A new linear spectral transformation associated with derivatives of Dirac linear functionals

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Abstract

In this contribution, we analyze the regularity conditions of a perturbation on a quasi-definite linear functional by the addition of Dirac delta functionals supported on N points of the unit circle or on its complement. We also deal with a new example of linear spectral transformation. We introduce a perturbation of a quasi-definite linear functional by the addition of the first derivative of the Dirac linear functional when its support is a point on the unit circle or two points symmetric with respect to the unit circle. Necessary and sufficient conditions for the quasi-definiteness of the new linear functional are obtained. Outer relative asymptotics for the new sequence of monic orthogonal polynomials in terms of the original ones are obtained. Finally, we prove that this linear spectral transform can be decomposed as an iteration of Christoffel and Geronimus linear transformations.

Keywords: Orthogonal polynomials on the unit circle, Hermitian linear functionals, quasi-definite linear functionals, Verblunsky parameters, Caratheodory functions, outer relative asymptotics.

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2010 MSC: 42C05

1. Introduction

Let consider the linear space of Laurent polynomials with complex coefficients $\Lambda = span\{z^k\}_{k\in\mathbb{Z}}$ as well as the linear subspace $\mathbb{P}$ of polynomials with complex coefficients. Let $\mathcal{L}$ be a linear functional in Λ such that

$$c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \bar{c}_{-n},$$

i.e. $\mathcal{L}$ is an Hermitian linear functional. In $\mathbb{P}$ we can associate with $\mathcal{L}$ a bilinear functional such that $\langle p(z), q(z) \rangle_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\overline{q}(z^{-1}) \right\rangle$. The set of complex numbers $\{c_k\}_{k\in\mathbb{Z}}$ are called the *moments* associated with $\mathcal{L}$, and the Gram matrix associated with $\mathcal{L}$ is the Toeplitz matrix

$$\mathbf{T} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}. \tag{1}$$

Let us denote by $\mathbf{T}_n$, the $(n+1)\times(n+1)$ principal leading submatrix of $\mathbf{T}$. If $\det(\mathbf{T}_n) \neq 0$ for every $n \geqslant 0$, then $\mathcal{L}$ is said to be a quasi-definite (or regular) linear functional. In such a case, there exists a family of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ satisfying

$$\left\langle \mathcal{L}, \Phi_n(z)\overline{\Phi}_m(z^{-1}) \right\rangle = \mathbf{k}_n \delta_{n,m}, \quad n, m \geqslant 0,$$

where $\mathbf{k}_n = \|\Phi_n(z)\|^2 \neq 0$, $n \geqslant 0$. $\{\Phi_n\}_{n\geqslant 0}$ is said to be the sequence of monic orthogonal polynomials (MOPS) with respect to $\mathcal{L}$. Furthermore, we have $\mathbf{k}_n = \det(\mathbf{T}_n)/\det(\mathbf{T}_{n-1})$, $n \geqslant 1$, with the convention $\mathbf{k}_0 = c_0$.

If $\det(\mathbf{T}_n) > 0$ for every $n \geqslant 0$, then $\mathcal{L}$ is said to be a positive definite linear functional, and the integral representation holds (see [8] and [11])

$$\langle \mathcal{L}, p(z) \rangle = \int_{\mathbb{T}} p(z)d\sigma(z),$$

where $p(z) \in \mathbb{P}$ and $d\sigma$ is a nontrivial positive measure supported on $\mathbb{T}$, which can be decomposed into $d\sigma = \sigma' \frac{d\theta}{2\pi} + d\sigma_s$, i.e. an absolutely continuous part with respect to the Lebesgue measure and a singular part. Unless otherwise noted, throughout the manuscript we will consider quasi-definite linear functionals.

The properties of $\{\Phi_n\}_{n\geqslant 0}$ have been extensively studied (see [8], [7], [17], [18], among others) . They satisfy

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \geqslant 0, \tag{2}$$

$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \quad n \geqslant 0, \tag{3}$$

the so-called forward and backward recurrence relations, where $\Phi_n^*(z) = z^n \bar{\Phi}_n(z^{-1})$ is the reversed polynomial and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are known as Verblunsky coefficients (they are also called either Schur or reflection parameters). It is important to notice that $|\Phi_n(0)| \neq 1$, $n \geqslant 1$ (for positive definite linear functionals, we have $|\Phi_n(0)| < 1$, $n \geqslant 1$). Furthermore, there is a one to one correspondence between a linear functional (or its corresponding measure), its sequence of moments, and its family of Verblunsky coefficients ([17]). The n-th reproducing kernel is given by

$$K_n(z,y) = \sum_{m=0}^{n} \frac{\Phi_m(z)\overline{\Phi_m(y)}}{\mathbf{k}_m} = \frac{\overline{\Phi_{n+1}^*(y)}\Phi_{n+1}^*(z) - \overline{\Phi_{n+1}(y)}\Phi_{n+1}(z)}{\mathbf{k}_{n+1}(1-\bar{y}z)}, \tag{4}$$

where the last identity holds if $z\bar{y} \neq 1$. It is known in the literature as Christoffel-Darboux formula. We denote by $K_n^{(j,k)}(z,y)$ the j-th (resp. k-th) derivative of $K_n(z,y)$ with respect to the variable z (resp. y).

In terms of the moments, we can define the function

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k. \tag{5}$$

If $\mathcal{L}$ is positive definite, F is an analytic function with positive real part in $\mathbb{D}$. Moreover, it has the integral representation

$$F(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\sigma(w),$$

where σ is the measure associated with $\mathcal{L}$. $F(z)$ is said to be the Carathéodory function associated with $\mathcal{L}$. For quasi-definite linear functionals, we will define $F(z)$ as (5).

Given a linear functional $\mathcal{L}$, the following perturbations have been studied in the last years (see [5], [6], [9], [10], [13], [14], [15] among others)

1. $\langle p(z), q(z)\rangle_{\mathcal{L}_C} = \langle (z-\alpha)p(z), (z-\alpha)q(z)\rangle_{\mathcal{L}}$, $\alpha \in \mathbb{C}$.

2. $\langle p(z), q(z)\rangle_{\mathcal{L}_G} = \left\langle \frac{p(z)}{z-\alpha}, \frac{q(z)}{z-\alpha}\right\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \bar{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}$, $\alpha \in \mathbb{C}, |\alpha| \neq 1$, $\boldsymbol{m} \in \mathbb{C}$.

3. $\langle p(z), q(z)\rangle_{\mathcal{L}_U} = \langle p(z), q(z)\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \bar{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}, \quad \boldsymbol{m} \in \mathbb{C}, |\alpha| > 1$.

The corresponding family of orthogonal polynomials, the Carathéodory function, and the associated Hessenberg matrix (the matrix representation of the multiplication operator in the canonical basis of the linear space of polynomials), as well as necessary and sufficient conditions for the regularity of the perturbed functionals have been deeply analyzed in the literature.

The above perturbations are called, respectively, Christoffel ($\mathcal{F}_C(\alpha)$), Geronimus ($\mathcal{F}_G(\alpha, \boldsymbol{m})$), and Uvarov ($\mathcal{F}_U(\alpha, \boldsymbol{m})$). They are related by

(*i*) $\mathcal{F}_C(\alpha) \circ \mathcal{F}_G(\alpha, \boldsymbol{m}) = \mathcal{I}$ (Identity transformation),

(*ii*) $\mathcal{F}_G \circ \mathcal{F}_C(\alpha) = \mathcal{F}_U(\alpha, \boldsymbol{m})$.

In particular, we will focus our attention in the Uvarov transformation. The simplest of this kind of transformations is defined by

$$\langle p(z), q(z)\rangle_{\mathcal{L}_U} = \langle p(z), q(z)\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\alpha)}, \quad \boldsymbol{m} \in \mathbb{R}, |\alpha| = 1, \tag{6}$$

i.e. the addition of a real mass on a point located on the unit circle, and it was analyzed in [5], where the authors obtained necessary and sufficient conditions for the regularity of $\mathcal{L}_U$, the relation between the corresponding families of orthogonal polynomials, Carathéodory functions, Hessenberg matrices, and Verblunsky coefficients. Later on, a generalization of this problem for positive definite linear functionals was studied in [20], where the author studied, among other properties, the asymptotic behavior of the Verblunsky parameters when N real masses are added on the unit circle.

If the mass points are located outside the unit circle, then the perturbation becomes

$$\langle p(z), q(z)\rangle_{\mathcal{L}_U} = \langle p(z), q(z)\rangle_{\mathcal{L}} + \boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \bar{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}, \quad \boldsymbol{m} \in \mathbb{C}, |\alpha| > 1, \tag{7}$$

where complex conjugates are considered in order to preserve the Hermitian character of $\mathcal{L}_U$. This perturbation was analyzed in [5].

It is not so difficult to show that, in terms of the moments, perturbations (6) and (7) can be expressed, respectively, as

$$\tilde{c}_k = c_k + \boldsymbol{m}\alpha^k, \quad k \in \mathbb{Z}, \tag{8}$$

$$\tilde{c}_k = c_k + \boldsymbol{m}\alpha^k + \bar{\boldsymbol{m}}\bar{\alpha}^{-k}, \quad k \in \mathbb{Z}. \tag{9}$$

Furthermore, in both cases the corresponding Carathéodory functions are related by

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)}, \tag{10}$$

where A, B, and D are polynomials whose coefficients depend on $\boldsymbol{m}$ and α (see [14]). The complex function $\widetilde{F}$ defined by (10) is said to be a *linear spectral transformation* of $F(z)$. In the case of measures supported on the real line, linear spectral transformations have been analyzed in [21], where the author proves that any transformation of the form (10) to a Stieltjes function can be expressed in terms of Christoffel and Geronimus transformations. Notice that in the cases described above, the class of linear transformations is quite rich and new examples appear. Indeed, in [3] a perturbation involving the addition of masses was studied. There, the authors considered the addition of a Lebesgue measure to a linear functional, i.e.

$$\langle p(z), q(z)\rangle_{\mathcal{L}_0} := \langle p(z), q(z)\rangle_{\mathcal{L}} + \boldsymbol{m}\int_{\mathbb{T}} p(z)\overline{q(z)}\frac{dz}{2\pi i z}, \quad \boldsymbol{m} \in \mathbb{R}. \tag{11}$$

Notice that only the first moment is perturbed, and thus $\tilde{c}_0 = c_0 + \boldsymbol{m}$, $\tilde{c}_k = c_k, k \in \mathbb{Z} \setminus \{0\}$. In other words, this is equivalent to perturb the main diagonal of the corresponding Toeplitz matrix by

$$\widetilde{\mathbf{T}} = \mathbf{T} + \boldsymbol{m}\mathbf{I}, \tag{12}$$

where $\mathbf{I}$ is the semi-infinite identity matrix. A particular case for $\boldsymbol{m} = 1$ was studied on [1] and the regularity conditions for (11), as well as an expression for the corresponding family of orthogonal polynomials, were obtained in [3].

The generalization of the previous perturbation to affect any subdiagonal of the Toeplitz matrix is defined by

$$\langle p(z), q(z)\rangle_{\mathcal{L}_j} := \langle p(z), q(z)\rangle_{\mathcal{L}} + m\left\langle z^j p(z), q(z)\right\rangle_{\mathcal{L}_\theta} + \bar{m}\left\langle p(z), z^j q(z)\right\rangle_{\mathcal{L}_\theta}, \tag{13}$$

where $\boldsymbol{m} \in \mathbb{C}$, and $\mathcal{L}_\theta$ is the linear functional associated with the Lebesgue measure. The corresponding analysis was developed in [4]. In terms of the Toeplitz matrix, we have

$$\widetilde{\mathbf{T}} = \mathbf{T} + \begin{pmatrix} 0 & \cdots & \boldsymbol{m} & 0 & \cdots \\ \vdots & 0 & \cdots & \boldsymbol{m} & \cdots \\ \bar{\boldsymbol{m}} & \vdots & \ddots & \vdots & \ddots \\ 0 & \bar{\boldsymbol{m}} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix},$$

i.e., only the $j-th$ sub-diagonal and upper-diagonal are perturbed.

It is not difficult to see that perturbations (11) and (13) can be expressed, in terms of the corresponding Carathéodory functions, as

$$\begin{aligned} F_0(z) &= F(z) + \boldsymbol{m}, \\ F_j(z) &= F(z) + 2\boldsymbol{m}z^j, \end{aligned}$$

so they are also linear spectral transformations in the sense of (10).

The aim of our contribution is to introduce two new examples of linear spectral transformations associated with the first derivative of the Dirac linear functional. The first one appears to when the support of the Dirac linear functional is a point in the unit circle. The second one corresponds to a Dirac linear functional supported in two symmetric points with respect to the unit circle. The structure of the manuscript is as follows.

In Section 2, an Uvarov perturbation of a quasi-definite linear functional by a Dirac linear functional supported on N points located either on the unit circle $\mathbb{T}$ or on its complement is introduced. Necessary and sufficient conditions for the regularity of the perturbed linear functional are deduced. In Section 3, we deal with the addition of a linear functional that is the derivative of a Dirac linear functional supported either on a point located on the unit circle $\mathbb{T}$ or on two points symmetric with respect to the unit circle. Both situations can be considered as limit cases of the previous one but the difficulties to deal with them yield a different approach. We prove the regularity of the perturbed linear functionals as well as the outer relative asymptotics of the new MOPS in terms of the initial MOPS. In Section 4, we prove that they are linear spectral transformations using the relation between the corresponding Carathéodory functions. Furthermore, we obtain their representation in terms of Christoffel and Geronimus transformations. Finally, in Section 5, some illustrative examples are presented.

2. Uvarov perturbation with N masses

Let us start our analysis with a generalization of the perturbation (6). Consider a quasi-definite functional $\mathcal{L}$ and let $\mathcal{L}_\Upsilon$ be the linear functional such that its associated bilinear functional satisfies

$$\langle p, q\rangle_{\mathcal{L}_\Upsilon} = \langle p, q\rangle_{\mathcal{L}} + \sum_{i=1}^{N} \boldsymbol{m}_i p(\alpha_i)\overline{q(\alpha_i)}, \tag{14}$$

where $\boldsymbol{m}_i \in \mathbb{R}\backslash\{0\}$ and $|\alpha_i| = 1$ for $i = 1, \ldots, N$. Using an analog method to the one used in [5], we can show

Proposition 1. *The following statements are equivalent.*

(*i*) *$\mathcal{L}_\Upsilon$ is a quasi-definite linear functional.*

(*ii*) *The matrix $\mathbf{R}_{n-1}^N + \mathbf{M}_N^{-1}$ is non singular, and*

$$\mathbf{k}_n + [\overline{\mathbf{\Phi}_n^N(\alpha)}]^t(\mathbf{R}_{n-1}^N + \mathbf{M}_N^{-1})^{-1}\mathbf{\Phi}_n^N(\alpha) \neq 0, \quad n \geqslant 1. \tag{15}$$

Moreover, the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_\Upsilon$ is given by

$$\Upsilon_n(z) = \Phi_n(z) - \mathbf{K}_{n-1}^N(z)(\mathbf{R}_{n-1}^N + \mathbf{M}_N^{-1})^{-1}\mathbf{\Phi}_n^N(\alpha), \quad n \geqslant 1, \tag{16}$$

with $\mathbf{K}_{n-1}^N(z) = [K_{n-1}(z, \alpha_1), K_{n-1}(z, \alpha_2), \ldots K_{n-1}(z, \alpha_N)]$, $\mathbf{M}_N = diag\{\boldsymbol{m}_1, \boldsymbol{m}_2, \ldots, \boldsymbol{m}_N\}$, $\mathbf{\Phi}_n^N(\alpha) = [\Phi_n(\alpha_1), \Phi_n(\alpha_2), \ldots, \Phi_n(\alpha_N)]^t$ *and*

$$\mathbf{R}_{n-1}^N = \begin{pmatrix} K_{n-1}(\alpha_1, \alpha_1) & K_{n-1}(\alpha_1, \alpha_2) & \cdots & K_{n-1}(\alpha_1, \alpha_N) \\ K_{n-1}(\alpha_2, \alpha_1) & K_{n-1}(\alpha_2, \alpha_2) & \cdots & K_{n-1}(\alpha_2, \alpha_N) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}(\alpha_N, \alpha_1) & K_{n-1}(\alpha_N, \alpha_2) & \cdots & K_{n-1}(\alpha_N, \alpha_N) \end{pmatrix}.$$

Proof. First, assume that $\mathcal{L}_\Upsilon$ is a quasi-definite linear functional and denote by $\{\Upsilon_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Thus, for $n \geqslant 1$,

$$\Upsilon_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z), \quad \text{where} \quad \lambda_{n,k} = -\frac{\sum_{i=1}^N m_i \Upsilon_n(\alpha_i)\overline{\Phi_k(\alpha_i)}}{\mathbf{k}_k}, n \geq 1.$$

Then, we have

$$\Upsilon_n(z) = \Phi_n(z) - \sum_{i=1}^{N} m_i \Upsilon_n(\alpha_i) K_{n-1}(z, \alpha_i). \tag{17}$$

In particular, for $j = 1, \ldots, N$, we have the following system of N linear equations and N unknowns $\Upsilon_n(\alpha_j)$, $j = 1, 2, ..., N$

$$\Upsilon_n(\alpha_j) = \Phi_n(\alpha_j) - \sum_{i=1}^{N} m_i \Upsilon_n(\alpha_i) K_{n-1}(\alpha_j, \alpha_i).$$

Therefore,

$$\begin{pmatrix} 1 + m_1 K_{n-1}(\alpha_1, \alpha_1) & m_2 K_{n-1}(\alpha_1, \alpha_2) & \cdots & m_N K_{n-1}(\alpha_1, \alpha_N) \\ m_1 K_{n-1}(\alpha_2, \alpha_1) & 1 + m_2 K_{n-1}(\alpha_2, \alpha_2) & \cdots & m_N K_{n-1}(\alpha_2, \alpha_N) \\ \vdots & \vdots & \ddots & \vdots \\ m_1 K_{n-1}(\alpha_N, \alpha_1) & m_2 K_{n-1}(\alpha_N, \alpha_2) & \cdots & 1 + m_N K_{n-1}(\alpha_N, \alpha_N) \end{pmatrix} \mathbf{\Upsilon}_n^N(\alpha) = \mathbf{\Phi}_n^N(\alpha),$$

where $\mathbf{\Upsilon}_n^N(\alpha) = [\Upsilon_n(\alpha_1), \Upsilon_n(\alpha_2), \ldots, \Upsilon_n(\alpha_N)]$. In other words, $(\mathbf{R}_{n-1}^N \mathbf{M}_N + \mathbf{I}_N)\mathbf{\Upsilon}_n^N(\alpha) = \mathbf{\Phi}_n^N(\alpha)$. Since $\mathcal{L}_\Upsilon$ is assumed to be quasi-definite, the matrix $(\mathbf{R}_{n-1}^N \mathbf{M}_N + \mathbf{I}_N)$ is non singular and, therefore, (16) follows from (17).

On other hand, assume (ii) holds. For $0 \leq k \leq n-1$, we have

$$\begin{aligned} \langle \Upsilon_n(z), \Phi_k(z) \rangle_{\mathcal{L}_\Upsilon} &= \left\langle \Phi_n(z) - \sum_{i=1}^{N} m_i \Upsilon_n(\alpha_i) K_{n-1}(z, \alpha_i), \Phi_k(z) \right\rangle + \sum_{i=1}^{N} m_i \Upsilon_n(\alpha_i) \overline{\Phi_k(\alpha_i)} \\ &= -\sum_{i=1}^{N} m_i \Upsilon_n(\alpha_i) \langle K_{n-1}(z, \alpha_i), \Phi_k(z) \rangle + \sum_{i=1}^{N} m_i \Upsilon_n(\alpha_i) \overline{\Phi_k(\alpha_i)} = 0, \end{aligned}$$

using the reproducing kernel property in the last expression. Furthermore,

$$\begin{aligned} \langle \Upsilon_n(z), \Phi_n(z) \rangle_{\mathcal{L}_\Upsilon} &= \mathbf{k}_n + \sum_{i=1}^{N} m_i \Upsilon_n(\alpha_i) \overline{\Phi_n(\alpha_i)} \\ &= \mathbf{k}_n + [\overline{\mathbf{\Phi}}_n^N(\alpha)]^t \mathbf{M}_N \mathbf{\Upsilon}_n^N \\ &= \mathbf{k}_n + [\overline{\mathbf{\Phi}_n^N(\alpha)}]^t (\mathbf{R}_{n-1}^N + \mathbf{M}_N^{-1})^{-1} \mathbf{\Phi}_n^N(\alpha) \neq 0, \end{aligned}$$

which proves that $\{\Upsilon_n\}_{n\geqslant 0}$ defined by (16) is the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_\Upsilon$. □

Remark 2. *Notice that for* $N = 1$*, the regularity condition for* $\mathcal{L}_\Upsilon$ *becomes* $1 + m_1 K_{n-1}(\alpha_1, \alpha_1) \neq 0$*,* $n \geq 0$*, as shown in [5].*

Evaluating (16) in $z = 0$, we get

Corollary 3. *For* $n \geqslant 1$*,*

$$\Upsilon_n(0) = \Phi_n(0) - \mathbf{K}_{n-1}^N(0)(\mathbf{R}_n^N + \mathbf{M}_N^{-1})^{-1}\mathbf{\Phi}_n^N(\alpha). \tag{18}$$

The previous expression allows us to obtain the Verblunsky coefficients associated with the perturbed polynomials directly, provided that the original Verblunsky coefficients are known.

Proposition 4. *For* $z \in \mathbb{D}$*, the Carathéodory function associated with* $\mathcal{L}_\Upsilon$ *is*

$$F_\Upsilon(z) = F(z) + \sum_{i=1}^{N} m_i \left(\frac{\alpha_i + z}{\alpha_i - z}\right).$$

Proof. Denoting $\tilde{c}_{-k} = \langle \mathcal{L}_\Upsilon, z^{-k} \rangle$, we have

$$\begin{aligned} F_\Upsilon(z) &= \tilde{c}_0 + 2\sum_{k=1}^{\infty} \tilde{c}_{-k} z^k \\ &= c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k + \sum_{i=1}^{N} m_i + 2\sum_{k=1}^{\infty}\sum_{i=1}^{N} m_i \bar{\alpha}_i^k z^k \\ &= F(z) + \sum_{i=1}^{N} m_i \left(\frac{\alpha_i + z}{\alpha_i - z}\right), \end{aligned}$$

i.e., $F_\Upsilon(z)$ has simple poles at $z = \alpha_i$. □

The next step is to consider a perturbation of the form (7), generalizing for N masses, i.e. to consider the linear functional $\mathcal{L}_\Omega$ such that its corresponding bilinear functional satisfies

$$\langle p, q \rangle_{\mathcal{L}_\Omega} = \langle p, q \rangle_{\mathcal{L}} + \sum_{i=1}^{N} (m_i p(\alpha_i)\overline{q}(\alpha_i^{-1}) + \overline{m}_i p(\bar{\alpha}_i^{-1})\overline{q(\alpha_i)}), \tag{19}$$

where $|\alpha_i| \neq 0, 1$ and $m_i \in \mathbf{C} \setminus \{0\}$, $1 \leqslant i \leqslant N$. By analogy with the previous case, we have the following result.

Proposition 5. *The following statements are equivalent.*

(i) $\mathcal{L}_\Omega$ *is a quasi-definite linear functional.*

(ii) *the matrix* $\mathbf{R}_{n-1}^{2N} + \mathbf{M}_{2N}^{-1}$ *is non singular, and*

$$\mathbf{k}_n + [\overline{\mathbf{\Phi}}_n^{2N}(\alpha)]^t \mathbf{M}_{2N} \mathbf{\Omega}_n^{2N}(\alpha) \neq 0, \quad n \geqslant 1. \tag{20}$$

Moreover, the corresponding sequence of monic polynomials orthogonal with respect to $\mathcal{L}_\Omega$ *is given by*

$$\Omega_n(z) = \Phi_n(z) - \mathbf{K}_{n-1}^{2N}(z)(\mathbf{R}_{n-1}^{2N} + \mathbf{M}_{2N}^{-1})^{-1}\mathbf{\Phi}_n^{2N}(\alpha), \quad n \geqslant 1, \tag{21}$$

with

$$\begin{aligned}
\mathbf{K}_{n-1}^{2N} &= [K_{n-1}(z,\alpha_1),\ldots K_{n-1}(z,\alpha_N), K_{n-1}(z,\overline{\alpha}_1^{-1}),\ldots,K_{n-1}(z,\overline{\alpha}_N^{-1})],\\
\mathbf{M}_{2N} &= diag\{m_1,\ldots,m_N,\overline{m}_1,\ldots,\overline{m}_N\},\\
\mathbf{\Phi}_n^{2N}(\alpha) &= [\Phi_n(\alpha_1),\ldots,\Phi_n(\alpha_N),\Phi_n(\overline{\alpha}_1^{-1}),\ldots,\Phi_n(\overline{\alpha}_N^{-1})]^t,\\
\mathbf{R}_{n-1}^{2N} &= \left(\begin{array}{c|c} R_{n-1}(\alpha_{1,N},\alpha_{1,N}) & R_{n-1}(\alpha_{1,N},\bar{\alpha}_{1,N}^{-1}) \\ \hline R_{n-1}(\bar{\alpha}_{1,N}^{-1},\alpha_{1,N}) & R_{n-1}(\bar{\alpha}_{1,N}^{-1},\bar{\alpha}_{1,N}^{-1}) \end{array}\right), \quad \textit{and}
\end{aligned}$$

$$R_{n-1}(\alpha_{1,N},\alpha_{1,N}) = \begin{pmatrix} K_{n-1}(\alpha_1,\alpha_1) & \cdots & K_{n-1}(\alpha_1,\alpha_N) \\ \vdots & \ddots & \vdots \\ K_{n-1}(\alpha_N,\alpha_1) & \cdots & K_{n-1}(\alpha_N,\alpha_N) \end{pmatrix}.$$

Proceeding as in the proof or Proposition 4, we obtain

Proposition 6. *For* $z \in \mathbb{D}$*,*

$$F_\Omega(z) = F(z) + \sum_{i=1}^{N}\left(m_i\frac{\alpha_i + z}{\alpha_i - z} + \bar{m}_i\frac{\bar{\alpha}_i^{-1} + z}{\bar{\alpha}_i^{-1} - z}\right).$$

i.e., $F_\Omega(z)$ has simple poles at $z = \alpha_i$ and $z = \bar{\alpha}_i^{-1}$.

3. Adding the derivative of a Dirac's delta

3.1. Mass point on the unit circle

Given an Hermitian linear functional $\mathcal{L}$, its derivative $D\mathcal{L}$ (see [19]) is defined by

$$\langle D\mathcal{L}, p(z)\rangle = -i\,\langle \mathcal{L}, zp'(z)\rangle\,, \tag{22}$$

where $p \in \Lambda$. Consider a perturbation of a linear functional $\mathcal{L}$ by the addition of a derivative of a Dirac's delta, i.e.

$$\left\langle \tilde{\mathcal{L}}, p(z) \right\rangle = \langle \mathcal{L}, p(z) \rangle + m \langle D\delta_\alpha, p(z) \rangle , \tag{23}$$

where $m \in \mathbb{R}$ and $|\alpha| = 1$. In terms of the associated bilinear functional,

$$\langle p(z), q(z) \rangle_{\tilde{\mathcal{L}}} = \langle p(z), q(z) \rangle_{\mathcal{L}} - im[\alpha p'(\alpha)\overline{q(\alpha)} - \bar{\alpha} p(\alpha)\overline{q'(\alpha)}]. \tag{24}$$

Our goal is to obtain the necessary and sufficient conditions for $\tilde{\mathcal{L}}$ to be a quasi-definite linear functional, as well as an expression for its corresponding family of orthogonal polynomials.

Proposition 7. *Assume $\mathcal{L}$ is a quasi-definite linear functional and denote by $\{\Phi_n\}_{n\geqslant 0}$ its corresponding MOPS. Let consider $\tilde{\mathcal{L}}$ as in* (24). *Then, the following statements are equivalent:*

(*i*) *$\tilde{\mathcal{L}}$ is quasi-definite.*

(*ii*) *The matrix $\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)$, with*

$$\mathbb{K}_{n-1}(\alpha,\alpha) = \begin{pmatrix} K_{n-1}(\alpha,\alpha) & K_{n-1}^{(0,1)}(\alpha,\alpha) \\ K_{n-1}^{(1,0)}(\alpha,\alpha) & K_{n-1}^{(1,1)}(\alpha,\alpha) \end{pmatrix}, \quad \mathbf{D}(\alpha) = \begin{pmatrix} 0 & -i\alpha \\ i\alpha^{-1} & 0 \end{pmatrix},$$

is non singular, and

$$\mathbf{k}_n + m[\overline{\mathbf{\Phi}_n(\alpha)}]^t[\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)]^{-1}\mathbf{\Phi}_n(\alpha) \neq 0, \quad n \geqslant 1. \tag{25}$$

Furthermore, the MOPS associated with $\tilde{\mathcal{L}}$ is given by

$$\Psi_n(z) = \Phi_n(z) - m \begin{pmatrix} K_{n-1}(z,\alpha) \\ K_{n-1}^{(0,1)}(z,\alpha) \end{pmatrix}^t [\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)]^{-1}\mathbf{\Phi}_n(\alpha), \tag{26}$$

where $\mathbf{\Phi}_n(z) = [\Phi_n(z), \Phi_n'(z)]^t$.

Proof. Assume $\tilde{\mathcal{L}}$ is quasi-definite and denote by $\{\Psi_n\}_{n\geqslant 0}$ its corresponding family of monic orthogonal polynomials. Let us consider the Fourier expansion

$$\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z),$$

where for $n \geq 1$

$$\begin{aligned}\lambda_{n,k} &= \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}}}{\mathbf{k}_k} \\ &= \frac{im[\alpha\Psi_n^{'}(\alpha)\overline{\Phi_k(\alpha)} - \bar{\alpha}\Psi_n(\alpha)\overline{\Phi_k^{'}(\alpha)}]}{\mathbf{k}_k}, 0 \leq k \leq n-1.\end{aligned}$$

Thus,

$$\begin{aligned}\Psi_n(z) &= \Phi_n(z) + \sum_{k=0}^{n-1} \frac{im[\alpha\Psi_n^{'}(\alpha)\overline{\Phi_k(\alpha)} - \bar{\alpha}\Psi_n(\alpha)\overline{\Phi_k^{'}(\alpha)}]}{\mathbf{k}_k}\Phi_k(z), \\ &= \Phi_n(z) + im\left[\alpha\Psi_n^{'}(\alpha)K_{n-1}(z,\alpha) - \bar{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right]. \qquad (27)\end{aligned}$$

Taking the derivative with respect to z in the previous expression and evaluating at $z = \alpha$, we obtain the linear system

$$\begin{aligned}\Psi_n(\alpha) &= \Phi_n(\alpha) + im\left[\alpha\Psi_n^{'}(\alpha)K_{n-1}(\alpha,\alpha) - \bar{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(\alpha,\alpha)\right], \\ \Psi_n^{'}(\alpha) &= \Phi_n^{'}(\alpha) + im\left[\alpha\Psi_n^{'}(\alpha)K_{n-1}^{(1,0)}(\alpha,\alpha) - \bar{\alpha}\Psi_n(\alpha)K_{n-1}^{(1,1)}(\alpha,\alpha)\right],\end{aligned}$$

which yields

$$\begin{pmatrix}\Phi_n(\alpha) \\ \Phi_n^{'}(\alpha)\end{pmatrix} = \begin{pmatrix}1 + im\bar{\alpha}K_{n-1}^{(0,1)}(\alpha,\alpha) & -im\alpha K_{n-1}(\alpha,\alpha) \\ im\bar{\alpha}K_{n-1}^{(1,1)}(\alpha,\alpha) & 1 - im\alpha K_{n-1}^{(1,0)}(\alpha,\alpha)\end{pmatrix}\begin{pmatrix}\Psi_n(\alpha) \\ \Psi_n^{'}(\alpha)\end{pmatrix}, \qquad (28)$$

and denoting $\mathbf{Q}(z) = [Q(z), Q^{'}(z)]^t$, we get

$$\mathbf{\Phi}_n(\alpha) = [\mathbf{I}_2 + m\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\alpha)]\mathbf{\Psi}_n(\alpha),$$

where we use the notation

$$\mathbb{K}_{n-1}(\alpha,\alpha) = \begin{pmatrix}K_{n-1}(\alpha,\alpha) & K_{n-1}^{(0,1)}(\alpha,\alpha) \\ K_{n-1}^{(1,0)}(\alpha,\alpha) & K_{n-1}^{(1,1)}(\alpha,\alpha)\end{pmatrix} \quad \text{and} \quad \mathbf{D}(\alpha) = \begin{pmatrix}0 & -i\alpha \\ i\bar{\alpha} & 0\end{pmatrix}.$$

Thus, the necessary condition for regularity is that $\mathbf{I}_2 + m\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\alpha)$ be non singular. Taking into account $\mathbf{D}^{-1}(\alpha) = \mathbf{D}(\alpha)$ we have the first part of our statement. Furthermore, from (27),

$$
\begin{aligned}
\Psi_n(z) &= \Phi_n(z) + m\left[K_{n-1}(z,\alpha), K_{n-1}^{(0,1)}(z,\alpha)\right]\begin{pmatrix} 0 & i\alpha \\ -i\bar{\alpha} & 0\end{pmatrix}\begin{pmatrix}\Psi_n(\alpha) \\ \Psi_n'(\alpha)\end{pmatrix} \\
&= \Phi_n(z) - m\begin{pmatrix} K_{n-1}(z,\alpha) \\ K_{n-1}^{(0,1)}(z,\alpha)\end{pmatrix}^t \mathbf{D}(\alpha)[\mathbf{I}_2 + m\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\alpha)]^{-1}\mathbf{\Phi}_n(\alpha) \\
&= \Phi_n(z) - m\begin{pmatrix} K_{n-1}(z,\alpha) \\ K_{n-1}^{(0,1)}(z,\alpha)\end{pmatrix}^t [\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)]^{-1}\mathbf{\Phi}_n(\alpha) \\
&= \Phi_n(z) - m\begin{pmatrix} K_{n-1}(z,\alpha) \\ K_{n-1}^{(0,1)}(z,\alpha)\end{pmatrix}^t [\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)]^{-1}\mathbf{\Phi}_n(\alpha).
\end{aligned}
$$

This yields (26). Conversely, if $\{\Psi_n\}_{n\geqslant 0}$ is given by (27), then, for $0 \leqslant k \leqslant n-1$,

$$
\begin{aligned}
\langle \Psi_n(z), \Psi_k(z)\rangle_{\tilde{\mathcal{L}}} &= \left\langle \Phi_n(z) + im\left[\alpha\Psi_n'(\alpha)K_{n-1}(z,\alpha) - \bar{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right], \Psi_k(z)\right\rangle_{\tilde{\mathcal{L}}} \\
&= \left\langle \Phi_n(z) + im\left[\alpha\Psi_n'(\alpha)K_{n-1}(z,\alpha) - \bar{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right], \Psi_k(z)\right\rangle_{\mathcal{L}} \\
&- im\left[\alpha\Psi_n'(\alpha)\overline{\Psi_k(\alpha)} - \bar{\alpha}\Psi_n(\alpha)\overline{\Psi_k'(\alpha)}\right] \\
&= 0.
\end{aligned}
$$

On the other hand, for $n \geqslant 1$,

$$
\begin{aligned}
\tilde{\mathbf{k}}_n &= \langle \Psi_n(z), \Psi_n(z)\rangle_{\tilde{\mathcal{L}}} = \langle \Psi_n(z), \Phi_n(z)\rangle_{\tilde{\mathcal{L}}} \\
&= \left\langle \Phi_n(z) + im\left[\alpha\Psi_n'(\alpha)K_{n-1}(z,\alpha) - \bar{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right], \Phi_n(z)\right\rangle_{\mathcal{L}} \\
&- im\left[\alpha\Psi_n'(\alpha)\overline{\Phi_n(\alpha)} - \bar{\alpha}\Psi_n(\alpha)\overline{\Phi_n'(\alpha)}\right] \\
&= \mathbf{k}_n - im\left[\alpha\Psi_n'(\alpha)\overline{\Phi_n(\alpha)} - \bar{\alpha}\Psi_n(\alpha)\overline{\Phi_n'(\alpha)}\right] \\
&= \mathbf{k}_n - im[\overline{\mathbf{\Phi}_n(\alpha)}]^t\begin{pmatrix} 0 & \alpha \\ -\overline{\alpha} & 0\end{pmatrix}\mathbf{\Psi}_n(\alpha) \\
&= \mathbf{k}_n + m[\overline{\mathbf{\Phi}_n(\alpha)}]^t[\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)]^{-1}\mathbf{\Phi}_n(\alpha) \neq 0,
\end{aligned}
$$

where we are again using the reproducing property of $K_{n-1}(z,\alpha)$. As a conclusion, $\{\Psi_n\}_{n\geqslant 0}$ is the MOPS with respect to $\tilde{\mathcal{L}}$. □

Notice that the addition of a Dirac's delta derivative (on a point of the unit circle) to a linear functional can be considered as the limit case of two equal masses with opposite sign, located on two nearby points located on the unit circle $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$, $0 \leqslant \theta_1, \theta_2 \leqslant 2\pi$, when $\theta_1 \to \theta_2$. Indeed, if we set $N = 2$ in the previous Section, then the 2×2 matrix in (15) becomes, on the limit, the 2×2 matrix in (25). As we will show later, the same occurs for the 4×4 matrix in (21) corresponding to masses located on two pairs of points outside the unit circle.

Remark 8. *Using the Christoffel-Darboux formula* (4)*, another way to express* (26) *is*

$$(z-\alpha)^2\Psi_n(z) = A(z,n,\alpha)\Phi_n(z) + B(z,n,\alpha)\Phi_n^*(z), \tag{29}$$

where $A(z,n,\alpha)$ and $B(z,n,\alpha)$ are polynomials of degree 2 and 1, respectively, in the variable z, given by

$$\begin{aligned} A(z,n,\alpha) &= (z-\alpha)^2 - \frac{m\alpha}{\mathbf{k}_n\Delta_{n-1}}\Big[[Y_{1,1}\Phi_n(\alpha) + Y_{1,2}\Phi_n'(\alpha)]\overline{\Phi_n(\alpha)}(z-\alpha) \\ &+ [Y_{2,1}\Phi_n(\alpha) + Y_{2,2}\Phi_n'(\alpha)][\Phi_n(\alpha)(z-\alpha) + \alpha\Phi_n(\alpha)z]\Big], \\ B(z,n,\alpha) &= \frac{m\alpha}{\mathbf{k}_n\Delta_{n-1}}\Big[[Y_{1,1}\Phi_n(\alpha) + Y_{1,2}\Phi_n'(\alpha)]\overline{\Phi_n^*(\alpha)} \\ &+ [Y_{2,1}\Phi_n(\alpha) + Y_{2,2}\Phi_n'(\alpha)][\overline{\Phi_n^*(\alpha)}'(z-\alpha) + \alpha\overline{\Phi_n^*(\alpha)}z]\Big], \end{aligned}$$

where $Y_{1,1} = mK_{n-1}^{(1,1)}(\alpha,\alpha)$, $Y_{1,2} = im\alpha K_{n-1}^{(0,1)}(\alpha,\alpha)$, $Y_{2,1} = -im\bar{\alpha}K_{n-1}^{(1,0)}(\alpha,\alpha)$, $Y_{2,2} = m\alpha K_{n-1}(\alpha,\alpha)$, and Δ_{n-1} is the determinant of the matrix $\mathbf{D}(\alpha) + im\mathbb{K}_{n-1}(\alpha,\alpha)$.

3.2. Asymptotic behavior

In this subsection, we will assume $\mathcal{L}$ is a positive definite linear functional, with an associated positive Borel measure σ. We are interested in the asymptotic behavior of the orthogonal polynomials associated with the addition of the derivative of a Dirac delta on the unit circle, i.e. the polynomials $\{\Psi_n\}_{n\geqslant 0}$ given in (29) (we will also assume that the regularity conditions hold). In particular, we will study its ratio asymptotics with respect to $\{\Phi_n\}_{n\geqslant 0}$. First, we will state a result that will be useful in our study.

Theorem 9. *[12] Let σ be a regular finite positive Borel measure supported on $(-\pi,\pi]$, i.e. $\lim_{n\to\infty}\kappa_n^{1/n} = 1$, $\kappa_n^2 = 1/\mathbf{k}_n$. Let $J \in (-\pi,\pi)$ be a compact subset such that σ is absolutely continuous in an open set containing J. Assume that σ' is positive and continuous at each point of J. Let l, j be non-negative integers. Then, uniformly for $\theta\in J$, $z = e^{i\theta}$,*

$$\lim_{n\to\infty}\frac{z^{l-j}}{n^{l+j}}\frac{K_n^{(l,j)}(z,z)}{K_n(z,z)} = \frac{1}{l+j+1}. \tag{30}$$

Proposition 10. *(Outer relative asymptotics). Let $\mathcal{L}$ be a positive definite linear functional, whose associated measure σ satisfies the conditions of Theorem 9. Let $\{\Phi_n\}_{n\geqslant 0}$ be the MOPS associated with $\mathcal{L}$ and $\{\Psi_n\}_{n\geqslant 0}$ the MOPS associated to $\widetilde{\mathcal{L}}$ defined as in* (24)*. Then, uniformly in $\mathbb{C}\setminus\overline{\mathbb{D}}$,*

$$\lim_{n\to\infty}\frac{\Psi_n(z)}{\Phi_n(z)}=1. \tag{31}$$

Proof. From the expression (29),

$$\frac{\Psi_n(z)}{\Phi_n(z)}=\frac{A(z,n,\alpha)}{(z-\alpha)^2}+\frac{B(z,n,\alpha)}{(z-\alpha)^2}\frac{\Phi_n^*(z)}{\Phi_n(z)}.$$

Since, for $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$ (see [17]),

$$\lim_{n\to\infty}\frac{\Phi_n^*(z)}{\Phi_n(z)}=0,$$

it suffices to show that, for $|\alpha|=1$,

$$\lim_{n\to\infty}\frac{A(z,n,\alpha)}{(z-\alpha)^2}=1.$$

Notice that $\lim_{n\to\infty}\Phi_n(\alpha)=O(1)$, $\lim_{n\to\infty}\Phi_n^{'}(\alpha)=O(n)$, $\lim_{n\to\infty}\Phi_n^*(\alpha)=O(1)$, $\lim_{n\to\infty}\Phi_n^{*'}(\alpha)=O(n)$, and $\lim_{n\to\infty}K_n(\alpha,\alpha)=O(n)$.

On the other hand, dividing the numerator and denominator of $\frac{A(z,n,\alpha)}{(z-\alpha)^2}-1$ by $n^2K_{n-1}(\alpha,\alpha)$, and using (30), we obtain

$$\lim_{n\to\infty}\frac{\Phi_n(\alpha)Y_{2,1}}{n^2K_{n-1}(\alpha,\alpha)}=O(1/n),\qquad \lim_{n\to\infty}\frac{\Phi_n^{'}(\alpha)Y_{2,2}}{n^2K_{n-1}(\alpha,\alpha)}=O(1/n),$$
$$\lim_{n\to\infty}\frac{\Phi_n(\alpha)Y_{1,1}}{n^2K_{n-1}(\alpha,\alpha)}=O(1),\qquad \lim_{n\to\infty}\frac{\Phi_n^{'}(\alpha)Y_{1,2}}{n^2K_{n-1}(\alpha,\alpha)}=O(1),$$

so that the numerator of $\frac{A(z,n,\alpha)}{(z-\alpha)^2}-1$ behaves as $\sim O(1)$. Similarity, one can shows that the denominator behaves as $\sim O(n)$, and therefore

$$\lim_{n\to\infty}\frac{A(z,n,\alpha)}{(z-\alpha)^2}=1.$$

The same arguments can be applied to $B(z,n,\alpha)$, what ensures the result. □

3.3. Mass points outside the unit circle

Now, consider an Hermitian linear functional $\hat{\mathcal{L}}$ such that its associated bilinear functional satisfies

$$\begin{aligned}\langle p(z), q(z)\rangle_{\hat{\mathcal{L}}} &= \langle p(z), q(z)\rangle_{\mathcal{L}} + im[\alpha^{-1}p(\alpha)\overline{q'(\bar{\alpha}^{-1})} - \alpha p'(\alpha)\overline{q(\bar{\alpha}^{-1})}] \\ &+ i\bar{m}[\bar{\alpha}p(\bar{\alpha}^{-1})\overline{q'(\alpha)} - \bar{\alpha}^{-1}p'(\bar{\alpha}^{-1})\overline{q(\alpha)}], \end{aligned} \tag{32}$$

with $m, \alpha \in \mathbb{C}$, $|\alpha| \neq 0$ and $|\alpha| \neq 1$ (see [2]). As above, we are interested on the regularity conditions for this linear functional and the corresponding family of orthogonal polynomials. Assuming that $\hat{\mathcal{L}}$ is a quasi-definite linear functional and following the method used in the proof of Proposition 7, we get

$$\begin{aligned}\Psi_n(z) &= \Phi_n(z) + im\left[\alpha\Psi_n'(\alpha)K_{n-1}(z,\bar{\alpha}^{-1}) - \alpha^{-1}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\bar{\alpha}^{-1})\right] \\ &+ i\bar{m}\left[\bar{\alpha}^{-1}\Psi_n'(\bar{\alpha}^{-1})K_{n-1}(z,\alpha) - \bar{\alpha}\Psi_n(\bar{\alpha}^{-1})K_{n-1}^{(0,1)}(z,\alpha)\right]. \end{aligned} \tag{33}$$

Evaluating the above expression and its first derivative in α and $\bar{\alpha}^{-1}$ we get the following linear systems

$$\begin{pmatrix}\Phi_n(\alpha)\\ \Phi_n'(\alpha)\end{pmatrix} = \begin{pmatrix}1 + im\alpha^{-1}K_{n-1}^{(0,1)}(\alpha,\bar{\alpha}^{-1}) & -im\alpha K_{n-1}(\alpha,\bar{\alpha}^{-1})\\ im\alpha^{-1}K_{n-1}^{(1,1)}(\alpha,\bar{\alpha}^{-1}) & 1 - im\alpha K_{n-1}^{(1,0)}(\alpha,\bar{\alpha}^{-1})\end{pmatrix}\begin{pmatrix}\Psi_n(\alpha)\\ \Psi_n'(\alpha)\end{pmatrix} \tag{34}$$

$$+ \begin{pmatrix}i\bar{m}\bar{\alpha}K_{n-1}^{(0,1)}(\alpha,\alpha) & -i\bar{m}\bar{\alpha}^{-1}K_{n-1}(\alpha,\alpha)\\ i\bar{m}\bar{\alpha}K_{n-1}^{(1,1)}(\alpha,\alpha) & -i\bar{m}\bar{\alpha}^{-1}K_{n-1}^{(1,0)}(\alpha,\alpha)\end{pmatrix}\begin{pmatrix}\Psi_n(\bar{\alpha}^{-1})\\ \Psi_n'(\bar{\alpha}^{-1})\end{pmatrix}, \tag{35}$$

$$\begin{pmatrix}\Phi_n(\bar{\alpha}^{-1})\\ \Phi_n'(\bar{\alpha}^{-1})\end{pmatrix} = \begin{pmatrix}im\alpha^{-1}K_{n-1}^{(0,1)}(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) & -im\alpha K_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})\\ im\alpha^{-1}K_{n-1}^{(1,1)}(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) & -im\alpha K_{n-1}^{(1,0)}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})\end{pmatrix}\begin{pmatrix}\Psi_n(\alpha)\\ \Psi_n'(\alpha)\end{pmatrix} \tag{36}$$

$$+ \begin{pmatrix}1 + i\bar{m}\bar{\alpha}K_{n-1}^{(0,1)}(\bar{\alpha}^{-1},\alpha) & -i\bar{m}\bar{\alpha}^{-1}K_{n-1}(\bar{\alpha}^{-1},\alpha)\\ i\bar{m}\bar{\alpha}K_{n-1}^{(1,1)}(\bar{\alpha}^{-1},\alpha) & 1 - i\bar{m}\bar{\alpha}^{-1}K_{n-1}^{(1,0)}(\bar{\alpha}^{-1},\alpha)\end{pmatrix}\begin{pmatrix}\Psi_n(\bar{\alpha}^{-1})\\ \Psi_n'(\bar{\alpha}^{-1})\end{pmatrix}, \tag{37}$$

which yield into the system of 4 linear equations with 4 unknowns

$$\begin{pmatrix}\mathbf{\Phi}_n(\alpha)\\ \mathbf{\Phi}_n(\bar{\alpha}^{-1})\end{pmatrix} = \begin{pmatrix}\mathbf{I}_2 + m\mathbb{K}_{n-1}(\alpha,\bar{\alpha}^{-1})\mathbf{D}(\alpha) & \bar{m}\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\bar{\alpha}^{-1})\\ m\mathbb{K}_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})\mathbf{D}(\alpha) & \mathbf{I}_2 + \bar{m}\mathbb{K}_{n-1}(\bar{\alpha}^{-1},\alpha)\mathbf{D}(\bar{\alpha}^{-1})\end{pmatrix}\begin{pmatrix}\mathbf{\Psi}_n(\alpha)\\ \mathbf{\Psi}_n(\bar{\alpha}^{-1})\end{pmatrix},$$

where $[\mathbf{Q}(z), \mathbf{R}(z)]^t = [Q(z), Q'(z), R(z), R'(z)]^t$. Thus, in order for $\hat{\mathcal{L}}$ to be a quasi-definite linear functional, we need the 4×4 matrix defined as above must be non singular. On the other hand,

$$\begin{pmatrix}\mathbf{\Psi}_n(\alpha)\\ \mathbf{\Psi}_n(\bar{\alpha}^{-1})\end{pmatrix} = \begin{pmatrix}\mathbf{I}_2 + m\mathbb{K}_{n-1}(\alpha,\bar{\alpha}^{-1})\mathbf{D}(\alpha) & \bar{m}\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\bar{\alpha}^{-1})\\ m\mathbb{K}_{n-1}(\bar{\alpha}^{-1},\bar{\alpha}^{-1})\mathbf{D}(\alpha) & \mathbf{I}_2 + \bar{m}\mathbb{K}_{n-1}(\bar{\alpha}^{-1},\alpha)\mathbf{D}(\bar{\alpha}^{-1})\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{\Phi}_n(\alpha)\\ \mathbf{\Phi}_n(\bar{\alpha}^{-1})\end{pmatrix}.$$

As a consequence, from (33), we get

$$\Psi_n(z) = \Phi_n(z) - m \begin{pmatrix} K_{n-1}(z, \bar{\alpha}^{-1}) \\ K_{n-1}^{(0,1)}(z, \bar{\alpha}^{-1}) \end{pmatrix}^t \mathbf{D}(\alpha) \mathbf{\Psi}_n(\alpha) - \bar{m} \begin{pmatrix} K_{n-1}(z, \alpha) \\ K_{n-1}^{(0,1)}(z, \alpha) \end{pmatrix}^t \mathbf{D}(\bar{\alpha}^{-1}) \mathbf{\Psi}_n(\bar{\alpha}^{-1}) \tag{38}$$

where $\mathbf{\Psi}_n(\alpha)$ and $\mathbf{\Psi}_n(\bar{\alpha}^{-1})$ can be obtained from the above linear system. Assuming that the regularity conditions hold, and following the method used in the proof of Proposition 7, is not difficult to show that $\{\Psi_n\}_{n\geqslant 0}$ defined as in (38) is the MOPS with respect to $\hat{\mathcal{L}}$.

On the other hand, it is possible to obtain a generalization of Proposition 10 for the MOPS associated with (32). As before, we can express (38) as in (29). Using the Christoffel-Darboux formula, we obtain

$$\Psi_n(z) = [1 + \tilde{A}(z, n, \alpha)]\Phi_n(z) + \tilde{B}(z, n, \alpha)\Phi_n^*(z),$$

with

$$\begin{aligned}
\tilde{A}(z, n, \alpha) &= im\bar{\alpha}^{-1} \frac{\overline{\Phi_n'(\bar{\alpha}^{-1})}(1 - \alpha^{-1}z) + z\overline{\Phi_n(\bar{\alpha}^{-1})}}{\mathbf{k}_n(1 - \alpha^{-1})^2} \Psi_n(\alpha) - im\alpha \frac{\overline{\Phi_n(\bar{\alpha}^{-1})}}{\mathbf{k}_n(1 - \alpha^{-1}z)} \Psi_n'(\alpha) \\
&+ i\bar{m}\alpha \frac{\overline{\Phi_n'(\alpha)}(1 - \bar{\alpha}z) + z\overline{\Phi_n(\alpha)}}{\mathbf{k}_n(1 - \bar{\alpha})^2} \Psi_n(\bar{\alpha}^{-1}) - i\bar{m}\bar{\alpha}^{-1} \frac{\overline{\Phi_n(\alpha)}}{\mathbf{k}_n(1 - \bar{\alpha}z)} \Psi_n'(\bar{\alpha}^{-1}), \\
\tilde{B}(z, n, \alpha) &= im\alpha \frac{\overline{\Phi_n^*(\bar{\alpha}^{-1})}}{\mathbf{k}_n(1 - \alpha^{-1}z)} \Psi_n'(\alpha) - im\bar{\alpha}^{-1} \frac{\overline{\Phi_n'^*(\bar{\alpha}^{-1})}(1 - \alpha^{-1}z) + z\overline{\Phi_n^*(\bar{\alpha}^{-1})}}{\mathbf{k}_n(1 - \alpha^{-1})^2} \Psi_n(\alpha) \\
&+ i\bar{m}\bar{\alpha}^{-1} \frac{\overline{\Phi_n^*(\alpha)}}{\mathbf{k}_n(1 - \bar{\alpha}z)} \Psi_n'(\bar{\alpha}^{-1}) - i\bar{m}\alpha \frac{\overline{\Phi_n'^*(\alpha)}(1 - \bar{\alpha}z) + z\overline{\Phi_n^*(\alpha)}}{\mathbf{k}_n(1 - \bar{\alpha})^2} \Psi_n(\bar{\alpha}^{-1}),
\end{aligned}$$

where the values of $\Psi_n(\alpha), \Psi_n'(\alpha), \Psi_n(\bar{\alpha}^{-1})$ and $\Psi_n'(\bar{\alpha}^{-1})$ can be obtained by solving the 4×4 linear system shown above. Denoting the entries of the 2×2 matrices in (34) - (37) by $\{b_{i,j}\}, \{c_{i,j}\}, \{a_{i,j}\}$ and $\{d_{i,j}\}$, respectively, we get

$$\begin{aligned}
\Psi_n(\alpha) &= [d_{1,1}\Phi_n(\alpha) + d_{1,2}\Phi_n'(\alpha) + c_{1,1}\Phi_n(\bar{\alpha}^{-1}) + c_{1,2}\Phi_n'(\bar{\alpha}^{-1})]/\Delta, \\
\Psi_n'(\alpha) &= [d_{2,1}\Phi_n(\alpha) + d_{2,2}\Phi_n'(\alpha) + c_{2,1}\Phi_n(\bar{\alpha}^{-1}) + c_{2,2}\Phi_n'(\bar{\alpha}^{-1})]/\Delta, \\
\Psi_n(\bar{\alpha}^{-1}) &= [a_{1,1}\Phi_n(\alpha) + a_{1,2}\Phi_n'(\alpha) + b_{1,1}\Phi_n(\bar{\alpha}^{-1}) + b_{1,2}\Phi_n'(\bar{\alpha}^{-1})]/\Delta, \\
\Psi_n(\bar{\alpha}^{-1}) &= [a_{2,1}\Phi_n(\alpha) + a_{2,2}\Phi_n'(\alpha) + b_{2,1}\Phi_n(\bar{\alpha}^{-1}) + b_{2,2}\Phi_n'(\bar{\alpha}^{-1})]/\Delta,
\end{aligned}$$

where Δ is the determinant of the 4×4 matrix. To get the asymptotic result, it suffices to show that $\tilde{A}(z, n, \alpha) \to 0$ and $\tilde{B}(z, n, \alpha) \to 0$ as $n \to \infty$. First, notice that, applying the corresponding derivatives to the Christoffel-Darboux formula, we obtain

$$K_{n-1}^{(0,1)}(z,y) = \frac{\overline{\Phi_n^{*\prime}(y)}\Phi_n^*(z) - \overline{\Phi_n'(y)}\Phi_n(z)}{\mathbf{k}_n(1-\bar{y}z)} + \frac{zK_{n-1}(z,y)}{1-\bar{y}z},$$
$$K_{n-1}^{(1,0)}(z,y) = \frac{\overline{\Phi_n^*(y)}\Phi_n^{*\prime}(z) - \overline{\Phi_n(y)}\Phi_n'(z)}{\mathbf{k}_n(1-\bar{y}z)} + \frac{\bar{y}K_{n-1}(z,y)}{1-\bar{y}z},$$
$$K_{n-1}^{(1,1)}(z,y) = \frac{\overline{\Phi_n^{*\prime}(y)}\Phi_n^{*\prime}(z) - \overline{\Phi_n'(y)}\Phi_n'(z)}{\mathbf{k}_n(1-\bar{y}z)} + \frac{zK_{n-1}^{(1,0)}(z,y) + \bar{y}K_{n-1}^{(0,1)}(z,y) + K_{n-1}(z,y)}{1-\bar{y}z}.$$

On the other hand, if $\mathcal{L}$ is positive definite, and its corresponding measure belongs to the Szegő class, then we have (see [17]) $\Phi_n(\alpha) = O(\alpha^n)$, $\Phi_n'(\alpha) = O(n\alpha^n)$, and

$$\frac{\Phi_n(\alpha)}{\Phi_n^*(\alpha)} \to 0, \quad |\alpha| < 1, \qquad \frac{\Phi_n^*(\alpha)}{\Phi_n(\alpha)} \to 0, \quad |\alpha| > 1.$$

Assume, without loss of generality, that $|\alpha| < 1$. Then (see [17]), $K_n(\alpha,\alpha) < \infty$ and $K_n(\bar{\alpha}^{-1},\bar{\alpha}^{-1}) = O(|\alpha|^{-2n})$, as well as $K_n(\alpha,\bar{\alpha}^{-1}) = K_n(\bar{\alpha}^{-1},\alpha) = O(n)$. Observe that, except for the entries containing $K_{n-1}(\alpha,\alpha)$ and their derivatives, all other entries of the 4×4 matrix diverge, and thus its determinant diverges much faster than any other term in the expressions for $\Psi_n(\alpha), \Psi_n'(\alpha), \Psi_n(\bar{\alpha}^{-1})$ and $\Psi_n'(\bar{\alpha}^{-1})$, so that $\tilde{A}(z,n,\alpha) \to 0$ and $\tilde{B}(z,n,\alpha) \to 0$ as $n \to \infty$. As a consequence,

Proposition 11. *(Outer relative asymptotics).Let $\mathcal{L}$ be a positive definite linear functional, whose associated measure σ satisfies the Szegő condition, i.e., $\sum_{n=1}^{\infty} |\Phi_n(0)|^2 < \infty$. Let $\{\Phi_n\}_{n\geqslant 0}$ be the MOPS associated with $\mathcal{L}$ and $\{\Psi_n\}_{n\geqslant 0}$ the MOPS associated to $\widetilde{\mathcal{L}}$ defined as in* (32)*. Then, uniformly in $\mathbb{C}\setminus\mathbb{T}$,*

$$\lim_{n\to\infty} \frac{\Psi_n(z)}{\Phi_n(z)} = 1. \tag{39}$$

Remark 12. *Spectral transformations defined by* (14)*,* (19)*,* (24)*, and* (32) *can be expressed by a superposition of transformations* (13)*, as follows*

(*i*) *First, we consider the generalized Uvarov perturbations* (14) *and* (19) *of a linear functional $\mathcal{L}$. The moments $\tilde{c}_k$, and $\hat{c}_k$, corresponding to the perturbed functionals $\mathcal{L}_\Upsilon$ and $\mathcal{L}_\Omega$, respectively, are given by*

$$\tilde{c}_k = \langle \mathcal{L}_\Upsilon, z^k\rangle = c_k + \sum_{i=1}^{N} m_i\alpha_i^k = c_k + M_k \quad k = 0,\pm1,\pm2,\dots$$
$$\hat{c}_k = \langle \mathcal{L}_\Upsilon, z^k\rangle = c_k + \sum_{i=1}^{N} m_i\alpha_i^k + \overline{m}_i\alpha_i^{-k} = c_k + \hat{M}_k \quad k = 0,\pm1,\pm2,\dots$$

Therefore, the Toeplitz matrices $\mathbf{T}_n(\mathcal{L}_\Upsilon)$ *and* $\mathbf{T}_n(\mathcal{L}_\Omega)$ *are*

$$\begin{aligned}\mathbf{T}_n(\mathcal{L}_\Upsilon) &= \mathbf{T}_n + \mathbf{M}_{n+1},\\ \mathbf{T}_n(\mathcal{L}_\Omega) &= \mathbf{T}_n + \mathbf{T}_n(\mathcal{L}_\Upsilon) + \hat{\mathbf{M}}_{n+1}.\end{aligned}$$

Notice that

$$\begin{aligned}\mathbf{M}_{n+1} &= M_0\mathbf{I}_{n+1} + M_1(\mathbf{Z}_{n+1} + \mathbf{Z}^t_{n+1}) + \cdots + M_N(\mathbf{Z}^N_{n+1} + (\mathbf{Z}^N_{n+1})^t)\\ \hat{\mathbf{M}}_{n+1} &= \hat{M}_0\mathbf{I}_{n+1} + \hat{M}_1(\mathbf{Z}_{n+1} + \mathbf{Z}^t_{n-1}) + \cdots + \hat{M}_N(\mathbf{Z}^N_{n+1} + (\mathbf{Z}^N_{n-1})^t)\end{aligned}$$

and $\mathbf{Z}_{n+1}$ *is the shift matrix with ones on the first upper-diagonal and zeros on the remaining entries.*
(ii) *Now, we take the perturbations* (24) *and* (32) *of a linear functional* $\mathcal{L}$*. In these cases, the moments* $\tilde{c}_k$ *and* $\hat{c}_k$*, for* $\tilde{\mathcal{L}}$ *and* $\hat{\mathcal{L}}$ *are, respectively,*

$$\begin{aligned}\tilde{c}_k &= \langle \tilde{\mathcal{L}}, z^k\rangle = c_k - imk\alpha^k = c_k + N_k \quad k = 0, \pm1, \pm2, \ldots\\ \tilde{c}_k &= \langle \hat{\mathcal{L}}, z^k\rangle = c_k + i\bar{m}k\bar{\alpha}^{-k} - imk\alpha^k = c_k + \hat{N}_k \quad k = 0, \pm1, \pm2, \ldots\end{aligned}$$

As a consequence, the Toeplitz matrices for $\tilde{\mathcal{L}}$ *and* $\hat{\mathcal{L}}$ *can be represented as a sum of matrices as in the previous cases. Thus, we get*

Proposition 13. *The perturbations* (14)*,* (19)*,* (24)*, and* (32) *can be expressed in terms of the subdiagonal perturbations* (13) *as*

$$\mathcal{L}_\Upsilon = \bigoplus_{j\in\mathbb{N}} \mathcal{L}_j(B_j), \quad with \quad B_j = M_j,$$

$$\mathcal{L}_\Omega = \bigoplus_{j\in\mathbb{N}} \mathcal{L}_j(B_j), \quad with \quad B_j = \hat{M}_j,$$

$$\tilde{\mathcal{L}} = \bigoplus_{j\in\mathbb{N}} \mathcal{L}_j(B_j), \quad with \quad B_j = N_j,$$

$$\hat{\mathcal{L}} = \bigoplus_{j\in\mathbb{N}} \mathcal{L}_j(B_j), \quad with \quad B_j = \hat{N}_j,$$

where B_j *is the mass associated with the perturbation.*

4. Carathéodory functions

First, we will assume that $|\alpha| = 1$. Consider the moments associated with $\tilde{\mathcal{L}}$. Notice that $\tilde{c}_0 = c_0$. For $k \geqslant 1$, we have $\tilde{c}_k = \left\langle z^k, 1 \right\rangle_{\tilde{\mathcal{L}}} = c_k - imk\alpha^k$. In a similar way, $\tilde{c}_{-k} = c_{-k} + imk\bar{\alpha}^k$. Therefore,

$$\begin{aligned}
\widetilde{F}(z) &= \tilde{c}_0 + 2\sum_{k=1}^{\infty} \tilde{c}_{-k} z^k = c_0 + 2\sum_{k=1}^{\infty} (c_{-k} + imk\bar{\alpha}^k) z^k = F(z) + 2im\sum_{k=1}^{\infty} k\bar{\alpha}^k z^k \\
&= F(z) + 2im\bar{\alpha}z\sum_{k=1}^{\infty} k(\bar{\alpha}z)^{(k-1)} = F(z) + 2imz\left(\sum_{k=1}^{\infty} (\bar{\alpha}z)^k\right)' \\
&= F(z) + 2imz\left(\frac{\bar{\alpha}z}{1-\bar{\alpha}z}\right)' = F(z) - \frac{2im}{1-\bar{\alpha}z} + \frac{2im}{(1-\bar{\alpha}z)^2} \\
&= F(z) + \frac{2im\alpha}{z-\alpha} + \frac{2im\alpha^2}{(z-\alpha)^2}.
\end{aligned}$$

This means that the resulting Carathéodory function is a perturbation of $F(z)$ by the addition of a rational function with a double pole at $z = \alpha$.

Now we will assume $|\alpha| > 1$ and let consider the moments associated with $\hat{\mathcal{L}}$. Notice that $\hat{c}_0 = c_0$. For $k \in \mathbb{N}$ we have, from (32),

$$\begin{aligned}
\hat{c}_k &= c_k - imk\alpha^k - i\bar{m}k\bar{\alpha}^{-k}, \\
\hat{c}_{-k} &= c_{-k} + i\bar{m}k\bar{\alpha}^k + imk\alpha^{-k},
\end{aligned}$$

and, as a consequence,

$$\begin{aligned}
\hat{F}(z) &= \hat{c}_0 + 2\sum_{k=1}^{\infty} \hat{c}_{-k} z^k \\
&= F(z) + 2i\bar{m}\sum_{k=1}^{\infty} k(\bar{\alpha}z)^k + 2im\sum_{k=1}^{\infty} k(\alpha^{-1}z)^k \\
&= F(z) + \frac{2i\bar{m}}{1-\bar{\alpha}z} + \frac{2i\bar{m}}{(1-\bar{\alpha}z)^2} + \frac{2im}{1-\alpha^{-1}z} + \frac{2im}{(1-\alpha^{-1}z)^2} \\
&= F(z) - \frac{2im\alpha}{z-\alpha} + \frac{2im\alpha^2}{(z-\alpha)^2} - \frac{2i\bar{m}\bar{\alpha}^{-1}}{z-\bar{\alpha}^{-1}} + \frac{2i\bar{m}\bar{\alpha}^{-2}}{(z-\bar{\alpha}^{-1})^2}. \qquad (40)
\end{aligned}$$

This means that the resulting Carathéodory function is a perturbation of the initial one by the addition of a rational function with two double poles at α and $\bar{\alpha}^{-1}$.

4.1. Connection to canonical transformations.

We will show that perturbations (32) can be expressed in terms of Christoffel and Geronimus transformations. The Carathéodory functions associated with $\mathcal{F}_C(\alpha)$ and $\mathcal{F}_G(\alpha, \boldsymbol{m})$ have the form (10), with (see [14])

$$\begin{aligned} A_C(z) &= D_G(z) = (z-\alpha)(1-\bar{\alpha}z), \\ D_C(z) &= A_G(z) = z, \\ B_C(z) &= -\bar{\alpha}c_0 z^2 + (\alpha c_{-1} - \bar{\alpha}c_1)z + \alpha c_0, \\ B_G(z) &= \bar{\alpha}\tilde{c}_0 z^2 + 2i\Im\mathfrak{m}(q_0)z - \alpha\tilde{c}_0, \end{aligned}$$

where q_0 is a free parameter that depends of the mass used in the Geronimus transformation. Now, consider the following product of transformations

$$\mathcal{F}_D = \mathcal{F}_{G_2}(\alpha, \boldsymbol{m}_2) \circ \mathcal{F}_{G_1}(\alpha, \boldsymbol{m}_1) \circ \mathcal{F}_{C_2}(\alpha) \circ \mathcal{F}_{C_1}(\alpha). \tag{41}$$

Is not difficult to show that $F_D(z)$, the Carathéodory function associated with $\mathcal{F}_D$, is given by

$$\begin{aligned} F_D(z) &= F(z) + \frac{B_{C_1}(z)}{D_{G_1}(z)} + \frac{B_{G_2}(z)}{D_{G_2}(z)} + \frac{B_{C_2}(z)A_{G_1}(z)}{D_{G_1}(z)D_{G_2}(z)} + \frac{B_{G_1}(z)A_{G_2}(z)}{D_{G_1}(z)D_{G_2}(z)} \\ &= F(z) + \frac{B_{C_1}(z) + B_{G_2}(z)}{(z-\alpha)(1-\bar{\alpha}z)} + \frac{z(B_{C_2}(z) + B_{G_1}(z))}{(z-\alpha)^2(1-\bar{\alpha}z)^2}. \end{aligned}$$

Assuming that all transformations are normalized, i.e., all of the first moments are equal to 1, and denoting $K_1 = \alpha c_{-1} - \bar{\alpha}c_1 + 2i\Im\mathfrak{m}(q_0^{(1)})$ and $K_2 = \alpha c_{-1} - \bar{\alpha}c_1 + 2i\Im\mathfrak{m}(q_0^{(2)})$, where $q_0^{(1)}$ and $q_0^{(2)}$ are the free parameters associated to $\mathcal{F}_{G_1}$ and $\mathcal{F}_{G_2}$, respectively, we obtain

$$\begin{aligned} F_D(z) &= F(z) + \frac{K_2 z}{(z-\alpha)(1-\bar{\alpha}z)} + \frac{K_1 z^2}{(z-\alpha)^2(1-\bar{\alpha}z)^2} \\ &= F(z) + \frac{K_2 z(z-\alpha)(1-\bar{\alpha}z) + K_1 z^2}{(z-\alpha)^2(1-\bar{\alpha}z)^2} \\ &= F(z) + \frac{L_1}{(z-\alpha)} + \frac{L_2}{(z-\alpha)^2} + \frac{L_3}{(z-\bar{\alpha}^{-1})} + \frac{L_4}{(z-\bar{\alpha}^{-1})^2}, \end{aligned} \tag{42}$$

for some constants L_1, L_2, L_3 and L_4 satisfying

$$\begin{aligned} -\bar{\alpha}K_2 &= L_1 + L_3, \\ (1+|\alpha|^2)K_2 + K_1 &= -(\alpha + 2\bar{\alpha}^{-1})L_1 + L_2 - (2\alpha + \bar{\alpha}^{-1})L_3 + L_4, \\ -\alpha K_2 &= (\bar{\alpha}^{-2} + 2\alpha\bar{\alpha}^{-1})L_1 - 2\bar{\alpha}^{-1}L_2 + (\alpha^2 + 2\bar{\alpha}^{-1}\alpha)L_3 - 2\alpha L_4, \\ 0 &= -\alpha\bar{\alpha}^{-2}L_1 + \bar{\alpha}^{-2}L_2 - \alpha^2\bar{\alpha}^{-1}L_3 + \alpha^2 L_4. \end{aligned}$$

Furthermore, comparing (40) and (42), we have $L_2 = -\alpha L_1$ and $L_4 = -\bar{\alpha}^{-1}L_3$. Solving the above system, we arrive at

$$L_1 = \frac{\alpha|\alpha|^2}{1-|\alpha|^2}K_2, \quad L_3 = -\frac{\alpha}{1-|\alpha|^2}K_2,$$

and thus, we conclude that transformation (41) is equivalent to $\hat{\mathcal{F}}(\alpha^{-1}, m)$, with

$$m = \frac{|\alpha|^2}{2i(1-|\alpha|^2)}K_2.$$

5. Examples

In this section, we study three examples that illustrate the behavior of the Verblunsky parameters for the MOPS associated to the perturbation (23). First, we study a perturbation to the Lebesgue measure $\sigma = \frac{d\theta}{2\pi}$ given by

$$d\tilde{\sigma} = \frac{d\theta}{2\pi} + m\delta_{\alpha}^{'},$$

where $m \in \mathbb{R}$ and $|\alpha| = 1$. It is very well known that $\Phi_n(z) = z^n$ is the $n-th$ monic orthogonal polynomial with respect to σ, and thus $\Psi_n(z)$, the $n-th$ monic orthogonal polynomial with respect to $\tilde{\sigma}$ can be obtained using (29). Indeed, evaluating these polynomials at $z = 0$, for the special case $\alpha = 1$, is not difficult to show that

$$\Psi_n(0) = \frac{\frac{n(n-1)(n+1)}{6} - \frac{in}{m}}{\frac{n^2(n-1)(n+1)}{12} - \frac{1}{m^2}}. \tag{43}$$

From the last expression, we are able to obtain the regularity condition in terms of the mass, by setting $|\Psi_n(0)| \neq 1$, $n \geqslant 1$. Notice that $|\Psi_n(0)| \to 0$, as can be seen from (43). Thus, there exists a nonnegative integer n_0, depending on m, such that $|\Psi_n(0)| < 1$ for $n \geqslant n_0$, but some of the preceding Verblunsky coefficients will be of modulus greater than 1, destroying the positivity of the perturbed functional. Indeed, from (25), we obtain that the positivity condition for this perturbation is

$$m^2 < \frac{12}{n^2(n^2-1)}, \quad n \geqslant 2.$$

Since the right side is a positive monotone decreasing sequence, we only have a positive definite case if $m = 0$. The following figure shows the Verblunsky coefficients for different values of m and $\alpha = 1$.

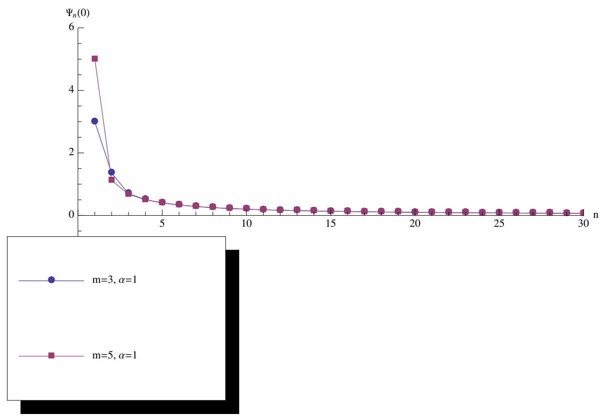

Figure 1. Behavior of Verblunsky coefficients for perturbations of the Lebesgue measure.

On the other hand, if we take $d\sigma = \frac{1-|\beta|^2}{|1-\beta z|^2}\frac{d\theta}{2\pi}$, the normalized Bernstein-Szegő measure, with $|\beta| < 1$, whose corresponding MOPS is given by $\Phi_n(z) = z^n - \beta z^{n-1}$, then the Verblunsky parameters associated with the perturbation (22) are shown below, for different values of m and α, as indicated in the figure.

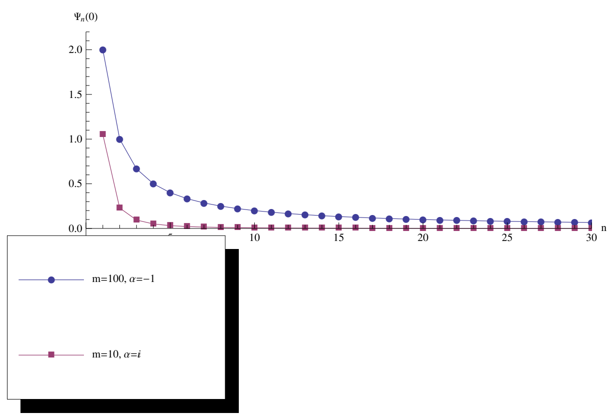

Figure 2. Behavior of Verblunsky coefficients for perturbations of the Bernstein-Szegő measure.

Finally, we exhibit the behavior of the Verblunsky parameters associated to the perturbation (22) for an absolutely continuous weight σ defined by the Féjer kernel as follows (see [16])

$$d\sigma = \frac{1}{N+1}\left|\frac{z^{N+1}-1}{z-1}\right|^2 \frac{d\theta}{2\pi}$$

whose MOPS $\Phi_n(z)$, $0 \leqslant n \leqslant N+1$, $N = 0, 1, 2, \ldots$ are given by

$$\begin{aligned} \Phi_0(z) &= 1 \\ \Phi_n(z) &= \frac{1}{2N-n+3} - \frac{2N-n+2}{2N-n+3} z^{n-1} + z^n, \qquad 1 \leqslant n \leqslant N+1. \end{aligned}$$

For $N = 30$ the following figure shows the behavior of the perturbed Verblunsky coefficients for several values of m and α.

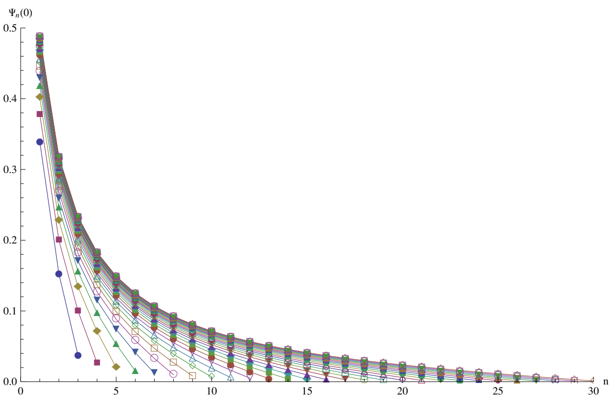

Figure 3. Behavior of Verblunsky coefficients for perturbations of the Féjer Kernel.

Acknowledgements

The authors thank the valuable comments of the anonymous referees. They greatly contributed to the improvement of this manuscript. The work of the first and third author have been supported by Comunidad de Madrid-Universidad Carlos III de Madrid, grant MTM2009-12740-C03-01.

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