A NONSYMMETRIC SECOND DEGREE SEMI-CLASSICAL FORM OF CLASS ONE

ATEF ALAYA, BELGACEM BOURAS, AND FRANCISCO MARCELLAN

Abstract. An orthogonal polynomial sequence with respect to a regular form (linear functional) u is said to be semi-classical if there exist a monic polynomial Φ and a polynomial Ψ, with deg $\Psi \geq 1$, such that $(\Phi u)' + \Psi u = 0$. Recently, all semi-classical monic orthogonal polynomial sequences of class one satisfying a three term recurrence relation with $\beta_n = (-1)^n \beta_0,\ n \geq 0, \beta_0 \in \mathbb{C} \setminus \{0\}$ have been determined (see *[A large family of semi-classical polynomials of class one*, Integral Transforms Spec. Funct. **18** (2007), 913-931).

In this paper, the sequences of the above family such that their corresponding Stieltjes function $S(u)(z) = -\sum_{n\geq 0} \langle u, x^n \rangle / z^{n+1}$ satisfies a quadratic relation of the form $BS^2(u) + CS(u) + D = 0$, where $B,\ C,\ D$ are polynomials, are described.

1. Introduction

A regular form in the linear space of polynomials with complex coefficients is said to be a second degree form if its corresponding Stieltjes function $S(u)(z) = -\sum_{n\geq 0} \langle u, x^n \rangle / z^{n+1}$ satisfies a quadratic relation $BS^2(u) + CS(u) + D = 0$, where $B,\ C,\ D$ are polynomials. Notice that every second degree form is semi-classical, i.e. there exist a monic polynomial Φ and a polynomial Ψ, with deg $\Psi \geq 1$, such that $(\Phi u)' + \Psi u = 0$. Furthermore, the family of second degree forms is invariant under elementary transformations of linear forms as association, perturbation, shift, multiplication and division by a polynomial, inversion, among others (see [14]).

A first example of second degree form are the Tchebychev forms of the first and the second kind, and more generally, the linear forms defined by the Bernstein-Szegő weights (see [5, 20]). This fact was pointed out in [15]. Later on, D. Beghdadi and P. Maroni determined in [7] all classical (i.e. semi-classical of class $s = 0$) second degree forms. A next natural step is to describe the second degree forms of class $s = 1$. This result was done only in the symmetric case (see [4]).

In this contribution we are dealing with second degree forms which are semi-classical of class $s = 1$ and such that their corresponding sequences of orthogonal polynomials satisfy a three term recurrence relation with coefficients $\beta_n = (-1)^n \beta_0,\ n \geq 0, \beta_0 \in \mathbb{C} \setminus \{0\}$.

The structure of the paper is as follows. In Section 2 we introduce the notations and definitions to be used in the sequel. In particular, we remind some basic background concerning semi-classical forms. In Section 3, we focus our attention

---

1991 *Mathematics Subject Classification.* 33C45; 42C05.

Key words and phrases. Orthogonal polynomials; Semi-classical forms; Second degree forms.

The work of the third author (FM) has been supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain, grant MTM2009-12740-C03-01.The work of the second author (BB) has been supported by Quassim University, grant SR-D-010-316.

on second degree forms. More precisely, all second degree semi-classical forms of class $s = 1$ such that their corresponding MOPS verify the so-mentioned recurrence relation, are determined. Finally, the polynomial coefficients of the second degree equation fulfilled by the corresponding formal Stieltjes function are deduced.

2. Notations and preliminaries

Let $\mathbb{P}$ be the linear space of polynomials in one real variable with complex coefficients and $\mathbb{P}'$ be its dual. For $u \in \mathbb{P}'$ and $p \in \mathbb{P}$, $\langle u, p\rangle$ means the action of the form u over the polynomial p. In particular, $(u)_n = \langle u, x^n\rangle,\ n \geq 0$ are the moments of u. For every c, $d \in \mathbb{C}$, f, p in $\mathbb{P}$ and u, v in $\mathbb{P}'$, we introduce the following linear forms:

$$\langle fu, p\rangle \quad = \quad \langle u, fp\rangle, \quad \langle u', p\rangle = -\langle u, p'\rangle, \langle uv, p\rangle = \langle v, up\rangle,\ \langle (x-c)^{-1}u, p\rangle = \langle u, \theta_c p\rangle,$$

where $\theta_c(p)(x) = \dfrac{p(x) - p(c)}{x - c}$ and $(up)(x) = \langle u_y, \dfrac{xp(x) - yp(y)}{x - y}\rangle$.
Here, $\langle u_y, .\rangle$ denotes the action of u over the polynomial on the y-variable.
The even part σu of a given form u is defined by

$$\langle \sigma u, p(x)\rangle = \langle u, p(x^2)\rangle.$$

Then [1, 15]

$$\sigma\big(f(x^2)u\big) \quad = \quad f(x)(\sigma u), \tag{2.1}$$

$$\sigma u' \quad = \quad 2\big(\sigma(xu)\big)'. \tag{2.2}$$

A linear form u is said to be regular (quasi-definite) if there exists a monic polynomial sequence (MPS, in short) $\{B_n\}_{n\geq 0}$, with $\deg B_n = n$, such that $\langle u, B_n B_m\rangle = 0,\ n \neq m$, and $\langle u, B_n^2\rangle \neq 0,\ n \geq 0$. In this case, the sequence $\{B_n\}_{n\geq 0}$, is called orthogonal with respect to u. In short, we will denote by MOPS a MPS that is orthogonal with respect to a regular linear form. In the sequel, we will assume that our regular forms u are normalized, i.e. $(u)_0 = 1$.
A MPS $\{B_n\}_{n\geq 0}$ is a MOPS if and only if it satisfies the three-term recurrence relation

$$\begin{aligned} &B_0(x) = 1, \quad B_1(x) = x - \beta_0, \\ &B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geq 0, \end{aligned} \tag{2.3}$$

with $\beta_n \in \mathbb{C}$ and $\gamma_{n+1} \neq 0,\ n \geq 0$. This is the so called Favard' s theorem (see [11, 21, 17]).
A MOPS $\{B_n\}_{n\geq 0}$ is said to be symmetric if in (2.3) we have $\beta_n = 0,\ n \geq 0$ [21].
In this work, we are dealing with the family of MOPS $\{B_n\}_{n\geq 0}$ satisfying the three-term recurrence relation:

$$\begin{aligned} &B_0(x) = 1, \quad B_1(x) = x - \beta_0, \\ &B_{n+2}(x) = \big(x - (-1)^{n+1}\beta_0\big)B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geq 0, \end{aligned} \tag{2.4}$$

with $\beta_0 \in \mathbb{C}$ and $\gamma_{n+1} \neq 0,\ n \geq 0$.
In the sequel, we need the following properties of a linear form u such that its corresponding MOPS $\{B_n\}_{n\geq 0}$ satisfies (2.4) :

$$\sigma(xu) = \beta_0 \sigma u, \tag{2.5}$$

$$\sigma(u^2) = (\sigma u)^2 + \beta_0^2 x^{-1}(\sigma u)^2, \tag{2.6}$$

$$\sigma\big(xf(x^2)u\big) = \beta_0 f(x)\sigma u. \tag{2.7}$$

The identities (2.5) and (2.6) are particular cases of formulas (2.46) and (1.13) in [18]. (2.7) is a direct consequence of (2.5) and (2.1).
In general, the even part of a regular form is not regular but, the even part σu of a regular form u such that its MOPS satisfies (2.4) is regular [18]. This also constitutes a generalization of the result given in [21] for symmetric forms.

A regular form u is said to be semi-classical if u satisfies a functional equation (Pearson equation)

$$(\Phi u)' + \Psi u = 0, \tag{2.8}$$

where Φ and Ψ are polynomials such that Φ is monic and $\deg(\Psi) \geq 1$.
The corresponding MOPS, $\{B_n\}_{n\geq 0}$, is said to be semi-classical (for more details, see[21] and the literature therein).
Furthermore, if for each zero c of Φ one has

$$|\Psi(c) + \Phi'(c)| + \big|\langle u, \theta_c\Psi + \theta_c^2(\Phi)\rangle\big| > 0, \tag{2.9}$$

then the integer $s := \max\big\{\deg(\Phi) - 2,\ \deg(\Psi) - 1\big\}$ will be called either the class of $\{B_n\}_{n\geq 0}$ or the class of u.
In the literature, the case $s = 0$ is described in [17]. This corresponds to the classical sequences (Hermite, Laguerre, Bessel, and Jacobi). The linear forms of class $s = 1$ are described in [9] where the corresponding Pearson equations as well as the integral representations for some of them are given. Unfortunately, the recurrence coefficients are only known for the symmetric forms [1] and some particular nonsymmetric cases obtained by doing some elementary operations on the classical linear forms such as the addition of a finite number of Dirac's masses [2], the product and the division of a form by a polynomial [8, 12, 13] among others (see [23, 16, 19, 21, 22]).

The semi-classical character of a linear form is invariant under a shift. Indeed, if u is a semi-classical form verifying (2.8) and (2.9), then the shifted form $\widehat{u}$ defined by

$$\widehat{u} = (h_{a^{-1}} \circ \tau_{-b})\, u, \tag{2.10}$$

where for all polynomial p

$$\langle \tau_b u, p\rangle = \langle u, \tau_{-b} p\rangle = \langle u, p(x+b)\rangle \quad \text{and} \quad \langle h_a u, p\rangle = \langle u, h_a p\rangle = \langle u, p(ax)\rangle,$$

is semi-classical. The form $\widehat{u}$ has the same class of u and satisfies [16, 21]:

$$(\widehat{\Phi}\widehat{u})' + \widehat{\Psi}\widehat{u} = 0, \tag{2.11}$$

with $\widehat{\Phi}(x) = a^{-t}\Phi(ax+b)$ and $\widehat{\Psi}(x) = a^{1-t}\Psi(ax+b),\ t = \deg\Phi$.

Proposition 2.1. [21] *Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to a form u. Then, u is semi-classical of class s and verifies* (2.8) *if and only if there exist two polynomial sequences $\{C_n\}_{n\geq 0}$ and $\{D_n\}_{n\geq 0}$ with coefficients depending on n such that* $\deg C_n \leq s+1,\ \deg D_n \leq s,\ n \geq 0$ *such that the following structure relation holds*

(2.12)
$$\Phi(x)B'_{n+1}(x) = \frac{1}{2}\big(C_{n+1}(x) - C_0(x)\big)B_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)B_n(x),\ n \geq 0.$$

Here

$$
(2.13)\qquad \begin{cases} C_{n+1}(x) &= -C_n(x)+2(x-\beta_n)D_n(x),\ \ n\geq 0,\\ \gamma_{n+1}D_{n+1}(x) &= -\Phi(x)+\gamma_n D_{n-1}(x)-(x-\beta_n)C_n(x)\\ & \quad +(x-\beta_n)^2 D_n(x),\ n\geq 0, \end{cases}
$$

with $D_{-1}(x)=0,\ C_0(z)=-\Psi(z)-\Phi'(z),$ *and* $D_0(z)=-(u\theta_0\Phi)'(z)-(u\theta_0\Psi)(z).$

Recently, we have determined in [10] all the semi-classical forms of class one that their corresponding MOPS, $\{B_n\}_{n\geq 0}$, satisfies (2.4). Up to an affine transformation in the variable (in order to have $\beta_0=1$), one and only one canonical (non symmetric) case appears.

Theorem 2.2. [10] *For a semi-classical form u of class $s=1$ fulfilling* (2.8) *such that the corresponding MOPS satisfies* (2.4) *with $\beta_0=1$, we get*

$$
\begin{cases} \Phi(x)=x(x^2-1),\\ \Psi(x)=-(e_0+3)x^2+x+(1-g_0). \end{cases}
$$

As a consequence,

$$
(2.14)\qquad \begin{cases} C_n(x)= & (2n+e_0)x^2+(-1)^{n+1}x\\ & \quad +\big[-2n+(-1)^n(e_0+g_0)-e_0\big],\ n\geq 0,\\ D_n(x)= & (2n+1+e_0)(x+(-1)^n),\ n\geq 0. \end{cases}
$$

Furthermore, for all $n\geq 1$,

$$
\gamma_n=-\frac{\big(2n+e_0+(-1)^{n+1}e_0\big)\big(2n+e_0+(-1)^{n+1}(e_0+2g_0)\big)}{4(2n-1+e_0)(2n+1+e_0)},
$$

assuming the regularity conditions

$$
\begin{cases} e_0\neq -(2n+1),\ n\geq 0, \quad g_0\neq 2n,\ n\geq 1,\\ e_0+g_0\neq -(2n+1),\ n\geq 0. \end{cases}
$$

3. Second degree forms

Definition 3.1. [4, 15] A regular form u is said to be a second degree form if there exist two polynomials B and C such that

$$
(3.1)\qquad B(z)S^2(u)(z)+C(z)S(u)(z)+D(z)=0,
$$

where $S(u)(z)$ is the formal Stieltjes function of the linear form u defined by $S(u)(z)=-\sum_{n\geq 0}\frac{(u)_n}{z^{n+1}}$ and D is a polynomial that depends on B,C and u. More precisely, one has $D(z)=(u\theta_0C)(z)-(u^2\theta_0^2B)(z)$.

The regularity of u means that $B\neq 0$, $C^2-4BD\neq 0$, and $D\neq 0$. In the sequel, we assume that B is a monic polynomial.

Proposition 3.2. *A regular form u is said to be of second degree if and only if there exist polynomials B and C such that the following relations hold*

$$
(3.2)\qquad B(x)u^2=xC(x)u,
$$

$$
(3.3)\qquad \langle u^2,\theta_0B\rangle=\langle u,C\rangle.
$$

The polynomials B and C in (3.2)-(3.3) or in (3.1) are not unique since we can multiply the above equations by an arbitrary polynomial to obtain new equations of the same type. If in (3.1) the polynomials B, C, and D are coprime, then the pair (B, C) is called a primitive pair. The primitive pair is unique.

Notice that every second degree form is semi-classical (see [15]). In [7] all the classical forms which are of second degree are determined. Hermite, Laguerre, and Bessel forms are not of second degree. Only some Jacobi linear forms are of second degree. Indeed,

Theorem 3.3. *Among the classical forms, only the Jacobi forms $\mathcal{J}(k-\frac{1}{2}, l-\frac{1}{2})$ with $k+l \geq 0, k, l \in \mathbb{Z}$ are of second degree.*
In this case, the Pearson equation is

$$\big((x^2-1)\mathcal{J}(k-\frac{1}{2}, l-\frac{1}{2})\big)' + \big(-(k+l+1)x + k - l\big)\mathcal{J}(k-\frac{1}{2}, l-\frac{1}{2}) = 0. \tag{3.4}$$

Later on, in [4] all the symmetric semi-classical forms of class one of second degree are determined. Indeed, let us denote by $\mathcal{I}(\alpha, \beta)$ the Generalized Gegenbauer form (see [11, page 156]). This form is symmetric and semi-classical of class $s = 1$. Furthermore, the coefficients of the three term recurrence relation are

$$\begin{cases} \gamma_{2n+1} = \dfrac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \ n \geq 0, \\ \gamma_{2n+2} = \dfrac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}, \ n \geq 0, \\ \alpha \neq -n, \ \ \beta \neq -n, \ \ \alpha+\beta \neq -(n+1), \ n \geq 1. \end{cases}$$

Moreover, the Pearson equation is

$$\big(x(x^2-1)\mathcal{I}(\alpha, \beta)\big)' + \big(-2(\alpha+\beta+2)x^2 + 2(\beta+1)\big)\mathcal{I}(\alpha, \beta) = 0. \tag{3.5}$$

In [4], the following result is deduced.

Theorem 3.4. [4] *Among the symmetric semi-classical forms of class $s = 1$, only the forms $\mathcal{I}(p-\frac{1}{2}, q-\frac{1}{2})$, with $p+q \geq 0$, $q \neq 0$, and p, $q \in \mathbb{Z}$, are of second degree.*

Remark 3.5.
For $q = 0$ and p, a nonnegative integer number, $\mathcal{I}(p-\frac{1}{2}, -\frac{1}{2})$ is the Gegenbauer form $\mathcal{J}(p-\frac{1}{2}, p-\frac{1}{2})$. So, it is of second degree (see [4], [7]). This form does not appear in Theorem 3.4 because it is of class $s = 0$.

Elementary transformations like association, perturbation, shift, multiplication and division by a polynomial, inversion, preserve the family of linear forms of second degree [7, 4, 14]. Moreover, we have

Lemma 3.6. [7] *Let u and v be two regular forms satisfying $M(x)u = N(x)v$, where $M(x)$ and $N(x)$ are two polynomials. If u is a second degree form satisfying* (3.1), *then v is also a second degree form such that*

$$\widetilde{B}(z)S^2(v)(z) + \widetilde{C}(z)S(v)(z) + \widetilde{D}(z) = 0, \tag{3.6}$$

with

$$\begin{cases} \widetilde{B}(z) = B(z)N^2(z), \\ \widetilde{C}(z) = N(z)\Big\{2B(z)\big((v\theta_0 N)(z) - (u\theta_0 M)(z)\big) + M(z)C(z)\Big\}, \\ \widetilde{D}(z) = B(z)\big((v\theta_0 N)(z) - (u\theta_0 M)(z)\big)^2 \\ \qquad\qquad + M(z)C(z)\big((v\theta_0 N)(z) - (u\theta_0 M)(z)\big) + M^2(z)D(z). \end{cases} \tag{3.7}$$

3.1. **Second degree semi-classical forms of class $s = 1$ such that the corresponding MOPS verifies (2.4).** The next result generalizes Proposition 3.2 in [4, page 6].

Proposition 3.7. *The even part σu of a second degree form u, such that the corresponding MOPS satisfies* (2.4), *is also of second degree.*

Proof. Let u be a second degree form satisfying (3.2)-(3.3).
Let consider the decomposition of the polynomials B and C in their even and odd parts

$$B(x) = B^e(x^2) + xB^o(x^2), \qquad C(x) = C^e(x^2) + xC^o(x^2).$$

Applying the operator σ in (3.2), we get

$$\sigma\big(B^e(x^2)u^2\big) + \sigma\big(B^o(x^2)xu^2\big) = \sigma\big(C^e(x^2)xu\big) + \sigma\big(x^2C^o(x^2)u\big).$$

Taking into account (2.5), (2.7) and (2.1), this relation becomes

$$\big[B^e(x) + \beta_0 B^o(x)\big]\sigma u^2 = \big[\beta_0 C^e(x) + xC^o(x)\big]\sigma u. \tag{3.8}$$

But, from (2.6) and the fact $x\big(x^{-1}u\big) = u$, we have

$$x\sigma(u^2) = (x + \beta_0^2)(\sigma u)^2. \tag{3.9}$$

Hence, (3.8) becomes

$$(x + \beta_0^2)\overline{B}(x)(\sigma u)^2 = x\overline{C}(x)\sigma u, \tag{3.10}$$

where $\quad \overline{B}(x) = B^e(x) + \beta_0 B^o(x) \quad$ and $\quad \overline{C}(x) = \beta_0 C^e(x) + xC^o(x)$.
On the other hand, using (2.6), we get

$$\begin{aligned}
\langle(\sigma u)^2, \theta_0\big((x + \beta_0^2)\overline{B}\big)\rangle &= \langle(\sigma u)^2, \overline{B} + \beta_0^2\theta_0\overline{B}\rangle \\
&= \langle\overline{B}(\sigma u)^2, 1\rangle + \beta_0^2\langle(\sigma u)^2, \theta_0\overline{B}\rangle \\
&= \langle\overline{B}\big[(\sigma u)^2 + \beta_0^2 x^{-1}(\sigma u)^2\big], 1\rangle = \langle\overline{B}\sigma u^2, 1\rangle.
\end{aligned}$$

Thus, from (3.8),

$$\langle(\sigma u)^2, \theta_0\big((x + \beta_0^2)\overline{B}\big)\rangle = \langle\sigma u, \overline{C}\rangle. \tag{3.11}$$

According to Proposition 3.2 and the regularity of the linear form σu , (3.10) and (3.11) the statement follows. ☐

In order to obtain the second degree semi-classical forms of class $s = 1$ such that their corresponding MOPS satisfies (2.4), first we will show some examples of them. Next, we will prove that these forms are (up to an affine transformation) the unique solutions of our problem.

The set of semi-classical forms of class $s = 1$ that their corresponding MOPS satisfies (2.4) is not empty. Indeed, let us consider $k,\ l \in \mathbb{Z}$, such that $k + l \geq 1$. If we $p = k + 1$ and $q = l - 1$, then, according to Theorem 3.4 and Remark 3.5, the normalized form

$$v = \mathcal{I}(p - \frac{1}{2}, q - \frac{1}{2}) = \mathcal{I}(k + \frac{1}{2}, l - \frac{3}{2}),$$

is a symmetric second degree semi-classical form of class $s \leq 1$. More precisely, if $q = 0$, then $s = 0$. Otherwise, $s = 1$. Notice that they satisfy (3.1) with polynomial coefficients explicitly given in [7, page 450] and [4, page 10]. In this paper, we denote

by B_1, C_1, and D_1 these polynomials.
Now, let us define the normalized form w by

$$\lambda(x-1)w = xv, \quad \text{where} \quad \lambda = \frac{1-2l}{2k+1}. \tag{3.12}$$

In [24, Example 4.1] it has been proved that w is regular. More precisely, w is semi-classical of class $\widetilde{s} = 1$ satisfying the Pearson equation (see [24, page 287])

$$\big(x(x^2-1)w\big)' + \big(-2(k+l+1)x^2 + x + 2l\big)w = 0. \tag{3.13}$$

The coefficients of the three term recurrence relation of the corresponding MOPS, $\{\widetilde{B}_n\}_{n\geq 0}$, are

$$\begin{cases} \widetilde{\beta}_n = (-1)^n, \\ \widetilde{\gamma}_{2n+1} = -\dfrac{(n+k+\frac{1}{2})(n+k+l)}{(2n+k+l)(2n+k+l+1)}, \\ \widetilde{\gamma}_{2n+2} = -\dfrac{(n+1)(n+l+\frac{1}{2})}{(2n+k+l+1)(2n+k+l+2)}, \ n \geq 0. \end{cases} \tag{3.14}$$

On the other hand, according to Lemma 3.6, the form w defined by (3.12) is also of second degree. Hence, w is a second degree semi-classical form of class one such that corresponding MOPS verifies (2.4)

In the sequel, we shall denote by $\mathcal{K}(k,l)$ the regular form w, which satisfies (3.13), where $k+l \geq 1$ and $k,\ l \in \mathbb{Z}$.

Next, we will show that the unique second degree and semi-classical of class one such that their corresponding MOPS satisfies (2.4) with $\beta_0 = 1$ are the $\mathcal{K}(k,l)$ ones.

Suppose, that u is a second degree form and semi-classical of class one such that corresponding MOPS satisfies (2.4) with $\beta_0 = 1$. So, from Theorem 2.2, the form u is a solution of

$$\big(x(x^2-1)u\big)' + \big(-(e_0+3)x^2 + x + 1 - g_0\big)u = 0, \tag{3.15}$$

with the regularity conditions

$$\begin{cases} e_0 \neq -(2n+1), \ n \geq 0, \quad g_0 \neq 2n, \ n \geq 1, \\ e_0 + g_0 \neq -(2n+1), \ n \geq 0. \end{cases}$$

Applying the operator σ, and using (2.1), (2.2), (2.5) as well as $\beta_0 = 1$, we get

$$2\big(x(x-1)\sigma u\big)' + \big(-(e_0+3)x + 2 - g_0\big)\sigma u = 0. \tag{3.16}$$

Thus, the form $\varpi = \big(h_{(-\frac{1}{2})^{-1}} \circ \tau_{-\frac{1}{2}}\big)(\sigma u)$ satisfies (2.11), with

$$\widehat{\Phi}(x) = x^2 - 1, \qquad \widehat{\Psi}(x) = -\frac{(e_0+3)}{2}x + \frac{e_0 + 2g_0 - 1}{2}.$$

Hence ϖ is the Jacobi form

$$\varpi = \mathcal{J}\Big(\frac{e_0+g_0-1}{2}, -\frac{g_0}{2}\Big). \tag{3.17}$$

But, from the assumption and Proposition 3.7, the form σu is of second degree as well as a solution of the Pearson equation (3.16). Hence, σu is a classical second degree form. The shifted form ϖ is of second degree too. From Theorem 3.3, there exist $k, l \in \mathbb{Z}$ such that if $k+l \geq 0$ then

$$\varpi = \mathcal{J}\Big(k - \frac{1}{2}, l - \frac{1}{2}\Big). \tag{3.18}$$

The comparison of (3.17) and (3.18) yields $k = \dfrac{e_0+g_0}{2}$ and $l = \dfrac{1-g_0}{2}$, with $e_0+1 \geq 0$. Thus, (3.15) becomes

$$\big(x(x^2-1)u\big)' + \big(-2(k+l+1)x^2 + x + 2l\big)u = 0.$$

Taking into account $k+l \geq 0$ and the regularity conditions, we get $k+l \geq 1$ and so $u = \mathcal{K}(k,l)$.
As a conclusion,

Theorem 3.8. *Among the semi-classical forms of class $s=1$ such that the corresponding MOPS $\{B_n\}_{n\geq 0}$ satisfies (2.4), with $\beta_0 = 1$, only the forms $\mathcal{K}(k,l)$ with $k+l \geq 1$, $k,l \in \mathbb{Z}$ are second degree forms.*

In the following table, we summarize the characteristic elements of the form $\mathcal{K}(p,q)$. For the calculation of C_n and D_n, we use (2.14).

Second degree semi-classical MOPS $\{B_n\}_{n\geq 0}$ of class $s=1$ with $\beta_n = (-1)^n,\ n \geq 0$.
$\Phi(x) = x(x^2-1), \quad \Psi(x) = -2(k+l+1)x^2 + x + 2l.$ $k+l \geq 1,\ k,l \in \mathbb{Z}.$ $\gamma_{2n+1} = -\dfrac{(n+k+l)(n+k+\frac{1}{2})}{(2n+k+l)(2n+k+l+1)},\ n \geq 0,$ $\gamma_{2n+2} = -\dfrac{(n+1)(n+l+\frac{1}{2})}{(2n+k+l+1)(2n+k+l+2)},\ n \geq 0.$ $C_n(x) = \big(2(n+k+l)-1\big)x^2 + (-1)^{n+1}x - 2(n+k+l-(-1)^n k) + 1,\ n \geq 0.$ $D_n(x) = 2(n+k+l)(x+(-1)^n).$ $k+l \geq 1,\ k,l \in \mathbb{Z}.$

Finally, we will give the polynomial coefficients in (3.1). From the definition of w (i.e. the relation $\big(3.12)\big)$, the fact that v satisfies (3.1), with corresponding polynomials $B_1(z)$, $C_1(z)$, and $D_1(z)$ given in [4, page 10] and [7, page 450], and from Lemma 3.6, we deduce that w satisfies (3.1), with

$$\begin{cases} B(z) = \lambda^2(z-1)^2 B_1(z), \\ C(z) = \lambda(z-1)\Big\{2(\lambda-1)B_1(z) + zC_1(z)\Big\}, \\ D(z) = (\lambda-1)^2 B_1(z) + (\lambda-1)zC_1(z) + z^2 D_1(z). \end{cases}$$

More precisely, if ρ denotes the normalization factor and $\mathcal{T}^{-1}$ is the inverse of the Tchebychev form of the first kind $\mathcal{T} := \mathcal{J}(-\frac{1}{2},-\frac{1}{2})$ (see [5, 20]), then

(i) If $k \geq 0$ and $l \geq 2$, then $\mathcal{I}(k+\frac{1}{2}, l-\frac{3}{2}) = \rho x^{2l}(x^2-1)^k \mathcal{T}^{-1}$, with $\rho = \big(\langle \mathcal{T}^{-1}, x^{2l}(x^2-1)^k\rangle\big)^{-1}$,

$$\begin{cases} B(z) = \lambda^2(z-1)^2 z^4, \\ C(z) = \lambda(z-1)\Big\{2(\lambda-1)z^4 - \rho z(\mathcal{T}^{-1}\xi^{2l-1}(\xi^2-1)^k)(z)\Big\}, \\ D(z) = (\lambda-1)^2 z^4 - \rho(\lambda-1)z(\mathcal{T}^{-1}\xi^{2l-1}(\xi^2-1)^l)(z) \\ \qquad - \rho^2 z^6\big\{(\mathcal{T}^{-1}\xi^{2l-1}(\xi^2-1)^k)^2(z) \\ \qquad + z^{2(l-2)}(z^2-1)^{2k+1}. \end{cases}$$

(ii) If $k+2 \geq l \geq 2$, then $x^{2(l-2)}\mathcal{I}(k+\frac{1}{2}, -l+\frac{1}{2}) = \rho(x^2-1)^k\mathcal{T}^{-1}$, with $\rho = \dfrac{\langle v, x^{2(l-2)}\rangle}{\langle \mathcal{T}^{-1}, (x^2-1)^k\rangle}$,

$$\begin{cases} B(z) = \lambda^2(z-1)^2 z^{4(l-1)}, \\ C(z) = \lambda(z-1)\Big\{2(\lambda-1)z^{4(l-1)} + 2z^{2l+1}\big\{\big(v\theta_0(\xi^{2(l-2)})\big)(z) \\ \qquad - \rho\big(\mathcal{T}^{-1}\theta_0\big((\xi^2-1)^k\big)(z)\big\}\Big\}, \\ D(z) = (\lambda-1)^2 z^{4(l-1)} \\ \qquad + 2(\lambda-1)z^{2l+1}\big\{\big(v\theta_0(\xi^{2(l-2)})\big)(z) - \rho\big(\mathcal{T}^{-1}\theta_0\big((\xi^2-1)^k\big)(z)\big\} \\ \qquad + z^6\big[\big(v\theta_0(\xi^{2(l-2)})\big)(z) - \rho\big(\mathcal{T}^{-1}\theta_0\big((\xi^2-1)^k\big)(z)\big]^2 \\ \qquad - \rho^2 z^2(z^2-1)^{2k+1}. \end{cases}$$

(iii) If $l \geq k+2 \geq 1$ and $l \neq 1$, then $(x^2-1)^{k+2}\mathcal{I}(-k-\frac{3}{2}, l-\frac{3}{2}) = \rho x^{2l}\mathcal{T}^{-1}$,

with $\rho = \dfrac{\langle v, (x^2-1)^{k+2}\rangle}{\langle \mathcal{T}^{-1}, x^{2l}\rangle}$,

$$\begin{cases} B(z) = \lambda^2(z-1)^{k+4}(z+1)^{k+2}, \\ C(z) = \lambda(z-1)\Big\{2(\lambda-1)(z^2-1)^{k+2} \\ \qquad + 2z(z^2-1)^{k+2}\big\{\big(v\theta_0\big((\xi^2-1)^{k+2}\big)\big)(z) - \rho\big(\mathcal{T}^{-1}\xi^{2l-1}\big)(z)\big\}\Big\}, \\ D(z) = (\lambda-1)^2(z^2-1)^{k+2} \\ \qquad + 2(\lambda-1)z(z^2-1)^{k+2}\big\{\big(v\theta_0\big((\xi^2-1)^{k+2}\big)\big)(z) - \rho\big(\mathcal{T}^{-1}\xi^{2l-1}\big)(z)\big\} \\ \qquad + z^2\big[\big(v\theta_0\big((\xi^2-1)^{k+2}\big)\big)(z) - \rho\big(\mathcal{T}^{-1}\xi^{2l-1}\big)(z)\big]^2 \\ \qquad - \rho^2 z^{4l-2}(z^2-1). \end{cases}$$

(iv) If $k \geq 0$ and $l = 1$, then $\mathcal{I}(k+\frac{1}{2}, -\frac{1}{2}) = \mathcal{J}(k+\frac{1}{2}, k+\frac{1}{2}) = \frac{1}{\rho}(x^2-1)^{k+1}\mathcal{T}$,
with $\rho = \langle \mathcal{T}, (x^2-1)^{k+1}\rangle$,

$$\begin{cases} B(z) = \lambda^2(z-1)^2, \\ C(z) = \lambda(z-1)\Big\{2(\lambda-1) - \frac{2}{\rho}z\Big(\mathcal{T}\theta_0\big((\xi^2-1)^{k+1}\big)\Big)(z)\Big\}, \\ D(z) = (\lambda-1)^2 - \frac{2}{\rho}(\lambda-1)z\Big(\mathcal{T}\theta_0\big((\xi^2-1)^{k+1}\big)\Big)(z) \\ \qquad + \frac{z^2}{\rho^2}\big\{\Big(\mathcal{T}\theta_0\big((\xi^2-1)^{k+1}\big)\Big)^2(z) - (z^2-1)^{2k+1}\big\}. \end{cases}$$

References

[1] J. Alaya, P. Maroni, *Symmetric Laguerre-Hahn forms of class* $s = 1$, Integral Transforms Spec. Funct. **4** (1996), 301-320.

[2] Álvarez-Nodarse, J. Arvesú, F. Marcellán, *Modifications of quasi-definite linear functionals via addition of delta and derivatives of delta Dirac functions*, Indag. Math. N. S. **15** (2004), 1-20

[3] M. Bachène, *Les polynômes semi-classiques de classe zéro et de classe un.* Thèse de Troisième Cycle, Université Pierre et Marie Curie, Paris. 1985.

[4] D. Beghdadi, *Second degree semi-classical forms of class* $s = 1$. *The symmetric case*, Appl. Numer. Math. **34** (2000), 1-11.

[5] D. Beghdadi, *Sur quelques formes du second degré et leurs inverses.* Thèse de l' Université Pierre et Marie Curie, Paris. 1995.

[6] D. Beghdadi, P. Maroni, *On the inverse problem of the product of a semiclassical form by a polynomial*, J. Comput. Appl. Math. **88** (1998), 377-399.

[7] D. Beghdadi, P. Maroni, *Second degree classical forms*, Indag. Math. N. S. **8** (1997), 439-452.

[8] S. Belmehdi, *On the left multiplication of a regular linear functional by a polynomial*, in *Orthogonal Polynomials and their Applications*, C. Brezinski et al. Editors, Annals Comput. Appl. Math. **9**. AG Baltzer, Basel, 1991. 196-176.

[9] S. Belmehdi, *On semi-classical linear functionals of class* $s = 1$. *Classification and integral representation*, Indag. Math. N. S. **3** (1992), 253-275.

[10] B. Bouras, A. Alaya, *A large family of semi-classical polynomials of class one*, Integral Transforms Spec. Funct. **18** (2007), 913-931.

[11] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[12] J. Dini, P. Maroni, *Sur la multiplication d'une forme semi-classique par un polynôme*, Publ. Sem. Math. Univ. d'Antananarivo **3** (1989), 76-89.

[13] J. Dini, P. Maroni, *The product of a linear form by a rational fraction*: *Application to Laguerre-Hahn forms*, Lect. Notes in Pure and Appl. Math. **117**, M. Dekker, New York, 1989. 131-138.

[14] F. Marcellán, E. Prianes, *Orthogonal polynomials and linear functionals of second degree*, in *Proceedings 3rd International Conference on Approximation and Optimization*, J. Guddat et al. Editors. Aport. Mat., Serie Comun. **24** (1998), 149-162.

[15] P. Maroni, *An introduction to second degree forms*, Adv. Comput. Math. **3** (1995), 59-88.

[16] P. Maroni, *Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques.* In *Orthogonal Polynomials and Their Applications*, M. Alfaro et al. Editors, Lecture Notes in Math. **1329**, Springer, Berlin, 1988. 279-290.

[17] P. Maroni, *Fonctions eulériennes. Polynômes orthogonaux classiques.* Techniques de l'ingénieur, A **154** (1994), 1-30.

[18] P. Maroni. *Sur la décomposition quadratique d'une suite de polynômes orthogonaux*, I. Rivista di Mat. Pura ed Appl. **6** (1990), 19-53.

[19] P. Maroni, *Sur la suite de polynômes orthogonaux associée à la forme* $u = \delta_c + \lambda(x-c)^{-1}L$, Period. Math. Hungar. **21** (1990), 223-248.

[20] P. Maroni, *Tchebychev forms and their perturbed as second degree forms*, Ann. Numer. Math. **2** (1995), 123-143.

[21] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques.* In *Orthogonal Polynomials and their Applications*, C. Brezinski et al. eds., Annals Comput. Appl. Math. **9**. AG Baltzer, Basel, 1991. 95-130.

[22] P. Maroni and I. Nicolau, *On the inverse problem of the product of a form by a polynomial: The cubic case,* Appl. Num. Math. **45** (2003), 419-451.

[23] G. Sansigre, G. Valent, *A large family of semi-classical polynomials: the perturbed Tchebychev*, J. Comput. Appl. Math. **57** (1995), 271-281.

[24] M. Sghaier, J. Alaya, *Orthogonal polynomials associated with some modifications of a linear form*, Meth. Appl. Anal. **11** (2004), 267-294.

Faculté des Sciences de Gabès, Département de Mathématiques, Cité Erriadh 6072, Zrig Gabès, Tunisia
E-mail address: Atef.Alaya@fsg.rnu.tn

Faculté des Sciences de Gafsa, Département de Mathématiques, Campus Universitaire Sidi Ahmed Zarroug 2112 Gafsa, Tunisia
E-mail address: Belgacem.Bouras@issatgb.rnu.tn

Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Spain
E-mail address: pacomarc@ing.uc3m.es