Perturbations on the subdiagonals of Toeplitz matrices.

K. Castillo[a], L. Garza[*,b], F. Marcellán[a]

[a] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911, Leganés, Spain.*
[b] *Universidad Autónoma de Tamaulipas, Carretera Sendero Nacional Km. 3, AP2005, Matamoros, México.*

Abstract

Let $\mathcal{L}$ be an Hermitian linear functional defined on the linear space of Laurent polynomials. It is very well known that the Gram matrix of the associated bilinear functional in the linear space of polynomials is a Toeplitz matrix. In this contribution we analyze some linear spectral transforms of $\mathcal{L}$ such that the corresponding Toeplitz matrix is the result of the addition of a constant in a subdiagonal of the initial Toeplitz matrix. We focus our attention in the analysis of the quasi-definite character of the perturbed linear functional as well as in the explicit expressions of the new monic orthogonal polynomial sequence in terms of the first one.

Key words: Hermitian Linear Functionals, Caratheodory functions, Laurent Polynomials, Orthogonal Polynomials, Toeplitz matrices.
2000 MSC: 42C05, 15A23.

1. Introduction

Let $\mathcal{L}$ be a linear functional in the linear space of Laurent polynomials with complex coefficients $\Lambda = span\{z^k\}_{k\in\mathbb{Z}}$, satisfying $c_n = \langle\mathcal{L}, z^n\rangle = \overline{\langle\mathcal{L}, z^{-n}\rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$. $\mathcal{L}$ is said to be a Hermitian linear functional. The complex numbers $\{c_n\}_{n\in\mathbb{Z}}$

*Corresponding author
Email addresses: kcastill@math.uc3m.es (K. Castillo), lgaona@uat.edu.mx (L. Garza), pacomarc@ing.uc3m.es (F. Marcellán)

are said to be the *moments* associated with $\mathcal{L}$ and the infinite matrix

$$\mathbf{T} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \quad (1)$$

is known in the literature as a Toeplitz matrix [1].

$\mathbb{P} = span\{z^k\}_{k\in\mathbb{N}}$ will denote the linear space of polynomials with complex coefficients. If $\mathbf{T}_n$, the $(n+1)\times(n+1)$ principal leading submatrix of $\mathbf{T}$, is nonsingular for every $n \geqslant 0$, then $\mathcal{L}$ is said to be quasi-definite and the existence of a sequence of monic polynomials, orthogonal with respect to $\mathcal{L}$, is guaranteed. On the other hand, if $\det \mathbf{T}_n > 0$, for every $n \geqslant 0$, then $\mathcal{L}$ is said to be positive definite and it has the following integral representation

$$\langle \mathcal{L}, p(z)\rangle = \int_{\mathbb{T}} p(z)d\sigma(z), \quad p \in \mathbb{P}, \quad (2)$$

where σ is a nontrivial positive Borel measure supported on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. In such a case, there exists a (unique) family of polynomials $\{\varphi_n\}_{n\geqslant 0}$, with $deg\,\varphi_n = n$ and positive leading coefficient, such that

$$\int_{\mathbb{T}} \varphi_n(z)\overline{\varphi_m(z)}d\sigma(z) = \delta_{m,n}. \quad (3)$$

$\{\varphi_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to σ. Denoting by κ_n the leading coefficient of $\varphi_n(z)$, $\Phi_n(z) = \varphi_n(z)/\kappa_n$ is the corresponding sequence of monic orthogonal polynomials. These polynomials satisfy the following forward and backward recurrence relations (see [1], [2], [3], [4])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \geqslant 0, \quad (4)$$

$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \quad n \geqslant 0, \quad (5)$$

where $\Phi_n^*(z) = z^n\bar{\Phi}_n(z^{-1})$ is the so-called reversed polynomial and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are known as Verblunsky, Schur or reflection parameters. It is important to notice that in the positive definite case we get $|\Phi_n(0)| < 1$, $n \geqslant 1$, and

$\mathbf{k}_n = \langle \Phi_n, \Phi_n \rangle_{\mathcal{L}} > 0, n \geqslant 0.$
Moreover,

$$\mathbf{k}_n = \frac{\det \mathbf{T}_n}{\det \mathbf{T}_{n-1}}, \qquad n \geq 1, \mathbf{k}_0 = c_0. \tag{6}$$

The n-th polynomial kernel $K_n(z, y)$ associated with $\{\Phi_n\}_{n\geqslant 0}$ is defined by

$$K_n(z, y) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(z)}{\mathbf{k}_j} = \frac{\overline{\Phi^*_{n+1}(y)}\Phi^*_{n+1}(z) - \overline{\Phi_{n+1}(y)}\Phi_{n+1}(z)}{\mathbf{k}_{n+1}(1 + \overline{y}z)}, \tag{7}$$

and it satisfies the so called reproducing property, i.e.

$$\int_{\mathbb{T}} K_n(z, y)\overline{p(z)}d\sigma(z) = \overline{p(y)} \tag{8}$$

for every polynomial p of degree at most n.

On the other hand,

$$\Phi^*_n(z) = \mathbf{k}_n K_n(z, 0), n \geq 0. \tag{9}$$

Furthermore, in terms of the moments, an analytic function can be defined by

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k. \tag{10}$$

If $\mathcal{L}$ is a positive definite functional, then (10) is analytic in $\mathbb{D}$ and its real part is positive in $\mathbb{D}$. In such a case, (10) is called Carathéodory function, and can be represented by the Riesz-Herglotz transform

$$F(z) = \int_{\mathbb{T}} \frac{w + z}{w - z} d\sigma(w),$$

where σ is the positive measure associated with $\mathcal{L}$. By extension, for a quasi-definite linear functional, we will call (10) its corresponding Carathéodory function. The measure σ can be decomposed in an absolutely continuous part with respect to the normalized Lebesgue measure and a singular part (see [3]). Thus, if we denote by σ' the Radon-Nikodym derivative of σ, then

$$d\sigma = \sigma' \frac{d\theta}{2\pi} + d\sigma_s, \tag{11}$$

where σ_s is the singular part of σ. The measure σ belongs to the Szegő class S if

$$\int_0^{2\pi} \log \sigma' \frac{d\theta}{2\pi} > -\infty. \tag{12}$$

(12) is equivalent to $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$.

Recently, the following perturbations of a linear functional (or its corresponding measure) have been studied. (see [5], [6], [7], [8]),

(i) $d\tilde{\sigma} = |z-\alpha|^2 d\sigma, \quad \alpha \in \mathbb{C}, \quad m \in \mathbb{C},$

(ii) $d\tilde{\sigma} = d\sigma + m\delta(z-\alpha) + \bar{m}\delta(z-\bar{\alpha}^{-1}), \quad \alpha \in \mathbb{C} \setminus \{0\}, \quad m \in \mathbb{C},$

(iii) $d\tilde{\sigma} = \frac{d\sigma}{|z-\alpha|^2} + m\delta(z-\alpha) + \bar{m}\delta(z-\bar{\alpha}^{-1}), \quad \alpha \in \mathbb{C} \setminus \{0\}, \quad |\alpha| \neq 1, \quad m \in \mathbb{C},$

the so-called Christoffel ($\mathcal{F}_C$), Uvarov ($\mathcal{F}_U$), and Geronimus ($\mathcal{F}_G$) transformations, respectively. Necessary and sufficient conditions for the positive definiteness (or quasi-definiteness) of the new linear functional have been analyzed. Special attention has been paid to the effect of such transformations on the corresponding families of orthogonal polynomials, the relation between their Hessenberg matrices (the matrix representation of the multiplication operator with respect to the orthogonal polynomial basis), and between their associated Carathéodory functions, which can be expressed in the previous cases as

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{C(z)}, \tag{13}$$

where A, B, and C are polynomials in the variable z.

Perturbations $(i) - (iii)$ are related by

(1) $\mathcal{F}_C(\alpha) \circ \mathcal{F}_G(\alpha, m) = \mathcal{I}$ (Identity transformation),

(2) $\mathcal{F}_G \circ \mathcal{F}_C(\alpha) = \mathcal{F}_U(\alpha, m)$,

and are called *linear* spectral transformations. They are the unit circle analog of the perturbations defined in [9] for measures supported on the real line. In [9], the author shows that the corresponding Stieltjes functions (the real line analog of the Carathéodory functions, given by $S(x) = \sum_{n=0}^{\infty} \mu_n x^{-(n+1)}$, where $\{\mu_n\}_{n\geqslant 0}$ are the moments associated with the measure on the real line) are related by

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{C(x)}, \tag{14}$$

for some polynomials A, B, and C. Furthermore, it is shown that *any* transformation of the form (14), for any polynomials A, B, and C, can be obtained as a product of Christoffel and Geronimus transformations. In other words, transformations of the form (14) constitute a group, whose generators are the Christoffel and Geronimus transformations. The analogous result for spectral transformations on the unit circle, as will be shown on the following Section, does not hold.

There is a one to one correspondence between measures supported on the interval $[-1, 1]$ and measures supported on $\mathbb{T}$, called the Szegő transformation (see [4]). Moreover, the corresponding Stieltjes and Carathéodory functions are related by

$$S(x) = \frac{2z}{1-z^2}F(z), \tag{15}$$

with $z = x - \sqrt{x^2-1}$. (15) can also be expressed as $F(z) = \sqrt{x^2-1}S(x)$, with $x = (z+z^{-1})/2$. Furthermore, in [10], the authors show that Christoffel, Uvarov, and Geronimus transformations for measures on $[-1, 1]$ result in the same kind of transformations for the corresponding measures on the unit circle, when the Szegő transformation is applied.

The structure of the manuscript is as follows. In Section 2 we analyze a diagonal perturbation of a Toeplitz matrix. Necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials with respect to the corresponding linear functional are discussed. In Section 3, an Hermitian perturbation of a Toeplitz matrix along a subdiagonal is introduced. Thus, necessary and sufficient conditions for the quasi-definiteness of the corresponding linear functional are stated. As a consequence, an expression of the monic orthogonal polynomials in terms of those associated with the first linear functional is given. Finally, in Section 4 an illustrative example of such a perturbation when the initial linear functional is given by the normalized Lebesgue measure is shown.

2. A diagonal perturbation of the Toeplitz matrix

Recently, a perturbation in terms of the moments associated with the linear functional was studied. In [11], the authors consider the perturbation

$$\langle p(z), q(z)\rangle_{\tilde{\mathcal{L}}} := \langle p(z), q(z)\rangle_{\mathcal{L}} + m\int_{\mathbb{T}} p(z)\overline{q(z)}\frac{dz}{2\pi iz}, \tag{16}$$

with $m \in \mathbb{R}$, $p, q \in \mathbb{P}$, and $\mathcal{L}$ is a positive definite linear functional. A special case with $m = 1$ was studied in [12]. It is easy to see that all of the moments associated with $\widetilde{\mathcal{L}}$ are equal to the moments of $\mathcal{L}$, up to for the first one, for which we get $\tilde{c}_0 = c_0 + m$. In other words, the Toeplitz matrix associated with the new linear functional is obtained by adding a mass m to the main diagonal of the original Toeplitz matrix. Thus, the corresponding Carathéodory function is

$$\widetilde{F}(z) = F(z) + m, \tag{17}$$

i,e., a linear spectral transformation of $F(z)$ with $A(z) = C(z)$ and $B(z) = mC(z)$. Since $B(z) = (\alpha - \bar{\alpha}z^2)(m_u + \bar{m}_u) - (1 - |\alpha|^2)(m_u - \bar{m}_u)z$ and $C(z) = (z - \alpha)(1 - \bar{\alpha}z)$ are the polynomials for the Uvarov transformation, it is clear that (17) can not be obtained for any value of α and m_u. Therefore, (17) is an example of a linear spectral transformation that can not be expressed as a product of Christoffel and Geronimus transformations.

For $m > 0$, it is clear that $\widetilde{\mathcal{L}}$ is also a positive definite linear functional, and therefore there exists a family of orthogonal polynomials with respect to it. The expression for these polynomials was obtained in [11], as follows

Proposition 1. *[11] Let $\mathcal{L}$ be a positive definite linear functional and denote by $\{\Phi_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Then, $\{\Psi_n\}_{n\geqslant 0}$, the sequence of monic polynomials orthogonal with respect to $\widetilde{\mathcal{L}}$ defined by* (16)*, is given by*

$$\Psi_n(z) = \Phi_n(z) - \mathbf{K}_{n-1}^t(z,0)(m^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_n)^{-1}\mathbf{\Phi}_n(0), \tag{18}$$

with $\mathbf{K}_{n-1}(z,0) = [K_{n-1}(z,0), K_{n-1}^{(0,1)}(z,0), \ldots, K_{n-1}^{(0,n-1)}(z,0)]^t$*,* $\mathbf{D}_n = diag\{\frac{1}{0!}, \ldots, \frac{1}{(n-1)!}\}$*,* $\mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi_n'(0), \ldots, \Phi_n^{(n-1)}(0)]^t$*,* $\mathbf{R}_n = \mathbf{P}_n\mathbf{P}_n^t$*, and*

$$\mathbf{P}_n = \begin{pmatrix} \varphi_0(0) & \varphi_1(0) & \cdots & \varphi_{n-1}(0) \\ 0 & \varphi_1'(0) & \cdots & \varphi_{n-1}'(0) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \varphi_{n-1}^{(n-1)}(0) \end{pmatrix}.$$

Furthermore, if $T_j(p(y);0)$ denotes the $j-th$ Taylor polynomial of $p(y)$ around $y = 0$, it can be shown that

Proposition 2. *[11]*

$$K_{n-1}^{(0,j)}(z,0) = j!\left[\frac{\Phi_n^*(z)}{\mathbf{k}_n}T_j^*(\Phi_n^*(z);0) - \frac{\Phi_n(z)}{\mathbf{k}_n}T_j^*(\Phi_n(z);0)\right].$$

From the previous Proposition, if we denote

$$\mathbf{T}(\Phi_n(z);0) = [T_0^*(\Phi_n(z);0), T_1^*(\Phi_n(z);0), \ldots, T_{n-1}^*(\Phi_n(z);0)]^t,$$

then (18) becomes

$$\Psi_n(z) \quad = \quad a_n(z)\Phi_n(z) + b_n(z)\Phi_n^*(z), \tag{19}$$

with

$$\begin{aligned} a_n(z) &= 1 + \frac{1}{\mathbf{k}_n}\mathbf{T}^t(\Phi_n(z);0)\mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\boldsymbol{\Phi}_n(0), \\ b_n(z) &= -\frac{1}{\mathbf{k}_n}\mathbf{T}^t(\Phi_n^*(z);0)\mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\boldsymbol{\Phi}_n(0). \end{aligned}$$

Furthermore, the corresponding measure, denoted by $\widetilde{\sigma}$, is

$$d\tilde{\sigma} = d\sigma + m\frac{d\theta}{2\pi}. \tag{20}$$

Denoting by $\|\cdot\|_{\tilde{\sigma}}$, $\|\cdot\|_{\sigma}$, and $\|\cdot\|_{\theta}$ the norms in the spaces $L^2_{\tilde{\sigma}}[0,2\pi]$, $L^2_{\sigma}[0,2\pi]$, and $L^2_{\theta}[0,2\pi]$, respectively, we obtain the following results, which are analogous to those obtained in [12] .

Proposition 3. *If*

$$\ell = \lim_{n\to\infty}\|\Phi_n\|^2_{\sigma} \quad \textit{and} \quad \widetilde{\ell} = \lim_{n\to\infty}\|\Psi_n\|^2_{\tilde{\sigma}}, \tag{21}$$

then

1. *det* $\widetilde{\mathbf{T}}_n \geqslant m$ *det* $\widetilde{\mathbf{T}}_{n-1}$.
2. $m + \|\Phi_n\|^2_{\sigma} \leqslant \|\Psi_n\|^2_{\tilde{\sigma}} \leqslant \tilde{c}_0 + m$.
3. $m + \ell \leqslant \widetilde{\ell} \leqslant \tilde{c}_0 + m$.
4. $\|\Psi_n\|^2_{\sigma} \leqslant \tilde{c}_0$.

Proof.

1. From (6),

$$\det\widetilde{\mathbf{T}}_n = \widetilde{\mathbf{k}}_n \det\widetilde{\mathbf{T}}_{n-1} = \|\Psi_n\|^2_{\tilde{\sigma}} \det\widetilde{\mathbf{T}}_{n-1}.$$

On the other hand,

$$\begin{aligned} \|\Psi_n\|^2_{\tilde{\sigma}} &= \|\Psi_n\|^2_{\sigma} + m\|\Psi_n\|^2_{\theta} \\ &\geqslant \|\Psi_n\|^2_{\sigma} + m\|z^n\|^2_{\theta} \\ &\geqslant m\|z^n\|^2_{\theta} = m. \end{aligned}$$

and the statement follows.

2. Applying the extremal property of the norm of the monic orthogonal polynomials,

$$\begin{aligned} m + \|\Phi_n\|_\sigma^2 &\leqslant m\|z^n\|_\theta^2 + \|\Psi_n\|_\sigma^2 \\ &\leqslant m\|\Psi_n\|_\theta^2 + \|\Psi_n\|_\sigma^2 = \|\Psi_n\|_{\tilde{\sigma}}^2 \\ &\leqslant \|z^n\|_{\tilde{\sigma}}^2 \\ &= \tilde{c}_0 + m. \end{aligned}$$

3. It follows from ii), taking limits when $n \to \infty$.
4. It is a straightforward consequence of

$$\|\Psi_n\|_\sigma^2 + m\|\Psi_n\|_\theta^2 \leq \tilde{c}_0 + m. \tag{22}$$

■

From (11),

$$d\tilde{\sigma} = (\sigma' + m)\frac{d\theta}{2\pi} + d\sigma_s,$$

and $\tilde{\sigma}' = \sigma' + m$ is the Radon-Nikodym derivative of the measure $\tilde{\sigma}$ with respect to the normalized Lebesgue measure.

Corollary 4.

1. *If σ belongs to the Szegő class S, then $\tilde{\sigma}$ belongs to the Szegő class S.*
2. *The absolutely continuous part of $\tilde{\sigma}$, $\tilde{\sigma}'$, satisfies $\frac{1}{\tilde{\sigma}'} \leq \frac{1}{m}$.*

Proof.

1. Applying the Szegő's Theorem ([3], [4]), we get

$$\tilde{\sigma} \in S \iff \widetilde{\ell} > 0,$$

so the result follows from (*iii*) in previous Proposition.

2. It is immediate from

$$\frac{1}{\tilde{\sigma}'} = \frac{1}{\sigma' + m} \leq \frac{1}{m}. \tag{23}$$

■

Finally, in the positive definite case, we will show other relations between the norms of the monic orthogonal polynomials in the following Proposition.

Proposition 5.

1. $||\Psi_n||^2_{\tilde{\sigma}} = ||\Phi_n||^2_{\sigma} + m\sum_{l=0}^{n} \frac{\Psi_n^{(l)}(0)\overline{\Phi_n^{(l)}(0)}}{l!^2}$.
2. $||\Psi_n||^2_{\theta} \leq ||\Phi_n||^2_{\theta}$.

Proof.

1. From (16),

$$\begin{aligned}
||\Psi_n||^2_{\tilde{\sigma}} &= \langle \Psi_n, \Psi_n \rangle_{\widetilde{\mathcal{L}}} = \langle \Psi_n, \Phi_n \rangle_{\widetilde{\mathcal{L}}} \\
&= \int \Psi_n(z)\overline{\Phi_n(z)}d\sigma(z) + m\int \Psi_n(z)\overline{\Phi_n(z)}\frac{dz}{2\pi iz} \\
&= ||\Phi_n||^2_{\sigma} + m\int \Psi_n(z)\overline{\Phi_n(z)}\frac{dz}{2\pi iz} \\
&= ||\Phi_n||^2_{\sigma} + m\sum_{l=0}^{n} \frac{\Psi_n^{(l)}(0)}{l!}\frac{\overline{\Phi_n^{(l)}(0)}}{l!}.
\end{aligned}$$

2. Using the extremal property of the norm of monic orthogonal polynomials,

$$\begin{aligned}
||\Psi_n||^2_{\tilde{\sigma}} &\leq ||\Phi_n||^2_{\tilde{\sigma}} \\
||\Psi_n||^2_{\sigma} + ||\Psi_n||^2_{\theta} &\leq ||\Phi_n||^2_{\sigma} + ||\Phi_n||^2_{\theta} \\
0 \leq ||\Psi_n||^2_{\sigma} - ||\Phi_n||^2_{\sigma} &\leq ||\Phi_n||^2_{\theta} - ||\Psi_n||^2_{\theta}
\end{aligned}$$

and the result follows. ■

For $m < 0$, we can generalize Proposition 1 and obtain necessary and sufficient conditions for the quasi-definiteness of the linear functional $\widetilde{\mathcal{L}}$. Indeed,

Proposition 6. *Using the notation of Proposition 1, the following statements are equivalent.*

(*i*) $\widetilde{\mathcal{L}}$ *is a quasi-definite linear functional.*

(*ii*) *The matrix* $(m^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_n)$ *is nonsingular, and*

$$0 \neq \mathbf{k}_0 + m, \tag{24}$$

$$0 \neq \mathbf{k}_n + \mathbf{\Phi}_n^t(0)m^{-1}(m^{-1}\mathbf{I}_n + \mathbf{R}_n\mathbf{D}_n^2)^{-1}\overline{\mathbf{\Phi}_n(0)}, \quad n \geqslant 1. \tag{25}$$

Moreover, $\{\Psi_n\}_{n\geqslant 0}$*, the family of monic polynomials orthogonal with respect to* $\widetilde{\mathcal{L}}$*, is given by* (19).

Proof. $(i) \to (ii)$. Set

$$\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z), \tag{26}$$

where, for $0 \leqslant k \leqslant n-1$,

$$\begin{aligned}\lambda_{n,k} &= \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}}}{\mathbf{k}_k} = \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\widetilde{\mathcal{L}}} - m\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}}{\mathbf{k}_k}, \\ &= -\frac{m}{\mathbf{k}_k}\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}.\end{aligned}$$

Thus,

$$\Psi_n(z) = \Phi_n(z) - m\int_{\mathbb{T}} \Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}, \tag{27}$$

$$= \Phi_n(z) - m\sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{l!}\frac{K_{n-1}^{(0,l)}(z,0)}{l!}, \tag{28}$$

and $K_n^{(s,l)}(z,y)$ denotes the $s-th$ ($resp.$ $l-th$) partial derivative of $K_n(z,y)$ with respect to the variable z ($resp.$ y). In particular, for $0 \leqslant s \leqslant n-1$ we get

$$\Psi_n^{(s)}(0) = \Phi_n^{(s)}(0) - m\sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{l!}\frac{K_{n-1}^{(s,l)}(0,0)}{l!}. \tag{29}$$

So, we have the following system of n linear equations and n unknowns which reads as

$$\begin{pmatrix} 1+m\frac{K_{n-1}^{(0,0)}(0,0)}{(0!)^2} & m\frac{K_{n-1}^{(0,1)}(0,0)}{(1!)^2} & \cdots & m\frac{K_{n-1}^{(0,n-1)}(0,0)}{(n-1!)^2} \\ m\frac{K_{n-1}^{(1,0)}(0,0)}{(0!)^2} & 1+m\frac{K_{n-1}^{(1,1)}(0,0)}{(1!)^2} & \cdots & m\frac{K_{n-1}^{(1,n-1)}(0,0)}{(n-1!)^2} \\ \vdots & \vdots & \ddots & \vdots \\ m\frac{K_{n-1}^{(n-1,0)}(0,0)}{(0!)^2} & m\frac{K_{n-1}^{(n-1,1)}(0,0)}{(1!)^2} & \cdots & 1+m\frac{K_{n-1}^{(n-1,n-1)}(0,0)}{(n-1!)^2} \end{pmatrix} \mathbf{\Psi}_n(0) = \mathbf{\Phi}_n(0), \tag{30}$$

where

$$\mathbf{\Psi}_n(0) = [\Psi_n(0), \Psi_n^{'}(0), \dots, \Psi_n^{(n-1)}(0)]^t, \quad \mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi_n^{'}(0), \dots, \Phi_n^{(n-1)}(0)]^t.$$

Denoting

$$\mathbf{R}_n = \begin{pmatrix} K_{n-1}^{(0,0)}(0,0) & K_{n-1}^{(0,1)}(0,0) & \cdots & K_{n-1}^{(0,n-1)}(0,0) \\ K_{n-1}^{(1,0)}(0,0) & K_{n-1}^{(1,1)}(0,0) & \cdots & K_{n-1}^{(1,n-1)}(0,0) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(n-1,0)}(0,0) & K_{n-1}^{(n-1,1)}(0,0) & \cdots & K_{n-1}^{(n-1,n-1)}(0,0) \end{pmatrix}, \tag{31}$$

and $\mathbf{D}_n = diag\{\frac{1}{0!}, \frac{1}{1!}, \ldots, \frac{1}{(n-1)!}\}$, (30) becomes

$$\mathbf{\Psi}_n(0) = m^{-1}(m^{-1}\mathbf{I}_n + \mathbf{R}_n\mathbf{D}_n^2)^{-1}\mathbf{\Phi}_n(0). \tag{32}$$

Since $\widetilde{\mathcal{L}}$ is quasi-definite, then $\{\Psi_n\}_{n\geqslant 0}$ is uniquely defined, so $(m^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_n)$ must be a nonsingular matrix to guarantee the existence and uniqueness of the solution of the previous linear system. Furthermore, if

$$\mathbf{K}_{n-1}(z,0) = [K_{n-1}(z,0), K_{n-1}^{(0,1)}(z,0), \ldots, K_{n-1}^{(0,n-1)}(z,0)]^t,$$

then (28) can be written

$$\begin{aligned} \Psi_n(z) &= \Phi_n(z) - m\mathbf{K}_{n-1}^t(z,0)\mathbf{D}_n^2\mathbf{\Psi}_n(0), \\ &= \Phi_n(z) - \mathbf{K}_{n-1}^t(z,0)(m^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_n)^{-1}\mathbf{\Phi}_n(0), \quad n \geqslant 1, \end{aligned}$$

which, using Proposition 2, yields (19). Moreover, (24) follows directly from (16) and

$$\begin{aligned} 0 &\neq \langle \Psi_n(z), \Phi_n(z)\rangle_{\widetilde{\mathcal{L}}} \\ &= \langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}} + m\,\langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}_0} \\ &= \mathbf{k}_n + m\sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{(l)!}\,\overline{\frac{\Phi_n^{(l)}(0)}{(l)!}} \\ &= \mathbf{k}_n + m\mathbf{\Psi}_n^t(0)\mathbf{D}_n^2\overline{\mathbf{\Phi}_n(0)}, \quad n \geqslant 1, \end{aligned}$$

so, from (32), (25) holds.

Conversely, assume $(m^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_n)$ is nonsingular and define $\{\Psi_n\}_{n\geqslant 0}$ by (18).

Then, for $0 \leqslant k \leqslant n-1$,

$$\begin{aligned}
\langle \Psi_n(z), \Phi_k(z) \rangle_{\widetilde{\mathcal{L}}} &= \langle \Psi_n(z), \Phi_k(z) \rangle_{\mathcal{L}} + m \sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{l!} \frac{\overline{\Phi_k^{(l)}(0)}}{l!} \\
&= -m \left\langle \sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(0,l)}(z,0)}{l!}, \Phi_k(z) \right\rangle_{\mathcal{L}} + m \sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{l!} \frac{\overline{\Phi_k^{(l)}(0)}}{l!} \\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\langle \Psi_n(z), \Phi_n(z) \rangle_{\widetilde{\mathcal{L}}} &= \langle \Psi_n(z), \Phi_n(z) \rangle_{\mathcal{L}} + m \sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{l!} \frac{\overline{\Phi_n^{(l)}(0)}}{l!} \\
&= \mathbf{k}_n + m \sum_{l=0}^{n-1} \frac{\Psi_n^{(l)}(0)}{l!} \frac{\overline{\Phi_n^{(l)}(0)}}{l!} \\
&= \mathbf{k}_n + m \boldsymbol{\Psi}_n^t(0) \mathbf{D}_n^2 \overline{\boldsymbol{\Phi}_n(0)} \neq 0,
\end{aligned}$$

since (25) holds. ∎

In the following Section, we generalize the previous perturbation, adding a mass m to any subdiagonal of the Toeplitz matrix.

3. A general perturbation of a Toeplitz matrix

Let $\mathcal{L}$ be an Hermitian linear functional defined in Λ. Let $\mathcal{L}_j$ be a linear functional such that its associated bilinear functional satisfies

$$\langle p(z), q(z) \rangle_{\mathcal{L}_j} := \langle p(z), q(z) \rangle_{\mathcal{L}} + m \left\langle z^j p(z), q(z) \right\rangle_{\mathcal{L}_\theta} + \bar{m} \left\langle p(z), z^j q(z) \right\rangle_{\mathcal{L}_\theta}, \qquad (33)$$

where $\boldsymbol{m} \in \mathbb{C}$, $p, q \in \mathbb{P}$, $j \in \mathbb{N}$, is a fixed number, and $\langle \cdot, \cdot \rangle_{\mathcal{L}_\theta}$ is the bilinear functional associated with the normalized Lebesgue measure on the unit circle. Assume $\mathcal{L}$ is a positive definite functional. Then, in terms of the corresponding measures, the above transformation can be expressed as

$$d\tilde{\sigma} = d\sigma + 2\Re\mathrm{e}(mz^j) \frac{d\theta}{2\pi}. \qquad (34)$$

From (33), notice that for every $k \in \mathbb{Z}$ we have

$$\begin{aligned} \tilde{c}_j &= c_j + \bar{m}, \\ \tilde{c}_{-j} &= c_{-j} + m, \\ \tilde{c}_k &= c_k, \qquad\qquad k \notin \{j, -j\}, \end{aligned}$$

and therefore $\mathcal{L}_j$ is also Hermitian. Furthermore, if $F_j(z)$ is the Carathéodory function associated with $\mathcal{L}_j$, then

$$F_j(z) = F(z) + 2mz^j, \tag{35}$$

i.e. a linear spectral transformation of $F(z)$.

On the other hand, $\widetilde{\mathbf{T}}$, the infinite Toeplitz matrix associated with $\mathcal{L}_j$, is

$$\widetilde{\mathbf{T}} = \mathbf{T} + m\mathbf{Z}^j + \bar{m}\left(\mathbf{Z}^t\right)^j,$$

where $\mathbf{Z}$ is the shift matrix with ones on the first lower-diagonal and zeros on the remaining entries, and $\mathbf{Z}^t$ is its transpose. Equivalently,

$$\widetilde{\mathbf{T}} = \mathbf{T} + \begin{pmatrix} 0 & \cdots & \bar{m} & 0 & \cdots \\ \vdots & 0 & \cdots & \bar{m} & \cdots \\ m & \vdots & \ddots & \vdots & \ddots \\ 0 & m & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$

Assume that $\mathcal{L}$ is a positive definite linear functional and denote by $\{\Phi_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. We now proceed to determine necessary and sufficient conditions for $\mathcal{L}_j$ to be a quasi-definite functional as well as the relation between $\{\Phi_n\}_{n\geqslant 0}$ and $\{\Psi_n\}_{n\geqslant 0}$, the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_j$.

Proposition 7. *The following statements are equivalent.*

(i) *$\mathcal{L}_j$ is a quasi-definite linear functional.*

(ii) *The matrix $\mathbf{I}_n + \mathbf{S}_n$ is nonsingular, and*

$$\tilde{\mathbf{k}}_n = \mathbf{k}_n + \mathbf{W}_n^t(0)(\mathbf{I}_n + \mathbf{S}_n)^{-1}\overline{\mathbf{Y}_n(0)} + \bar{m}\frac{\overline{\Phi_n^{(n-j)}(0)}}{(n-j)!} \neq 0, \quad n \geqslant 1, \tag{36}$$

with $\mathbf{W}_n(0) = [\mathbf{\Phi}_n(0) - \bar{m}n!\mathbf{C}_{(0,n-1;n)}]$, $\mathbf{Y}_n(0) = \Big[\bar{m}\frac{\Phi_n^{(j)}(0)}{j!}, \ldots, \bar{m}\frac{\Phi_n^{(2j-1)}(0)}{(2j-1)!}, \bar{m}\frac{\Phi_n^{(2j)}(0)}{(2j)!} + m\frac{\Phi_n^{(0)}(0)}{(0)!}, \ldots, \bar{m}\frac{\Phi_n^{(n)}(0)}{(n)!} + m\frac{\Phi_n^{(n-2j)}(0)}{(n-2j)!}, m\frac{\Phi_n^{(n-2j+1)}(0)}{(n-2j+1)!}, \ldots, m\frac{\Phi_n^{(n-j-1)}(0)}{(n-j-1)!}\Big]^t$, $\mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi_n'(0), \ldots, \Phi_n^{(n-1)}(0)]^t$ *and*

$$\mathbf{S}_n = \left(\begin{array}{c|c|c} m\mathbf{A}_{(0,j-1;0,j-1)} & \mathbf{B}_{(0,j-1;j,n-j-1)} & \bar{m}\mathbf{C}_{(0,j-1;n-j,n-1)} \\ \hline m\mathbf{A}_{(j,n-j-1;0,j-1)} & \mathbf{B}_{(j,n-j-1;j,n-j-1)} & \bar{m}\mathbf{C}_{(j,n-j-1;n-j,n-1)} \\ \hline m\mathbf{A}_{(n-j,n-1;0,j-1)} & \mathbf{B}_{(n-j,n-1;j,n-j-1)} & \bar{m}\mathbf{C}_{(n-j,n-1;n-j,n-1)} \end{array}\right),$$

where $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are matrices whose elements are given by*

$$\begin{aligned} a_{s,l} &= \frac{K_{n-1}^{(s,l+j)}(0,0)}{(l)!(l+j)!}, \\ b_{s,l} &= m\frac{K_{n-1}^{(s,l+j)}(0,0)}{(l)!(l+j)!} + \bar{m}\frac{K_{n-1}^{(s,l-j)}(0,0)}{(l)!(l-j)!}, \\ c_{s,l} &= \frac{K_{n-1}^{(s,l-j)}(0,0)}{(l)!(l-j)!}. \end{aligned}$$

Moreover, $\{\Psi_n\}_{n\geqslant 0}$*, the corresponding sequence of monic polynomials orthogonal with respect to* $\mathcal{L}_j$*, is given by*

$$\Psi_n(z) = A_n(z)\Phi_n(z) + B_n(z)\Phi_n^*(z), \quad n \geqslant 1, \tag{37}$$

with

$$\begin{aligned} A_n(z) &= 1 + \frac{1}{\mathbf{k}_n}\mathbf{W}_n^t(0)(\mathbf{I}_n + \mathbf{S}_n)^{-1}\mathbf{D}_n\mathcal{T}(\Phi_n(z);0) + \bar{m}\frac{T_{n-j}^*(\Phi_n(z);0)}{\mathbf{k}_n}, \\ B_n(z) &= -\frac{1}{\mathbf{k}_n}\mathbf{W}_n^t(0)(\mathbf{I}_n + \mathbf{S}_n)^{-1}\mathbf{D}_n\mathcal{T}(\Phi_n^*(z);0) - \bar{m}\frac{T_{n-j}^*(\Phi_n^*(z);0)}{\mathbf{k}_n}, \quad \textit{and} \end{aligned}$$

$\mathcal{T}(\Phi_n(z);0) = [mT_j^*(\Phi_n(z);0), \ldots, mT_{2j-1}^*(\Phi_n(z);0), mT_{2j}^*(\Phi_n(z);0) + \bar{m}T_0^*(\Phi_n(z);0), \ldots, mT_{n-1}^*(\Phi_n(z);0) + \bar{m}T_{n-2j-1}^*(\Phi_n(z);0), \bar{m}T_{n-2j}^*(\Phi_n(z);0), \ldots, \bar{m}T_{n-j-1}^*(\Phi_n(z);0)]^t$.

Remark 8. *The case* $\mathcal{L}_0$ $(j = 0)$ *reduces to the linear functional analyzed in the previous Section with mass* $\mathfrak{Re}[m]$*. In such a case,* $\tilde{\mathbf{k}}_0 = \mathbf{k}_0 + \mathfrak{Re}[m]$*. On the other hand, for* $j \geqslant 1$*, it follows from* (33) *that* $\tilde{\mathbf{k}}_l = \mathbf{k}_l$ *for* $0 \leqslant l \leqslant j-1$*. In other words, we only need* (36) *for* $n \geqslant j$*. Notice that, for a given* j*, the polynomials of degree* $n < j$ *remain unchanged. In such a case,* (36) *and* (37) *still hold, with the convention that the negative derivatives are zero.*

Proof. Let us write
$$\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z), \tag{38}$$
where, for $0 \leqslant k \leqslant n-1$,
$$\begin{aligned}
\lambda_{n,k} &= \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}}}{\mathbf{k}_k} \\
&= \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}_j} - m\int_{\mathbb{T}} y^j\Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy} - \bar{m}\int_{\mathbb{T}} y^{-j}\Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}}{\mathbf{k}_k} \\
&= -\frac{m}{\mathbf{k}_k}\int_{\mathbb{T}} y^j\Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy} - \frac{\bar{m}}{\mathbf{k}_k}\int_{\mathbb{T}} y^{-j}\Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}.
\end{aligned}$$
Therefore,
$$\Psi_n(z) = \Phi_n(z) - m\int_{\mathbb{T}} y^j\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} - \bar{m}\int_{\mathbb{T}} y^{-j}\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}.$$

Taking into account that
$$\begin{aligned}
y^j\Psi_n(y) &= \sum_{l=0}^{n} \frac{\Psi_n^{(l)}(0)}{l!}y^{l+j} \\
&= \sum_{l=j}^{n+j} \frac{\Psi_n^{(l-j)}(0)}{(l-j)!}y^{l},
\end{aligned}$$
and, for $|y| = 1$,
$$K_{n-1}(z,y) = \sum_{l=0}^{n-1} \frac{K_{n-1}^{(0,l)}(z,0)}{l!}\frac{1}{y^l},$$
we obtain
$$\begin{aligned}
\int_{\mathbb{T}} y^j\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} &= \sum_{l=j}^{n-1} \frac{\Psi_n^{(l-j)}(0)}{(l-j)!}\frac{K_{n-1}^{(0,l)}(z,0)}{(l)!} \\
&= \sum_{l=0}^{n-j-1} \frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!}.
\end{aligned}$$

In an analog way,

$$\int_{\mathbb{T}} y^{-j}\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} = \sum_{l=0}^{n-j}\frac{\Psi_n^{(l+j)}(0)}{(l+j)!}\frac{K_{n-1}^{(0,l)}(z,0)}{(l)!} = \sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}.$$

Thus, we get

$$\Psi_n(z) = \Phi_n(z) - m\sum_{l=0}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!} - \bar{m}\sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}, \quad (39)$$

or, equivalently,

$$\Psi_n(z) = \Phi_n(z) - m\sum_{l=0}^{j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!} - \bar{m}\sum_{l=n-j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}, - \sum_{l=j}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\left[m\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!} + \bar{m}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}\right]. \quad (40)$$

In particular, for $0 \leqslant s \leqslant n$, we have

$$\Psi_n^{(s)}(0) = \Phi_n^{(s)}(0) - m\sum_{l=0}^{j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(s,l+j)}(0,0)}{(l+j)!} - \bar{m}\sum_{l=n-j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(s,l-j)}(0,0)}{(l-j)!}, - \sum_{l=j}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\left[m\frac{K_{n-1}^{(s,l+j)}(0,0)}{(l+j)!} + \bar{m}\frac{K_{n-1}^{(s,l-j)}(0,0)}{(l-j)!}\right],$$

i.e., we obtain a system of $n+1$ linear equations and $n+1$ unknowns as follows

$$\Psi_n^{(s)}(0) = \Phi_n^{(s)}(0) - m\sum_{l=0}^{j-1}a_{s,l}\Psi_n^{(l)}(0) - \sum_{l=j}^{n-j-1}b_{s,l}\Psi_n^{(l)}(0) - \bar{m}\sum_{l=n-j}^{n}c_{s,l}\Psi_n^{(l)}(0),$$

where

$$a_{s,l} = \frac{K_{n-1}^{(s,l+j)}(0,0)}{(l)!(l+j)!},$$
$$b_{s,l} = m\frac{K_{n-1}^{(s,l+j)}(0,0)}{(l)!(l+j)!} + \bar{m}\frac{K_{n-1}^{(s,l-j)}(0,0)}{(l)!(l-j)!},$$
$$c_{s,l} = \frac{K_{n-1}^{(s,l-j)}(0,0)}{(l)!(l-j)!}.$$

Thus, if $\mathbf{M}_{(s_1,s_2;l_1,l_2)} = (m_{s,i})_{s_1\leqslant s\leqslant s_2; l_1\leqslant l\leqslant l_2}$ and $\mathbf{I}_k$ is the $k\times k$ identity matrix, then

$$(\mathbf{I}_{n+1}+\mathbf{S}_{n+1})\begin{pmatrix}\Psi_n^{(0)}(0)\\ \vdots \\ \Psi_n^{(n)}(0)\end{pmatrix} = \begin{pmatrix}\Phi_n^{(0)}(0)\\ \vdots \\ \Phi_n^{(n)}(0)\end{pmatrix},$$

where

$$\mathbf{S}_{n+1} = \left(\begin{array}{c|c|c} m\mathbf{A}_{(0,j-1;0,j-1)} & \mathbf{B}_{(0,j-1;j,n-j-1)} & \bar{m}\mathbf{C}_{(0,j-1;n-j,n-1)} \\ \hline m\mathbf{A}_{(j,n-j-1;0,j-1)} & \mathbf{B}_{(j,n-j-1;j,n-j-1)} & \bar{m}\mathbf{C}_{(j,n-j-1;n-j,n-1)} \\ \hline m\mathbf{A}_{(n-j,n;0,j-1)} & \mathbf{B}_{(n-j,n;j,n-j-1)} & \bar{m}\mathbf{C}_{(n-j,n;n-j,n-1)} \end{array}\right).$$

Notice that the entries in the last row of the above matrix vanish, which is consistent with the fact that $\Psi_n^{(n)}(0) = \Phi_n^{(n)}(0) = n!$. Therefore, if we denote

$$\mathbf{\Psi}_n(0) = [\Psi_n(0), \Psi_n'(0), \ldots, \Psi_n^{(n-1)}(0)]^t,$$

$$\mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi_n'(0), \ldots, \Phi_n^{(n-1)}(0)]^t,$$

then the above $(n+1)\times(n+1)$ linear system can be reduced to a $n\times n$ linear system as follows

$$(\mathbf{I}_n+\mathbf{S}_n)\mathbf{\Psi}_n(0) = \mathbf{\Phi}_n(0) - \bar{m}n!\mathbf{C}_{(0,n-1;n)} := \mathbf{W}_n(0). \tag{41}$$

Since $\mathcal{L}_j$ is a quasi-definite linear functional, there exists a unique family of monic polynomials orthogonal with respect to $\mathcal{L}_j$. Therefore, the matrix $\mathbf{I}_n+\mathbf{S}_n$ is nonsingular, according to the existence and uniqueness of the solution of the above linear system. As a consequence, if

$$\mathbb{K}_{n-1}(z,0) = \begin{pmatrix} m\frac{K_{n-1}^{(0,j)}(z,0)}{j!} \\ \vdots \\ m\frac{K_{n-1}^{(0,2j-1)}(z,0)}{(2j-1)!} \\ m\frac{K_{n-1}^{(0,2j)}(z,0)}{(2j)!} + \bar{m}\frac{K_{n-1}^{(0,0)}(z,0)}{(0)!} \\ \vdots \\ m\frac{K_{n-1}^{(0,n-1)}(z,0)}{(n-1)!} + \bar{m}\frac{K_{n-1}^{(0,n-2j-1)}(z,0)}{(n-2j-1)!} \\ \bar{m}\frac{K_{n-1}^{(0,n-2j)}(z,0)}{(n-2j)!} \\ \vdots \\ \bar{m}\frac{K_{n-1}^{(0,n-j-1)}(z,0)}{(n-j-1)!} \end{pmatrix}, \tag{42}$$

then (40) becomes

$$\Psi_n(z) = \Phi_n(z) - \mathbf{\Psi^t}_n(0)\mathbf{D}_n\mathbb{K}_{n-1}(z,0) - \bar{m}\frac{K_{n-1}^{(0,n-j)}(z,0)}{(n-j)!}.$$

Thus, from (41) and Proposition 2, (37) holds. Furthermore,

$$\begin{aligned}
0 &\neq \langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}_j} \\
&= \langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}} + m\left\langle z^j\Psi_n(z), \Phi_n(z)\right\rangle_{\mathcal{L}_\theta} + \bar{m}\left\langle \Psi_n(z), z^j\Phi_n(z)\right\rangle_{\mathcal{L}_\theta} \\
&= \mathbf{k}_n + m\sum_{l=0}^{n-j}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_n^{(l+j)}(0)}}{(l+j)!} + \bar{m}\sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_n^{(l-j)}(0)}}{(l-j)!} \\
&= \mathbf{k}_n + \mathbf{\Psi}_n^t(0)\overline{\mathbf{Y}_n(0)} + \bar{m}\frac{\overline{\Phi_n^{(n-j)}(0)}}{(n-j)!},
\end{aligned}$$

so (36) follows.

For the converse, assume $\mathbf{I}_n + \mathbf{S}_n$ is nonsingular for every $n \geq 1$ and define $\{\Psi_n\}_{n\geqslant 0}$ as in (37). We will show that $\{\Psi_n\}_{n\geqslant 0}$ is orthogonal with respect to $\mathcal{L}_j$. Indeed, for $0 \leqslant k \leqslant n-1$ and taking into account (39), we get

$$\begin{aligned}
\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}_j} &= \langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}} + m\left\langle z^j\Psi_n(z), \Phi_k(z)\right\rangle_{\mathcal{L}_\theta} + \bar{m}\left\langle \Psi_n(z), z^j\Phi_k(z)\right\rangle_{\mathcal{L}_\theta} \\
&= \langle \Phi_n(z), \Phi_k(z)\rangle_{\mathcal{L}} - m\left\langle \sum_{l=0}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!}, \Phi_k(z)\right\rangle_{\mathcal{L}} \\
&- \bar{m}\left\langle \sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}, \Phi_k(z)\right\rangle_{\mathcal{L}} + m\sum_{l=0}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_k^{(l+j)}(0)}}{(l+j)!} \\
&+ \bar{m}\sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_k^{(l-j)}(0)}}{(l-j)!} \\
&= 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\widetilde{\mathbf{k}}_n &= \langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}_j} \\
&= \langle \Psi_n(z), \Phi_n(z)\rangle_{\mathcal{L}} + m\left\langle z^j\Psi_n(z), \Phi_n(z)\right\rangle_{\mathcal{L}_\theta} + \bar{m}\left\langle \Psi_n(z), z^j\Phi_n(z)\right\rangle_{\mathcal{L}_\theta} \\
&= \mathbf{k}_n + m\sum_{l=0}^{n-j}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_n^{(l+j)}(0)}}{(l+j)!} + \bar{m}\sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_n^{(l-j)}(0)}}{(l-j)!} \\
&= \mathbf{k}_n + \mathbf{\Psi}_n^t(0)\overline{\mathbf{Y}_n(0)} + \bar{m}\frac{\overline{\Phi_n^{(n-j)}(0)}}{(n-j)!} \\
&\neq 0,
\end{aligned}$$

since (36) is assumed. Thus, we conclude that $\mathcal{L}_j$ is quasi-definite. ∎

From the previous result, evaluating (37) in $z = 0$, we get

Corollary 9. *The family of Verblunsky parameters associated with $\mathcal{L}_j$ is*

$$\Psi_n(0) = A_n(0)\Phi_n(0) + B_n(0), \quad n \geqslant 1. \tag{43}$$

Finally, applying the Szegő transformation to (35), we get

$$\begin{aligned}
\sqrt{x^2-1}\widetilde{S}(x) &= \sqrt{x^2-1}S(x) + 2mz^j, \\
\widetilde{S}(x) &= S(x) + 2m\frac{(x-\sqrt{x^2-1})^j}{\sqrt{x^2-1}},
\end{aligned}$$

and thus $\widetilde{S}(x)$, the Stieltjes function for the corresponding perturbed measure on the real line, can not be expressed as a linear spectral transform of $S(x)$, since square roots appear for any value of j. Therefore, we conclude that a perturbation on the moments of a measure supported on $\mathbb{T}$ does not yield a linear spectral transformation of the corresponding Stieltjes function. Conversely, if we consider a similar perturbation of the moments of a measure on the real line, then

$$\widetilde{S}(x) = S(x) + \frac{m}{x^{j+1}}.$$

Applying the Szegő transformation,

$$\begin{aligned}
\widetilde{F}(z) &= F(z) + m\frac{1-z^2}{2z(x)^{j+1}} \\
&= F(z) + 2^j m\frac{1-z^2}{z(z+z^{-1})^{j+1}} \\
&= F(z) + 2^j m\frac{(1-z^2)z^j}{(z^2+1)^{j+1}},
\end{aligned}$$

which is a linear spectral transformation of $F(z)$. In the special case when $j = 0$,

$$\widetilde{F}(z) = F(z) - m\frac{z^2 - 1}{z^2 + 1}.$$

As a conclusion, the study of linear spectral transformations on the unit circle is far more complicated than the real line case. It remains an open problem to determine the generators of the group of linear spectral transformations defined by (13). As oppose to the real line case, they are not the Christoffel and Geronimus transforms, as we have shown.

4. Example

We present an example of the previous transformation when σ is the normalized Lebesgue measure and $j = 1$, i.e. the transformation

$$\langle p, q \rangle_{\mathcal{L}_1} = \langle p, q \rangle_\theta + m \langle zp, q \rangle_\theta + \bar{m} \langle p, zq \rangle_\theta \,, \tag{44}$$

where $m \in \mathbb{C}$. Our goal is to obtain necessary and sufficient conditions for $\mathcal{L}_1$ to be a positive definite (quasi-definite) functional. As a consequence, we will deduce its corresponding family of orthogonal polynomials as well as the sequence of Verblunsky parameters. Notice that in this case, $\Phi_n(z) = z^n$, $n \geqslant 0$, as well as $\mathbf{k}_n = 1$, $n \geqslant 0$. Thus, we get

$$K_{n-1}^{(0,l)}(z, 0) = l!z^l, \quad 0 \leqslant l \leqslant n - 1.$$

So, $\mathbb{K}_{n-1}(z, 0) = (mz, mz^2 + \bar{m}z^0, \ldots, mz^{n-1} + \bar{m}z^{n-3}, \bar{m}z^{n-2})^t$ and $\mathbf{\Phi}_n(0) = [0, \ldots, 0]^t$, $n \geqslant 1$. On the other hand, we have

$$K_{n-1}^{(s,l)}(0, 0) = \begin{cases} s!l! & \text{if} \quad s = l, \\ 0 & \text{otherwise}, \end{cases}$$

and, therefore,

$$\begin{aligned} a_{s,l} &= \frac{K_{n-1}^{(s,l+1)}(0,0)}{(l)!(l+1)!} = \delta_{s,s-1}, \\ b_{s,l} &= m\frac{K_{n-1}^{(s,l+1)}(0,0)}{(l)!(l+1)!} + \bar{m}\frac{K_{n-1}^{(s,l-1)}(0,0)}{(l)!(l-1)!} = m\delta_{s,s-1} + \bar{m}\delta_{s,s+1}, \\ c_{s,l} &= \frac{K_{n-1}^{(s,l-1)}(0,0)}{(l)!(l-1)!} = \delta_{s,s+1}, \end{aligned}$$

where $\delta_{s,l}$ is the Kronecker's delta. Thus,

$$\mathbf{I}_n + \mathbf{S}_n \quad = \quad \begin{pmatrix} 1 & \bar{m} & & & \\ m & 1 & \ddots & & \\ & \ddots & \ddots & \bar{m} & \\ & & m & 1 & \bar{m} \\ & & & m & 1 \end{pmatrix}, \quad n \geqslant 2.$$

Notice that, for $n \geqslant 2$, $\mathbf{I}_n + \mathbf{S}_n$ is $\widetilde{\mathbf{T}}_{n-1}$, the $n \times n$ Toeplitz matrix associated with $\mathcal{L}_1$. Thus, we need to establish the conditions on m for $\widetilde{\mathbf{T}}_{n-1}$ be nonsingular. Since $\widetilde{\mathbf{T}}_{n-1}$ is Hermitian, their eigenvalues $\{\lambda_k\}_{k=1}^n$ are real numbers. Moreover, $\widetilde{\mathbf{T}}_{n-1}$ is quasi-definite if and only if $\lambda_k \neq 0$, for every $1 \leqslant k \leqslant n$ (see [13]). From Theorem 2.4 in [14], the eigenvalues of $\widetilde{\mathbf{T}}_{n-1}$ are

$$\lambda_k = 1 + 2|m| \cos \frac{\pi k}{n+1}, \quad k = 1, \dots, n.$$

Thus, $\mathcal{L}_1$ is a quasi-definite linear functional if and only if

$$|m| \neq -\left(2 \cos \frac{\pi k}{n+1}\right)^{-1}, \quad k = 1, \dots, n, \quad \textit{and} \quad n \geq 1,$$

or, equivalently,

$$\frac{\pi k}{\cos^{-1}\left(-\frac{1}{2|m|}\right)} \notin \mathbb{N}. \tag{45}$$

Assuming that (45) holds, $\{\Psi_n\}_{n\geqslant 0}$ can be obtained using (37), since all elements are known. Since $\mathbb{K}_{n-1}(0,0) = (0, \bar{m}, 0, \dots, 0)^t$ and $\mathbf{W}_n = [0, \dots, 0, -\bar{m}(n-1)!)]$, the sequence of Verblunsky parameters can be computed using (43). It is not difficult to see that

$$\Psi_n(0) = -\bar{m}^2(n-1)!\ell_{n,2},$$

where $\ell_{i,j} = (\mathbf{I}_n + \mathbf{S}_n)^{-1}_{i,j}$. An explicit expression for $\ell_{n,2}$ can be obtained using the method described in [15]. Indeed,

$$\ell_{n,2} = \frac{(-1)^{n+2} m^{n-2}}{\theta_n}$$

where θ_n is the solution of the recurrence relation

$$\theta_i = \theta_{i-1} - |m|^2 \theta_{i-2}, \qquad i = 2, \dots, n.$$

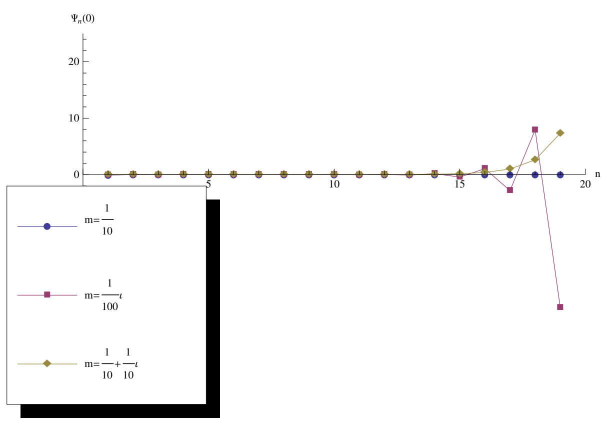
with initial conditions $\theta_0 = \theta_1 = 1$. Thus,

$$\theta_n = \left(2^{-(n+1)} + \frac{2^{-(n+1)}}{\sqrt{1-|m|^2}}\right)\left(\left(1 - \sqrt{1-|m|^2}\right)^n + \left(1 + \sqrt{1-|m|^2}\right)^n\right),$$

and therefore we get

$$\Psi_n(0) = \frac{(-1)^{n+3} m^{n-2} \bar{m} (n-1)!}{\left(2^{-(n+1)} + \frac{2^{-(n+1)}}{\sqrt{1-|m|^2}}\right)\left(\left(1 - \sqrt{1-|m|^2}\right)^n + \left(1 + \sqrt{1-|m|^2}\right)^n\right)}. \tag{46}$$

Finally, the following figure shows the corresponding Verblunsky coefficients for different values of m. From (46), notice that $|\Psi_n(0)|$ grows as n increases, and it grows faster for values $|m| > 1$. The figure shows that using small values of m, the first Verblunsky coefficients are small (close to zero), but then begin to grow as n increases. Since (46) is an increasing function, we deduce that the functional $\mathcal{L}_1$ defined by (44), for the values of m shown on the figure, is quasi-definite.



Acknowledgements

The authors thank the valuable comments from the anonymous referee. They contributed to improve the presentation of the manuscript. The work of the authors has been partially supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain, grant MTM2009-12740-C03-01.

References

[1] U. Grenander, G. Szegő, Toeplitz Forms and their Applications, University of California Press, Berkeley 1958, Chelsea, New York, 2^{nd} edition, 1984.

[2] Y. L. Geronimus, Polynomials orthogonal on a circle and their applications, Amer. Math. Soc. Transl. Series, Amer. Math. Soc. Providence, RI. 1 (1962) 1–78.

[3] B. Simon, Orthogonal Polynomials on the Unit Circle, Amer. Math. Soc. Coll. Publ. Series, vol. 54, Amer. Math. Soc. Providence, RI, 2005.

[4] G. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. Series. vol 23, Amer. Math. Soc., Providence, RI, 4^{th} edition, 1975.

[5] L. Daruis, J. Hernández, F. Marcellán, Spectral transformations for Hermitian Toeplitz matrices, J. Comput. Appl. Math. 202 (2007) 155–176.

[6] L. Garza, J. Hernández, F. Marcellán, Orthogonal polynomials and measures on the unit circle. The Geronimus transformations, J. Comput. Appl. Math. 233 (2007) 1220–1231.

[7] F. Marcellán, J. Hernández, Christoffel transforms and Hermitian linear functionals , Mediterr. J. Math. 2 (2005) 451–458.

[8] F. Marcellán, Polinomios ortogonales no estándar. Aplicaciones en Análisis Numérico y Teoría de Aproximación, Rev. Acad. Colomb. Ciencias Exactas, Físicas y Naturales 30 (2006) 563–579, in Spanish.

[9] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67–83.

[10] L. Garza, J. Hernández, F. Marcellán, Spectral transformations of measures supported on the unit circle and the Szegő transformation, Numer. Algorithms 49 (2008) 169–185.

[11] K. Castillo, L. Garza, F. Marcellán, Linear spectral transformations, Hessenberg matrices and orthogonal polynomials, Rend. Circ. Mat. Palermo, (II) 82 (2010) 3–26.

[12] A. Cachafeiro, F. Marcellán, C. Pérez, Lebesgue perturbation of a quasi-definite Hermitian functional. The positive definite case, Linear Alg. Appl. 369 (2003) 235–250.

[13] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1990.

[14] A. Böttcher, S. M. Grudsky, Matrix Spectral Properties of Banded Toeplitz Matrices, SIAM, Philadelphia, PA, 2005.

[15] R. Usmani, Inversion of a tridiagonal Jacobi matrix, Linear Algebra Appl. 212/213 (1994) 413–414.