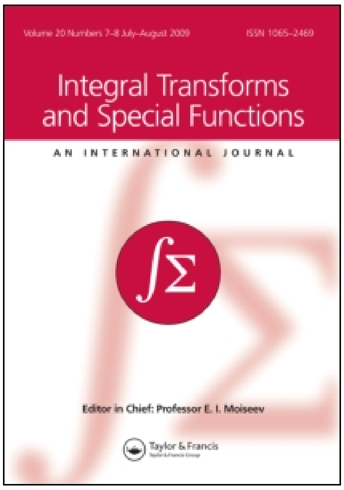

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Lowering operators associated with D-Laguerre–Hahn polynomials

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In this paper, a new lowering operator $\mathbf{D}_u$ with a linear functional u as a parameter is introduced in the linear space $\mathbb{P}$ of all polynomials in one variable with complex coefficients. It is given by

$$\mathbf{D}_u(p)(x) = p'(x) + \left\langle u_y, \frac{p(x) - p(y)}{x - y} \right\rangle, \quad p \in \mathbb{P}.$$

The concept of $\mathbf{D}_u$-semiclassical polynomial sequence is defined. We show that such a sequence belongs to the family of **D**-Laguerre–Hahn polynomials. This allows us to provide some characterizations of a $\mathbf{D}_u$-semiclassical polynomial sequence in terms of a distributional equation that the linear functional satisfies as well as a structure relation. An illustrative example is considered.

Keywords: linear functionals; quasi-orthogonality; orthogonal polynomials; Laguerre–Hahn polynomials; Appell polynomials

AMS Subject Classifications: 33C45; 42C05

1. Introduction

Let $\mathbb{P}$ be the linear space of polynomials with complex coefficients. Let $\mathbb{P}'$ be its dual space. $(u)_n := \langle u, x^n \rangle,\ n \geq 0$, will denote the moments of $u \in \mathbb{P}'$ with respect to the sequence $\{x^n\}_{n\geq 0}$. In the sequel, we set $\mathbb{P}'_M := \{u \in \mathbb{P}',\ (u)_0 \neq -n,\ n \geq 1\}$. When $(u)_0 = 1$, the linear functional u is said to be normalized. Let us define the following operations on $\mathbb{P}'$, (see [6,12]). For any $c \in \mathbb{C}$, $p, q \in \mathbb{P}$ and $u, v \in \mathbb{P}'$, we have

$$\langle qu, p \rangle = \langle u, qp \rangle, \quad \langle u', p \rangle = -\langle u, p' \rangle,$$
$$\langle \delta_c, p \rangle = p(c), \quad \text{(Dirac delta at point } c \in \mathbb{C}), \quad \delta := \delta_0,$$

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$$\langle uv, p\rangle = \langle v, up\rangle, \quad \text{where } (up)(x) = \left\langle u_y, \frac{xp(x) - yp(y)}{x - y}\right\rangle,$$

$$\langle (x-c)^{-1}u, p\rangle = \langle u, \theta_c(p)\rangle = \left\langle u, \frac{p(x) - p(c)}{x - c}\right\rangle.$$

Here, $\langle u_y, p(x, y)\rangle$ means the action of u on the polynomial $p(x, y)$ with respect to the variable y. We also denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

The following lemma will be useful in the sequel.

Lemma 1.1 ([6]) *For any $u, v \in \mathbb{P}', f, g \in \mathbb{P}$ and $c \in \mathbb{C}$, we have*

$$(x - c)((x - c)^{-1}u) = u, \tag{1.1}$$

$$(fu)' = fu' + f'u, \tag{1.2}$$

$$f(x^{-1}u) = x^{-1}(fu) + \langle u, \theta_0 f\rangle\delta, \tag{1.3}$$

$$f(uv) = (fu)v + x(u\theta_0 f)(x)v, \tag{1.4}$$

$$(u\theta_0(fg))(x) = g(x)(u\theta_0 f)(x) + ((fu)\theta_0 g)(x), \tag{1.5}$$

$$\langle u^2, \theta_0(fg)\rangle = \langle u, f(u\theta_0 g) + g(u\theta_0 f)\rangle. \tag{1.6}$$

A non-zero $v \in \mathbb{P}'$ is said to be weakly regular if for a $\phi \in \mathbb{P}$ such that $\phi v = 0$, then $\phi = 0$ [10]. Clearly, δ_c is not a weakly regular functional for every $c \in \mathbb{C}$, since $(x - c)\delta_c = 0$.

Let $\{B_n\}_{n\geq 0}$ be a monic polynomial sequence (MPS), $\deg B_n = n$, and let $\{w_n\}_{n\geq 0}$ be its dual sequence defined by $\langle w_n, B_m\rangle = \delta_{n,m},\ n,\ m \geq 0$, where $\delta_{n,m}$ is the Kronecker delta. The MPS $\{B_n\}_{n\geq 0}$ is said to be orthogonal (MOPS) with respect to $w \in \mathbb{P}'$, if $\langle w, B_n B_m\rangle = 0,\ n \neq m$ and $\langle w, B_n^2\rangle \neq 0, n \geq 0$. In this case, w is said to be quasi-definite (regular) [5]. Note that $w = (w)_0 w_0$, with $(w)_0 \neq 0$. Referring to [10], quasi-definite linear functionals are weakly regular. Note that, in general, the converse is not true.

Proposition 1.2 ([12]) *An MPS $\{B_n\}_{n\geq 0}$ with dual sequence $\{w_n\}_{n\geq 0}$ is orthogonal with respect to w_0 if and only if one of the following statements hold.*

(i) $w_n = \langle w_0, B_n^2\rangle^{-1} B_n w_0,\ n \geq 0.$
(ii) $\{B_n\}_{n\geq 0}$ *satisfies a three-term recurrence relation* (*TTRR*).

$$B_0(x) = 1, \quad B_1(x) = x - \beta_0,$$
$$B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geq 0, \tag{1.7}$$

where $\beta_n = \langle w_0, xB_n^2\rangle / \langle w_0, B_n^2\rangle \in \mathbb{C}$ *and* $\gamma_{n+1} = \langle w_0, B_{n+1}^2\rangle / \langle w_0, B_n^2\rangle \in \mathbb{C}^*,\ n \geq 0.$

Recall that $w \in \mathbb{P}'$ is quasi-definite if and only if for each non-negative integer n the Hankel determinant $\Delta_n(w) = \det((w)_{i+j})_{0\leq i,j\leq n} \neq 0,\ n \geq 0$ [5].

For an MOPS $\{B_n\}_{n\geq 0}$, we can define the sequence of associated polynomials of the first kind $\{B_n^{(1)}\}_{n\geq 0}$ as follows [5]: $B_n^{(1)}(x) = w_0\theta_0 B_{n+1}(x)$, $n \geq 0$. The sequence $\{B_n^{(1)}\}_{n\geq 0}$ is also a MOPS and satisfies the shifted TTRR

$$B_0^{(1)}(x) = 1, \quad B_1^{(1)}(x) = x - \beta_1,$$
$$B_{n+2}^{(1)}(x) = (x - \beta_{n+2})B_{n+1}^{(1)}(x) - \gamma_{n+2}B_n^{(1)}(x), \ n \geq 0.$$

An MOPS $\{B_n\}_{n\geq 0}$ with respect to w_0 is said to be of **D**-Laguerre–Hahn if w_0 satisfies the following distributional equation (see [3,4,8]):

$$(\Phi w_0)' + (\Psi_1 w_0) + B(x^{-1}w_0^2) = 0, \tag{1.8}$$

where $(\Phi, \Psi_1, B) \in \mathbb{P}^3$ and Φ is a monic polynomial. In this case, w_0 is said to be a **D**-Laguerre–Hahn linear functional. We can associate with (Φ, Ψ_1, B) the non-negative integer number: $\max(\deg(\Phi) - 2, \ \deg(B) - 2, \ \deg(\Psi_1) - 1)$. Obviously, a **D**-Laguerre–Hahn linear functional w_0 satisfies an infinite number of equations like (1.8). Indeed, it is enough to multiply by any $\chi \in \mathbb{P}$ on both sides of (1.8) in such a way that $(\chi\Phi w_0)' + (\chi\Psi_1 - \chi'\Phi)w_0 + (\chi B)(x^{-1}w_0^2) = 0$. So, we can associate with w_0 a non-negative integer s, the so-called class of w_0, which is defined by $s := \min(\max(\deg(\Phi) - 2, \ \deg(B) - 2, \ \deg(\Psi_1) - 1))$, where the minimum is taken over all possible choices of polynomials (Φ, Ψ_1, B) satisfying (1.8). It is well known (see [1]) that a **D**-Laguerre–Hahn linear functional w_0 satisfying (1.8) is of class s, if and only if for any zero c of Φ, we have $|\Psi_1(c) + \Phi'(c)| + |B(c)| + |\langle w_0, \ \theta_c\Psi_1 + \theta_c^2\Phi + w_0(\theta_0\theta_c B)\rangle| > 0$.

When B vanishes, $\{B_n\}_{n\geq 0}$ or w_0 is said to be **D**-semiclassical. In particular, a **D**-semiclassical polynomial sequence (respectively, a linear functional) of class zero is said to be **D**-classical. By means of a suitable affine transformation, there exist four canonical classical cases, the well-known families of Hermite, Laguerre, Bessel and Jacobi MOPS, respectively (see [2,3,10]).

If $B \neq 0$, then $\{B_n\}_{n\geq 0}$ or w_0 is said to be of strict **D**-Laguerre–Hahn. In the same way, by means of a suitable affine transformation, there exist four families of **D**-Laguerre–Hahn polynomials of class zero. Each of them includes two subfamilies: a singular family and a non-singular one. For more details, see [4].

PROPOSITION 1.3 ([7]) *Let $\{B_n\}_{n\geq 0}$ be an MOPS with respect to w_0, satisfying* (1.7). *The following statements are equivalent.*

(i) w_0 *is a **D**-Laguerre–Hahn linear functional of class s satisfying* (1.8).
(ii) $\{B_n\}_{n\geq 0}$ *satisfies a first structure relation* (*FSR*)

$$\Phi(x)B'_{n+1}(x) - B(x)B_n^{(1)}(x) = \frac{C_{n+1}(x) - C_0(x)}{2}B_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)B_n(x), \quad n \geq 0, \tag{1.9}$$

where C_n and D_n are polynomials with coefficients depending on n, such that $\deg C_n \leq s + 1$, $\deg D_n \leq s$, *and given by*

$$C_{n+1}(x) = -C_n(x) + 2(x - \beta_n)D_n(x),$$
$$\gamma_{n+1}D_{n+1}(x) = -\Phi(x) + \gamma_n D_{n-1}(x) - (x - \beta_n)C_n(x) + (x - \beta_n)^2 D_n(x),$$

for every $n \geq 0$, and the initial conditions

$$C_0(x) = -\Phi'(x) - \Psi_1(x),$$
$$D_0(x) = -(w_0\theta_0\Phi)'(x) - w_0\theta_0(\Psi_1)(x) - (w_0^2\theta_0^2 B)(x), \quad D_{-1}(x) = 0.$$

Using the orthogonality of $\{B_n\}_{n\geq 0}$, (1.9) can be written as follows:

$$\Phi(x)B'_{n+1}(x) - B(x)B_n^{(1)}(x) = \sum_{\nu=n-s}^{n+d} \lambda_{n,\nu} B_\nu(x), \quad n \geq s, \tag{1.10}$$

where $d = \max(\deg \Phi, \deg B)$ and s is the class of the linear functional w_0.

In this paper, we are dealing with the non-singular **D**-Laguerre–Hahn MOPS of class zero analogous to the Hermite MOPS that we denote by $\{\hat{H}_n(.;\xi)\}_{n\geq 0}$, where $\xi = (\tau, \lambda, \rho) \in \mathbb{C}^3$, with $\tau \neq -n,\ n \geq 1$, and $\rho \neq 0$. Such a polynomial sequence is orthogonal with respect to a unique monic linear functional $\mathcal{H}(\xi)$. Recall that $\{\hat{H}_n(.;\xi)\}_{n\geq 0}$ appears in the description of the Laguerre–Hahn MOPS of class zero [4]. In the sequel, the following properties of $\{\hat{H}_n(.;\xi)\}_{n\geq 0}$ will be needed.

$$\text{(TTRR)} \begin{cases} \hat{H}_0(x;\xi) = 1,\ \hat{H}_1(x;\xi) = x - \lambda, \\ \hat{H}_{n+2}(x;\xi) = x\hat{H}_{n+1}(x;\xi) - \gamma_{n+1}\hat{H}_n(x;\xi), \quad n \geq 0, \end{cases} \tag{1.11}$$

where $\gamma_1 = (\rho/2)(\tau+1)$, $\gamma_{n+1} = (\frac{1}{2})(n+\tau+1),\ n \geq 1$.

$$\begin{aligned} \text{(FSR)} \quad \hat{H}'_{n+1}(x;\xi) - B(x)\hat{H}_n^{(1)}(x;\xi) &= -2\rho^{-1}[(\rho-1)x+\lambda]\hat{H}_{n+1}(x;\xi), \\ (n+\tau+1)\hat{H}_n(x;\xi) &+ (\tau+1)(\rho-1)\delta_{n,0}, \quad n \geq 0, \end{aligned} \tag{1.12}$$

where $B(x) = 2\rho^{-1}(\rho-1)x^2 + 2\rho^{-1}\lambda(2-\rho)x + 1 - \rho(\tau+1) - 2\rho^{-1}\lambda^2$.

Note that $\{\hat{H}_n(.\,;\xi)\}_{n\geq 0}$ is a generalization of the Hermite MOPS $\{\hat{H}_n\}_{n\geq 0}$, which appears when $\xi = (0,0,1)$. It is well known that $\{\hat{H}_n\}_{n\geq 0}$ is an Appell sequence with respect to the usual derivative operator $\mathbf{D} = (\mathrm{d}/\mathrm{d}x)$ or, simply, a **D**-AMPS, since $\mathbf{D}(\hat{H}_{n+1}) = (n+1)\hat{H}_n,\ n \geq 0$ [9]. Let us show that $\{\hat{H}_n(.;\xi)\}_{n\geq 0}$ satisfies a similar property with respect to a new operator. Indeed, from the expression of $B(x)$ already quoted and (1.11), we obtain $(B\mathcal{H}(\xi))\theta_0\hat{H}_{n+1}(x;\xi) = -2\rho^{-1}((\rho-1)x+\lambda)\hat{H}_{n+1}(x;\xi) + B(x)\hat{H}_n^{(1)}(x;\xi) + (\tau+1)(\rho-1)\delta_{n,0}, n \geq 0$. Accordingly, $\hat{H}'_{n+1}(x;\xi) - (B\mathcal{H}(\xi))\theta_0\hat{H}_{n+1}(x;\xi) = (n+\tau+1)\hat{H}_n(x;\xi)$, $n \geq 0$, by (1.12). If we set $\mathcal{U}(\xi) := -B\mathcal{H}(\xi) \in \mathbb{P}'$, then $(\mathcal{U}(\xi))_0 = -\tau \neq -n,\ n \geq 1$, and we can write $\hat{H}'_{n+1}(x;\xi) + \mathcal{U}(\xi)\theta_0\hat{H}_{n+1}(x;\xi) = (n+\tau+1)\ \hat{H}_n(x;\xi),\ \ n \geq 0$. Hence, by introducing the operator $\mathbf{D}_{\mathcal{U}(\xi)}(p)(x) := p'(x) + \mathcal{U}(\xi)\theta_0 p(x)$, for all $p \in \mathbb{P}$, it follows that

$$\mathbf{D}_{\mathcal{U}(\xi)}\hat{H}_{n+1}(x;\xi) = (n+\tau+1)\hat{H}_n(x;\xi), \quad n \geq 0. \tag{1.13}$$

Thus, $\{\hat{H}_n(.,\xi)\}_{n\geq 0}$ is a $\mathbf{D}_{\mathcal{U}(\xi)}$-AMPS. Besides, from (1.11), it follows that

$$\mathbf{D}^2_{\mathcal{U}(\xi)}\hat{H}_{n+1}(x;\xi) - 2x\mathbf{D}_{\mathcal{U}(\xi)}\hat{H}_{n+1}(x;\xi) = -2(n+\tau+1)\hat{H}_{n+1}(x;\xi), \quad n \geq 2.$$

From a more general point of view, for a given $u \in \mathbb{P}'_M$, let $\mathbf{D}_u : \mathbb{P} \longrightarrow \mathbb{P}$ be the linear operator defined by

$$\mathbf{D}_u(p)(x) = p'(x) + \left\langle u_y, \frac{p(x)-p(y)}{x-y} \right\rangle, p \in \mathbb{P}, \quad (\mathbf{D}_0 = \mathbf{D}).$$

The aim of this work is to introduce the concept of $\mathbf{D}_u$-semiclassical MOPS and to show that they belong to the **D**-Laguerre–Hahn class. A necessary and sufficient condition for a **D**-Laguerre–Hahn MOPS to be a $\mathbf{D}_u$-semiclassical MOPS will be given.

The structure of the paper is as follows. Section 2 contains some properties of the operator $\mathbf{D}_u$ as well as the definition of the $\mathbf{D}_u$-semiclassical polynomials. In Section 3, we give a characterization of a $\mathbf{D}_u$-semiclassical linear functional by a linear distributional equation. In Section 4, first we characterize a $\mathbf{D}_u$-semiclassical MOPS by means of an FSR, i.e. a finite-type relation between the polynomials and the monic polynomials of its first $\mathbf{D}_u$-derivative [11]. Second, we will prove that every $\mathbf{D}_u$-semiclassical linear functional belongs to the family of **D**-Laguerre–Hahn linear functionals. Finally, an example of $\mathbf{D}_u$-semiclassical MOPS is presented.

2. The $\mathbf{D}_u$-semiclassical polynomials

2.1. *Some properties of the operator* $\mathbf{D}_u$

Among the properties of the linear operator $\mathbf{D}_u$, we emphasize the following one related to the lowering character of such operator. Indeed,we have

$$\mathbf{D}_u(1) = 0, \quad \mathbf{D}_u(x) = (u)_0 + 1, \quad \mathbf{D}_u(x^n) = (n + (u)_0)x^{n-1} + \sum_{\nu=0}^{n-2} (u)_{n-\nu-1}x^\nu, \quad n \geq 2.$$

The assumption $u \in \mathbb{P}'_M$ is related to the fact that $\deg(\mathbf{D}_u(p)) = \deg(p) - 1$, for every polynomial p.

The transpose ${}^t\mathbf{D}_u$ of $\mathbf{D}_u$ is defined as follows. For all $(p, w) \in \mathbb{P} \times \mathbb{P}'$, $\langle {}^t\mathbf{D}_u(w), p\rangle = \langle w, \mathbf{D}_u(p)\rangle = \langle w, p' + u\theta_0 p\rangle = \langle -w' + x^{-1}(uw), p\rangle$. Thus, ${}^t\mathbf{D}_u(w) = -w' + x^{-1}(uw)$, $w \in \mathbb{P}'$. For the sake of simplicity, we will denote $\mathbf{D}_u := -{}^t\mathbf{D}_u$, and then we have: $\langle \mathbf{D}_u(w), p\rangle = -\langle w, \mathbf{D}_u(p)\rangle$, $(p, w) \in \mathbb{P} \times \mathbb{P}'$.

The linear operator $\mathbf{D}_u$ is one-to-one. Indeed, if $\mathbf{D}_u(w) = 0$, $w \in \mathbb{P}'$, then $\langle w, \mathbf{D}_u(x^{n+1})\rangle = 0$, $n \geq 0$. Hence, $w = 0$, since we have $\deg(\mathbf{D}_u(x^{n+1})) = n$.

The proof of the following operational rules is a straightforward exercise.

PROPOSITION 2.1 *For $v \in \mathbb{P}'$ and $(f, g) \in \mathbb{P}^2$, we have*

$$\boldsymbol{D}_u(fg) = \boldsymbol{D}_u(f)g + f\boldsymbol{D}_u(g) + u\theta_0(fg) - (u\theta_0 f)g - f(u\theta_0 g), \tag{2.1}$$

$$\boldsymbol{D}_u(fv) = f\boldsymbol{D}_u(v) + \boldsymbol{D}_u(f)v + (v\theta_0 f)u - (u\theta_0 f)v. \tag{2.2}$$

For an MPS $\{B_n\}_{n\geq 0}$ and $u \in \mathbb{P}'_M$, we can define

$$B_n^{[1]}(x; u) := (n + (u)_0 + 1)^{-1}(\mathbf{D}_u B_{n+1})(x), \quad n \geq 0. \tag{2.3}$$

Clearly, $\{B_n^{[1]}(.; u)\}_{n\geq 0}$ is an MPS, $\deg B_n^{[1]}(.; u) = n$, $n \geq 0$. If $\{w_n^{[1]}(u)\}_{n\geq 0}$ denotes the dual sequence of $\{B_n^{[1]}(.; u)\}_{n\geq 0}$, then we have the following.

LEMMA 2.2

$$\boldsymbol{D}_u(w_n^{[1]}(u)) = -(n + (u)_0 + 1)w_{n+1}, \quad n \geq 0. \tag{2.4}$$

Proof Let $n \geq 0$ be a fixed integer. Since $\langle w_n^{[1]}(u), B_m^{[1]}(.; u)\rangle = \delta_{n,m}$, then $\langle \mathbf{D}_u(w_n^{[1]}(u)), B_{m+1}\rangle = -(m + (u)_0 + 1)\delta_{n,m} = -(n + (u)_0 + 1)\delta_{n,m}$, $m \geq 0$. In addition, $\langle \mathbf{D}_u(w_n^{[1]}(u)), 1\rangle = 0$. Hence, $\mathbf{D}_u(w_n^{[1]}(u)) = -(n + (u)_0 + 1)w_{n+1}$. ■

2.2. *The D_u-semiclassical MOPS*

We start by recalling the notion of the quasi-orthogonality and some of its useful results (see [5,13]).

Definition 2.3 *Let $v \in \mathbb{P}'$ and σ is a non-negative integer. A MPS $\{B_n\}_{n\geq 0}$ is said to be quasi-orthogonal of order σ with respect to v, if*

$$\langle v, x^m B_n(x)\rangle = 0, \quad 0 \leq m \leq n-\sigma-1, \quad n \geq \sigma+1, \tag{2.5}$$

$$\exists r \geq \sigma, \quad \langle v, x^{r-\sigma} B_r(x)\rangle \neq 0. \tag{2.6}$$

The MPS $\{B_n\}_{n\geq 0}$ is said to be strictly quasi-orthogonal of order σ with respect to v, if it satisfies (2.5), *and*

$$\langle v, x^{n-\sigma} B_n\rangle \neq 0, \quad n \geq \sigma. \tag{2.7}$$

Remark 2.4

(i) Here, v is not assumed to be quasi-definite. In general, a quasi-orthogonal sequence can not be strictly quasi-orthogonal.
(ii) In particular, a strictly quasi-orthogonal sequence of order zero with respect to v is an orthogonal sequence with respect to v.

In order to give a suitable definition of a $\mathbf{D}_u-$semiclassical MOPS, we need the following results.

Lemma 2.5 *Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to w_0. Then, for any integer $\sigma \geq 0$ there exists $A_{\sigma+1} \in \mathbb{P}$, $\deg A_{\sigma+1} = \sigma+1$, such that*

$$\langle w_0, A_{\sigma+1}\rangle = 0, \tag{2.8}$$

$$\langle w_0, xA_{\sigma+1}\rangle \neq -n,\ n \geq 0. \tag{2.9}$$

Proof Assume that $\{B_n\}_{n\geq 0}$ is a MOPS with respect to w_0. So, w_0 is quasi-definite and hence $\Delta_1(w_0) \neq 0$, i.e. $(w_0)_2 - (w_0)_1^2 \neq 0$.

If $\sigma = 0$, we can take $A_1(x) = \lambda(x-(w_0)_1)$, with $\lambda \neq -n((w_0)_2-(w_0)_1^2)^{-1}$, $n \geq 0$. We get $\langle w_0, A_1\rangle = \lambda((w_0)_1-(w_0)_1) = 0$ and $\langle w_0, xA_1\rangle = \lambda((w_0)_2-(w_0)_1^2) \neq -n,\ n \geq 0$.

If $\sigma \geq 1$, we can take $A_{\sigma+1}(x) = x^{\sigma+1} + \theta(x-(w_0)_1) - (w_0)_{\sigma+1}$, with $\theta \neq (-n - (w_0)_{\sigma+2} + (w_0)_{\sigma+1}(w_0)_1)((w_0)_2-(w_0)_1^2)^{-1}$, $n \geq 0$. Then $\langle w_0, A_{\sigma+1}\rangle = 0$, and $\langle w_0, xA_{\sigma+1}\rangle = (w_0)_{\sigma+2} + \theta((w_0)_2-(w_0)_1^2) - (w_0)_{\sigma+1}(w_0)_1 \neq -n,\ n \geq 0$. ■

Proposition 2.6 *For any MOPS $\{B_n\}_{n\geq 0}$ with respect to w_0 and any integer $\sigma \geq 0$, there exists $A_{\sigma+1} \in \mathbb{P}$, $\deg A_{\sigma+1} = \sigma+1$, such that for any $c \in \mathbb{C}$, the MPS $\{B_n^{[1]}(.;u)\}_{n\geq 0}$, where $u = -\delta_c + (x-c)A_{\sigma+1}w_0$, is quasi-orthogonal of order σ with respect to δ_c.*

Proof Assume that $\{B_n\}_{n\geq 0}$ is a MOPS with respect to w_0 and σ a non-negative integer. From Lemma 2.5, there exists $A_{\sigma+1} \in \mathbb{P}$, $\deg A_{\sigma+1} = \sigma+1$ such that (2.8) and (2.9) hold. Given $c \in \mathbb{C}$ and let $u = -\delta_c + (x-c)A_{\sigma+1}w_0 \in \mathbb{P}'$. Since, $(u)_0 = -1 + \langle w_0, xA_{\sigma+1}\rangle - c\langle w_0, A_{\sigma+1}\rangle = -1 + \langle w_0, xA_{\sigma+1}\rangle \neq -n,\ n \geq 1$, then $\{B_n^{[1]}(.;u)\}_{n\geq 0}$ is an MPS, $\deg B_n^{[1]}(.;u) = n,\ n \geq 0$.

Next, let us show that $B_n^{[1]}(c;u) = 0,\ n \geq \sigma+1$, and $B_\sigma^{[1]}(c;u) \neq 0$, i.e. $\langle \delta_c, x^m B_n^{[1]}(x;u)\rangle = 0,\ 0 \leq m \leq n-\sigma-1,\ n \geq \sigma+1$, and $\langle \delta_c, B_\sigma^{[1]}(x;u)\rangle \neq 0$, and then $\{B_n^{[1]}(.;u)\}_{n\geq 0}$ is quasi-orthogonal of order σ with respect to δ_c.

Indeed, by (2.3), $B_n^{[1]}(c;u) = (n+(u)_0+1)^{-1}\langle \delta_c' - x^{-1}(u\delta_c), B_{n+1}\rangle$. Since $\delta_c^2 = -x\delta_c'$ and $((x-c)A_{\sigma+1}w_0)\delta_c = xA_{\sigma+1}w_0$, then $u\delta_c = x(\delta_c' + A_{\sigma+1}w_0)$, and hence $x^{-1}(u\delta_c) = \delta_c' +$

$A_{\sigma+1}w_0$. So, $B_n^{[1]}(c;u) = -(n+(u)_0+1)^{-1}\langle w_0, A_{\sigma+1}B_{n+1}\rangle$, $n \geq 0$. Using the orthogonality of $\{B_n\}_{n\geq 0}$, we get $B_n^{[1]}(c;u) = 0$, $n \geq \sigma+1$, and $B_\sigma^{[1]}(c;u) = -K\langle w_0, B_{\sigma+1}^2\rangle/(\sigma+(u)_0+1) \neq 0$, where K is the leading coefficient of $A_{\sigma+1}$. ■

To avoid analogous situations to those found in the above proposition, we give the following definition of a $\mathbf{D}_u$-semiclassical MOPS.

DEFINITION 2.7 *Let $u \in \mathbb{P}'_M$. An MPS $\{B_n\}_{n\geq 0}$ is said to be $\boldsymbol{D}_u$-semiclassical, when it is orthogonal and such that the MPS $\{B_n^{[1]}(.;u)\}_{n\geq 0}$ given by* (2.3) *is quasi-orthogonal of order σ with respect to a weak-regular linear functional v.*

The quasi-definite linear functional w_0 corresponding to $\{B_n\}_{n\geq 0}$ is also said to be $\boldsymbol{D}_u$-semiclassical.

As a straightforward consequence, the non-singular **D**-Laguerre–Hahn MOPS of class zero analogous to the Hermite polynomial sequence, is $\mathbf{D}_{\mathcal{U}(\xi)}$-semiclassical, according to (1.13) and Remark 2.4(ii).

3. Characterization of a $\mathbf{D}_u$-semiclassical linear functional by means of a distributional equation

To give a characterization of a $\mathbf{D}_u$-semiclassical linear functional by a distributional equation, we will need some information concerning u and v.

LEMMA 3.1 *Under the assumption of the previous definition, we have the following.*

(i) *There exists a non-zero $\phi \in \mathbb{P}$, $\deg\phi = t \leq \sigma+2$, such that $v = \phi w_0$.*
(ii) *If $(v)_0 \neq 0$, then there exists $\hat{B} \in \mathbb{P}$, $\deg \hat{B} \leq \sigma+2$, such that $u = \hat{B}w_0$.*

Proof From the assumption, there exist a weak-regular linear functional v and a non-negative integer $r \geq \sigma$, such that

$$\langle v, x^m B_n^{[1]}(x;u)\rangle = 0, \quad 0 \leq m \leq n-\sigma-1,\ n \geq \sigma+1, \tag{3.1}$$

$$\langle v, x^{r-\sigma} B_r^{[1]}(x;u)\rangle \neq 0. \tag{3.2}$$

Applying $\mathbf{D}_u$ on both sides of (1.7) and using (2.3) and (2.1), we obtain

$$\begin{aligned} B_{n+1} + \langle u, B_{n+1}\rangle &= (n+(u)_0+2)B_{n+1}^{[1]}(x;u) + (n+(u)_0+1)\beta_{n+1}B_n^{[1]}(x;u) \\ &\quad + (n+(u)_0)\gamma_{n+1}B_{n-1}^{[1]}(x;u) - (n+(u)_0+1)xB_n^{[1]}(x;u), \quad n \geq 0. \end{aligned} \tag{3.3}$$

Then, $\langle v, x^m B_{n+1}\rangle + \langle u, B_{n+1}\rangle(v)_m = 0$, $n \geq m+\sigma+2$, $m \geq 0$, due to (3.1) and (3.3). So, $\langle x^m v + (v)_m u, B_n\rangle = 0$, $n \geq \sigma+3+m$, $m \geq 0$. Using the orthogonality of $\{B_n\}_{n\geq 0}$, there exists a polynomial sequence $\{\Omega_{m+\sigma+2}(x)\}_{n\geq 0}$, $\deg\Omega_{m+\sigma+2} \leq m+\sigma+2$, such that

$$x^m v + (v)_m u = \Omega_{m+\sigma+2}w_0, \quad m \geq 0, \tag{3.4}$$

where $\Omega_{m+\sigma+2}(x) = \sum_{\nu=0}^{m+\sigma+2}\langle x^m v + (v)_m u, B_\nu\rangle\langle w_0, B_\nu^2\rangle^{-1}B_\nu(x)$, $m \geq 0$.

For $m = 0$, (3.4) gives

$$v + (v)_0 u = \Omega_{\sigma+2} w_0. \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

$$((v)_m - (v)_0 x^m) u = (\Omega_{m+\sigma+2} - x^m \Omega_{\sigma+2}) w_0, \quad m \geq 0. \tag{3.6}$$

So, we need to discuss two cases.

(A) $(v)_0 = 0$. Directly, (i) holds, with $\phi = \Omega_{\sigma+2}$, due to (3.5) with $(v)_0 = 0$, and the weak-regularity of v to obtain $\deg \phi \geq 0$. In this case, (3.6) becomes

$$(v)_m u = (\Omega_{m+\sigma+2} - x^m \Omega_{\sigma+2}) w_0, \quad m \geq 0. \tag{3.7}$$

From the weak-regularity of v, there exists an integer $k \geq 1$ such that

$$k = \min\{m \geq 1 \mid (v)_m \neq 0\}. \tag{3.8}$$

Thus, for $m = k$, (3.7) yields

$$u = \hat{B} w_0, \tag{3.9}$$

where $\hat{B}(x) = (v)_k^{-1} (\Omega_{k+\sigma+2}(x) - x^k \Omega_{\sigma+2}(x))$.

Besides, by (3.9), (3.7) and the quasi-definiteness of w_0, we obtain

$$\Omega_{m+\sigma+2}(x) = x^m \Omega_{\sigma+2}(x) + (v)_m \hat{B}(x), \quad m \geq 0. \tag{3.10}$$

Hence, if $(v)_0 = 0$ then (ii) holds.

(B) $(v)_0 \neq 0$. There exists an integer $l \geq 2$ such that $(v)_l \neq ((v)_0)^{1-l} (v)_1^l$. Otherwise, if we set $c = (v)_1 (v)_0^{-1}$, then $(v)_m = (v)_0 c^m = 0$, $m \geq 0$, i.e. $v = (v)_0 \delta_c$. This contradicts the weak-regularity of v.

By taking successively $m = 1$ and $m = l$ in (3.6), we obtain

$$((v)_1 - (v)_0 x) u = (\Omega_{\sigma+3} - x \Omega_{\sigma+2}) w_0, \tag{3.11}$$

$$((v)_l - (v)_0 x^l) u = (\Omega_{l+\sigma+2} - x^l \Omega_{\sigma+2}) w_0. \tag{3.12}$$

Since, $(v)_l \neq (v)_0^{1-l} (v)_1^l$, then $(v)_l - (v)_0 x^l$ and $(v)_1 - (v)_0 x$ are coprimes. From Bézout's identity, there exist two polynomials $E(x)$ and $F(x)$ such that

$$((v)_l - (v)_0 x^l) E(x) + ((v)_1 - (v)_0 x) F(x) = 1. \tag{3.13}$$

By (3.11)–(3.13), we can easily deduce that

$$u = \hat{B} w_0, \tag{3.14}$$

where $\hat{B}(x) = E(x)(\Omega_{\sigma+3}(x) - x \Omega_{\sigma+2}(x)) + F(x)(\Omega_{l+\sigma+2}(x) - x^l \Omega_{\sigma+2}(x))$.

Hence, if $(v)_0 \neq 0$, then (ii) holds.

By inserting (3.14) in (3.6) and using the quasi-definiteness of w_0, we get $((v)_m - (v)_0 x^m) \hat{B}(x) = \Omega_{m+\sigma+2}(x) - x^m \Omega_{\sigma+2}(x)$, $m \geq 0$. The consideration of the degrees on both sides of the previous formula yields $\deg \hat{B} \leq \sigma + 2$. Substituting (3.14) into (3.5), we get $v = \phi w_0$ with $\phi(x) = \Omega_{\sigma+2}(x) - (v)_0 \hat{B}(x)$, and $0 \leq \deg \phi \leq \sigma + 2$. Hence, if $(v)_0 \neq 0$, then (i) holds. ■

Next we will give a characterization of a $\mathbf{D}_u$-semiclassical MOPS.

THEOREM 3.2 *Let $u \in \mathbb{P}'_M$ and let $\{B_n\}_{n\geq 0}$ be an MOPS with respect to w_0. The following statements are equivalent.*

(i) $\{B_n\}_{n\geq 0}$ *is* $\boldsymbol{D}_u$*-semiclassical.*
(ii) *There exists* $(\Phi, \Psi, \hat{B}) \in \mathbb{P}^3$, Φ *monic,* $\deg \Phi = t$, $\deg \Psi = p \geq 1$, *and* $\deg \hat{B} \leq k + \sigma + 2$, *where* $k = \min\{m \geq 0 \mid (\Phi w_0)_m \neq 0\}$, *and* $\sigma = \max(t-2, p-1)$, *such that* w_0 *is a solution of the distributional equation:*

$$\boldsymbol{D}_u(\Phi w_0) + \Psi w_0 = 0 \quad \text{where } u = \hat{B} w_0. \tag{3.15}$$

Proof (i) $\Rightarrow$ (ii). From Lemma 3.1(i), let us write $\phi(x) = \lambda\Phi(x)$, where λ is a normalization constant and Φ is a monic polynomial with $\deg \Phi = t \leq \sigma + 2$. From (2.3), we get

$$\langle \mathbf{D}_u(\Phi w_0), B_{n+1}\rangle = -\lambda^{-1}(n + (u)_0 + 1)\langle v, B_n^{[1]}(x; u)\rangle, \quad n \geq 0. \tag{3.16}$$

But, we have $\langle v, B_n^{[1]}(x; u)\rangle = 0$, $n \geq \sigma + 1$, according to (3.1), with $m = 0$. As a consequence, since $v \neq 0$, there exists an integer p, $1 \leq p \leq \sigma + 1$, such that $\langle v, B_{p-1}^{[1]}(x; u)\rangle \neq 0$, $\langle v, B_n^{[1]}(x; u)\rangle = 0$, $n \geq p$. Using (3.16), it follows that $\langle \mathbf{D}_u(\Phi w_0), B_p\rangle \neq 0$ and $\langle \mathbf{D}_u(\Phi w_0), B_n\rangle = 0$, $n \geq p + 1$. So, from the orthogonality of $\{B_n\}_{n\geq 0}$, there exists a polynomial $\Psi(x) = \sum_{\nu=1}^{p}(\nu + (u)_0)\lambda^{-1}\langle v, B_{\nu-1}^{[1]}(.; u)\rangle\langle w_0, B_\nu^2\rangle^{-1} B_\nu(x)$, $\deg \Psi = p \geq 1$, such that $\mathbf{D}_u(\Phi w_0) + \Psi w_0 = 0$. Hence, (ii) holds.

Before showing that (ii) $\Rightarrow$ (i), we can determine the nature of the quasi-orthogonality, i.e. if it is strict or not.

For every integer $n \geq \sigma$, let us consider,

$$(n + (u)_0 + 1)\langle v, x^{n-\sigma} B_n^{[1]}(x; u)\rangle = -\lambda\langle \mathbf{D}_u(x^{n-\sigma}\Phi w_0), B_{n+1}\rangle. \tag{3.17}$$

But, from (2.2), we can write $\mathbf{D}_u(x^{n-\sigma}\Phi w_0) = x^{n-\sigma}\mathbf{D}_u(\Phi w_0) + \Phi\mathbf{D}_u(x^{n-\sigma})w_0 + [((\Phi w_0)\theta_0 x^{n-\sigma})\hat{B} - ((\hat{B} w_0)\theta_0 x^{n-\sigma})\Phi]w_0$. So, in view of (3.15), we obtain

$$\mathbf{D}_u(x^{n-\sigma}\phi w_0) = \Lambda_{n+1} w_0, \quad n \geq \sigma \tag{3.18}$$

where

$$\begin{aligned}\Lambda_{n+1}(x) = &-x^{n-\sigma}\Psi(x) + \Phi(x)\mathbf{D}_u(x^{n-\sigma}) \\ &+ ((\Phi w_0)\theta_0 x^{n-\sigma})(x)\hat{B}(x) - ((\hat{B} w_0)\theta_0 x^{n-\sigma})(x)\Phi(x).\end{aligned} \tag{3.19}$$

Note that $\deg \Lambda_{n+1} \leq n + 1$, $n \geq \sigma$. Indeed, we will analyse the two previous situations.

(A) $(v)_0 = 0$. By Lemma 3.1, $\deg \Phi = t \leq \sigma + 2$, $\deg \Psi = p \leq \sigma + 1$ and $\deg \hat{B} = \hat{r} \leq k + \sigma + 2$, where k is given by (3.8).

Note that $\deg((\Phi w_0)\theta_0 x^{n-\sigma}) = n - \sigma - k - 1$, because we have $v = \lambda\Phi w_0$, $(v)_0 = \cdots = (v)_{k-1} = 0$ and $(v)_k \neq 0$. Since $\deg \mathbf{D}_u(x^{n-\sigma}) = n - \sigma - 1$, and $\deg((\hat{B} w_0)\theta_0 x^{n-\sigma}) \leq n - \sigma - 1$, then $\deg \Lambda_{n+1} \leq n + 1$, $n \geq \sigma$.

(B) $(v)_0 \neq 0$. By Lemma 3.1, $\deg \Phi = t \leq \sigma + 2$, $\deg \Psi = p \leq \sigma + 1$ and $\deg \hat{B} = \hat{r} \leq \sigma + 2$. Besides, $\deg((\Phi w_0)\theta_0 x^{n-\sigma}) = n - \sigma - 1$, because we have $(\Phi w_0)_0 = \lambda^{-1}(v)_0 \neq 0$. Therefore, since $\deg \mathbf{D}_u(x^{n-\sigma}) = n - \sigma - 1$ and $\deg(\hat{B} w_0)\theta_0 x^{n-\sigma} \leq n - \sigma - 1$, we obtain $\deg \Lambda_{n+1} \leq n + 1$, $n \geq \sigma$.

By inserting (3.18) in (3.17), we obtain

$$(n+(u)_0+1)\langle v, x^{n-\sigma}B_n^{[1]}(x;u)\rangle = -\lambda\langle w_0, \Lambda_{n+1}B_{n+1}(x)\rangle, \quad n\geq\sigma. \tag{3.20}$$

Denoting by ρ_n the coefficient of the monomial x^{n+1} in $\Lambda_{n+1}(x)$, $n\geq\sigma$, and using (3.19) we obtain

$$\rho_n = (n-\sigma)\frac{\Phi^{(\sigma+2)}(0)}{(\sigma+2)!} - \frac{\Psi^{(\sigma+1)}(0)}{(\sigma+1)!} + (v)_0\frac{\hat{B}^{(\sigma+2)}(0)}{(\sigma+2)!\lambda}, \quad n\geq\sigma. \tag{3.21}$$

Hence, (3.20) can be written as

$$(n+(u)_0+1)\langle v, x^{n-\sigma}B_n^{[1]}(x;u)\rangle = -\lambda\rho_n\langle w_0, B_{n+1}^2\rangle, \quad n\geq\sigma. \tag{3.22}$$

For $n=r$ in (3.22), where r is given by (3.2), $(r\geq\sigma)$, since we have $\langle v, x^{r-\sigma}B_r^{[1]}(x;u)\rangle\neq 0$, then $\rho_r\neq 0$ and so that $\deg\Lambda_{r+1}=r+1$. Thus,

$$\sigma = \max(t-2, p-1). \tag{3.23}$$

Indeed, first note that we have $\sigma\geq\max(t-2,p-1)$. If we assume that $\sigma>\max(t-2,p-1)$, then the analysis of the degrees of the polynomial Λ_{r+1}, yields $\deg\Lambda_{r+1}\leq r$. This is a contradiction. Hence, (3.23) holds.

As a consequence, we can analyse two situations.

When $t=\sigma+2$, then

$$\rho_n = n-\sigma-\frac{\Psi^{(\sigma+1)}(0)}{(\sigma+1)!} + (v)_0\frac{\hat{B}^{(\sigma+2)}(0)}{(\sigma+2)!\lambda}, \quad n\geq\sigma. \tag{3.24}$$

The MPS $\{B_n^{[1]}(x;u)\}_{n\geq 0}$ is strictly quasi-orthogonal of order σ if and only if $-\Psi^{(\sigma+1)}(0)/(\sigma+1)! + (v)_0(\hat{B}^{(\sigma+2)}(0))/\,((\sigma+2)!\lambda)\neq -n,\ n\geq 0$.

When $t<\sigma+2$, then

$$\rho_n = -\frac{\Psi^{(\sigma+1)}(0)}{(\sigma+1)!} + (v)_0\frac{\hat{B}^{(\sigma+2)}(0)}{(\sigma+2)!\lambda} = \rho_r\neq 0, \quad n\geq\sigma. \tag{3.25}$$

Thus, the MPS $\{B_n^{[1]}(x;u)\}_{n\geq 0}$ is strictly quasi-orthogonal of order σ.

(ii) $\Rightarrow$ (i). We have $(n+(u)_0+1)\langle\Phi w_0, x^m B_n^{[1]}(x;u)\rangle = -\langle\mathbf{D}_u(x^m\Phi w_0), B_{n+1}\rangle$, for all non-negative integers m and n. From (2.2) and (3.15) as above, we have $\mathbf{D}_u(x^m\Phi w_0)=\Lambda_{m+\sigma+1}w_0$, with $\Lambda_{m+\sigma+1}(x) = -x^m\Psi(x)+\Phi(x)\mathbf{D}_{u,\tau}(x^m) + ((\Phi w_0)\theta_0 x^m)\hat{B}(x) - ((\hat{B}w_0)\theta_0 x^m)\Phi(x)$, and where $\deg\Lambda_{m+\sigma+1}\leq m+\sigma+1$, $m\geq 0$. Thus,

$$(n+(u)_0+1)\langle\Phi w_0, x^m B_n^{[1]}(x;u)\rangle = -\langle w_0, \Lambda_{m+\sigma+1}B_{n+1}\rangle. \tag{3.26}$$

Using the orthogonality of $\{B_n\}_{n\geq 0}$, we obtain

$$\langle\Phi w_0, x^m B_n^{[1]}(x;u)\rangle = 0, \quad 0\leq m\leq n-\sigma-1,\ n\geq\sigma+1. \tag{3.27}$$

For $m=n-\sigma$, in (3.26) we get

$$(n+(u)_0+1)\langle\Phi w_0, x^{n-\sigma}B_n^{[1]}(x;u)\rangle = -\rho_n\langle w_0, B_{n+1}^2\rangle, \quad n\geq\sigma, \tag{3.28}$$

where $\rho_n = (n-\sigma)(\Phi^{(\sigma+2)}(0))/((\sigma+2)!) - (\Psi^{(\sigma+1)}(0))/((\sigma+1)!) + (\Phi w_0)_0(\hat{B}^{(\sigma+2)}(0))/((\sigma+2)!)$, $n\geq\sigma$.

Note that there exists an integer $r \geq \sigma$ such that $\langle \Phi w_0, x^{r-\sigma} B_r^{[1]}(x;u)\rangle \neq 0$. Otherwise, $\langle \Phi w_0, x^{n-\sigma} B_n^{[1]}(x;u)\rangle = 0,\ \ n \geq \sigma$. Then, according to (3.28) we get $\rho_n = 0,\ \ n \geq \sigma$. This requires that $t \leq \sigma + 1 = p$ and $-(\sigma+2)\Psi^{(\sigma+1)}(0) + (\Phi w_0)_0 \hat{B}^{(\sigma+2)}(0) = 0$. So, $\langle \Phi w_0, x^m B_n^{[1]}(x;u)\rangle = 0,\ 0 \leq m \leq n-\sigma,\ n \geq \sigma$. In this way, there exists $(l,r) \in N^2$, with $1 \leq l \leq \sigma$ and $r \geq \sigma - l$, such that

$$\langle \Phi w_0, x^m B_n^{[1]}(x;u)\rangle = 0, \quad 0 \leq m \leq n-\sigma+l-1,\ n \geq \sigma - l + 1, \tag{3.29}$$

$$\langle \Phi w_0, x^{r-\sigma+l} B_r^{[1]}(x;u)\rangle \neq 0. \tag{3.30}$$

Otherwise, $\langle \Phi w_0, x^m B_n^{[1]}(x;u)\rangle = 0$, $0 \leq m \leq n$. In particular, for $m = 0$, $\langle \Phi w_0, B_n^{[1]}(x;u)\rangle = 0,\ \ n \geq 0$, i.e. $\Phi w_0 = 0$, where $\Phi \neq 0$. This contradicts the quasi-definiteness of the linear functional w_0.

Hence, from (3.29) and (3.30), $\{B_n^{[1]}(x;u)\}_{n\geq 0}$ will be quasi-orthogonal of order $\sigma' = \sigma - l = p - 1 - l$ with respect to $v = \Phi w_0$.

In addition, $(p + (u)_0)\langle \Phi w_0, B_{p-1}^{[1]}(.;u)\rangle = \langle \mathbf{D}_u(\Phi w_0), B_p\rangle = -\langle \Psi w_0, B_p\rangle \neq 0$ and $\langle \Phi w_0, B_n^{[1]}(x;u)\rangle = 0$, $n \geq \sigma - l + 1$, due to (3.29), where $m = 0$. Since $\langle \Phi w_0, B_{p-1}^{[1]}(x;u)\rangle \neq 0$, then $\sigma - l + 1 \geq p$. But $\sigma = p - 1$ and, as a consequence, $p - l \geq p$, i.e. $l \leq 0$. This contradicts the fact that $l \geq 1$.

Finally, there exists an integer $r \geq \sigma$ such that $\langle \Phi w_0, x^m B_n^{[1]}(x;u)\rangle = 0$, $0 \leq m \leq n - \sigma - 1$ and $\langle \Phi w_0, x^{r-\sigma} B_r^{[1]}(x;u)\rangle \neq 0$. Hence, (i) holds. ■

4. Another characterization of a $\mathbf{D}_u$-semiclassical MOPS

The following result allows us to characterize the $\mathbf{D}_u$-semiclassical monic orthogonal polynomial sequence by a first structure relation.

THEOREM 4.1 *Let $u \in \mathbb{P}'_M$ and $\{B_n\}_{n\geq 0}$ be an MOPS with respect to w_0. The following statements are equivalent.*

(i) $\{B_n\}_{n\geq 0}$ *is $\mathbf{D}_u$-semiclassical.*
(ii) *There exists $(r,\sigma) \in \mathbb{N}^2$, with $0 \leq \sigma \leq r$, such that*

$$\Phi(x) B_n^{[1]}(x;u) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} B_\nu(x),\ \ n \geq \sigma, \tag{4.1}$$

$$\lambda_{r,r-\sigma} \neq 0. \tag{4.2}$$

Proof (i) $\Rightarrow$ (ii). First, we always have $\Phi B_n^{[1]}(.;u) = \sum_{\nu=0}^{n+t} \lambda_{n,\nu} B_\nu,\ \ n \geq 0$, with $t = \deg \Phi$ and $\lambda_{n,\nu} = \langle w_0, B_\nu \Phi B_n^{[1]}(.;u)\rangle \langle w_0, B_\nu^2\rangle^{-1}$, $0 \leq \nu \leq n+t$, $n \geq 0$. But, $\{B_n^{[1]}(x;u)\}_{n\geq 0}$ is quasi-orthogonal of order $\sigma = \max(t-2, p-1)$ with respect to Φw_0, according to Definition 2.7 and Theorem 3.2. Therefore, there exists an integer $r \geq \sigma$ such that $\langle \Phi w_0, B_\nu B_n^{[1]}(x;u)\rangle = 0,\ 0 \leq \nu \leq n-\sigma-1,\ n \geq \sigma+1$ and $\langle \Phi w_0, B_{r-\sigma} B_r^{[1]}(x;u)\rangle \neq 0$. Thus, we have $\lambda_{n,\nu} = 0$, $0 \leq \nu \leq n-\sigma-1,\ n \geq \sigma+1$, and $\lambda_{r,r-\sigma} \neq 0$. Hence, (4.1) and (4.2) hold, with $\lambda_{r,r-\sigma} = \langle w_0, B_{r-\sigma}^2\rangle^{-1} \langle \Phi w_0, B_{r-\sigma} B_r^{[1]}(x;u)\rangle \neq 0$.

(ii) $\Rightarrow$ (i). From the assumption and the orthogonality of $\{B_n\}_{n\geq 0}$, we get $\langle \Phi w_0, B_m B_n^{[1]}(.;u)\rangle = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} \langle w_0, B_\nu B_m\rangle = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} \langle w_0, B_\nu^2\rangle \delta_{m,\nu},\ \ n \geq \sigma$. Then, $\langle \Phi w_0, B_m B_n^{[1]}(.;u)\rangle = 0$, $0 \leq m \leq n-\sigma-1,\ \ n \geq \sigma+1$ and $\langle \Phi w_0, B_{r-\sigma} B_r^{[1]}(.;u)\rangle = \lambda_{r,r-\sigma} \langle w_0, B_{r-\sigma}^2\rangle \neq 0$. Thus, $\{B_n^{[1]}(x;u)\}_{n\geq 0}$ is quasi-orthogonal of order σ with respect to Φw_0. ■

The following result shows that any $\mathbf{D}_u$-semiclassical MOPS is a $\mathbf{D}$-Laguerre–Hahn MOPS.

THEOREM 4.2 *Let $u \in \mathbb{P}'_M$ and $\{B_n\}_{n\geq 0}$ be an MOPS with respect to w_0. The following statements are equivalent.*

(i) $\{B_n\}_{n\geq 0}$ *is a $\mathbf{D}_u$-semiclassical MOPS.*
(ii) *There exists* $(\Phi, \Psi, \hat{B}) \in \mathbb{P}^3$, Φ *monic*, $\deg \Phi = t$, $\deg \Psi = p \geq 1$, $\deg \hat{B} \leq k + \sigma + 2$, *with* $k = \min\{m \geq 0 : (\Phi w_0)_m \neq 0\}$, *and* $\sigma = \max(t-2, p-1)$, *such that* $\hat{B} w_0 \in \mathbb{P}'_M$, *and where w_0 satisfies*

$$(\Phi w_0)' + \Psi_1 w_0 - (\Phi \hat{B})(x^{-1} w_0^2) = 0, \tag{4.3}$$

where

$$\Psi_1(x) = \Psi(x) + \Phi(x)(w_0\theta_0\hat{B})(x) + \hat{B}(x)(w\theta_0\Phi)(x). \tag{4.4}$$

Proof (i) $\Rightarrow$ (ii). Assuming (i). By Theorem 3.2(ii), there exists $(\Phi, \Psi, \hat{B}) \in \mathbb{P}$, Φ monic, $\deg \Phi = t$, $\deg \Psi = p \geq 1$, $\deg \hat{B} \leq k + \sigma + 2$, with $\sigma = \max(t-2, p-1)$ and $k = \min\{m \geq 0 \mid (\Phi w_0)_m \neq 0\}$, such that $\mathbf{D}_u(\Phi w_0) + \Psi w_0 = 0$, with $u = \hat{B} w_0$. But, $\mathbf{D}_u(\Phi w_0) = (\Phi w_0)' - x^{-1} u(\Phi w_0)$, then

$$(\Phi w_0)' - x^{-1}((\hat{B} w_0)(\Phi w_0)) + \Psi w_0 = 0. \tag{4.5}$$

Using (1.4) we can write

$$(\hat{B} w_0)(\Phi w_0) = \Phi \hat{B} w_0^2 - x F w_0, \tag{4.6}$$

where

$$F(x) := \Phi(x)(w_0\theta_0\hat{B})(x) + \hat{B}(x)(w_0\theta_0\Phi)(x). \tag{4.7}$$

From (4.6), (1.3) and (1.6), we get

$$x^{-1}((\hat{B} w_0)(\Phi w_0)) = (\Phi \hat{B}) x^{-1} w_0^2 - F w. \tag{4.8}$$

Then, (4.5) becomes $(\Phi w_0)' + (F + \Psi) w_0 - (\Phi \hat{B})(x^{-1} w_0^2) = 0$. Hence, (4.3) holds, with $\Psi_1(x) = F(x) + \Psi(x)$.

(ii) $\Rightarrow$ (i). Assuming (ii), let $F(x) = \Phi(x)(w_0\theta_0\hat{B})(x) + \hat{B}(x)(w_0\theta_0\Phi)(x)$. Setting $u = \hat{B} w_0$, $\Psi(x) = \Psi_1(x) - F(x)$ and using (4.3), (4.4) and (4.8), we get $\mathbf{D}_u(\Phi w_0) + \Psi w_0 = (\Phi \hat{B})(x^{-1} w_0^2) - x^{-1}((\hat{B} w_0)(\Phi w_0)) - F w_0 = 0$. Hence, by Theorem 3.2(i) holds. ∎

COROLLARY 4.3 *A $\mathbf{D}$-Laguerre–Hahn linear functional w_0 satisfying $(\Phi w_0)' + (\Psi_1 w_0) + B(x^{-1} w_0^2) = 0$, with $(\Phi, \Psi_1, B) \in \mathbb{P}^3$ and Φ monic, is $\mathbf{D}_u$-semiclassical if and only if the following conditions hold.*

(i) $\Phi(x)$ *divides* $B(x)$. *We will write* $B(x) = -\Phi(x)\hat{B}(x)$.
(ii) $\deg \hat{B} \leq k + \sigma + 2$, *where* $k = \min\{m \geq 0 \mid (\Phi w)_m \neq 0\}$, $\sigma = \max(\deg(\Phi) - 2, \deg(\Psi) - 1)$, *and* $\Psi = \Psi_1 - \Phi(w_0\theta_0\hat{B}) - \hat{B}(w_0\theta_0\Phi)$.
(iii) $(\hat{B} w_0)_0 \neq -n,\ n \geq 1$.

In this case, $u = \hat{B} w_0$.

As an example, let us consider the singular $\mathbf{D}$-Laguerre–Hahn MOPS of class zero analogous to the Laguerre's one which we denote by $\{l_n(.;\xi)\}_{n\geq 0}$ with $\xi = (\alpha, \lambda, \rho)$ (see [1,4]). We will denote by $\ell(\xi)$ the corresponding normalized linear functional. In Table 1, we summarize the characteristic elements of the MOPS $\{l_n(.;\xi)\}_{n\geq 0}$.

Table 1. The singular Laguerre–Hahn sequences of class zero analogous to Laguerre's ones with parameter $\xi = (\alpha, \lambda, \rho)$.

$\Phi(x) = x,\ \Psi_1(x) = -x + \alpha - 1$
$B(x) = x^2 + (2(1-\alpha) - \lambda)x + \alpha(\alpha + \lambda - 1) - \rho$
$\beta_0 = \alpha + \lambda - 1, \quad \beta_{n+1} = 2n + \alpha + 1,\ n \geq 0$
$\gamma_1 = \rho,\ \gamma_{n+1} = n(n+\alpha),\ n \geq 1$
$C_0(x) = x - \alpha,\ C_{n+1}(x) = -x + 2n + \alpha,\ n \geq 0$
$D_0(x) = 0,\ D_{n+1}(x) = -1,\ n \geq 0$
$\lambda \in \mathbb{C},\ \rho \neq 0,\ \alpha \neq -n,\ n \geq 1$

Table 2. The singular Laguerre–Hahn sequences of class zero analogous to the Laguerre's ones with parameter $\xi^* = (\alpha, \lambda, \alpha(\alpha + \lambda - 1))$.

$\Phi(x) = x,\ \Psi(x) = x - (\alpha + \lambda - 1), \quad \hat{B}(x) = -x + 2(\alpha - 1) + \lambda$
$\beta_0 = \alpha + \lambda - 1, \quad \beta_{n+1} = 2n + \alpha + 1,\ n \geq 0$
$\gamma_1 = \alpha(\alpha + \lambda - 1), \quad \gamma_{n+1} = n(n+\alpha),\ n \geq 1$
$C_0(x) = x - \alpha,\ C_{n+1}(x) = -x + 2n + \alpha,\ n \geq 0$
$D_0(x) = 0,\ D_{n+1}(x) = -1,\ n \geq 0$
$\lambda \neq 1 - \alpha,\ \alpha \neq -n,\ n \geq 0$

Using Corollary 4.3, the condition (i) is equivalent to $\rho = \alpha(\alpha + \lambda - 1)$. Then, $\hat{B}(x) = -x + 2(\alpha - 1) + \lambda$, $\Psi(x) = x - (\alpha - 1 + \lambda)$, and $\sigma = 0$. Besides, $(xw_0)_0 = \alpha + \lambda - 1 \neq 0$, $\rho = \alpha(\alpha + \lambda - 1) \neq 0$, then $k = 0$. Therefore, (ii) holds because $\deg \hat{B} = 1 \leq k + \sigma + 2 = 2$. Finally, since $(\hat{B}\ell(\xi))_0 = \alpha - 1$, the condition (iii) is equivalent to $\alpha \neq -n,\ n \geq 0$. Thus, the MOPS $\{l_n(.;\xi^*)\}_{n\geq 0}$, where $\xi^* = (\alpha, \lambda, \alpha(\alpha + \lambda - 1))$, with $\alpha \neq -n,\ n \geq 0$, and $\alpha + \lambda - 1 \neq 0$ is $\mathbf{D}_{\mathcal{U}(\xi^*)}$-semiclassical, with $\mathcal{U}(\xi^*) = \hat{B}\ell(\xi^*)$ and $\hat{B}(x) = -x + 2(\alpha - 1) + \lambda$. The linear functional $\ell(\xi^*)$ satisfies: $\mathbf{D}_{\mathcal{U}(\xi^*)}(x\ell(\xi^*)) + (x - \alpha - \lambda + 1)\ell(\xi^*) = 0$. Also, $\ell(\xi^*)$ is a $\mathbf{D}$-Laguerre–Hahn linear functional of class zero, since it satisfies: $(x\ell(\xi^*))' + (-x + \alpha - 1)\ell(\xi^*) - x(-x + 2(\alpha - 1) + \lambda)(x^{-1}(\ell(\xi^*))^2) = 0$.

In summary, we have the following.

Since the $\mathbf{D}$-Laguerre–Hahn MOPS $\{l_n(.;\xi^*)\}_{n\geq 0}$ satisfies an FSR, then from Table 2 and (1.9), we get

$$\begin{aligned} xl'_{n+1}(x;\xi^*) + x\hat{B}(x)l_n^{(1)}(x;\xi^*) &= (-x + n + \alpha)l_{n+1}(x;\xi^*) \\ &\quad + n(n+\alpha)l_n(x;\xi^*), \quad n \geq 1. \end{aligned} \tag{4.9}$$

But, from the orthogonality of $\{l_n(.;\xi^*)\}_{n\geq 0}$ with respect to $\ell(\xi^*)$, we get

$$\begin{aligned} (\mathcal{U}(\xi^*)\theta_0 l_{n+1}(.;\xi^*))(x) &= l_{n+1}(x;\xi^*) + \hat{B}(x)(\ell(\xi^*)\theta_0 l_{n+1}(.;\xi^*))(x), \\ &= l_{n+1}(x;\xi^*) + \hat{B}(x)l_n^{(1)}(x;\xi^*), \quad n \geq 0. \end{aligned}$$

Then, (4.9) becomes $xl_n^{[1]}(x;\xi^*) = l_{n+1}(x;\xi^*) + n\, l_n(x;\xi^*),\ n \geq 1$, where $\{l_n^{[1]}(.;\xi^*)\}_{n\geq 0}$ is the $\mathbf{D}_{\mathcal{U}(\xi^*)}$-derivative sequence of $\{l_n(.;\xi^*)\}_{n\geq 0}$, defined by

$$(n+\alpha)l_n^{[1]}(x;\xi^*) = l'_{n+1}(x;\xi^*) + (\mathcal{U}(\xi^*)\theta_0 l_{n+1}(.;\xi^*))(x), \quad n \geq 0.$$

In addition, it is easy to see that $xl_0^{[1]}(x;\xi^*) = l_1(x;\xi^*) + (\alpha + \lambda - 1)\, l_0(x;\xi^*)$. Hence, the FSR becomes

$$xl_n^{[1]}(x;\xi^*) = l_{n+1}(x;\xi^*) + \vartheta_n l_n(x;\xi^*), \quad n \geq 0, \tag{4.10}$$

where $\vartheta_n = n,\ n \geq 1$ and $\vartheta_0 = \alpha + \lambda - 1$.

By (4.10) and the orthogonality of $\{l_n(.;\xi^*)\}_{n\geq 0}$ with respect to $\ell(\xi^*)$, we get

$$\langle x\ell(\xi^*), x^m l_n^{[1]}(x;\xi^*)\rangle = 0, \quad 0 \leq m \leq n-1,\ n \geq 1,$$
$$\langle x\ell(\xi^*), x^n l_n^{[1]}(x;\xi^*)\rangle = \vartheta_n \langle \ell(\xi^*), l_n^2(x;\xi^*)\rangle \neq 0, \quad n \geq 0.$$

Equivalently, $\langle (\alpha+\lambda-1)^{-1}x \quad \ell(\xi^*), l_m^{[1]}(.;\xi^*) l_n^{[1]}(.;\xi^*)\rangle = r_n^{[1]}(\xi^*)\delta_{n,m}, \quad n,m \geq 0,$ with $r_n^{[1]}(\xi^*) = (\alpha+\lambda-1)^{-1}\vartheta_n\langle \ell(\xi^*), l_n^2(x;\xi^*)\rangle = n!\Gamma(\alpha)^{-1}\Gamma(n+\alpha) \neq 0,\ n \geq 0.$

Thus, $\{l_n^{[1]}(.;\xi^*)\}_{n\geq 0}$ is orthogonal with respect to $\ell_0^{[1]}(\xi^*) = (\alpha+\lambda-1)^{-1}x\ell(\xi^*)$. The MOPS $\{l_n^{[1]}(.;\xi^*)\}_{n\geq 0}$ satisfies the following TTRR:

$$l_0^{[1]}(x;\xi^*) = 1, \quad l_1^{[1]}(x;\xi^*) = x - \beta_0^{[1]},$$
$$l_{n+2}^{[1]}(x;\xi^*) = (x-\beta_{n+1}^{[1]}) l_{n+1}^{[1]}(x;\xi^*) - \gamma_{n+1}^{[1]} l_n^{[1]}(x;\xi^*), \quad n \geq 0, \tag{4.11}$$

where $\gamma_{n+1}^{[1]} = r_{n+1}^{[1]}(\xi^*)(r_n^{[1]}(\xi^*))^{-1} = (n+1)(n+\alpha) \neq 0,\ n \geq 0,$ and from (4.10) and the orthogonality of $\{l_n(.;\xi^*)\}_{n\geq 0}$ with respect to $\ell(\xi^*)$, we get

$$\beta_n^{[1]} = \frac{\langle \ell(\xi^*), [l_{n+1}(.;\xi^*)]^2\rangle + \vartheta_n^2\langle \ell(\xi^*), [l_n(.;\xi^*)]^2\rangle}{\vartheta_n\langle \ell(\xi^*), (l_n(.;\xi^*))^2\rangle} = \frac{\gamma_{n+1}}{\vartheta_n} + \vartheta_n, \quad n \geq 0.$$

Thus, $\beta_0^{[1]} = 2\alpha+\lambda-1,\ \beta_{n+1}^{[1]} = 2n+\alpha+2,\ n \geq 0.$

Note that $\{l_n^{[1]}(.;\xi^*)\}_{n\geq 0}$ is a **D**-Laguerre–Hahn MOPS of class zero. More precisely, it is a non-singular **D**-Laguerre–Hahn MOPS analogous of the Laguerre's one [1,4]. Recall that the non-singular **D**-Laguerre–Hahn MOPS analogous to the Laguerre's polynomial sequence denoted by $\{\mathcal{L}_n(.;\varpi)\}_{n\geq 0}$ with $\varpi = (\lambda,\rho,\tau,\alpha)$, satisfies the following TTRR:

$$\mathcal{L}_0(x;\varpi) = 1, \quad \mathcal{L}_1(x;\varpi) = x - \tilde{\beta}_0,$$
$$\mathcal{L}_{n+2}(x;\varpi) = (x-\tilde{\beta}_{n+1})\mathcal{L}_{n+1}(x;\varpi) - \tilde{\gamma}_{n+1}\mathcal{L}_n(x;\varpi), \quad n \geq 0,$$

where

$$\tilde{\beta}_0 = 2\tau+\alpha+\lambda+1, \quad \tilde{\beta}_{n+1} = 2(n+\tau+1)+\alpha+1,\ n \geq 0,$$
$$\tilde{\gamma}_1 = \rho(\tau+1)(\tau+\alpha+1), \quad \tilde{\gamma}_{n+1} = (n+\tau+1)(n+\tau+\alpha+1),\ n \geq 1,$$

with $\tau+\alpha \neq -(n+1)$ and $\tau \neq -(n+1),\ n \geq 0.$

As a straightforward consequence, we get $l_n^{[1]}(x;\xi^*) = \mathcal{L}_n(x;\varpi^*),\ n \geq 0$, where $\varpi^* = (\alpha+\lambda-1,\ 1,\ 0,\ \alpha-1)$.

We conclude our work stating two finite-type relations linking $\{l_n(.;\xi^*)\}_{n\geq 0}$ to $\{\mathcal{L}_n(.;\varpi^*)\}_{n\geq 0}$, which are known in the literature as second structure relations (see [10,12]). By inserting (4.10) in (3.3), where $B_n(x) = l_n(x;\xi^*)$, $n \geq 0$, and using the fact that $\langle \mathcal{U}(\xi^*), l_{n+1}(x;\xi^*)\rangle = -\delta_{n,0},\ n \geq 0,$ we obtain

$$l_{n+1}(x;\xi^*) + \upsilon_n l_n(x;\xi^*) = \mathcal{L}_{n+1}(x;\varpi^*) + \frac{(n+1)(2n+\alpha+1)}{n+\alpha+1}\mathcal{L}_n(x;\varpi^*)$$
$$+ \frac{(n+\alpha-1)}{n+\alpha+1}\mathcal{L}_{n-1}(x;\varpi^*) + (\alpha+1)^{-1}\delta_{n,0}, \quad n \geq 0,$$

where $\upsilon_n = n(n+\alpha)(n+\alpha+1)^{-1}$.

Finally, substituting (4.11) into (3.3) where $B_n(x) = l_n(x;\xi^*)$, $n \geq 0$, and using the fact that $\langle \mathcal{U}(\xi^*), l_{n+1}(x;\xi^*)\rangle = -\delta_{n,0},\ n \geq 0$, we obtain

$$l_n(x;\xi^*) = \mathcal{L}_n(x;\varpi^*) + (n+\alpha-1)\mathcal{L}_{n-1}(x;\varpi^*), \quad n \geq 0,\ \mathcal{L}_{-1}(x;\varpi^*) = 0.$$

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