of the coefficient a_3 (2.19) becomes

$$\begin{cases} (x^4 u)' + \big(-2(\alpha x + 1)x^2\big)u = 0, \\ (-2\alpha + 1)(u)_2 - 2(u)_1 \neq 0. \end{cases} \tag{2.20}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) has a zero of multiplicity four and it is a double zero of Ψ.

Case B. Set
$\tilde{\Phi}(x) = \tilde{c}_3 x^3 + \tilde{c}_2 x^2 + \tilde{c}_1 x + \tilde{c}_0, \quad |\tilde{c}_3| + |\tilde{c}_2| + |\tilde{c}_1| + |\tilde{c}_0| \neq 0,$
$\tilde{\Psi} = \tilde{a}_3 x^3 + \tilde{a}_2 x^2 + \tilde{a}_1 x + \tilde{a}_0$, $\tilde{a}_3 \neq 0$.
We will analyze the following situations:
$\mathbf{B}_1$. $\tilde{\Phi}$ has three simple zeros α_1, α_2, and α_3.
$\tilde{u}$ and u satisfy, respectively,

$$\left(\prod_{i=1}^{3}(x - \alpha_i)\tilde{u}\right)' + (\tilde{a}_3 x^3 + \tilde{a}_2 x^2 + \tilde{a}_1 x + \tilde{a}_0)\tilde{u} = 0,$$

$$\left(\prod_{i=1}^{3}\left(x - \frac{\alpha_i - b}{a}\right)u\right)' + \left(\tilde{a}_3 a x^3 + \frac{1}{2}\tilde{\Psi}''(b)x^2 + a^{-1}\tilde{\Psi}'(b)x + a^{-2}\tilde{\Psi}(b)\right)u = 0. \tag{2.21}$$

Let a and b be such that $\alpha_1 - b = a$, $\alpha_2 - b = -a$, and $\alpha_3 - b = ac$. Then (2.21) becomes

$$\big((x^2 - 1)(x - c)u\big)' + (a_3 x^3 + a_2 x^2 + a_1 x + a_0)u = 0. \tag{2.22}$$

By a skilful choice of the coefficients a_3, a_2, a_1, and a_0, we get

$$\begin{cases} \big((x^2 - 1)(x - c)u\big)' + \big(\lambda x^3 - (\lambda c + \alpha + \beta + \gamma + 3)x^2 \\ \qquad + (c(\alpha + \beta + 2) + \beta - \alpha - \lambda)x + \gamma + c(\alpha + \lambda - \beta) + 1\big)u = 0, \\ \big\{|\alpha| + |\lambda(u)_2 - (\lambda(c - 1) + \alpha + \beta + \gamma + 2)(u)_1 + c(\alpha + \beta - \lambda + 1) - 2\alpha - \gamma - 1|\big\} \\ \times\big\{|\beta| + |\lambda(u)_2 - (\lambda(c + 1) + \alpha + \beta + \gamma + 2)(u)_1 + c(\alpha + \beta + \lambda + 1) + 2\beta + \gamma + 1|\big\} \\ \times\big\{|\gamma| + |\lambda(u)_2 - (\alpha + \beta + \gamma + 2)(u)_1 + \beta - \alpha - \lambda - c\gamma|\big\} \neq 0. \end{cases} \tag{2.23}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) has three simple zeros.

$\mathbf{B}_2$. $\tilde{\Phi}$ has two different zeros and one of them has multiplicity two i.e. $\alpha_1 = \alpha_2 \neq \alpha_3$.
We choose a and b such that $\alpha_1 - b = \alpha_2 - b = 0$ and $\alpha_3 - b = a$, then (2.21) becomes

$$\Big(x^2(x - 1)u\Big)' + (a_3 x^3 + a_2 x^2 + a_1 x + a_0)u = 0. \tag{2.24}$$

An appropriate choice of the coefficients a_3, a_2, a_1, and a_0, transforms (2.24) in

$$\begin{cases} \big(x^2(x - 1)u\big)' + \big(\lambda x^3 - (\alpha + \beta + \lambda + 2)x^2 + (\beta + 2\gamma + 1)x - 2\gamma\big)u = 0, \\ \big\{|\gamma| + |\lambda(u)_2 - (\alpha + \beta + \lambda + 1)(u)_1 - (\beta + 2\gamma)|\big\} \\ \times\big\{|\alpha| + |\lambda(u)_2 - (\alpha + \beta + 1)(u)_1 + 2\gamma - \alpha|\big\} \neq 0. \end{cases} \tag{2.25}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) has two different zeros: one is simple and the other one has multiplicity two.

B$_3$. $\tilde{\Phi}$ has one triple zero $\alpha_1 = \alpha_2 = \alpha_3$.
We choose a and b in such a way that $\alpha_1 = b$ and $\tilde{a}_3 a = 1$. Then (2.21) reads as

$$(x^3 u)' + (x^3 + a_2 x^2 + a_1 x + a_0)u = 0. \tag{2.26}$$

By a suitable choice of the coefficients a_2, a_1, and a_0, (2.26) can be written as

$$\begin{cases} (x^3 u)' + \big(x^3 - (\alpha+2)x^2 + \beta x + 2\gamma\big)u = 0, \\ |\gamma| + |(u)_2 - (\alpha+1)(u)_1 + \beta| \neq 0. \end{cases} \tag{2.27}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) has a triple zero.

B$_4$. $\tilde{\Phi}$ is a quadratic polynomial
$\tilde{\Phi}(x) = (x - \alpha_1)(x - \alpha_2)$,
$\tilde{\Psi} = \tilde{a}_3 x^3 + \tilde{a}_2 x^2 + \tilde{a}_1 x + \tilde{a}_0$, $\tilde{a}_3 \neq 0$.
We need to discuss the following situations:
B$_{4.1}$. $\tilde{\Phi}$ has two simple zeros, $\alpha_1 \neq \alpha_2$.
$\tilde{u}$ and u satisfy, respectively,

$$\big((x-\alpha_1)(x-\alpha_2)\tilde{u}\big)' + (\tilde{a}_3 x^3 + \tilde{a}_2 x^2 + \tilde{a}_1 x + \tilde{a}_0)\tilde{u} = 0,$$

$$\left(\prod_{i=1}^{2}\left(x - \frac{\alpha_i - b}{a}\right)u\right)' + \left(a^2\tilde{a}_3 x^3 + \frac{a}{2}\tilde{\Psi}"(b)x^2 + \tilde{\Psi}'(b)x + a^{-1}\tilde{\Psi}(b)\right)u = 0. \tag{2.28}$$

If we choose a and b such that $\alpha_1 - b = a$ and $\alpha_2 - b = -a$, then (2.28) can be written as

$$\big((x^2-1)u\big)' + (a_3 x^3 + a_2 x^2 + a_1 x + a_0)u = 0. \tag{2.29}$$

By an appropriate choice of the coefficients a_3, a_2, a_1, and a_0, (2.29) becomes

$$\begin{cases} \big((x^2-1)u\big)' + \big(2\lambda x^3 + \alpha x^2 - (2\lambda+\beta+\gamma+2)x - (\alpha+\beta-\gamma)\big)u = 0, \\ \big\{|\beta| + |2\lambda(u)_2 + (2\lambda+\alpha)(u)_1 + \alpha - \beta - \gamma - 1|\big\} \\ \times\big\{|\gamma| + |2\lambda(u)_2 + (\alpha - 2\lambda)(u)_1 - (\alpha+\beta+\gamma+1)|\big\} \neq 0. \end{cases} \tag{2.30}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) is a quadratic polynomial with two simple zeros.

B$_{4.2}$. $\tilde{\Phi}$ has a double zero, $\alpha_1 = \alpha_2$.
If we choose a and b such that $\alpha_1 = b$ and $a^2\tilde{a}_3 = 2$. Thus we obtain

$$(x^2 u)' + (2x^3 + a_2 x^2 + a_1 x + a_0)u = 0. \tag{2.31}$$

By a skilful choice of the coefficients a_2, a_1, and a_0, (2.31) becomes

$$\begin{cases} (x^2 u)' + \big(2x^3 - 2\lambda x^2 - (2\alpha+3)x + \beta\big)u = 0, \\ |\beta| + |(u)_2 - \lambda(u)_1 - \alpha - 1| \neq 0. \end{cases} \tag{2.32}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) is a quadratic polynomial with a double zero.

$\mathbf{B}_5$. $\tilde{\Phi}(x) = x - \alpha_1$
$\tilde{u}$ and u satisfy, respectively

$$\big((x-\alpha_1)\tilde{u}\big)' + (\tilde{a}_3x^3 + \tilde{a}_2x^2 + \tilde{a}_1x + \tilde{a}_0)\tilde{u} = 0,$$

$$\left(\left(x - \frac{\alpha_1 - b}{a}\right)u\right)' + \left(a^3\tilde{a}_3x^3 + \frac{a^2}{2}\tilde{\Psi}''(b)x^2 + a\tilde{\Psi}'(b)x + \tilde{\Psi}(b)\right)u = 0. \tag{2.33}$$

If we choose a and b such that $\alpha_1 = b$ and $a^3\tilde{a}_3 = 3\lambda \neq 0$, then we have

$$(xu)' + (3\lambda x^3 + a_2x^2 + a_1x + a_0)u = 0. \tag{2.34}$$

By an appropriate choice of the coefficients a_2, a_1, and a_0, (2.34) reads as

$$\begin{cases} (xu)' + (3\lambda x^3 + 2\beta x^2 + \alpha x - \gamma - 1)u = 0, \\ |\gamma| + |3\lambda(u)_2 + 2\beta(u)_1 + \alpha| \neq 0. \end{cases} \tag{2.35}$$

This is the irreducible distributional equation of class two, when Φ in (2.1) is a linear polynomial.

$\mathbf{B}_6$. $\tilde{\Phi}$ is a constant.
After the displacement, we get

$$u' + \big(a^4\tilde{a}_3x^3 + a^3\tilde{\Psi}''(b)x^2 + a^2\tilde{\Psi}'(b)x + a\tilde{\Psi}(b)\big)u = 0. \tag{2.36}$$

If we choose a and b such that $a^4\tilde{a}_3 = 4$ and $a^3\tilde{\Psi}''(b) = 3\alpha$, then (2.36) can be written as

$$u' + (4x^3 + 3\alpha x^2 + 2\lambda x + \beta)u = 0. \tag{2.37}$$

Since Φ is a non zero constant, condition (1.7) is satisfied, and (2.37) is the irreducible distributional equation of class two, when Φ in (2.1) is a constant.

3. Integral representation of semiclassical linear functionals of class two

Let u be a semiclassical linear functional of class s satisfying (2.1) and let assume $(u)_0 = 1$.
Our aim will be to obtain an integral representation of u

$$\langle u, f(x)\rangle = \int_{\mathcal{C}} f(x)U(x)\,dx, \quad (u)_0 = \int_{\mathcal{C}} U(x)\,dx = 1. \tag{3.1}$$

Notice that in the above representation there are two different elements, which must be chosen as circumstances demand. Namely,

(i) The function $U(x)$, which is known as a generalized weight function of u.

(ii) The complex path of integration $\mathcal{C}$.

In order that (3.1) yields the representation of a solution of the equation (2.1) the following conditions must hold (see [11], [12], and [14])

$$\big(\Phi(x)U(x)\big)' + \Psi(x)U(x) = 0, \tag{3.2}$$

$$\Phi(x)U(x)f(x)|_{\mathcal{C}} = 0, \quad f \in \mathcal{P}. \tag{3.3}$$

We will consider the above fourteen canonical functional equations and, in each case, an integral representation of the corresponding linear functionals will be given.

Case A.

$\mathbf{A}_1$. The equation (2.7) is equivalent to the fact that the moments $(u)_n$ of the linear functional u satisfy the linear difference equation

$$\begin{aligned}-(\alpha+\beta+\gamma+\rho+n+4)(u)_{n+3}+[d(\alpha+\beta+\gamma+n+3)+c(\alpha+\beta+\rho+n+3)+\alpha-\beta](u)_{n+2}\\ +[(\beta-\alpha)(c+d)-cd(\alpha+\beta+n+2)+\gamma+\rho+n+2](u)_{n+1}\\ -[cd(\beta-\alpha+c(\rho+n+1)+d(\gamma+n+1)](u)_n+ncd(u)_{n-1}=0,\ n\geq 0,\end{aligned} \quad (3.4)$$

with $(u)_{-1}=0$ and $(u)_0=1$. Then the set of solutions is a linear space of dimension at most three.
On the other hand from (2.7)-(3.2) we get

$$\begin{aligned}\big((x^2-1)(x-c)(x-d)U(x)\big)'+\big(-(\alpha+\beta+\gamma+\rho+4)x^3\\ +(d(\alpha+\beta+\gamma+3)+c(\alpha+\beta+\rho+3)+\alpha-\beta)x^2+(-cd(\alpha+\beta+2)\\ +(\beta-\alpha)(c+d)+\gamma+\rho+2)x-cd(\beta-\alpha)-c(\rho+1)-d(\gamma+1)\big)U(x)=0.\end{aligned}$$

Thus,

$$\frac{U'(x)}{U(x)}=\frac{\beta}{x-1}+\frac{\alpha}{x+1}+\frac{\gamma}{x-c}+\frac{\rho}{x-d},$$

and, as a consequence,

$$U(x)=(1-x)^{\beta}(x+1)^{\alpha}|c-x|^{\gamma}|x-d|^{\rho} \quad (3.5)$$

is the solution at some intervals depending on c and d (see below).
On the other hand, if $\alpha\beta\gamma\rho\neq 0$, α, β, γ, $\rho>-1$, then the conditions (1.7) and (3.3) hold in the following situations:
$\mathbf{s}_1$: c, $d\in]-1,1[$, $c<d$. Then, u is represented by

$$\langle u,f\rangle=\int_{-1}^{1}f(x)(1-x)^{\beta}(x+1)^{\alpha}|c-x|^{\gamma}|x-d|^{\rho}\left(A\chi_{[-1,c]}(x)+B\chi_{[c,d]}(x)+C\chi_{[d,1]}(x)\right)dx, \quad (3.6)$$

since from an integration by parts we deduce that the linear functionals u_1, u_2, and u_3 such that

$$\langle u_1,f\rangle=\int_{-1}^{c}U(x)f(x)dx,\ \langle u_2,f\rangle=\int_{c}^{d}U(x)f(x)dx,\text{ and }\langle u_3,f\rangle=\int_{d}^{1}U(x)f(x)dx,$$

are solutions of (2.7). The same is true for any constants A , B, and C.

$$\langle Au_1+Bu_2+Cu_3,f\rangle=\int_{-1}^{1}U(x)(A\chi_{[-1,c]}+B\chi_{[c,d]}+C\chi_{[d,1]})f(x)dx$$

where $\chi_{[a,b]}(x)=1$ when $x\in[a,b]$ and zero otherwise (A,B, and C will be chosen in such a way that the linear functional is normalized, i.e. $(u)_0=1$).

We will proceed in a similar way in the cases below.
$\mathbf{s}_2$: $c<-1$ and $d>1$. u is represented by

$$\langle u,f\rangle=\int_{c}^{d}f(x)|1-x|^{\beta}|x+1|^{\alpha}(x-c)^{\gamma}(d-x)^{\rho}\left(A_1\chi_{[c,-1]}(x)+B_1\chi_{[-1,1]}(x)+C_1\chi_{[1,d]}(x)\right)dx. \quad (3.7)$$

$\mathbf{s}_3$: c, $d>1$ and $c<d$. u is represented by

$$\langle u,f\rangle=\int_{-1}^{d}f(x)|1-x|^{\beta}(x+1)^{\alpha}|x-c|^{\gamma}(d-x)^{\rho}\left(A_2\chi_{[-1,1]}(x)+B_2\chi_{[1,c]}(x)+C_2\chi_{[c,d]}(x)\right)dx. \quad (3.8)$$

s$_4$: $c,\ d < -1$ and $c < d$. u is represented by

$$\langle u, f\rangle = \int_c^1 f(x)(1-x)^{\beta}|x+1|^{\alpha}(x-c)^{\gamma}|d-x|^{\rho}\left(A_3\chi_{[c,d]}(x) + B_3\chi_{[d,-1]}(x) + C_3\chi_{[-1,1]}(x)\right)\,dx. \quad (3.9)$$

Remark 1. The condition $\alpha\beta\gamma\rho \neq 0$ is simple and sufficient to ensure that the condition of irreducibility (1.7) is satisfied (see second equation in (2.7)). Indeed, for every parameter $\alpha,\ \beta,\ \gamma,\ \rho > -1$, satisfying (2.7), we obtain the integral representations given above.

Particular case:

If $\gamma = \rho$, $\alpha = \beta$ and $c = -d$, then from (3.4) we have

$$(\beta+\gamma+2)(u)_3 - [d^2(\beta+1)+\gamma+1](u)_1 = 0,$$

$$(\beta+\gamma+n+2)(u)_{2n+3} - [d^2(\beta+n+1)+\gamma+n+1](u)_{2n+1} + nd^2(u)_{2n-1} = 0, \quad , n \geq 1.$$

Thus, if $(u)_1 = 0$, we obtain $(u)_{2n+1} = 0,\ n \geq 0$. Then u is a symmetric linear functional of class two and satisfies (see [7])

$$\begin{cases} \left((x^2-1)(x^2-d^2)u\right)' + 2x\left(-(\beta+\gamma+2)x^2 + d^2(\beta+1)+\gamma+1\right)u = 0, \\ \left\{|\beta| + |-(2\beta+2\gamma+3)(u)_2 + d^2(2\beta+1)|\right\} \\ \times\left\{|\gamma| + |-(2\beta+2\gamma+3)(u)_2 + 2\gamma(1-d^2)+1|\right\} \neq 0. \end{cases}$$

If $d \in]0,1[$, then we obtain

$$\langle u, f\rangle = \int_{-1}^1 f(x)(1-x^2)^{\beta}|x^2-d^2|^{\gamma}\left(A + (B-A)\chi_{[-d,d]}(x)\right)dx, \quad (3.10)$$

and if $d > 1$,then we have

$$\langle u, f\rangle = \int_{-d}^d f(x)|1-x^2|^{\beta}(d^2-x^2)^{\gamma}\left(A_1 + (B_1-A_1)\chi_{[-1,1]}(x)\right)dx. \quad (3.11)$$

In the other cases, we will proceed following the same steps and the techniques described above.

A$_2$. From (2.9),

$$-(\alpha+2\beta+\rho+n+5)(u)_{n+3} + [c(\alpha+2\beta+n+4)+2\beta+\gamma+\rho+n+4](u)_{n+2} - [c(2\beta+\gamma+n+3)+\gamma](u)_{n+1} + c\gamma(u)_n = 0,\ n \geq 0, \quad (3.12)$$

with $(u)_0 = 1$. Then the set of solutions is a linear space of dimension at most three.

On the other hand, from (2.9)-(3.2) we get

$$\frac{U'}{U} = \frac{2\beta+1}{x} - \frac{\gamma}{x^2} + \frac{\alpha}{x-1} + \frac{\rho}{x-c},$$

and, as a consequence,

$$U(x) = x^{2\beta+1}(1-x)^{\alpha}|x-c|^{\rho}e^{\frac{\gamma}{x}}, \quad (3.13)$$

is the solution at some intervals depending on c.

For instance, if $\alpha\gamma\rho \neq 0,\ \alpha,\ \beta,\ \rho > -1$, the conditions (1.7) and (3.3) hold in the following situations:

s$_1$: $c \in]0,1[$ and $\gamma < 0$. u is represented by

$$\langle u, f\rangle = \int_0^1 f(x)x^{2\beta+1}(1-x)^{\alpha}|x-c|^{\rho}e^{\frac{\gamma}{x}}\left(A\chi_{[0,c]}(x) + B\chi_{[c,1]}(x)\right)dx. \quad (3.14)$$

s$_2$: $c > 1$ and $\gamma < 0$. u is represented by

$$\langle u, f\rangle = \int_0^c f(x)x^{2\beta+1}|1-x|^{\alpha}(c-x)^{\rho}e^{\frac{\gamma}{x}}\left(A_1\chi_{[0,1]}(x) + B_1\chi_{[1,c]}(x)\right)dx. \quad (3.15)$$

$\mathbf{s}_3$: $c<0$ and $\gamma>0$. u is represented by

$$\langle u, f\rangle = \int_c^0 f(x)x^{2\beta+1}(1-x)^{\alpha}(x-c)^{\rho}e^{\frac{\gamma}{x}}\,dx. \tag{3.16}$$

Remark 2. For every parameter α, β, $\rho>-1$, and γ such that (2.9) is verified, we also obtain the integral representations given above, except when $\gamma=0$. For this case see below. (Since in some other cases, we have similar comments to Remark 1. or Remark 2., we will not present them in the sequel.)

Particular cases:

- If $\gamma=0$ and $\rho=1$, then (2.9) becomes

$$\begin{cases} \big(x^2(x-1)(x-c)u\big)' + \big(x(x-c)(-(\alpha+2\beta+2)x+2\beta+1) - 2(2x-c)(x^2-x)\big)u = 0, \\ \big(-(\alpha+2\beta+5)(u)_2 + (c(\alpha+2\beta+3)+2\beta+4)(u)_1 - 2c(\beta+1)\big) \\ \times\big\{|\alpha| + |-(\alpha+2\beta+5)(u)_2 + (c(\alpha+2\beta+3)-\alpha)(u)_1 + (c-1)\alpha|\big\} \neq 0. \end{cases}$$

Hence

$$u = kx(x-c)(h_{\frac{1}{2}}\circ\tau_1)\mathcal{J}(2\beta,\alpha), \tag{3.17}$$

where k is a normalization term and $\mathcal{J}(a,b)$ is the Jacobi linear functional satisfying [14]

$$\big((x^2-1)\mathcal{J}(a,b)\big)' + \big(-(a+b+2)x+a-b\big)\mathcal{J}(a,b) = 0,$$

and $a+1\neq -n$, $b+1\neq -n$, $a+b+2\neq -n$, $n\geq 0$.

Notice that the linear functional u defined by (3.17) is regular if and only if the MOPS $\{P_n^{(2\beta,\alpha)}\}_{n\geq 0}$ satisfies (see [8])

$$P_n^{(2\beta,\alpha)}(0)P_{n+1}^{(2\beta,\alpha)}(c) - P_n^{(2\beta,\alpha)}(c)P_{n+1}^{(2\beta,\alpha)}(0) \neq 0,\ n\geq 0.$$

Here $\{P_n^{(2\beta,\alpha)}\}_{n\geq 0}$ is the MOPS with respect to $(h_{\frac{1}{2}}\circ\tau_1)\mathcal{J}(2\beta,\alpha)$.

- When $\gamma=0$ and $\beta=-\frac{1}{2}$, then (2.9) becomes

$$\begin{cases} \big(x^2(x-1)(x-c)u\big)' + \big(-(\alpha+\rho+4)x^3 + (c(\alpha+3)+\rho+3)x^2 - 2cx\big)u = 0, \\ \big(-(\alpha+\rho+3)(u)_2 + (c(\alpha+2)+\rho+2)(u)_1 - c\big) \\ \times\big\{|\alpha| + |-(\alpha+\rho+3)(u)_2 + (c(\alpha+2)-\alpha)(u)_1 + (c-1)\alpha|\big\} \\ \times\big\{|\rho| + |-(\alpha+\rho+3)(u)_2 + (\rho(1-c)+2)(u)_1 + c\rho(1-c)|\big\} \neq 0. \end{cases}$$

Therefore

$$u = (1-k)(h_{\frac{c-1}{2}}\circ\tau_{\frac{c+1}{c-1}})\mathcal{J}(\alpha,\rho) + k\delta_0. \tag{3.18}$$

The linear functional u defined by (3.18) is regular if and only if k satisfies (see [10])

$$1 + k\sum_{m=0}^{n} \frac{(P_m^{(\alpha,\rho)})^2(0)}{\Big\langle (h_{\frac{c-1}{2}}\circ\tau_{\frac{c+1}{c-1}})\mathcal{J}(\alpha,\rho), (P_m^{(\alpha,\rho)})^2 \Big\rangle} \neq 0,\ n\geq 0,$$

where $\{P_n^{(\alpha,\rho)}\}_{n\geq 0}$ is the MOPS with respect to $(h_{\frac{c-1}{2}}\circ\tau_{\frac{c+1}{c-1}})\mathcal{J}(\alpha,\rho)$.

In this case,

$$\langle u,\, f\rangle = \Lambda\int_c^1 (1-x)^{\alpha}(x-c)^{\rho}f(x)\,dx + kf(0),\ \ c\in]-\infty,1[\backslash\{0\}$$

or

$$\langle u,\, f\rangle = \Lambda\int_1^c (x-1)^{\alpha}(c-x)^{\rho}f(x)\,dx + kf(0),\ \ c\in]1,+\infty[$$

where

$$\Lambda = (1-k)\frac{\Gamma(\alpha+\rho+2)}{|c-1|^{\alpha+\rho+1}\Gamma(\alpha+1)\Gamma(\rho+1)},\ \alpha,\,\rho > -1.$$

• If $\gamma = 0$ and $\rho = -1$, then (2.9) becomes

$$\begin{cases} \big(x^2(x-1)(x-c)u\big)' + x(x-c)\big(-(\alpha++2\beta+4)x+2\beta+3\big)u = 0, \\ \big(-(\alpha+2\beta+3)(u)_2 + (c(\alpha+2\beta+3)+2\beta+2)(u)_1 - 2c(\beta+1)\big) \\ \times\big\{|\alpha| + |-(\alpha+2\beta+3)(u)_2 + (c(\alpha+2\beta+3)-\alpha)(u)_1 + (c-1)\alpha|\big\} \neq 0. \end{cases}$$

Hence

$$x(x-c)u = k(h_{\frac{1}{2}} \circ \tau_1)\mathcal{J}(2\beta+2,\alpha). \tag{3.19}$$

The linear functional u defined by (3.19) is regular if and only if the MOPS $\{P_n^{(2\beta+2,\alpha)}\}_{n\geq 0}$ satisfies (see [5])

$$(u)_1(c-(u)_1) \neq 0,$$

$$\begin{vmatrix} P_{n+1}^{(2\beta+2,\alpha)}(0\,,-\frac{1}{(u)_1-c}) & P_n^{(2\beta+2,\alpha)}(0\,,-\frac{1}{(u)_1-c}) \\ P_{n+1}^{(2\beta+2,\alpha)}(c\,,-\frac{1}{(u)_1}) & P_n^{(2\beta+2,\alpha)}(c\,,-\frac{1}{(u)_1}) \end{vmatrix} \neq 0,\ n \geq 0,$$

where $\{P_n^{(2\beta+2,\alpha)}\}_{n\geq 0}$ is the MOPS with respect to $(h_{\frac{1}{2}} \circ \tau_1)\mathcal{J}(2\beta+2,\alpha)$ and $\{P_n(.,\mu)\}_{n\geq 0}$ is the co-recursive MOPS of $\{P_n\}_{n\geq 0}$ satisfying (see [6])

$$P_n(x,\mu) = P_n(x) - \mu P_{n-1}^{(1)}(x),$$

where $\{P_n^{(1)}\}_{n\geq 0}$ is the sequence of associated polynomials of the first kind for the sequence $\{P_n\}_{n\geq 0}$. (see [14])

From (1.4) and (3.19), we obtain

$$u = k(x(x-c))^{-1}(h_{\frac{1}{2}} \circ \tau_1)\mathcal{J}(2\beta+2,\alpha) + \delta_0 + \frac{(u)_1}{c}(\delta_c - \delta_0),$$

then $k = (u)_2 - c(u)_1$.

• If $\gamma = 0$, $\alpha = \rho$ and $c = -1$, then from (3.12) we obtain for $n \geq 0$

$$(2\alpha+2\beta+2n+5)(u)_{2n+3} - (2\beta+\alpha+2n+3)(u)_{2n+1} = 0,$$

$$(\alpha+\beta+n+3)(u)_{2n+4} - (\beta+n+2)(u)_{2n+2} = 0.$$

Thus, if $(u)_1 = 0$ and $(u)_2 \neq 0$ we obtain by induction

$$(u)_{2n+1} = 0, \quad (u)_{2n+2} = \frac{\Gamma(\alpha+\beta+3)\Gamma(n+\beta+2)}{\Gamma(\beta+2)\Gamma(n+\alpha+\beta+3)}(u)_2, \quad n \geq 0.$$

Then u is a symmetric linear functional of class two and satisfies (see [7, 16])

$$\begin{cases} \big(x^2(x^2-1)u\big)' + x\big(-(2\alpha++2\beta+5)x^2+2\beta+3\big)u = 0, \\ \big(-(\alpha+\beta+2)(u)_2+\beta+1\big) \times \big((\alpha+\beta+2)(u)_2+\alpha\big) \times \big\{|\alpha| + |(\alpha+\beta+2)(u)_2+\alpha|\big\} \neq 0. \end{cases}$$

Then,

$$u = (1-k)\mathcal{G}(\alpha,\beta) + k\delta_0 \tag{3.20}$$

where $\mathcal{G}(\alpha,\beta)$ is the Generalized Gegenbauer linear functional defined by the Pearson equation [6]

$$\big(x(x^2-1)\mathcal{G}(\alpha,\beta)\big)' + 2\big(-(\alpha+\beta+2)x^2+\beta+1\big)\mathcal{G}(\alpha,\beta) = 0.$$

Notice that the linear functional u, defined by (3.20) is regular if and only if k satisfies (see [10])

$$1 + k\sum_{m=0}^{n} \frac{(S_m^{(\alpha,\beta)})^2(0)}{\Big\langle \mathcal{G}(\alpha,\beta), (S_m^{(\alpha,\beta)})^2 \Big\rangle} \neq 0,\ n \geq 0,$$

where $\{S_n^{(\alpha,\beta)}\}_{n\geq 0}$ is the MOPS with respect to $\mathcal{G}(\alpha,\beta)$.
Finally, from (3.20) for $\alpha,\ \beta > -1$ we get

$$\langle u,\ f\rangle = (1-k)\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} f(x)|x|^{2\beta+1}(1-x^2)^{\alpha}\,dx + kf(0). \tag{3.21}$$

$\mathbf{A}_{3.1}$. From (2.11), we obtain

$$\begin{aligned} -(\alpha+\beta+n+4)(u)_{n+3} &+ (\beta+\gamma+\rho-\alpha)(u)_{n+2} + (\alpha+\beta+2\gamma-2\rho+2n+4)(u)_{n+1} \\ &+ (\alpha+\gamma+\rho-\beta)(u)_n - n(u)_{n-1} = 0,\ n\geq 0, \end{aligned} \tag{3.22}$$

with $(u)_{-1}=0$ and $(u)_0=1$. Thus, the set of solutions is a linear space of dimension at most three.
On the other hand, from (2.11)-(3.2) we get

$$\frac{U'(x)}{U(x)} = \frac{\alpha}{x-1} - \frac{\gamma}{(x-1)^2} + \frac{\beta}{x+1} - \frac{\rho}{(x+1)^2},$$

and, as a consequence,

$$U(x) = (1-x)^{\alpha}(x+1)^{\beta}e^{\frac{\rho}{x+1}-\frac{\gamma}{1-x}}. \tag{3.23}$$

Notice that if $\gamma\rho \neq 0$, $\alpha,\ \beta > -1$ and $\gamma \geq 0$, $\rho \leq 0$, then the conditions (1.7) and (3.3) hold. Then u is represented by

$$\langle u, f\rangle = A\int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta}e^{\frac{-\gamma}{1-x}+\frac{\rho}{x+1}}\,dx. \tag{3.24}$$

Particular cases:

• If $\gamma=\rho=0$, thus from (2.11) we get

$$\begin{cases} \big((x^2-1)^2u\big)' + (x^2-1)\big(-(\alpha+\beta+2)x+(\beta-\alpha)-2x\big)u = 0, \\ \big(-(\alpha+\beta+3)(u)_2 - 2(\alpha+1)(u)_1+\beta-\alpha+1\big)\times\big(-(\alpha+\beta+3)(u)_2+2(\beta+1)(u)_1+\alpha-\beta+1\big)\neq 0. \end{cases}$$

Then

$$u = (1-k_1-k_2)\mathcal{J}(\beta,\alpha) + k_1\delta_1 + k_2\delta_{-1}. \tag{3.25}$$

To ensure the regularity of u, defined by (3.25) k_1 and k_2 must verify (see [2])

$$\begin{vmatrix} 1+k_1K_n(1,1) & k_2K_n(1,-1) \\ k_1K_n(1,-1) & 1+k_2K_n(-1,-1) \end{vmatrix} \neq 0,\ \ n\geq 0,$$

where $K_n(x,y) := \sum_{m=0}^{n}\frac{P_m^{(\beta,\alpha)}(x)P_m^{(\beta,\alpha)}(y)}{\langle \mathcal{J}(\beta,\alpha),\ (P_m^{(\beta,\alpha)})^2\rangle}$ is the nth reproducing Kernel associated with the linear functional $\mathcal{J}(\beta,\alpha)$ and $\{P_n^{(\beta,\alpha)}\}_{n\geq 0}$ is the MOPS with respect to $\mathcal{J}(\beta,\alpha)$.
Finally, from (3.25) we have

$$\langle u, f\rangle = (1-k_1-k_2)A\int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx + k_1f(1) + k_2f(-1).$$

where $A = \dfrac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}$, $\alpha,\,\beta > -1$. This is the so-called Koornwinder linear functional (see [9]).

• If $\alpha = \beta$ and $\gamma = -\rho$, then (3.22) becomes

$$-(\beta+2)(u)_3 + (\beta+\gamma+2)(u)_1 = 0,$$

$$-(\beta+n+2)(u)_{2n+3} + (\beta+\gamma+2n+2)(u)_{2n+1} + n(u)_{2n-1} = 0,\ n \geq 1.$$

Thus, if $(u)_1 = 0$, then by induction we get $(u)_{2n+1} = 0,\ n \geq 0$. As a consequence, u is a symmetric linear functional of class two and verifies (see [7])

$$\begin{cases} \big((x^2-1)^2u\big)' + 2x\big((\beta+2)x^2 + (\beta+2\gamma+2)\big)u = 0, \\ |\gamma| + |-(2\beta+3)(u)_2 + 4\gamma + 1| \neq 0, \end{cases} \tag{3.26}$$

as well as

$$\langle u,\, f\rangle = A\int_{-1}^{1}(1-x^2)^{\beta}e^{-\frac{2\gamma}{1-x^2}}f(x)\,dx, \quad \beta > -1, \gamma > 0. \tag{3.27}$$

• If $\gamma = 0$, then (3.26) becomes

$$\begin{cases} \big((x^2-1)^2u\big)' - 2(\beta+2)x(x^2-1)u = 0, \\ (2\beta+3)(u)_2 \neq 1. \end{cases}$$

Thus

$$u = (1-2k)\mathcal{G}(\beta) + k(\delta_{-1} + \delta_1), \tag{3.28}$$

where $\mathcal{G}(\beta)$ is the Gegenbauer linear functional that satisfies (see [14])

$$\big((x^2-1)\mathcal{G}(\beta)\big)' - 2(\beta+1)x\mathcal{G}(\beta) = 0,$$

with $\beta + 1 \neq -n,\ 2\beta + 1 \neq -n,\ n \geq 0$.

Finally, from (3.28) we have

$$\langle u,\, f\rangle = (1-2k)K_2\int_{-1}^{1}(1-x^2)^{\beta}f(x)\,dx + k(f(1) + f(-1)).$$

where $K_2 = \dfrac{\Gamma(2\beta+2)}{2^{2\beta+1}\Gamma^2(\beta+1)}$, $\beta > -1$

$\mathbf{A}_{3.2}$. From (2.13), we get

$$-(\alpha+\beta+n+2)(u)_{n+3} + (\beta+\gamma+n+1)(u)_{n+2} + (2\lambda-\gamma)(u)_{n+1} - 2\lambda(u)_n = 0, n \geq 0 \tag{3.29}$$

with $(u)_0 = 1$. Then the set of solutions is a linear space of dimension at most three.

On the other hand, from (2.13)-(3.2) we have

$$\frac{U'(x)}{U(x)} = \frac{\beta-2}{x} - \frac{\gamma}{x^2} - \frac{2\lambda}{x^3} + \frac{\alpha}{x-1},$$

so

$$U(x) = x^{\beta-2}(1-x)^{\alpha}e^{\frac{\lambda}{x^2}+\frac{\gamma}{x}}. \tag{3.30}$$

Finally, it is straightforward to check that for $\alpha\lambda \neq 0$, $\alpha,\,\beta > -1$, and $\lambda,\,\gamma < 0$, the conditions (1.7) and

(3.3) hold.

Then u is represented by

$$\langle u,f\rangle = A\int_0^1 f(x)x^{\beta-2}(1-x)^{\alpha}e^{\frac{\lambda}{x^2}+\frac{\gamma}{x}}\,dx. \tag{3.31}$$

PARTICULAR CASE:

If $\gamma=\lambda=0$, then (2.13) becomes

$$\begin{cases} \Big(x^3(x-1)u\Big)^{'} + [x^2(-(\alpha+\beta)x+\beta-1)-2x^2(x-1)]u=0, \\ \Big(-(\alpha+\beta+1)(u)_2+\beta(u)_1\Big)\times\Big\{|\alpha|+|(\alpha+\beta+1)(u)_2+\alpha(u)_1+\alpha|\Big\}\neq 0. \end{cases}$$

Therefore,

$$u=(1-k_1)(h_{\frac{1}{2}}\circ\tau_1)\mathcal{J}(\beta-2,\alpha)+k_1\delta_0+k_2\delta_0^{'}. \tag{3.32}$$

To ensure the regularity of u, defined by (3.32) k_1 and k_2 must verify (see [2] and [3])

$$\Delta_n=\begin{vmatrix} 1+k_1K_n^{(0,0)}(0,0)-k_2K_n^{(0,1)}(0,0) & -k_2K_n^{(0,0)}(0,0) \\ k_1K_n^{(1,0)}(0,0)-k_2K_n^{(1,1)}(0,0) & 1-k_2K_n^{(0,1)}(0,0)\end{vmatrix}\neq 0,\ n\geq 0,$$

where $K_n^{(i,j)}(x,y):=\sum_{m=0}^{n}\frac{P_m^{(i)}(x)P_m^{(j)}(y)}{\langle (h_{\frac{1}{2}}\circ\tau_1)\mathcal{J}(\beta-2,\alpha),\,P_m^2\rangle}$, $P_m^{(i)}(x):=\frac{d^i}{dx^i}P_m(x)$, and $\{P_n\}_{n\geq 0}$ is the MOPS with respect to $(h_{\frac{1}{2}}\circ\tau_1)\mathcal{J}(\beta-2,\alpha)$. In this case

$$\langle u,\,f\rangle=\frac{(1-k_1)\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta-1)}\int_0^1 x^{\beta-2}(1-x)^{\alpha}f(x)+k_1f(0)-k_2f^{'}(0),\ \alpha>-1,\ \beta>1.$$

$\mathbf{A}_{4.1}$. The equation (2.16) is equivalent to the fact that the moments $(u)_n$ of u satisfy

$$-(\alpha+2\gamma+n-1)(u)_{n+3}+\beta(u)_{n+2}-2\gamma(u)_{n+1}+3(u)_n=0,\ n\geq 0, \tag{3.33}$$

with $(u)_0=1$. Then, the set of solutions is a linear space of dimension at most three.
On the other hand, from (2.16)-(3.2) we get

$$\frac{U^{'}(x)}{U(x)}=\frac{\alpha}{x}+\frac{\beta}{x^2}-\frac{2\gamma}{x^3}+\frac{3}{x^4}.$$

A priori there is no a real path such that $x^4U(x)f(x)|_{\mathcal{C}}=0,\quad f\in\mathcal{P}$.
$\mathbf{A}_{4.2}$. From (2.18), we obtain

$$-(n+4\alpha-1)(u)_{n+3}+2\beta(u)_{n+2}-4(u)_{n+1}=0,\quad n\geq 0, \tag{3.34}$$

with $(u)_0=1$. Then the set of solutions is a linear space of dimension at most two.
On the other hand, from (2.18)-(3.2) we get

$$\frac{U^{'}(x)}{U(x)}=\frac{4\alpha-5}{x}-\frac{2\beta}{x^2}+\frac{4}{x^3}.$$

A priori there is no a real path such that $x^4U(x)f(x)|_{\mathcal{C}}=0,\quad f\in\mathcal{P}$.

PARTICULAR CASES:

• From (2.18) we have

$$u=\lambda\mathcal{W}+(1-\lambda)\delta_0,$$

where $\mathcal{W}$ is the semiclassical linear functional of class one satisfying (see [4, page 260])

$$(x^3\mathcal{W})' + \big(-(4\alpha-2)x^2 + 2\beta x - 4\big)\mathcal{W} = 0.$$

• If $\beta = 0$, then from (3.34) we have

$$(2n+4\alpha-1)(u)_{2n+3} + 4(u)_{2n+1} = 0, \quad (n+2\alpha)(u)_{2n+4} + 2(u)_{2n+2} = 0,\ n \geq 0.$$

But, with $(u)_1 = 0$ and $(u)_2 \neq 0$ we obtain by induction

$$(u)_{2n+1} = 0, \quad (u)_{2n+2} = (-2)^n \frac{\Gamma(2\alpha)}{\Gamma(2\alpha+n)}(u)_2,\ \alpha > 0, \quad n \geq 0.$$

Therefore u is a symmetric linear functional of class two such that (see [7, 16])

$$\begin{cases} (x^4u)' - x\big((4\alpha-1)x^2+4\big)u = 0, \\ (\alpha+2)(u)_2 + 4 \neq 0. \end{cases}$$

$\mathbf{A}_{4.3}$. From (2.20), we obtain

$$(n+2\alpha)(u)_{n+3} + 2(u)_{n+2} = 0, \quad n \geq 0, \tag{3.35}$$

with $(u)_0 = 1$. Then, $(u)_{n+2} = (-2)^n \dfrac{\Gamma(2\alpha)}{\Gamma(2\alpha+n)}(u)_2$, $n \geq 0$, and $(u)_1$ is an arbitrary parameter.
In this case

$$x^2u = k\mathcal{B}(\alpha) \tag{3.36}$$

where k is a normalization term and $\mathcal{B}(\alpha)$ is the Bessel linear functional that satisfies (see [14])

$$(x^2\mathcal{B}(\alpha))' - 2(\alpha x+1)\mathcal{B}(\alpha) = 0,$$

with $\alpha \neq -\frac{n}{2}$, $n \geq 0$.
The linear functional u defined by (3.36) is regular if and only if (see [13])

$$\langle u, P_{n+1}^{(\alpha)}\rangle\langle u, xP_n^{(\alpha)}\rangle - \langle u, P_n^{(\alpha)}\rangle\langle u, xP_{n+1}^{(\alpha)}\rangle \neq 0,\ n \geq 0,$$

where $\{P_n^{(\alpha)}\}_{n\geq 0}$ is the Bessel MOPS.
From (1.3) and (3.36), we obtain

$$u = kx^{-2}\mathcal{B}(\alpha) + \delta_0 - (u)_1\delta_0'.$$

Notice that $k = (u)_2$.

Case B.

$\mathbf{B}_1$. From (2.23), we have

$$\lambda(u)_{n+3} - (\lambda c + \alpha + \beta + \gamma + n + 3)(u)_{n+2} + [c(\alpha+\beta+n+2) - \alpha - \lambda](u)_{n+1} + [c(\alpha+\lambda-\beta) + \gamma + n + 1](u)_n - cn(u)_{n-1} = 0,\ n \geq 0, \tag{3.37}$$

with $(u)_{-1} = 0$ and $(u)_0 = 1$. Then, the set of solutions is a linear space of dimension at most three.
On the other hand, from (2.23)-(3.2) we get

$$\frac{U'(x)}{U(x)} = -\lambda + \frac{\alpha}{x-1} + \frac{\beta}{x+1} - \frac{\gamma}{x-c},$$

hence

$$U(x) = (1-x)|^{\alpha}|1+x|^{\beta}|x-c|^{\gamma}e^{-\lambda x},\ , \tag{3.38}$$

is the solution at some intervals which depend on c.
Furthermore, for $\alpha\beta\gamma \neq 0$, α, β, $\gamma > -1$, in the following situations (1.7) and (3.3) hold

$\mathbf{s}_1$: $c \in]-1,1[$ and $\lambda < 0$. u is represented by

$$\langle u, f\rangle = \int_{-\infty}^{1} f(x)(1-x)^{\alpha}|1+x|^{\beta}|x-c|^{\gamma}e^{-\lambda x}\Big(A\chi_{]-\infty,-1]}(x) + B\chi_{[-1,c]}(x) + C\chi_{[c,1]}(x)\Big)\,dx. \quad (3.39)$$

$\mathbf{s}_2$: $c \in]-1,1[$ and $\lambda > 0$. u is represented by

$$\langle u, f\rangle = \int_{-1}^{+\infty} f(x)|1-x|^{\alpha}(1+x)^{\beta}|c-x|^{\gamma}e^{-\lambda x}\Big(A_1\chi_{[-1,c]}(x) + B_1\chi_{[c,1]}(x) + C_1\chi_{[1,+\infty[}(x)\Big)\,dx. \quad (3.40)$$

$\mathbf{s}_3$: $c < -1$, $\lambda > 0$. u is represented by

$$\langle u, f\rangle = \int_{c}^{+\infty} f(x)|x-1|^{\alpha}|1+x|^{\beta}(x-c)^{\gamma}e^{-\lambda x}\Big(A_2\chi_{[c,-1]}(x) + B_2\chi_{[-1,1]}(x) + C_2\chi_{[1,+\infty[}(x)\Big)\,dx. \quad (3.41)$$

$\mathbf{s}_4$: $c < -1$, $\lambda < 0$, u is represented by

$$\langle u, f\rangle = \int_{-\infty}^{1} f(x)(1-x)^{\alpha}|1+x|^{\beta}|x-c|^{\gamma}e^{-\lambda x}\Big(A_3\chi_{[-\infty,c]}(x) + B_3\chi_{[c,-1]}(x) + C_3\chi_{[-1,1]}(x)\Big)\,dx. \quad (3.42)$$

$\mathbf{s}_5$: $\lambda > 0$, $c > 1$. u is represented by

$$\langle u, f\rangle = \int_{-1}^{+\infty} f(x)|x-1|^{\alpha}(x+1)^{\beta}|x-c|^{\gamma}e^{-\lambda x}\Big(A_4\chi_{[-1,1]}(x) + B_4\chi_{[1,c]}(x) + C_4\chi_{[c,+\infty[}(x)\Big)\,dx. \quad (3.43)$$

$\mathbf{s}_6$: $\lambda < 0$, $c > 1$. u is represented by

$$\langle u, f\rangle = \int_{-\infty}^{c} f(x)|x-1|^{\alpha}|x+1|^{\beta}(c-x)^{\gamma}e^{-\lambda x}\Big(A_5\chi_{]-\infty,-1]}(x) + B_5\chi_{[-1,1]}(x) + C_5\chi_{[1,c]}(x)\Big)\,dx. \quad (3.44)$$

$\mathbf{B}_2$. The equation (2.25) is equivalent to the fact that the moments $(u)_n$ of the linear functional u satisfy

$$\lambda(u)_{n+3} - (\alpha+\beta+\lambda+n+2)(u)_{n+2} + (\beta+2\gamma+n+1)(u)_{n+1} - 2\gamma(u)_n = 0,\ n \geq 0, \quad (3.45)$$

with $(u)_0 = 1$. Then the set of solutions is a linear space of dimension at most three.
On the other hand, from (2.25)-(3.2) we obtain

$$\frac{U'(x)}{U(x)} = -\lambda + \frac{\beta-1}{x} - \frac{\gamma}{x^2} + \frac{\alpha}{x-1},$$

hence

$$U(x) = x^{\beta-1}|x-1|^{\alpha}e^{-\lambda x + \frac{\gamma}{x}}, \quad (3.46)$$

is the solution in $]-\infty,0]$, $[0,1]$, and $[1,+\infty[$.
Furthermore, for $\alpha\gamma \neq 0$, $\alpha > -1$, $\beta > 0$, (1.7) and (3.3) hold in the following situations:
$\mathbf{s}_1$: $\lambda > 0$ and $\gamma < 0$. u is represented by

$$\langle u, f\rangle = \int_{0}^{+\infty} f(x)x^{\beta-1}|x-1|^{\alpha}e^{-\lambda x+\frac{\gamma}{x}}\Big(A\chi_{[0,1]}(x) + B\chi_{[1,+\infty[}(x)\Big)\,dx. \quad (3.47)$$

$\mathbf{s}_2$: $\lambda < 0$ and $\gamma > 0$. u is represented by

$$\langle u, f\rangle = A_1\int_{-\infty}^{0} f(x)|x|^{\beta-1}(1-x)^{\alpha}e^{-\lambda x+\frac{\gamma}{x}}\,dx. \quad (3.48)$$

$\mathbf{s}_3$: $\lambda \in \mathbb{R}$ and $\gamma < 0$. u is represented by

$$\langle u, f\rangle = A_2\int_{0}^{1} f(x)|x|^{\beta-1}(1-x)^{\alpha}e^{-\lambda x+\frac{\gamma}{x}}\,dx. \quad (3.49)$$

Particular cases:

• If $\gamma = 0$ and $\beta = -1$, then (2.25) becomes

$$\begin{cases} \Big(x^2(x-1)u\Big)' + x^2(\lambda x - \alpha - \lambda - 1)u = 0, \\ \Big(\lambda(u)_2 - (\alpha+\lambda)(u)_1 + 1\Big) \times \Big\{|\alpha| + |\lambda(u)_2 - \alpha(u)_1 - \alpha|\Big\} \neq 0. \end{cases}$$

Therefore

$$x^2 u = k(h_{\lambda^{-1}} \circ \tau_\lambda)\mathcal{L}(\alpha), \tag{3.50}$$

where k is normalization term and $\mathcal{L}(a)$ is the Laguerre linear functional satisfying (see [8])

$$\big(x\mathcal{L}(a)\big)' + (x - a - 1)\mathcal{L}(a) = 0, \;\; a \neq -n-1, \; n \geq 0.$$

From (1.3) and (3.50) we get

$$u = kx^{-2}(h_{\lambda^{-1}} \circ \tau_\lambda)\mathcal{L}(\alpha) + \delta_0 - (u)_1\delta_0'.$$

Notice that $k = (u)_2$.
The linear functional u defined by (3.50) is regular if and only if (see [13])

$$\langle u, P_{n+1}^{(\alpha)}\rangle\langle u, xP_n^{(\alpha)}\rangle - \langle u, xP_{n+1}^{(\alpha)}\rangle\langle u, P_n^{(\alpha)}\rangle \neq 0, \; n \geq 0,$$

where $\{P_n^{(\alpha)}\}_{n\geq 0}$ is the MOPS with respect to $(h_{\lambda^{-1}} \circ \tau_\lambda)\mathcal{L}(\alpha)$

• If $\gamma = 0$ and $\beta = 1$, then (2.25) becomes

$$\begin{cases} \Big(x^2(x-1)u\Big)' + \Big(x^2(\lambda x - \alpha - \lambda - 1) - 2x(x-1)\Big)u = 0, \\ \Big(\lambda(u)_2 - (\alpha+\lambda+2)(u)_1 - 1\Big) \times \Big\{|\alpha| + |\lambda(u)_2 - (\alpha+2)(u)_1 - \alpha|\Big\} \neq 0. \end{cases}$$

Hence

$$u = (1-k)(h_{\lambda^{-1}} \circ \tau_\lambda)\mathcal{L}(\alpha) + k\delta_0. \tag{3.51}$$

The linear functional u defined by (3.51) is regular if and only if (see [10])

$$1 + k\sum_{m=0}^{n} \frac{\big(P_m^{(\alpha)}\big)^2(0)}{\Big\langle (h_{\lambda^{-1}} \circ \tau_\lambda)\mathcal{L}(\alpha), \big(P_m^{(\alpha)}\big)^2 \Big\rangle} \neq 0, \; n \geq 0.$$

In this case

$$\langle u, \, f\rangle = (1-k)\frac{\lambda^{\alpha+1}e^{\lambda}}{\Gamma(\alpha+1)} \int_1^{sign(\lambda)\infty} |x-1|^{\alpha} e^{-\lambda x} f(x)\, dx + kf(0), \;\; \alpha > -1.$$

• If $\gamma = 0$ and $\beta = 3$, then (2.25) becomes

$$\begin{cases} \Big(x^2(x-1)u\Big)' + \Big(x^2(\lambda x - \alpha - \lambda - 1) - 4x(x-1)\Big)u = 0, \\ \Big(\lambda(u)_2 - (\alpha+\lambda+4)(u)_1 - 3\Big) \times \Big\{|\alpha| + |\lambda(u)_2 - (\alpha+4)(u)_1 - \alpha|\Big\} \neq 0. \end{cases}$$

Hence

$$u = kx^2(h_{\lambda^{-1}} \circ \tau_\lambda)\mathcal{L}(\alpha), \tag{3.52}$$

where k is a normalization term.

The linear functional u defined by (3.52) is regular if and only if (see [8])

$$P_n^{(\alpha)}(0)\left(P_{n+1}^{(\alpha)}\right)'(0)-\left(P_n^{(\alpha)}\right)'(0)P_{n+1}^{(\alpha)}(0)\neq 0,\ n\geq 0.$$

B$_3$. From (2.27), we obtain

$$(u)_{n+3}-(\alpha+n+2)(u)_{n+2}+\beta(u)_{n+1}+2\gamma(u)_n=0,\quad n\geq 0, \tag{3.53}$$

with $(u)_0=1$. Then the set of solutions is a linear space of dimension at most three.
On the other hand, from (2.27)-(3.2) we get

$$\frac{U'(x)}{U(x)}=-1+\frac{\alpha-1}{x}-\frac{\beta}{x^2}-\frac{2\gamma}{x^3},$$

hence

$$U(x)=x^{\alpha-1}e^{-x+\frac{\beta}{x}+\frac{\gamma}{x^2}}. \tag{3.54}$$

On the other hand, for $\alpha>0$ and $\beta,\,\gamma<0$, (1.7) and (3.3) hold. Then $[0,+\infty[$ is an admissible integration path and u is represented by

$$\langle u,f\rangle=A\int_0^{+\infty}f(x)x^{\alpha-1}e^{-x+\frac{\beta}{x}+\frac{\gamma}{x^2}}\,dx. \tag{3.55}$$

Particular case:
If $\gamma=\beta=0$, then (2.27) becomes

$$\begin{cases}(x^3u)'+[x^2(x-\alpha)-2x^2]u=0,\\ (u)_2-(\alpha+1)(u)_1\neq 0.\end{cases}$$

Therefore

$$u=(1-k_1)\mathcal{L}(\alpha-1)+k_1\delta_0+k_2\delta_0'. \tag{3.56}$$

The linear functional u defined by (3.56) is regular if and only if (see [2] and [3])

$$\Delta_n=\begin{vmatrix}1+k_1K_n^{(0,1)}(0,0)-k_2K_n^{(0,1)}(0,0) & -k_2K_n^{(0,0)}(0,0)\\ k_1K_n^{(0,1)}(0,0)-k_2K_n^{(1,1)}(0,0) & 1-k_2K_n^{(0,1)}(0,0)\end{vmatrix}\neq 0,\ n\geq 0,$$

where $K_n^{(i,j)}(x,y)=\displaystyle\sum_{m=0}^{n}\frac{P_m^{(i)}(x)P_m^{(j)}(y)}{\langle\mathcal{L}(\alpha-1),\,P_m^2\rangle}$, $n\geq 0$, and $\{P_n\}_{n\geq 0}$ is the Laguerre MOPS. In this case

$$\langle u,\,f\rangle=\frac{1-k_1}{\Gamma(\alpha)}\int_0^{+\infty}x^{\alpha-1}e^{-x}f(x)\,dx+k_1f(0)-k_2f'(0),\ \alpha>0.$$

B$_{4.1}$. From (2.30), we have

$$2\lambda(u)_{n+3}+\alpha(u)_{n+2}-(n+2\lambda+\beta+\gamma+2)(u)_{n+1}-(\alpha+\beta-\gamma)(u)_n+n(u)_{n-1}=0,n\geq 0 \tag{3.57}$$

with $(u)_0=1$. Then the set of solutions is a linear space of dimension at most three.
On the other hand from (2.30)-(3.2) we get

$$\frac{U'(x)}{U(x)}=-2\lambda x-\alpha+\frac{\beta}{x-1}+\frac{\gamma}{x+1},$$

hence

$$U(x)=|x-1|^{\beta}|x+1|^{\gamma}e^{-\lambda x^2-\alpha x}, \tag{3.58}$$

is the solution in $]-\infty,-1]$, $[-1,1]$, and $[1,+\infty[$.
If $\beta\gamma\neq 0$, β, and $\gamma>-1$ then (1.7) and (3.3) hold in the following situations:
$\mathbf{s}_1$: $\lambda>0$. u is represented by

$$\langle u,f\rangle=\int_{-\infty}^{+\infty} f(x)|x-1|^{\beta}|x+1|^{\gamma}e^{-\lambda x^2-\alpha x}\Big(A\chi_{]-\infty,-1]}(x)+B\chi_{[-1,1]}(x)+C\chi_{[1,+\infty[}(x)\Big)\,dx. \quad (3.59)$$

$\mathbf{s}_2$: $\lambda,\alpha\in\mathbb{R}$. u is represented by

$$\langle u,f\rangle=A_1\int_{-1}^{1} f(x)|x-1|^{\beta}|x+1|^{\gamma}e^{-\lambda x^2-\alpha x}\,dx. \quad (3.60)$$

Particular case:
If $\alpha=0$ and $\gamma=\beta$, then from (3.57)

$$\lambda(u)_3-(\lambda+\beta+1)(u)_1=0,$$
$$\lambda(u)_{2n+3}-(n+\lambda+\beta+1)(u)_{2n+1}+n(u)_{2n-1}=0,\ n\geq 1.$$

But if $(u)_1=0$, then by induction we get $(u)_{2n+1}=0$, $n\geq 0$, i.e. u is a symmetric linear functional of class two satisfying (see [7])

$$\begin{cases} \big((x^2-1)u\big)'+2x\big(\lambda x^2-(\lambda+\beta+1)\big)u=0,\\ |\beta|+|2\lambda(u)_2-4\beta-1|\neq 0\end{cases}$$

and, as a consequence,

$$\langle u,f\rangle=\int_{-\infty}^{+\infty} f(x)|x^2-1|^{\beta}e^{-\lambda x^2}\Big(A+(B-A)\chi_{[-1,1]}(x)\Big)\,dx, \beta>-1, \lambda>0, \quad (3.61)$$

or

$$\langle u,f\rangle=A_1\int_{-1}^{1} f(x)|x^2-1|^{\beta}e^{-\lambda x^2}\,dx, \beta>-1, \lambda\in\mathbb{R}. \quad (3.62)$$

$\mathbf{B}_{4.2}$. Using (2.32), we have

$$2(u)_{n+3}-2\lambda(u)_{n+2}-(n+2\alpha+3)(u)_{n+1}+\beta(u)_n=0,\quad n\geq 0, \quad (3.63)$$

with $(u)_0=1$. Then the set of solutions is a linear space of dimension at most three.
On the other hand from (2.32)-(3.2) we have

$$\frac{U'(x)}{U(x)}=-2x+2\lambda+\frac{2\alpha+1}{x}-\frac{\beta}{x^2},$$

hence

$$U(x)=|x|^{2\alpha+1}e^{-x^2+2\lambda x+\frac{\beta}{x}}, \quad (3.64)$$

is a solution in $\mathbb{R}^*$.
If $\alpha>-1$ and $\beta\neq 0$, (1.7) and (3.3) hold. Then u is represented by

$$\langle u,f\rangle=A\int_{0}^{sign(\beta)\infty} f(x)|x|^{2\alpha+1}e^{-x^2+2\lambda x+\frac{\beta}{x}}\,dx. \quad (3.65)$$

Particular cases:
• If $\beta=0$ and $\alpha=-\frac{1}{2}$, then (2.32) becomes

$$\begin{cases} (x^2u)'+[x^2(2x-2\lambda)-2x]u=0,\\ 2(u)_2-2\lambda(u)_1+1\neq 0.\end{cases}$$

Therefore

$$u = (1-k)\tau_\lambda \mathcal{H} + k\delta_0, \tag{3.66}$$

where $\mathcal{H}$ is the Hermite linear functional that satisfies (see [6])

$$\mathcal{H}' + 2x\mathcal{H} = 0.$$

The linear functional u defined by (3.66) is regular if and only if $1 + k\sum_{m=0}^{n} \frac{P_m^2(0)}{\langle \tau_\lambda \mathcal{H}, P_m^2 \rangle} \neq 0$, $n \geq 0$, where $\{P_n\}_{n\geq 0}$ is the MOPS with respect to $\tau_\lambda \mathcal{H}$ (see [10]). In this case

$$\langle u,\, f\rangle = (1-k)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(x-\lambda)^2} f(x)\, dx + kf(0).$$

• If $\beta = 0$ and $\alpha = \frac{1}{2}$, then (2.32) becomes

$$\begin{cases} (x^2 u)' + \Big(x^2(2x-2\lambda) - 4x\Big)u = 0, \\ 2(u)_2 - 2\lambda(u)_1 - 3 \neq 0. \end{cases}$$

Therefore

$$u = kx^2 \tau_\lambda \mathcal{H}, \tag{3.67}$$

where k is a normalization term. The linear functional u, defined by (3.67), is regular if and only if (see [3])

$$P_n(0)P_{n+1}'(0) - P_n'(0)P_{n+1}(0) \neq 0,\; n \geq 0,$$

where $\{P_n\}_{n\geq 0}$ is the MOPS with respect to $\tau_\lambda \mathcal{H}$.

• If $\lambda = \beta = 0$, then from (3.63) we obtain

$$2(u)_{2n+3} - (2n+2\alpha+3)(u)_{2n+1} = (u)_{2n+4} - (n+\alpha+2)(u)_{2n+2} = 0, \quad n \geq 0.$$

If $(u)_1 = 0$ and $(u)_2 \neq 0$, then we obtain by induction

$$(u)_{2n+1} = 0, \quad (u)_{2n+2} = \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+2)}(u)_2, \quad n \geq 0.$$

Then u is a symmetric linear functional of class two and satisfies (see [7] and [16])

$$\begin{cases} (x^2 u)' + x\big(2x^2 - (2\alpha+3)\big)u = 0, \\ (u)_2 \neq \alpha + 1. \end{cases}$$

Therefore

$$u = (1-k)\mathcal{H}(\alpha + \frac{1}{2}) + k\delta_0, \tag{3.68}$$

where $\mathcal{H}(\alpha)$ is the generalized Hermite linear functional that satisfies the Pearson equation (see [6])

$$\big(x\mathcal{H}(\alpha)\big)' + \big(2x^2 - (2\alpha+1)\big)\mathcal{H}(\alpha) = 0.$$

u is a regular functional if and only if $1 + k\sum_{m=0}^{n} \frac{P_m^2(0)}{\Big\langle \mathcal{H}(\alpha+\frac{1}{2}), (P_m^2 \Big\rangle} \neq 0$, $n \geq 0$, where $\{P_n\}_{n\geq 0}$ is the MOPS with respect to $\mathcal{H}(\alpha + \frac{1}{2})$.

Finally, from (3.68) u is represented by

$$\langle u, f\rangle = \frac{1-k}{\Gamma(\alpha+1)}\int_{-\infty}^{+\infty} f(x)|x|^{2\alpha+1}e^{-x^2}\, dx + kf(0),\;\; \alpha > -1. \tag{3.69}$$

$\mathbf{B}_5$. The equation (2.35) is equivalent to the fact that the moments $(u)_n$ of u satisfy

$$3\lambda(u)_{n+3} + 2\beta(u)_{n+2} + \alpha(u)_{n+1} - (\gamma + n - 1)(u)_n = 0,\ n \geq 0, \tag{3.70}$$

with $(u)_0 = 1$. Then the set of solutions is a linear space of dimension at most three.
On the other hand, from (2.35)-(3.2) we obtain

$$\frac{U'(x)}{U(x)} = -3\lambda x^2 - 2\beta x - \alpha + \frac{\gamma}{x},$$

hence

$$U(x) = x^\gamma e^{-\lambda x^3 - \beta x^2 - \alpha x}, \tag{3.71}$$

is a solution in $]-\infty, 0]$ and $[0, +\infty[$
If $\gamma \neq 0$, $\gamma > -1$, and $\lambda \neq 0$ then (1.7) and (3.3) hold. Thus u is represented by

$$\langle u, f\rangle = A\int_0^{sign(\lambda)\infty} f(x)x^\gamma e^{-\lambda x^3 - \beta x^2 - \alpha x}\, dx. \tag{3.72}$$

$\mathbf{B}_6$. From (2.37), we have

$$4(u)_{n+3} + 3\alpha(u)_{n+2} + 2\lambda(u)_{n+1} + \beta(u)_n - n(u)_{n-1} = 0,\ n \geq 0, \tag{3.73}$$

with $(u)_0 = 1$. Then the set of solutions is a linear space of dimension at most three.
On the other hand, from (2.37)-(3.2) we get

$$\frac{U'(x)}{U(x)} = -4x^3 - 3\alpha x^2 - 2\lambda x - \beta,$$

hence

$$U(x) = e^{-x^4 - \alpha x^3 - \lambda x^2 - \beta x}, \tag{3.74}$$

is the solution in $\mathbb{R}$. Furthermore (1.7) and (3.3) hold. Then u is represented by

$$\langle u, f\rangle = A\int_{-\infty}^{+\infty} f(x)e^{-x^4 - \alpha x^3 - \lambda x^2 - \beta x}\, dx. \tag{3.75}$$

Particular case:
If $\alpha = \beta = 0$, then from (3.73), we get

$$2(u)_3 + \lambda(u)_1 = 0,$$

$$2(u)_{2n+3} + \lambda(u)_{2n+1} - n(u)_{2n-1} = 0,\ n \geq 1.$$

But if $(u)_1 = 0$, we obtain by induction $(u)_{2n+1} = 0$, $n \geq 0$. Therefore u is a symmetric linear functional of class two satisfying the Pearson equation (see [7])

$$u' + 2x(2x^2 + \lambda)u = 0,$$

and

$$\langle u, f\rangle = A\int_{-\infty}^{+\infty} f(x)e^{-x^4 - \lambda x^2}\, dx. \tag{3.76}$$

This is an example of a Freud linear functional (see [7] and [15]).

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