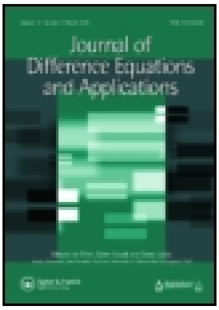

Journal of Difference Equations and Applications

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gdea20

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Version of record first published: 15 Aug 2012

To cite this article: F. Marcellán, M. Sghaier & M. Zaatra (2012): Semi-classical linear functionals of class three: the symmetric case, Journal of Difference Equations and Applications, DOI:10.1080/10236198.2012.712969

To link to this article: http://dx.doi.org/10.1080/10236198.2012.712969

Journal of Difference Equations and Applications
iFirst article, 2012, 1–17

Semi-classical linear functionals of class three: the symmetric case

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(*Received 10 January 2012; final version received 11 July 2012*)

In this paper, we obtain all the symmetric semi-classical linear functionals of class three taking into account the irreducible expression of the corresponding Pearson equation. We focus our attention on their integral representations. Thus, some linear functionals very well known in the literature, associated with perturbations of semi-classical linear functionals of class one at most, appear as well as new linear functionals which have not been studied.

Keywords: orthogonal polynomials; semi-classical linear functionals; integral representations

Mathematics Subject Classification (2000): 42C05; 33C45

1. Introduction

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. The elements of $\mathcal{P}'$ will be called either linear functionals or linear forms. We denote by $\langle u,f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \geq 0$ the moments of u. For any linear functional u and any polynomial h let $Du = u'$, hu, δ_c and $(x-c)^{-1}u$ be the linear functionals defined by $\langle u',f\rangle := -\langle u,f'\rangle$, $\langle hu,f\rangle := \langle u, hf\rangle$, $\langle \delta_c,f\rangle := f(c)$ and $\langle (x-c)^{-1}u,f\rangle := \langle u, \theta_c f\rangle$, where $(\theta_c f)(x) = (f(x)-f(c))/(x-c)$, $c \in \mathbb{C}, f \in \mathcal{P}$.

It is straightforward to prove that for $c, d \in \mathbb{C}, c \neq d$ and $u \in \mathcal{P}'$ (see [11])

$$(x-c)^{-1}((x-c)u) = u - (u)_0\delta_c, \tag{1.1}$$

$$((x-d)(x-c))^{-1}((x-d)(x-c)u) = u + \frac{1}{c-d}\big((d(u)_0 - (u)_1)\delta_c - (c(u)_0 - (u)_1)\delta_d\big). \tag{1.2}$$

Let us define the operator $\sigma : \mathcal{P} \to \mathcal{P}$ by $(\sigma f)(x) := f(x^2)$. Then, we define the even part σu of a linear functional u by $\langle \sigma u,f\rangle = \langle u, \sigma f\rangle$.

Therefore, we have [10]

$$f(x)\sigma u = \sigma\big(f(x^2)u\big), \tag{1.3}$$

$$\sigma u' = 2(\sigma(xu))'. \tag{1.4}$$

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ISSN 1023-6198 print/ISSN 1563-5120 online

http://dx.doi.org/10.1080/10236198.2012.712969
http://www.tandfonline.com

The linear functional u is said to be regular (quasi-definite) if there exists a sequence $\{P_n\}_{n\geq 0}$ of polynomials with $\deg P_n = n, n \geq 0$, such that

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m}, \quad n, m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

We can always assume that each P_n is monic, i.e. $P_n(x) = x^n +$ lower degree terms. Then the sequence $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to u (monic orthogonal polynomial sequence (MOPS) in short). It is a very well-known fact that the sequence $\{P_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (see, for instance, the monograph by Chihara [6])

$$P_{n+2}(x) = (x - \xi_{n+1})P_{n+1}(x) - \rho_{n+1}P_n(x), \quad n \geq 0, \quad P_1(x) = x - \xi_0, \quad P_0(x) = 1, \tag{1.5}$$

with $(\xi_n, \rho_{n+1}) \in \mathbb{C} \times \mathbb{C}^*, n \geq 0$. By convention, we set $\rho_0 = (u)_0 = 1$.

A linear functional u is called symmetric if $(u)_{2n+1} = 0, n \geq 0$. The conditions $(u)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding MOPS, $\{P_n\}_{n\geq 0}$, satisfies the recurrence relation (1.5) with $\xi_n = 0, n \geq 0$ (see [6]).

A regular linear functional u is said to be positive definite if $\langle u, f\rangle > 0$ for all $f \in \mathcal{P}$ such that $f(x) \geq 0$, for every $x \in \mathbb{R}$ and $f \neq 0$ or, equivalently, its MOPS satisfies (1.5) with $\xi_n \in \mathbb{R}$ and $\rho_n \in \mathbb{R}_+^*$ for all $n \geq 1$ (see [11]).

From a structural point of view, a very important family of regular linear functionals has been exhaustively analysed in the literature during the last two decades. Let us recall that a linear functional $\tilde{u}$ is said to be semi-classical when it is regular and there exist two polynomials $\tilde{\Phi}$, a monic polynomial and $\tilde{\Psi}$, $\deg \tilde{\Psi} \geq 1$, such that

$$(\tilde{\Phi}\tilde{u})' + \tilde{\Psi}\tilde{u} = 0. \tag{1.6}$$

This is the Pearson equation associated with the linear functional. The class of the semi-classical linear functional $\tilde{u}$ is defined as the non-negative number $\tilde{s} = \max(\deg \tilde{\Psi} - 1, \deg \tilde{\Phi} - 2)$ if and only if the following condition is satisfied:

$$\prod_c \left(\left|\tilde{\Phi}'(c) + \tilde{\Psi}(c)\right| + \left|\langle \tilde{u}, \theta_c\tilde{\Psi} + \theta_c^2\tilde{\Phi}\rangle\right|\right) > 0, \tag{1.7}$$

where c belongs to the set of zeros of $\tilde{\Phi}$. Notice that this condition means that the polynomials $\tilde{\Phi}$ and $\tilde{\Phi}'(x) + \tilde{\Psi}(x)$ are coprime as well as equation (1.6) is irreducible (see [11]).

The corresponding MOPS $\{P_n\}_{n\geq 0}$ is said to be semi-classical of class $\tilde{s}$. When $\tilde{s} = 0$, $\tilde{u}$ is a classical linear functional (Hermite, Laguerre, Jacobi and Bessel). See [6,11].

Semi-classical linear functionals associated with weight functions were considered first by Shohat [16] in the framework of the existence of sequences of orthogonal polynomials satisfying second-order linear differential equations with polynomial coefficients (holonomic equations). Later on P Maroni and coworkers have extensively studied such a kind of linear functionals with a special emphasis on their structure properties. For instance, [11] constitutes a relevant survey on the subject. Many examples appearing in the literature, mainly related to spectral perturbations of classical linear functionals, are semi-classical but a constructive theory of such linear functionals remains open.

Indeed, the classification of semi-classical linear functionals according to some criteria of optimal information from their Pearson equation plays a central role in the constructive theory of such linear functionals. In [4], Belmehdi makes use of this approach to provide a full description of all semi-classical linear functionals of class $s = 1$. In [1], the classification of symmetric semi-classical linear functionals of such a class is given.

Recently, semi-classical linear functionals of class $s = 2$ are completely described by Marcellán et al. [7,9]. Finally, a complete study of the class of the symmetric companion u of a linear functional v in terms of the class of v is done in [3]. Notice that no examples of symmetric semi-classical linear functionals of class >2 have been analysed in the literature. The aim of our contribution is to cover this gap.

As a first step, it is natural to deal with the description of the semi-classical linear functionals of class $s = 3$. In this contribution, we focus our attention on the symmetric case.

The structure of the manuscript is as follows. In Section 2, the irreducible canonical Pearson equations associated with symmetric semi-classical linear functionals of class $s = 3$ are obtained. Thus, seven irreducible canonical cases appear. In Section 3, the integral representation of such linear functionals is given, with special emphasis on the positive definite case.

2. Irreducible canonical functional equations

First of all, let us recall the following result:

PROPOSITION 2.1. *Let* u *be a symmetric semi-classical linear functional of class* s, *satisfying* (1.6) [1]. *If* s *is an even non-negative integer number, then* Φ *is an even polynomial function and* Ψ *is an odd polynomial function. If* s *is an odd non-negative integer number, then* Φ *is an odd polynomial function and* Ψ *is an even polynomial function.*

In the sequel, we will assume that the linear functional $\tilde{u}$ is symmetric and semi-classical of class $\tilde{s} = 3$. Then, according to the above proposition $\tilde{u}$ satisfies (1.6) with

$$\tilde{\Phi}(x) = \tilde{c}_5 x^5 + \tilde{c}_3 x^3 + \tilde{c}_1 x, \quad \tilde{\Psi}(x) = \tilde{a}_4 x^4 + \tilde{a}_2 x^2 + \tilde{a}_0, \quad |\tilde{c}_5| + |\tilde{a}_4| \neq 0.$$

As a consequence, the moments $(\tilde{u})_n$ of the linear functional $\tilde{u}$ satisfy the linear difference equation

$$(\tilde{a}_4 - 2n\tilde{c}_5)(u)_{2n+4} + (\tilde{a}_2 - 2n\tilde{c}_3)(u)_{2n+2} + (\tilde{a}_0 - 2n\tilde{c}_1)(u)_{2n} = 0, \quad n \geq 0,$$

with $(u)_0 = 1$ and $(u)_{2n+1} = 0$, $n \geq 0$. Then, the set of solutions is a linear space of dimension at most two.

The linear functional $h_a\tilde{u}$ (dilation of $\tilde{u}$) is defined by

$$\langle h_a\tilde{u}, f\rangle := \langle \tilde{u}, h_a f\rangle := \langle \tilde{u}, f(ax)\rangle, \quad f \in \mathcal{P}. \tag{2.1}$$

The semi-classical character of a linear functional is preserved by a dilation. Indeed, the dilated linear functional $u = \left(h_{a^{-1}}\right)\tilde{u}, a \in \mathbb{C}^*$ satisfies

$$(\Phi u)' + \Psi u = 0, \tag{2.2}$$

with

$$\Phi(x) = a^{-t}\tilde{\Phi}(ax), \quad \Psi(x) = a^{1-t}\tilde{\Psi}(ax), \quad t = \deg \tilde{\Phi}. \tag{2.3}$$

The sequence $\{\hat{P}_n(x)\}_{n\geq 0}$, where $\hat{P}_n(x) = a^{-n}P_n(ax)$, is orthogonal with respect to u and fulfils (1.5) with

$$\hat{\xi}_n = 0, \quad \hat{\rho}_{n+1} = \frac{\rho_{n+1}}{a^2}, \quad n \geq 0. \tag{2.4}$$

As we can see, a dilation h_a does not modify the nature of a symmetric semi-classical linear functional. This process will be applied to the Pearson equation satisfied by a symmetric semi-classical linear functional of class $s = 3$. In the sequel, $a \in \mathbb{C}^*$ denotes an arbitrary complex number. A convenient choice of a, according to the expression of $\tilde{\Phi}$, allows us to re-locate the zeros of $\tilde{\Phi}$ in the complex plane. In this way, (2.1) can be either reduced to some situations which appear in the literature or yield new linear functionals not yet studied. Thus, we get canonical distributional equations of class three in a simple way, which becomes a pattern for the family of equations that can be reduced using a shifting.

According to Proposition 2.1, we will analyse two situations.

a. deg $\tilde{\Phi} = 5$ and $2 \leq$ deg $\tilde{\Psi} \leq 4$.
b. deg $\tilde{\Psi} = 4$ and $1 \leq$ deg $\tilde{\Phi} \leq 3$.

Case A. Set

$$\tilde{\Phi}(x) = x\prod_{i=1}^{2}\left(x^2 - \alpha_i^2\right),$$

$$\tilde{\Psi} = \tilde{a}_4x^4 + \tilde{a}_2x^2 + \tilde{a}_0, \quad |\tilde{a}_4| + |\tilde{a}_2| \neq 0.$$

Then u satisfies

$$x\left(\prod_{i=1}^{2}\left(x^2 - \alpha_i^2a^{-2}\right)u\right)' + \left(\tilde{a}_4x^4 + a^{-2}\tilde{a}_2x^2 + a^{-4}\tilde{a}_0\right)u = 0. \tag{2.5}$$

We will discuss the following cases.

A_1. $\tilde{\Phi}$ has five simple zeros.

We choose a such that $\alpha_1 = a \neq 0$ and $\alpha_2 = ac$ with $c \notin \{-1, 0, 1\}$. Thus, (2.5) reduces to

$$\left(x(x^2 - 1)(x^2 - c^2)u\right)' + \left(a_4x^4 + a_2x^2 + a_0\right)u = 0. \tag{2.6}$$

The rational function $-((\Phi' + \Psi)/\Phi)$ has five simple poles $0, 1, -1, c$ and $-c$. We denote by

$$\alpha = -\frac{\Phi'(c) + \Psi(c)}{\Phi'(c)}, \quad \beta = -\frac{\Phi'(1) + \Psi(1)}{\Phi'(1)} \quad \text{and} \quad 2\gamma = -\frac{\Phi'(0) + \Psi(0)}{\Phi'(0)}$$

the corresponding residues. Then, after some straightforward calculation, we obtain

$$\begin{pmatrix} a_4 \\ a_2 \\ a_0 \end{pmatrix} = M\begin{pmatrix} 2\alpha + 2 \\ 2\beta + 2 \\ 2\gamma + 1 \end{pmatrix},$$

where

$$M = \begin{pmatrix} -1 & -1 & -1 \\ 1 & c^2 & c^2 + 1 \\ 0 & 0 & -c^2 \end{pmatrix}.$$

Notice that det $M = c^2(c^2 - 1) \neq 0$. This means that this change of parameters is bijective. (We have the same kind of relation between the new parameters and the old parameters in the other cases that we will study in the sequel.)

Now, changing the parameters in (2.6), according to the condition of irreducibility (1.7) we get

$$\begin{cases} (x(x^2-1)(x^2-c^2)u)' + (-(2\alpha+2\beta+2\gamma+5)x^4 \\ +(2\alpha+2\gamma+3+c^2(2\beta+2\gamma+3))x^2 - c^2(2\gamma+1))u = 0, \\ \gamma\{|\beta| + |-(2\alpha+2\beta+2\gamma+3)(u)_2 + c^2(2\beta+2\gamma+1) - 2\beta|\} \\ \times\{|\alpha| + |(2\alpha+2\beta+2\gamma+3)(u)_2 + 2\alpha c^2 - 2\alpha - 2\gamma - 1|\} \neq 0. \end{cases} \tag{2.7}$$

This is the irreducible Pearson equation satisfied by a linear functional of class three when Φ in (2.1) has five simple zeros.

We will proceed in a similar way in the cases listed below.

A$_2$. $\tilde{\Phi}$ has three different zeros and two of them are double zeros.

We choose a such that $\alpha_1 = \alpha_2 = a$. Then (2.5) can be written as

$$\left(x(x^2-1)^2u\right)' + \left(a_4x^4 + a_2x^2 + a_0\right)u = 0. \tag{2.8}$$

By an appropriate choice of the coefficients a_4, a_2 and a_0, (2.8) becomes

$$\begin{cases} (x(x^2-1)^2u)' + \left(-(2\alpha+2\beta+5)x^4 + 2(2\alpha+\beta+\gamma+3)x^2 - 2\alpha - 1\right)u = 0, \\ \alpha\{|\gamma| + |-(2\alpha+2\beta+3)(u)_2 + 2\alpha + 2\gamma + 1|\} \neq 0. \end{cases} \tag{2.9}$$

This is the corresponding irreducible Pearson equation of a linear functional of class three when Φ in (2.1) has three different zeros and two of them are of multiplicity two.

A$_3$. $\tilde{\Phi}$ has three different zeros and one of them has multiplicity three. In such a situation $\alpha_1 = 0$ and $\alpha_2 \neq 0$.

We choose a such that $\alpha_2 = a$. Then (2.5) becomes

$$\left(x^3(x^2-1)u\right)' + \left(a_4x^4 + a_2x^2 + a_0\right)u = 0. \tag{2.10}$$

From a suitable choice of the coefficients a_4, a_2 and a_0, (2.10) yields

$$\begin{cases} \left(x^3(x^2-1)u\right)' + 2\left(-(\alpha+2\beta+3)x^4 + (2\beta-\gamma+2)x^2 + \gamma\right)u = 0, \\ \gamma\{|\alpha| + |(\alpha+2\beta+2)(u)_2 + \alpha + \gamma|\} \neq 0. \end{cases} \tag{2.11}$$

This is an irreducible Pearson equation of a linear functional of class three, when Φ in (2.1) has three different zeros: two of them are simple and the other one has multiplicity three.

A$_4$. $\tilde{\Phi}$ has one zero of multiplicity five, i.e. $\alpha_1 = \alpha_2 = 0$.

(2.5) becomes

$$(x^5u)' + \left(\tilde{a}_4x^4 + a^{-2}\tilde{a}_2x^2 + a^{-4}\tilde{a}_0\right)u = 0. \tag{2.12}$$

Let a be such that $a^{-4}\tilde{a}_0 = -8$. Then (2.12) reduces to

$$(x^5u)' + \left(a_4x^4 + a_2x^2 - 8\right)u = 0. \tag{2.13}$$

By a skilful choice of the coefficients a_4 and a_2, (2.13) can be written as

$$(x^5u)' + 2(-(\alpha+3)x^4 + \beta x^2 - 4)u = 0, \tag{2.14}$$

since $\Phi'(0) + \Psi(0) = -8 \neq 0$. This is the irreducible Pearson equation for a functional of class three, when Φ in (2.1) has a zero of multiplicity five.

Case B. Set

$$\tilde{\Phi}(x) = \tilde{c}_3 x^3 + \tilde{c}_1 x, \quad |\tilde{c}_3| + |\tilde{c}_1| \neq 0,$$
$$\tilde{\Psi} = \tilde{a}_4 x^4 + \tilde{a}_2 x^2 + \tilde{a}_0, \quad \tilde{a}_4 \neq 0.$$

We will analyse the following situations:

B_1. $\tilde{\Phi}$ has three simple zeros $0, \alpha_1$ and $-\alpha_1$.

$\tilde{u}$ and u satisfy, respectively,

$$\left(x\left(x^2 - \alpha_i^2\right)\tilde{u}\right)' + \left(\tilde{a}_4 x^4 + \tilde{a}_2 x^2 + \tilde{a}_0\right)\tilde{u} = 0,$$
$$\left(x\left(x^2 - \alpha_1^2 a^{-2}\right)u\right)' + \left(\tilde{a}_4 a^2 x^4 + \tilde{a}_2 x^2 + a^{-2}\tilde{a}_0\right)u = 0. \tag{2.15}$$

Let a be such that $\alpha_1 = a$. Then (2.15) becomes

$$\left(x(x^2 - 1)u\right)' + \left(a_4 x^4 + a_2 x^2 + a_0\right)u = 0. \tag{2.16}$$

By an appropriate choice of the coefficients a_4, a_2 and a_0, with $\lambda \neq 0$ we get

$$\begin{cases} \left(x(x^2 - 1)u\right)' + \left(2\lambda x^4 - (2\lambda + 2\alpha + 2\beta + 3)x^2 + 2\alpha + 1\right)u = 0, \\ \alpha\{|\beta| + |2\lambda(u)_2 - 2\alpha - 2\beta - 1|\} \neq 0. \end{cases} \tag{2.17}$$

This is the irreducible Pearson equation for a linear functional of class three, when Φ in (2.1) has three simple zeros.

B_2. $\tilde{\Phi}$ has one triple zero $\alpha_1 = 0$.

We choose a in such a way that $\tilde{a}_4 a^2 = 2$. Then (2.15) reads as

$$\left(x^3 u\right)' + \left(2x^4 + a_2 x^2 + a^{-2}a_0\right)u = 0. \tag{2.18}$$

By a suitable choice of the coefficients a_2 and a_0, (2.18) can be written as

$$\begin{cases} (x^3 u)' + 2(x^4 - (\alpha + 2)x^2 + \beta)u = 0, \\ \beta \neq 0. \end{cases} \tag{2.19}$$

This is the irreducible Pearson equation for a linear functional of class three, when Φ in (2.1) has a zero of multiplicity three.

B_3. $\tilde{\Phi}(x) = x$.

$\tilde{u}$ and u satisfy, respectively,

$$(x\tilde{u})' + \left(\tilde{a}_4 x^4 + \tilde{a}_2 x^2 + \tilde{a}_0\right)\tilde{u} = 0,$$
$$(xu)' + \left(a^4 \tilde{a}_4 x^4 + a^2 \tilde{a}_2 x^2 + \tilde{a}_0\right)u = 0. \tag{2.20}$$

If we choose a such that $a^4 \tilde{a}_4 = 4$, then we have

$$(xu)' + \left(4x^4 + a_2 x^2 + a_0\right)u = 0. \tag{2.21}$$

By an appropriate choice of the coefficients a_2 and a_0, (2.21) reads as

$$\begin{cases} (xu)' + (4x^4 - 4\lambda x^2 - 4\mu - 1)u = 0, \\ \mu \neq 0. \end{cases} \tag{2.22}$$

This is the irreducible Pearson equation for a linear functional of class three, when Φ in (2.1) is a linear polynomial.

3. Integral representation of symmetric semi-classical linear functionals of class $s = 3$

Let u be a symmetric semi-classical linear functional of class s satisfying (2.1) and let us assume $(u)_0 = 1$.

Our aim is to obtain an integral representation of u

$$\langle u, f(x)\rangle = \int_{-\infty}^{+\infty} f(x)U(x)\,\mathrm{d}x, \tag{3.1}$$

where we assume that function U is absolutely continuous on $\mathbb{R}$ and it decays as fast as its derivative U'. From (2.1), we get

$$\int_{-\infty}^{+\infty}((\Phi U)' + \Psi U)f(x)\,\mathrm{d}x - \left[\Phi(x)U(x)f(x)\right]_{-\infty}^{+\infty} = 0, \quad f \in \mathcal{P}.$$

Hence, from the assumptions on U, the following conditions hold:

$$[\Phi(x)U(x)f(x)]_{-\infty}^{+\infty} = 0, \quad f \in \mathcal{P}, \tag{3.2}$$

$$\int_{-\infty}^{+\infty}((\Phi U)' + \Psi U)f(x)\,\mathrm{d}x = 0, \quad f \in \mathcal{P}. \tag{3.3}$$

Condition (3.3) implies

$$(\Phi U)' + \Psi U = wg, \tag{3.4}$$

where w is arbitrary and g is a locally integrable function with rapid decay representing the null-linear functional (see [12])

$$\int_{-\infty}^{+\infty} x^n g(x) = 0, \quad n \geq 0. \tag{3.5}$$

Conversely, if U is a solution of (3.4) verifying the hypothesis above as well as

$$\int_{-\infty}^{+\infty} U(x)\,\mathrm{d}x \neq 0, \tag{3.6}$$

then (3.2) and (3.3) are fulfilled and (3.1) defines a linear functional u which is a solution of (2.1).

If $w = 0$, then (3.4) becomes

$$\frac{U'}{U} = -\frac{\Phi' + \Psi}{\Phi}. \tag{3.7}$$

We will consider the above seven canonical functional equations and, in each case, an integral representation of the corresponding linear functionals is given.

Case A.

A_1. From (2.7) to (3.7), we get

$$\frac{U'(x)}{U(x)} = \frac{2\gamma}{x} + \frac{2\beta x}{x^2 - 1} + \frac{2\alpha x}{x^2 - c^2},$$

and, as a consequence,

$$U(x) = |x^2 - 1|^{\beta}|x^2 - c^2|^{\alpha}|x|^{2\gamma} \tag{3.8}$$

is the solution at some intervals depending on c (see below).

On the other hand, if $\alpha\beta\gamma \neq 0$, $\alpha, \beta > -1$ and $\gamma > -1/2$, then the conditions (1.7) and (3.2) hold in the following situations:

s_1: $c \in]0,1[$. In such a case, u is represented by

$$\langle u, f\rangle = \int_{-1}^{1} f(x)|x|^{2\gamma}(1-x^2)^{\beta}|c^2-x^2|^{\alpha}(A\chi_{[-c,c]}(x) + B\chi_{[-1,1]}(x))\,\mathrm{d}x, \tag{3.9}$$

since from an integration by parts we deduce that the linear functionals u_1 and u_2 defined by

$$\langle u_1, f\rangle = \int_{-c}^{c} U(x)f(x)\,\mathrm{d}x \quad \text{and} \quad \langle u_2, f\rangle = \int_{-1}^{1} U(x)f(x)\,\mathrm{d}x,$$

are solutions of (2.7). The same result holds for any linear combination of u_1 and u_2 with coefficients A and B

$$\langle Au_1 + Bu_2, f\rangle = \int_{-1}^{1} U(x)\big(A\chi_{[-c,c]} + B\chi_{[-1,1]}\big)f(x)\,\mathrm{d}x,$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$, i.e. $\chi_{[a,b]}(x) = 1$ when $x \in [a,b]$ and zero otherwise. A and B are chosen in such a way that $(u)_0 = 1$.

We will proceed in a similar way in the case below.

s_2: $c > 1$. In such a case, we get

$$\langle u, f\rangle = \int_{-c}^{c} f(x)|x|^{2\gamma}|1-x^2|^{\beta}(c^2-x^2)^{\alpha}(A_1\chi_{[-1,1]}(x) + B_1\chi_{[-c,c]}(x))\,\mathrm{d}x. \tag{3.10}$$

Remark 1. The condition $\alpha\beta\gamma \neq 0$ is sufficient to ensure that the condition of irreducibility (1.7) is satisfied (see second equation in (2.7)). Indeed, for every parameter $\alpha, \beta > -1$ and $\gamma > -1/2$ satisfying (2.7), we obtain the integral representations given above.

Particular cases:

- If $\alpha = -1$, then (2.7) becomes

$$\begin{cases} (x(x^2-1)(x^2-c^2)u)' + (-(2\beta+2\gamma+3)x^2 + 2\gamma+1)(x^2-c^2)u = 0, \\ \gamma\{|\beta| + |-(2\beta+2\gamma+1)(u)_2 + c^2(2\beta+2\gamma+1) - 2\beta|\} \neq 0. \end{cases}$$

Hence,

$$(x^2-c^2)u = k\mathcal{GG}\left(\beta, \gamma - \frac{1}{2}\right), \tag{3.11}$$

where $\mathcal{GG}(a,b)$ is the normalized generalized Gegenbauer linear functional defined by the Pearson equation (see [6])

$$(x(x^2-1)\mathcal{GG}(a,b))' + 2(-(a+b+2)x^2 + b + 1)\mathcal{GG}(a,b) = 0.$$

Here, k is a constant such that $(u)_0 = 1$. Notice that the linear functional u defined by (3.11) is regular if and only if the MOPS $P_n^{(\beta,\gamma-1/2)}(x), n \geq 0$ satisfies (see [5])

$$\begin{vmatrix} P_{n+1}^{(\beta,\gamma-1/2)}\left(-c;\frac{k}{c}\right) & P_{n}^{(\beta,\gamma-1/2)}\left(-c;\frac{k}{c}\right) \\ P_{n+1}^{(\beta,\gamma-1/2)}\left(c;-\frac{k}{c}\right) & P_{n}^{(\beta,\gamma-1/2)}\left(c;-\frac{k}{c}\right) \end{vmatrix} \neq 0, \quad n \geq 0,$$

where $P_n^{(\beta,\gamma-1/2)}(x), n \geq 0$ are the monic polynomials orthogonal with respect to $\mathcal{GG}(\beta, \gamma - 1/2)$ and $\{P_n(\cdot\,;\mu)\}_{n\geq 0}$ is the co-recursive MOPS of $\{P_n\}_{n\geq 0}$ (see [6])

$$P_n(x;\mu) = P_n(x) - \mu P_{n-1}^{(1)}(x),$$

where $\{P_n^{(1)}\}_{n\geq 0}$ is the sequence of associated polynomials of the first kind for the sequence $\{P_n\}_{n\geq 0}$ (see [11]).

From (1.2) and (3.11), taking into account that $(u)_0 = 1$, we obtain

$$u = k(x^2 - c^2)^{-1}\mathcal{GG}\left(\beta, \gamma - \frac{1}{2}\right) + \frac{1}{2}(\delta_c + \delta_{-c}).$$

The action of the linear functionals defined in (3.11) over the polynomial x^{2n} yields

$$(u)_{2n+2} = c^{2n+2} + \big((u)_2 - c^2\big)\sum_{l=0}^{n}\frac{\Gamma(\beta+\gamma+3/2)\Gamma(l+\gamma+1/2)}{\Gamma(\gamma+1/2)\Gamma(l+\beta+\gamma+3/2)}c^{2n-2l}, \quad (u)_{2n+1} = 0, \quad n \geq 0.$$

- When $\gamma = 1/2$ and $c^2 = -1$, then from (2.7) we obtain

$$\begin{cases} (x(x^4-1)u)' + 2(-(\alpha+\beta+3)x^4 + (\alpha-\beta)x^2 + 1)u = 0, \\ \{|\beta| + |(\alpha+\beta+2)(u)_2 + 2\beta + 1|\} \times \{|\alpha| + |(\alpha+\beta+2)(u)_2 - 2\alpha - 1|\} \neq 0. \end{cases}$$

Therefore, according to (1.3)–(1.4)

$$\sigma u = \mathcal{J}(\alpha, \beta), \quad \sigma(xu) = 0, \tag{3.12}$$

where $\mathcal{J}(\alpha,\beta)$ is the normalized Jacobi linear functional satisfying (see [11])

$$((x^2-1)\mathcal{J}(\alpha,\beta))' + (-(\alpha+\beta+2)x + \alpha - \beta)\mathcal{J}(\alpha,\beta) = 0,$$

and $\alpha + 1 \neq -n, \beta + 1 \neq -n, \alpha + \beta + 2 \neq -n, n \geq 0$.

Notice that the linear functional u defined by (3.12) is regular if and only if $P_n^{(\alpha,\beta)}(0) \neq 0, n \geq 0$, where $P_n^{(\alpha,\beta)}(x), n \geq 0$ are the monic polynomials orthogonal with respect to $\mathcal{J}(\alpha,\beta)$. For more information about the linear functionals defined by a relation of type (3.12), see [3].

From (3.12), applying both linear functionals over x^n we get

$$(u)_{2n} = \sum_{l=0}^{n} 2^{l-1}\binom{n}{l}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(l+\alpha+\beta+2)}F_{n,l}(\alpha,\beta), \quad (u)_{2n+1} = 0, \quad n \geq 0,$$

with

$$F_{n,l}(\alpha,\beta) = (-1)^{n-l}\frac{\Gamma(l+\alpha+1)}{\Gamma(\alpha+1)} + (-1)^l\frac{\Gamma(l+\beta+1)}{\Gamma(\beta+1)}.$$

- If $\gamma = -1/2$ and $c^2 = -1$, then from (2.7) we get

$$\begin{cases} (x(x^4-1)u)' + 2(-(\alpha+\beta+2)x^2 + \alpha - \beta)x^2 u = 0, \\ \{|\beta| + |(\alpha+\beta+1)(u)_2 + 2\beta|\} \times \{|\alpha| + |(\alpha+\beta+1)(u)_2 - 2\alpha|\} \neq 0. \end{cases}$$

Then,

$$x\sigma u = k\mathcal{J}(\alpha,\beta), \quad \sigma(xu) = 0. \tag{3.13}$$

Here, k is a normalization constant. The linear functional u defined by (3.13) is regular if and only if $P_n^{(\alpha,\beta)}(0; -k) \neq 0, n \geq 0$ (see [14]). From (3.13), applying both linear functionals over x^n we obtain

$$(u)_{2n+2} = -\lambda \sum_{l=0}^{n} 2^{l-1} \binom{n}{l} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(l+\alpha+\beta+2)} F_{n,l}(\alpha,\beta), \quad (u)_{2n+1} = 0, \quad n \geq 0.$$

In the other cases, we will proceed following the same steps and the techniques described above.

Remark 2. Equation (3.13) is a particular case of a more general one considered in [14].

A_2. From (2.9) to (3.7), we have

$$\frac{U'(x)}{U(x)} = \frac{2\alpha}{x} + \frac{2\beta x}{x^2-1} - \frac{2\gamma x}{(x^2-1)^2},$$

so

$$U(x) = |x|^{2\alpha}|1-x^2|^{\beta} e^{\gamma/(x^2-1)}. \tag{3.14}$$

Finally, it is straightforward to check that for $\alpha\gamma \neq 0, \alpha > -1/2, \beta > -1$ and $\gamma > 0$, conditions (1.7) and (3.2) hold.

Then,

$$\langle u, f\rangle = k\int_{-1}^{1} f(x)|x|^{2\alpha}(1-x^2)^{\beta} e^{\gamma/(x^2-1)}\,dx. \tag{3.15}$$

Here, k is a constant such that $(u)_0 = 1$.

Remark 3. For every parameter $\alpha > -1/2, \beta > -2$ and $\gamma > 0$ such that (1.7) holds, we also obtain the integral representations given above, except when $\gamma \leq 0$. For $\gamma = 0$, see below.

Particular case:

If $\gamma = 0$, thus from (2.9) we get

$$\begin{cases} (x(x^2-1)^2 u)' + (x^2-1)(-(2\beta+2\alpha+5)x^2+2\alpha+1)u = 0, \\ \alpha(-(2\alpha+2\beta+3)(u)_2 + 2\alpha+1) \neq 0. \end{cases}$$

Assuming that the generalized Gegenbauer linear functional and u are normalized, then

$$u = (1-2M)\mathcal{GG}\left(\beta+1, \alpha-\frac{1}{2}\right) + M(\delta_1+\delta_{-1}). \tag{3.16}$$

The above linear functional u is regular if and only if M satisfies (see [2])

$$\begin{vmatrix} 1+MK_n(1,1) & MK_n(1,-1) \\ MK_n(1,-1) & 1+MK_n(-1,-1) \end{vmatrix} \neq 0, \quad n \geq 0,$$

where

$$K_n(x,y) := \sum_{m=0}^{n} \frac{P_m^{(\beta+1,\alpha-(1/2))}(x) P_m^{(\beta+1,\alpha-(1/2))}(y)}{\left\langle \mathcal{GG}(\beta+1,\alpha-(1/2)), \left(P_m^{(\beta+1,\alpha-(1/2))}\right)^2 \right\rangle}$$

is the n-th reproducing Kernel associated with the linear functional $\mathcal{GG}(\alpha,\beta)$ and $P_n^{(\beta+1,\alpha-(1/2))}(x)$, $n \geq 0$ are the monic polynomials orthogonal with respect to $\mathcal{GG}(\beta+1,\alpha-(1/2))$.

Finally, from (3.16) we have

$$\langle u,f\rangle = (1-2M)\Lambda\int_{-1}^{1} f(x)|x|^{2\alpha-1}(1-x^2)^{\beta+1}\,\mathrm{d}x + M(f(1)+f(-1)).$$

When $f(x)=x^{2n}$, the above expression becomes

$$(u)_{2n} = 2M + (1-2M)\frac{\Gamma(\alpha+\beta+(3/2))\Gamma(n+\alpha+(1/2))}{\Gamma(\alpha+(1/2))\Gamma(n+\alpha+\beta+(3/2))}, \quad (u)_{2n+1}=0, \quad n\geq 0,$$

where

$$\Lambda = \frac{\Gamma(\alpha+\beta+(3/2))}{\Gamma(\beta+1)\Gamma(\alpha+(1/2))}, \quad \alpha > -\frac{1}{2}, \quad \beta > -1.$$

This linear functional generalizes the so-called Koornwinder polynomials [8].

A_3. From (2.11) to (3.7), we obtain

$$\frac{U'}{U} = \frac{4\beta+1}{x} + \frac{2\gamma}{x^3} + \frac{2\alpha x}{x^2-1},$$

and, as a consequence,

$$U(x) = |x|^{4\beta+1}|1-x^2|^{\alpha}\mathrm{e}^{((-\gamma)/x^2)}. \tag{3.17}$$

Notice that if $\alpha\gamma \neq 0, \alpha, 2\beta > -1$ and $\gamma > 0$, then the conditions (1.7) and (3.2) hold. Thus, u is represented by

$$\langle u,f\rangle = k\int_{-1}^{1} f(x)|x|^{4\beta+1}(1-x^2)^{\alpha}\mathrm{e}^{((-\gamma)/x^2)}\,\mathrm{d}x. \tag{3.18}$$

Here, k is a constant such that $(u_0)=1$.

Particular case:

If $\alpha=-1$, then (2.11) becomes

$$\begin{cases} (x^3(x^2-1)u)' + 2(-2(\beta+1)x^2-\gamma)(x^2-1)u = 0, \\ \gamma \neq 0. \end{cases}$$

Hence,

$$(x^2-1)u = k\left(h_{(2\sqrt{\gamma})}\hat{v}(2\beta+1)\right), \tag{3.19}$$

where $\hat{v}(\nu)$ is symmetric semi-classical linear functional of class one that satisfies (see [1,15])

$$\left(x^3\hat{v}(\nu)\right)' + \left(-2(\nu+1)x^2 - \frac{1}{2}\right)\hat{v}(\nu) = 0.$$

Here, k is a constant such that $(u)_0=1$. Notice that the linear functional u defined by (3.19) is regular if and only if the MOPS $P_n^{(2\beta+1)}(x), n\geq 0$, satisfies (see [5])

$$\begin{vmatrix} P_{n+1}^{(2\beta+1)}(-1;k) & P_n^{(2\beta+1)}(-1;k) \\ P_{n+1}^{(2\beta+1)}(1;-k) & P_n^{(2\beta+1)}(1;-k) \end{vmatrix} \neq 0, \quad n\geq 0,$$

where $P_n^{(2\beta+1)}(x), n\geq 0$ are the monic polynomials orthogonal with respect to $h_{(2\sqrt{\gamma})}\hat{v}(2\beta+1)$.

From (1.2) and (3.19), and taking into account $(u)_0 = 1$ we get

$$u = k(x^2 - 1)^{-1}\big(h_{(2\sqrt{\gamma})}\hat{v}(2\beta + 1)\big) + \frac{1}{2}(\delta_1 + \delta_{-1}).$$

Applying both linear functionals to x^n, we get

$$(u)_{2n+2} = 1 + \big((u)_2 - 1\big)\sum_{l=0}^{n}(-1)^l 2^{3l}\gamma^l \frac{\Gamma(4\beta + 2)}{\Gamma(l + 4\beta + 2)}, \quad (u)_{2n+1} = 0, \quad n \geq 0.$$

Moreover, applying the operator σ for (3.19) and using (1.1)–(1.3), we obtain

$$\sigma u = k(x - 1)^{-1}(h_{\gamma/2}\mathcal{B}(\beta + 1)) + \delta_1,$$

where $\mathcal{B}(a)$ is the Bessel linear functional that satisfies (see [11])

$$(x^2\mathcal{B}(a))' + (-2ax - 2)\mathcal{B}(a) = 0.$$

Therefore, we deduce that the linear functional u, defined by (3.19), is the symmetrized of a Bessel-type linear functional (see [3]).

A_4. In this case, it is not possible to choose $w = 0$. Indeed, from (2.14) and (3.7) (i.e. (3.4) with $w = 0$), we obtain

$$\frac{U'(x)}{U(x)} = \frac{2(\alpha + 1)}{x} - \frac{2\beta}{x^3} + \frac{8}{x^5}.$$

A priori there is no real path $\mathcal{C}$ such that $x^5U(x)f(x)|_{\mathcal{C}} = 0, \quad f \in \mathcal{P}$. This is the analogue of Bessel in the classical case (see [12] for more information).

Thus, we handle it differently with the choice $w \neq 0$ in (3.4).

From (2.14) to (3.4), we get

$$(x^5U)' + \{-2(\alpha + 3)x^4 + 2\beta x^2 - 8\}U = wg(x). \tag{3.20}$$

For instance, let $g(x) = |x|s(x^2), x \in \mathbb{R}$, where s is the classical Stieltjes function [12]

$$s(x) = \begin{cases} 0, & x \leq 0, \\ \mathrm{e}^{-x^{1/4}}\sin x^{1/4}, & x > 0. \end{cases} \tag{3.21}$$

$g(x) = |x|s(x^2)$ is an even representation of the null form.

A possible solution of (3.20) is the even function

$$U(x) = \begin{cases} 0, & x = 0, \\ w|x|^{2\alpha+1}\mathrm{e}^{(\beta/x^2)-(2/x^4)}\int_{|x|}^{+\infty} t^{-2\alpha-5}\mathrm{e}^{(2/t^4)-(\beta/t^2)}s(t^2)\,\mathrm{d}t, & \beta \leq 0, x \in \mathbb{R} - \{0\}. \end{cases} \tag{3.22}$$

First, condition (3.2) is fulfilled since

$$|x^5U(x)| \leq |w||x|^{2\alpha+6}\mathrm{e}^{(\beta/x^2)-(2/x^4)}\int_{|x|}^{+\infty} t^{-2\alpha-5}\mathrm{e}^{(2/t^4)-(\beta/t^2)}\mathrm{e}^{-t^{1/2}}\,\mathrm{d}t = o\Big(\mathrm{e}^{-(1/2)|x|^{1/2}}\Big),$$

$$|x| \longrightarrow +\infty.$$

Furthermore, when $x \longrightarrow +\infty$

$$|U(x)| \leq |w|x^{2\alpha+1}\int_{x}^{+\infty} t^{-2\alpha-5}\mathrm{e}^{-t^{1/2}}\,\mathrm{d}t = o\Big(\mathrm{e}^{-(1/2)x^{1/2}}\Big),$$

and when $x \to +0$

$$|U(x)| \leq |w| x^{2\alpha+1} \mathrm{e}^{(\beta/x^2)-(2/x^4)} \int_x^1 t^{-2\alpha-5} \mathrm{e}^{(2/t^4)-(\beta/t^2)} \,\mathrm{d}t + o(1).$$

Applying l'Hospital's rule to the ratio

$$\lim_{x\to+0} \frac{\int_x^1 t^{-2\alpha-5}\, \mathrm{e}^{(2/t^4)-(\beta/t^2)}\,\mathrm{d}t}{x^{-2\alpha-1}\, \mathrm{e}^{(2/x^4)-(\beta/x^2)}} = \lim_{x\to+0} \frac{x}{(2\alpha+1)x^4 - 2\beta x^2 + 8} = 0,$$

we get $\lim_{x\to+0} U(x) = 0 = U(0)$.

Consequently, $U \in L_1$.

Condition (3.6) now becomes

$$\int_{-\infty}^{+\infty} U(x)\,\mathrm{d}x = 2w \int_0^{+\infty} \xi^{-2\alpha-5}\, \mathrm{e}^{(2/\xi^4)-(\beta/\xi^2)} s(\xi^2) \left(\int_0^{\xi} x^{2\alpha+1}\, \mathrm{e}^{(\beta/x^2)-(2/x^4)}\,\mathrm{d}x \right) \mathrm{d}\xi$$

$$= wS_\alpha \neq 0, \tag{3.23}$$

with

$$S_\alpha = 4 \int_0^{+\infty} t^{-4\alpha-9}\, \mathrm{e}^{(2/t^8)-(\beta/t^4)}\, \mathrm{e}^{-t} \varphi_{\alpha-(1/2)}(t^2) \sin t\,\mathrm{d}t, \tag{3.24}$$

$$\varphi_\alpha(t) = \int_0^t x^{2\alpha+2}\, \mathrm{e}^{(\beta/x^2)-(2/x^4)}\,\mathrm{d}x. \tag{3.25}$$

Therefore, for α such that $S_\alpha \neq 0$,

$$\langle u, f \rangle = S_\alpha^{-1} \int_{-\infty}^{+\infty} \frac{1}{x^4} \int_{|x|}^{+\infty} \left(\frac{|x|}{t} \right)^{2\alpha+5} \exp\left(\frac{\beta}{x^2} - \frac{\beta}{t^2} \right) \exp\left(\frac{2}{t^4} - \frac{2}{x^4} \right) s(t^2)\,\mathrm{d}t\, f(x)\,\mathrm{d}x. \tag{3.26}$$

Remark 4. Applying the operator σ to (2.14) and using (1.3)–(1.4), we get

$$(x^3 \sigma u)' + (-(\alpha+3)x^2 + \beta x - 4)\sigma u = 0. \tag{3.27}$$

Moreover, since the linear form u is symmetric and regular, then σu is regular.

According to (3.27) and (1.7), the linear functional σu is semi-classical of class $s = 1$.

From (3.26), we get

$$\langle \sigma u, f(x) \rangle = \langle u, f(x^2) \rangle$$

$$= S_\alpha^{-1} \int_{-\infty}^{+\infty} \frac{1}{x^4} \int_{|x|}^{+\infty} \left(\frac{|x|}{t} \right)^{2\alpha+5} \exp\left(\frac{\beta}{x^2} - \frac{\beta}{t^2} \right) \exp\left(\frac{2}{t^4} - \frac{2}{x^4} \right) s(t^2)\,\mathrm{d}t\, f(x^2)\,\mathrm{d}x.$$

After a change of variables, we get

$$\langle \sigma u, f(x) \rangle = S_\alpha^{-1} \int_0^{+\infty} x^\alpha \exp\left(\frac{\beta}{x} - \frac{2}{x^2} \right) \int_{\sqrt{x}}^{+\infty} t^{-2\alpha-5} \exp\left(\frac{2}{t^4} - \frac{\beta}{t^2} \right) s(t^2)\,\mathrm{d}t\, f(x)\,\mathrm{d}x. \tag{3.28}$$

The integral representation (3.28) does not exist in the list given in [4].

Case B.

B_1. From (2.17) to (3.7), we have

$$\frac{U'(x)}{U(x)} = -2\lambda x + \frac{2\alpha}{x} + \frac{2\beta x}{x^2 - 1}.$$

Thus,

$$U(x) = |x|^{2\alpha}|1 - x^2|^{\beta}\,\mathrm{e}^{-\lambda x^2} \tag{3.29}$$

is the solution in $]-\infty, +\infty[$ and $[-1, 1]$.

Furthermore, for $\alpha\beta \neq 0, \beta > -1$ and $\alpha > -1/2$, (1.7) and (3.2) hold in the following situations:

s_1 : If $\lambda > 0$, then

$$\langle u, f\rangle = \int_{-\infty}^{+\infty} |x|^{2\alpha}|1 - x^2|^{\beta}\,\mathrm{e}^{-\lambda x^2}(A\chi_{[-1,1]} + B\chi_{]-\infty,+\infty[})f(x)\,\mathrm{d}x. \tag{3.30}$$

s_2 : If $\lambda \leq 0$, then

$$\langle u, f\rangle = \int_{-1}^{1} |x|^{2\alpha}(1 - x^2)^{\beta}\,\mathrm{e}^{-\lambda x^2}f(x)\,\mathrm{d}x. \tag{3.31}$$

Particular cases:

- If $\beta = -1$, then (2.17) becomes

$$\begin{cases} (x(x^2-1)u)' + (2\lambda x^2 - 2\alpha - 1)(x^2 - 1)u = 0, \\ \alpha \neq 0. \end{cases}$$

Hence,

$$(x^2 - 1)u = k\left(h_{(\sqrt{\lambda})^{-1}}\mathcal{H}(2\alpha)\right), \tag{3.32}$$

where $\mathcal{H}(2a+1)$ is the generalized Hermite linear functional that satisfies (see [1,6])

$$(x\mathcal{H}(2a+1))' + (2x^2 - 2(a+1))\mathcal{H}(2a+1) = 0.$$

The above linear functional u is regular if and only if the MOPS $P_n^{(2\alpha)}(x), n \geq 0$ satisfies (see [5])

$$\begin{vmatrix} P_{n+1}^{(2\alpha)}(-1;k) & P_n^{(2\alpha)}(-1;k) \\ P_{n+1}^{(2\alpha)}(1;-k) & P_n^{(2\alpha)}(1;-k) \end{vmatrix} \neq 0, \quad n \geq 0,$$

where $P_n^{(2\alpha)}(x), n \geq 0$ are the monic polynomials orthogonal with respect to $h_{(\sqrt{\lambda})^{-1}}\mathcal{H}(2\alpha)$.

From (3.32), applying both linear functionals over x^{2n} we get

$$(u)_{2n+2} = 1 + ((u)_2 - 1)\sum_{l=0}^{n} \lambda^{-l}\frac{\Gamma(l + \alpha + (1/2))}{\Gamma(\alpha + (1/2))}, \quad (u)_{2n+1} = 0, \quad n \geq 0.$$

Moreover, applying the operator σ to (3.32) and using (1.1)–(1.3), we obtain

$$\sigma u = k(x-1)^{-1}\left(h_{\lambda^{-1}}\mathcal{L}\left(\alpha - \frac{1}{2}\right)\right) + \delta_1,$$

where $\mathcal{L}(a)$ is the Laguerre linear functional that satisfies (see [11])

$$(x\mathcal{L}(a))' + (x - a - 1)\mathcal{L}(a) = 0.$$

Therefore, we deduce that the linear functional u, defined by (3.32), is the symmetrized of a Laguerre-type linear functional (see [3]).

- If $\alpha = 1/2$, then from (2.17) we obtain

$$\begin{cases} (x(x^2-1)u)' + 2(\lambda x^4 - (\lambda+\beta+2)x^2+1)u = 0, \\ |\beta| + |\lambda(u)_2 - \beta - 1| \neq 0. \end{cases}$$

Therefore, a simple computation from the integral representation for U given in (3.29) yields

$$\sigma u = (h_{\lambda^{-1}} \circ t_\lambda)\mathcal{L}(\beta), \quad \sigma(xu) = 0, \tag{3.33}$$

where the form $t_\lambda u$ (translation of u) is defined by

$$\langle t_\lambda u, f(x)\rangle := \langle u, f(x+\lambda)\rangle, \quad f \in \mathcal{P}.$$

Notice that the linear functional u defined by (3.33) is regular if and only if $P_n^{(\beta)}(0) \neq 0, n \geq 0$, where $P_n^{(\beta)}(x), n \geq 0$ are the monic polynomials orthogonal with respect to $(h_{\lambda^{-1}} \circ t_\lambda)\mathcal{L}(\beta)$.

From (3.33) if we apply both linear functionals over x^n, then we get

$$(u)_{2n} = \lambda^{-n} n! \sum_{\nu+\mu=n} \frac{\lambda^\mu \Gamma(\nu+\beta+1)}{\nu!\mu!\Gamma(\beta+1)}, \quad (u)_{2n+1} = 0, \quad n \geq 0.$$

B_2. From (2.19) to (3.7), we obtain

$$\frac{U'(x)}{U(x)} = -2x + \frac{2\alpha+1}{x} - \frac{2\beta}{x^3}.$$

Thus,

$$U(x) = |x|^{2\alpha+1} \mathrm{e}^{-x^2+(\beta/x^2)}$$

is the solution in $]-\infty, +\infty[$.

If $\beta < 0$ and $\alpha > -1$, then (1.7) and (3.2) hold. Thus,

$$\langle u, f\rangle = k\int_{-\infty}^{+\infty} f(x)|x|^{2\alpha+1} \mathrm{e}^{-x^2+\beta/(x^2)}\,\mathrm{d}x.$$

Here, k is a normalization constant in order $u_0 = 1$. The generalized Hermite linear functional appears when $\beta = 0$ (see [6]).

B_3. From (2.22) to (3.7), we get

$$\frac{U'(x)}{U(x)} = -4x^3 + 4\lambda x + \frac{4\mu}{x}.$$

Thus,

$$U(x) = |x|^{4\mu} \mathrm{e}^{-x^4+2\lambda x^2}$$

is the solution in $\mathbb{R}$. If $\mu \neq 0$ and $\mu > -1/4$, then (1.7) and (3.2) hold. As a consequence, u is represented by

$$\langle u, f\rangle = k\int_{-\infty}^{+\infty} f(x)|x|^{4\mu} \mathrm{e}^{-x^4+2\lambda x^2}\,\mathrm{d}x.$$

Here, k is a normalization constant such that $(u)_0 = 1$. This is an example of Freud linear functional (see [13]).

Particular case:

When $\lambda = 0$ and $\mu = -(1/4)$, thus from (2.22) we get

$$(xu)' + 4x^4u = 0.$$

Then

$$x\sigma u = k\mathcal{H}, \quad \sigma(xu) = 0, \tag{3.34}$$

where $\mathcal{H}$ is the normalized Hermite linear functional, $((\mathcal{H})_0 = 1)$, with Pearson equation [6]

$$\mathcal{H}' + 2x\mathcal{H} = 0$$

and k is a normalization constant such that $(u)_0 = 1$. The linear functional u defined by (3.34) is regular (see [14]).

From (1.3), if in (3.34) we apply both functionals to the canonical basis $\{x^n\}_{n\geq 0}$, then the moments of the linear functional u are

$$(u)_{4n+2} = k\frac{(2n)!}{2^{2n}n!}, \quad (u)_{4n+4} = (u)_{2n+1} = 0, \quad n \geq 0.$$

Indeed, this sequence was analysed in (Example 2, [14]).

Acknowledgements

The work of the first author (FM) has been supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain, Grant MTM2009-12740-C03-01. The authors thank the referees for their careful revision of the manuscript. Their comments and suggestions have contributed to improve the presentation.

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