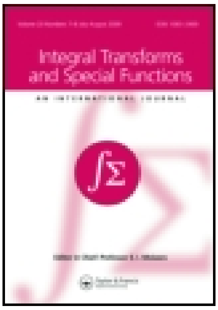

Integral Transforms and Special Functions
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gitr20

Asymptotic behaviour of Sobolev orthogonal polynomials on the unit circle

Kenier Castillo [a] , Luis E. Garza [b] & Francisco Marcellán [a]
[a] Departamento de Matemáticas, Universidad Carlos III de Madrid, Ave. de la Universidad 30, Leganés, Spain
[b] Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo No. 340, Colima, Colima, México

Available online: 31 Jan 2012

Integral Transforms and Special Functions
iFirst, 2012, 1–16

Taylor & Francis
Taylor & Francis Group

Asymptotic behaviour of Sobolev orthogonal polynomials on the unit circle

Kenier Castillo[a]*, Luis E. Garza[b] and Francisco Marcellán[a]

[a]*Departamento de Matemáticas, Universidad Carlos III de Madrid, Ave. de la Universidad 30, Leganés, Spain;* [b]*Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo No. 340, Colima, Colima, México*

(*Received 25 October 2011; final version received 12 December 2011*)

In this contribution, we deal with analytic properties of sequences of polynomials orthogonal with respect to a Sobolev-type inner product

$$\langle f, g\rangle_S := \int_{\mathbb{T}} f(z)\overline{g(z)}\,\mathrm{d}\mu(z) + \lambda f^{(j)}(\alpha)\overline{g^{(j)}(\alpha)}, \quad \alpha \in \mathbb{C},\ \lambda \in \mathbb{R}^+\backslash\{0\},\ j \in \mathbb{N},$$

where μ is a non-trivial probability measure supported on the unit circle. We focus our attention on the outer relative asymptotics of these polynomials in terms of those associated with the measure μ. The behaviour of their zeros in terms of the parameter λ is studied in some illustrative examples.

Keywords: probability measures on the unit circle; orthogonal polynomials; Sobolev inner products; outer relative asymptotics; zeros

AMS Subject Classifications: 42C05, 33C47

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc, and let $\mathrm{d}\mu$ be a non-trivial probability measure supported on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It is a very well-known fact (see [8,16,17], among others) that there exists a sequence of polynomials $\{\phi_n\}_{n\geqslant 0}$ with $\deg \phi_n = n$ such that

$$\int_{\mathbb{T}} \phi_m(z)\overline{\phi_n(z)}\,\mathrm{d}\mu(z) = \delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker delta. $\{\phi_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to the measure $\mathrm{d}\mu$. The polynomial $K_n(z, y) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(y)}$ is called the

*Corresponding author. Email: kcastill@math.uc3m.es

ISSN 1065-2469 print/ISSN 1476-8291 online

http://dx.doi.org/10.1080/10652469.2011.649751
http://www.tandfonline.com

reproducing kernel associated to $\{\phi_n\}_{n\geqslant 0}$. It has an explicit expression

$$K_n(z,y)=\frac{\phi_{n+1}^*(z)\overline{\phi_{n+1}^*(y)}-\phi_{n+1}(z)\overline{\phi_{n+1}(y)}}{1-\bar{y}z},$$

called the Christoffel–Darboux formula. We will denote by $K_n^{(k,j)}(z,y)$ the kth (resp. jth) derivative of $K_n(z,y)$ with respect to the variable z (resp. y). The corresponding sequence of monic orthogonal polynomials will be denoted by $\{\Phi_n\}_{n\geqslant 0}$. They satisfy the following forward and backward recurrence relations (see [17]):

$$\Phi_{n+1}(z)=z\Phi_n(z)+\Phi_{n+1}(0)\Phi_n^*(z),\quad n\geqslant 0,$$
$$\Phi_{n+1}(z)=(1-|\Phi_{n+1}(0)|^2)z\Phi_n(z)+\Phi_{n+1}(0)\Phi_{n+1}^*(z),\quad n\geqslant 0,$$

where $\Phi_n^*(z)=z^n\overline{\Phi_n(1/\bar{z})}$ is the reversed polynomial and $\{\Phi_n(0)\}_{n\geqslant 1}$ are the so-called Verblunsky (reflection) coefficients.

The measure $\mathrm{d}\mu$ can be decomposed into a part that is purely absolutely continuous with respect to the Lebesgue measure and a singular part, i.e. $\mathrm{d}\mu(\theta)=\mu'(\theta)(\mathrm{d}\theta/2\pi)+\mathrm{d}\mu_s(\theta)$. In the literature, two important families of measures are considered.

DEFINITION 1.1 *μ belongs to the Nevai class $\mathcal{N}$ (see* [13] *and* [14]*), if*

$$\lim_{n\to\infty}|\phi_{n+1}(0)|=0. \tag{1}$$

DEFINITION 1.2 *μ belongs to the Szegő class $\mathcal{S}$, (see* [8]*), if*

$$\int_{-\pi}^{\pi}\log\mu'(\theta)\,\mathrm{d}\theta>-\infty. \tag{2}$$

Note that $\mathcal{S}\subset\mathcal{N}$. Furthermore, $\mu\in\mathcal{S}$ if and only if $\sum_{n=0}^{\infty}|\phi_{n+1}(0)|^2<+\infty$ (see [16]). On the other hand, if $\mu\in\mathcal{S}$, then the Szegő function, $D(z,\mu')$ is defined by

$$D(z,\mu')=\exp\left(\frac{1}{4\pi}\int\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\log\mu'(\theta)\right)\mathrm{d}\theta,\quad |z|<1.$$

Otherwise, we set $D(z,\mu')\equiv 0$.

The relation between these classes can be viewed using the results in [12]. If $\mu\in\mathcal{S}$, then it has a normal L^2-derivative behaviour, i.e.

$$\lim_{n\to\infty}\left(\int\frac{|\phi_n'(\mathrm{e}^{\imath\theta})|^2}{n^2}\,\mathrm{d}\mu(\theta)\right)^{1/2}=1,$$

and then $\mu\in\mathcal{N}$.

On the other hand, if $\mu\in\mathcal{N}$,

$$\left|\frac{\Phi_n(z)}{\Phi_{n-1}(z)}-z\right|\leqslant|\Phi_n(0)|,\quad |z|\geqslant 1.$$

Thus, $\lim_{n\to\infty}(\Phi_n(z)/\Phi_{n-1}(z))=z$, uniformly in compact subsets of $|z|>1$. This asymptotics was obtained under much weaker conditions. A well-known result (see [15]) says that any measure that satisfies a Lipschitz condition with some positive exponent, i.e.

$$\frac{\mathrm{d}\mu(z)}{|\mathrm{d}z|}=\mu'(z)\quad\text{with }\mu'(z)>0\quad\text{a.e. for }z\in\mathbb{T},$$

satisfies (1). Then, the Lipschitz condition is a sufficient condition for a measure in order to belong to the $\mathcal{N}$ class.

In this paper, we will consider the following discrete Sobolev inner product for nontrivial probability measures supported on the unit circle

$$\langle f, g\rangle_S := \int_{\mathbb{T}} f(z)\overline{g(z)}\,\mathrm{d}\mu(z) + \lambda f^{(j)}(\alpha)\overline{g^{(j)}(\alpha)}, \quad \alpha \in \mathbb{C},\ \lambda \in \mathbb{R}^{+}\backslash\{0\},\ j \in \mathbb{N}, \tag{3}$$

where f and g are in the Sobolev space

$$W^{j,2}[\mathbb{T};\mu] = \{f \in \mathcal{C}_j(\mathbb{T}) \cap L^2[\mathbb{T};\mu] : f^{(j)} \in L^2[\mathbb{T};\mu]\}.$$

Here, $\mathcal{C}_j(\mathbb{T})$ denotes the function space containing all functions $f : \mathbb{T} \to \mathbb{C}$ such that $f \in C^{j-2}$ and $f^{(j-1)}$ is absolutely continuous on $\mathbb{T}$. Equation (3) is a particular case of the more general discrete Sobolev inner product

$$\langle f, g\rangle = \int_{\Gamma} f(z)\overline{g(z)}\,\mathrm{d}\mu(z) + f(Z)Ag(Z)^{\mathrm{t}}, \tag{4}$$

where $f(Z) = (f(z_1), \dots, f^{(l_1)}(z_1), \dots, f(z_m), \dots, f^{(l_m)}(z_m))$, Γ is any rectifiable Jordan curve on the complex plane, A is a $M \times M$ positive definite Hermitian matrix, with $M = l_1 + \cdots + l_m + m$, and $z_i \in \mathbb{C}$. Note that, since A is a positive definite matrix, there exists a family of polynomials orthogonal with respect to (4). Most of the literature existing on the subject has been devoted to the study of the asymptotic behaviour of the family of orthogonal polynomials with respect to (4). For instance, in [2], the authors focus the attention on the case where the points $z_i \notin \Gamma$, and obtain (among other results) the asymptotic behaviour of the ratio of the leading coefficients of both families of polynomials, as well as the relative asymptotics of their derivatives. On the other hand, in [3,6,10], the authors consider the case when Γ is the unit circle, and $|z_i| > 1$, and develop a similar analysis. Finally, Foulquié Moreno *et al.* [7] deals with the case $\Gamma = \mathbb{T}$, but the authors consider points $|z_i| = 1$.

The structure of the manuscript is as follows. Section 2 contains some preliminary results concerning asymptotic properties of a family of orthogonal polynomials with respect to a nontrivial probability measure supported on the unit circle and their derivatives, which will be useful in the analysis of the asymptotic properties of the family of polynomials orthogonal with respect to (3). Section 3 deals with the relations between the families of orthogonal polynomials, moments, and Toeplitz-like matrices associated with (3) and μ, respectively. In Section 4, we present some asymptotic results for the polynomials orthogonal with respect to (3). We focus our attention on the asymptotics of the ratio of the leading coefficients and the relative asymptotics of the polynomials, for $\alpha \in \mathbb{C}$. We also state some asymptotics for their norms. Although most of the results of this section are known, some of the proofs presented here are new. Finally, in Section 5, we analyse the location of the zeros of the polynomials orthogonal with respect to (3). Some examples for two illustrative case of nontrivial measures are analyzed. Note that there is not a well-stated theory for zeros of these Sobolev orthogonal polynomials, in contrast with the case of the real line [1,11].

2. Asymptotic properties of the derivatives of ϕ_n

We state the following preliminary result, which we will use in the sequel.

LEMMA 2.1 [9] *Let p and q be two polynomials in $\mathbb{P}$ with degrees at least k. Then*

$$\frac{p^{(k)}(z)}{q^{(k)}(z)} = \frac{q^{(k-1)}(z)}{q^{(k)}(z)}\left(\frac{p^{(k-1)}(z)}{q^{(k-1)}(z)}\right)' + \frac{p^{(k-1)}(z)}{q^{(k-1)}(z)}.$$

Using the previous lemma, the outer ratio asymptotics for the derivatives of orthogonal polynomials are deduced.

PROPOSITION 2.2 *If $\mu \in \mathcal{N}$, then*

$$\lim_{n\to\infty} \frac{\phi_{n+1}^{(j)}(z)}{\phi_n^{(j)}(z)} = z \quad \textit{and} \quad \lim_{n\to\infty} \frac{\phi_n^{(j)}(z)}{\phi_n^{(j+1)}(z)} = 0, \quad j = 0, 1, \ldots$$

uniformly in $\mathbb{C}\backslash\overline{\mathbb{D}}$.

Proof According to Lemma 2.1,

$$\frac{\phi_{n+1}^{(j)}(z)}{\phi_n^{(j)}(z)} = \frac{\phi_n^{(j-1)}(z)}{\phi_n^{(j)}(z)} \left(\frac{\phi_{n+1}^{(j-1)}(z)}{\phi_n^{(j-1)}(z)}\right)' + \frac{\phi_{n+1}^{(j-1)}(z)}{\phi_n^{(j-1)}(z)}. \tag{5}$$

Using induction in j, we obtain

$$\lim_{n\to\infty} \left(\frac{\phi_{n+1}^{(j-1)}}{\phi_n^{(j-1)}}\right)'(z) = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\phi_n^{(j-1)}(z)}{\phi_n^{(j)}(z)} = 0$$

uniformly in $\mathbb{C}\backslash\overline{\mathbb{D}}$. Therefore, if n tends to infinity in (5), the result follows. ■

COROLLARY 2.3 *If $\mu \in \mathcal{N}$, then*

$$\lim_{n\to\infty} \frac{\phi_n^{*(j)}(z)}{\phi_n^{(j)}(z)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{K_{n-1}^{(l,r)}(z,\zeta)}{\phi_n^{(k)}(z)\overline{\phi_n^{(j)}(\zeta)}} = 0, \quad l < k \in \mathbb{N},\ r < j \in \mathbb{N}$$

uniformly in $\mathbb{C}\backslash\overline{\mathbb{D}}$.

Proof From Lemma 2.1, we have

$$\frac{\phi_n^{*(j)}(z)}{\phi_n^{(j)}(z)} = \frac{\phi_n^{(j-1)}(z)}{\phi_n^{(j)}(z)} \left(\frac{\phi_n^{*(j-1)}(z)}{\phi_n^{(j-1)}(z)}\right)' + \frac{\phi_n^{*(j-1)}(z)}{\phi_n^{(j-1)}(z)}. \tag{6}$$

Using a similar argument as in the proof of the previous theorem, the first statement follows. The second statement is a straightforward consequence of $\lim_{n\to\infty}(\phi_n^{*(j)}(z)/\phi_n^{(j)}(z)) = 0$ and Proposition 2.2. ■

The following result was proved in [6] using a different method, and has been generalized for rectifiable curves in [2]. We show here another proof of the same result.

PROPOSITION 2.4 *If $\mu \in \mathcal{N}$, then uniformly in* $\mathbb{C}\backslash\overline{\mathbb{D}}$,

$$\lim_{n\to\infty} \frac{K_{n-1}^{(k,j)}(z,\zeta)}{\phi_n^{(k)}(z)\overline{\phi_n^{(j)}(\zeta)}} = \frac{1}{z\overline{\zeta} - 1}, \quad k, j = 0, 1, \ldots.$$

Proof From the Christoffel–Darboux formula, we obtain

$$\phi_n^*(z)\overline{\phi_n^{*(j)}(\zeta)} - \phi_n(z)\overline{\phi_n^{(j)}(\zeta)} = (1 - z\bar{\zeta})K_{n-1}^{(0,j)}(z,\zeta) - jzK_{n-1}^{(0,j-1)}(z,\zeta),$$

and, as a consequence,

$$\begin{aligned}\phi_n^{*(k)}(z)\overline{\phi_n^{*(j)}(\zeta)} - \phi_n^{(k)}(z)\overline{\phi_n^{(j)}(\zeta)} &= (1 - z\bar{\zeta})K_{n-1}^{(k,j)}(z,\zeta) - k\bar{\zeta}K_{n-1}^{(k-1,j)}(z,\zeta)\\ &\quad - j[zK_{n-1}^{(k,j-1)}(z,\zeta) + kK_{n-1}^{(k-1,j-1)}(z,\zeta)].\end{aligned}$$

Thus, dividing by $\phi_n^{(k)}(z)\overline{\phi_n^{(j)}(\zeta)}$ and using the previous corollary when n tends to infinity, the result follows. ■

Remark 1 Note that

$$\lim_{n\to\infty} \frac{K_{n-1}^{(j,j)}(\alpha,\alpha)}{|\phi_n^{(j)}(\alpha)|^2} = \frac{1}{|\alpha|^2 - 1}, \quad |\alpha| > 1, \ j = 1, 2, \ldots$$

Let $\psi_n(z)$ be the nth degree orthonormal polynomials with respect to the Christoffel transform of a measure $\mathrm{d}\mu$, given by $\mathrm{d}\mu_1 = |z-\alpha|^2\mathrm{d}\mu$. Let us denote by $\mathcal{K}_n(z,\zeta)$ its associated reproducing kernel. Then [5]

$$(z-\alpha)\psi_{n-1}(z) = \sqrt{\frac{K_{n-1}(\alpha,\alpha)}{K_n(\alpha,\alpha)}}\left(\phi_n(z) - \frac{\phi_n(\alpha)}{K_{n-1}(\alpha,\alpha)}K_{n-1}(z,\alpha)\right). \tag{7}$$

Furthermore, using the last expression, we obtain the following:

LEMMA 2.5

$$(z-\alpha)\overline{(y-\alpha)}\mathcal{K}_{n-1}(z,y) = K_n(z,y) - \frac{K_n(z,\alpha)K_n(y,\alpha)}{K_n(\alpha,\alpha)}. \tag{8}$$

Proof Let us denote by $\mathbb{P}_n$ the linear space of polynomials with complex coefficients of degree at most n. The system $\{(z-\alpha)\psi_k(z) : k = 0, 1, \ldots, n-1\} \cup \{K_n(z,\alpha)\}$ is an orthogonal basis of $\mathbb{P}_n(\mathrm{d}\mu)$. This means that in $L^2(\mathrm{d}\mu_1)$ we have $\mathbb{P}_n = (z-\alpha)\mathbb{P}_{n-1} \oplus_\perp \mathrm{span}\,(K_n(z,\alpha))$. Taking into account that for a fixed ζ

$$(z-y)\overline{(\zeta-y)}\mathcal{K}_{n-1}(z,y) - K_n(z,y) \in \mathbb{P}_n,$$

from (8), we obtain

$$(z-\alpha)\overline{(y-\alpha)}\mathcal{K}_{n-1}(z,y) - K_n(z,y) = \sum_{k=0}^{n-1}\lambda_{n,k}(z-\alpha)\psi_k(z) + \lambda_{n,n}K_n(z,\alpha). \tag{9}$$

From the reproducing property of the kernel polynomial, for $k < n$, we obtain

$$\begin{aligned}\lambda_{n,k} &= \int_{\mathbb{T}}(z-\alpha)\overline{(y-\alpha)}\mathcal{K}_{n-1}(z,y)\overline{(z-\alpha)\psi_k(z)}\,\mathrm{d}\mu - \int_{\mathbb{T}}K_n(z,y)\overline{(z-\alpha)\psi_k(z)}\,\mathrm{d}\mu\\ &= \int_{\mathbb{T}}\overline{(y-\alpha)}\mathcal{K}_{n-1}(z,y)\overline{\psi_k(z)}\,\mathrm{d}\mu_1 - \overline{(y-\alpha)\psi_k(y)}\\ &= \overline{(y-\alpha)\psi_k(y)} - \overline{(y-\alpha)\psi_k(y)} = 0.\end{aligned}$$

On the other hand, if $z = \alpha$ in (9), then $\lambda_{n,n} = -K_n(y,\alpha)/K_n(\alpha,\alpha)$ and the result follows. ■

Using the previous lemma, we can give a new proof for the following result [14].

PROPOSITION 2.6 *If* $\mu \in \mathcal{N}$ *and* $\alpha \in \mathbb{T}$, *then* $\lim_{n\to\infty} |\phi_n^{(j)}(\alpha)|^2/K_n^{(j,j)}(\alpha,\alpha) = 0$ *for* $j = 0, 1, \ldots$

Proof Taking derivatives of order j in (7), we obtain

$$(z-\alpha)\psi_{n-1}^{(j)}(z) + j\psi_{n-1}^{(j-1)}(z) = \sqrt{\frac{K_{n-1}(\alpha,\alpha)}{K_n(\alpha,\alpha)}}\left(\phi_n^{(j)}(z) - \frac{\phi_n(\alpha)}{K_{n-1}(\alpha,\alpha)}K_n^{(j,0)}(z,\alpha)\right). \tag{10}$$

On the other hand, from (8)

$$\begin{aligned}
&(z-\alpha)\overline{(y-\alpha)}\mathcal{K}_{n-1}^{(j,j)}(z,y) + j(z-\alpha)\mathcal{K}_{n-1}^{(j,j-1)}(z,y) + j\overline{(y-\alpha)}\mathcal{K}_{n-1}^{(j-1,j)}(z,y) + j^2\mathcal{K}_{n-1}^{(j-1,j-1)}(z,y)\\
&\quad = K_n^{(j,j)}(z,y) - \frac{K_n^{(j,0)}(z,\alpha)K_n^{(j,0)}(z,\alpha)}{K_n(\alpha,\alpha)}.
\end{aligned}$$

Evaluating the above expression in $z = \alpha$ and $y = \alpha$, we get

$$j^2\mathcal{K}_{n-2}^{(j-1,j-1)}(\alpha,\alpha) = K_{n-1}^{(j,j)}(\alpha,\alpha) - \frac{|K_{n-1}^{(j,0)}(\alpha,\alpha)|^2}{K_{n-1}(\alpha,\alpha)} \leqslant K_{n-1}^{(j,j)}(\alpha,\alpha).$$

The evaluation of (10) at $z = \alpha$ yield

$$\frac{|\psi_n^{(j-1)}(\alpha)|^2}{j\mathcal{K}_{n-2}^{(j-1,j-1)}(\alpha,\alpha)} \geqslant \frac{K_{n-1}(\alpha,\alpha)}{K_n(\alpha,\alpha)}\frac{\left|\phi_n^{(j)}(\alpha) - (\phi_n(\alpha)/K_{n-1}(\alpha,\alpha))K_n^{(j,0)}(\alpha,\alpha)\right|^2}{K_{n-1}^{(j,j)}(\alpha,\alpha)}. \tag{11}$$

Under the assumptions of the theorem, $\lim_{n\to\infty}(|\phi_n(\alpha)|^2/K_{n-1}(\alpha,\alpha)) = 0$. Thus, $\lim_{n\to\infty}(K_{n-1}(\alpha,\alpha)/K_n(\alpha,\alpha)) = 1$. Furthermore, using the Cauchy–Schwarz inequality, we obtain $|K_{n-1}^{(j,0)}(\alpha,\alpha)|^2 \leq K_{n-1}^{(j,j)}(\alpha,\alpha)K_{n-1}(\alpha,\alpha)$ and, therefore,

$$\frac{|\phi_n(\alpha)|}{K_{n-1}(\alpha,\alpha)}\frac{|K_{n-1}^{(j,0)}(\alpha,\alpha)|}{\sqrt{K_{n-1}^{(j,j)}(\alpha,\alpha)}} \leqslant \frac{|\phi_n(\alpha)|}{\sqrt{K_{n-1}(\alpha,\alpha)}}.$$

Note that the right-hand side of the previous inequality vanishes when n tends to infinity, so the limit of the left-hand side also vanishes. Thus, taking limits in (11) when n tends to infinity, we obtain

$$\lim_{n\to\infty}\frac{|\psi_n^{(j-1)}(\alpha)|^2}{j\mathcal{K}_{n-2}^{(j-1,j-1)}(\alpha,\alpha)} \geqslant \lim_{n\to\infty}\frac{|\phi_n^{(j)}(\alpha)|^2}{K_{n-1}^{(j,j)}(\alpha,\alpha)}.$$

On the other hand (see [10]), if $\mu \in \mathcal{N}$, then $\mu_1 \in \mathcal{N}$, and thus $\lim_{n\to\infty}(|\psi_n(\alpha)|^2/\mathcal{K}_{n-1}(\alpha,\alpha)) = 0$. Setting $j = 1$ in the previous inequality, we obtain $\lim_{n\to\infty}(|\phi_n'(\alpha)|^2/K_{n-1}^{(1,1)}(\alpha,\alpha)) = 0$. The result follows by induction. ■

LEMMA 2.7 *If* $\mu \in \mathcal{S}$, *then*, *uniformly in* $\mathbb{C} \setminus \mathbb{D}$,

$$\lim_{n\to\infty}\frac{\phi_n^{(j)}(z)}{n(n-1)\cdots(n-j+1)z^{n-j}} = D(z), \quad j = 0, 1, \ldots.$$

Proof Taking $p(z) = \phi_n(z)$ and $q(z) = z^n$ in Lemma 2.1, the result follows. ■

3. Relations between orthonormal families

Now, we turn our attention to the study of the Sobolev inner product (3). Let us denote by $\{\varphi_n\}_{n\geqslant 0}$ its corresponding sequence of orthonormal polynomials. The following result states the relation between the sequences $\{\varphi_n\}_{n\geqslant 0}$ and $\{\phi_n\}_{n\geqslant 0}$.

PROPOSITION 3.1 *Let* $\phi_n(z) = \alpha_n z^n + \cdots$ *and* $\varphi_n(z) = \beta_n z^n + \cdots$ *with* $\alpha_n, \beta_n > 0$. *Then*, $\{\varphi_n\}_{n\geqslant 0}$ *is the sequence of polynomials orthonormal with respect to* (3) *if and only if*

$$\varphi_n(z) = \frac{\beta_n}{\alpha_n}\phi_n(z) - \lambda\varphi_n^{(j)}(\alpha)K_{n-1}^{(0,j)}(z,\alpha), \quad j = 0, 1, \ldots, \tag{12}$$

with

$$\frac{\beta_n}{\alpha_n} = \sqrt{\frac{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)}} \quad \textit{and} \quad \varphi_n^{(j)}(\alpha) = \frac{\phi_n^{(j)}(\alpha)}{\sqrt{(1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha))(1+\lambda K_n^{(j,j)}(\alpha,\alpha))}}. \tag{13}$$

Proof Assume that $\{\varphi_n\}_{n\geqslant 0}$ is the sequence of polynomials orthonormal with respect to (3). The Fourier expansion of $\varphi_n(z)$ in terms of the orthonormal basis $\{\phi_n\}_{n\geqslant 0}$ is

$$\varphi_n(z) = \sum_{k=0}^{n} \lambda_{n,k}\phi_k(z), \tag{14}$$

where the coefficients $\lambda_{n,k}$ are given

$$\lambda_{n,k} = \langle \varphi_n(z), \phi_k(z)\rangle_S - \lambda\varphi_n^{(j)}(\alpha)\overline{\phi_k^{(j)}(\alpha)} = -\lambda\varphi_n^{(j)}(\alpha)\overline{\phi_k^{(j)}(\alpha)}, \tag{15}$$

if $0 \leqslant k \leqslant n-1$. If $k = n$, we can compare the leading coefficients to find that $\lambda_{n,n} = \beta_n/\alpha_n$, and (12) follows. From (14), we also obtain

$$\int_{\partial\mathbb{D}} |\varphi_n(z)|^2 \mathrm{d}\mu(z) = \sum_{k=0}^{n} |\lambda_{n,k}|^2 = \left(\frac{\beta_n}{\alpha_n}\right)^2 + \lambda^2|\varphi_n^{(j)}(\alpha)|^2 K_{n-1}^{(j,j)}(\alpha,\alpha). \tag{16}$$

On the other hand,

$$\int_{\partial\mathbb{D}} |\varphi_n(z)|^2 \mathrm{d}\mu(z) = 1 - \lambda|\varphi_n^{(j)}(\alpha)|^2. \tag{17}$$

Using (16) and (17), we obtain

$$|\varphi_n^{(j)}(\alpha)|^2 = \frac{1-(\beta_n/\alpha_n)^2}{\lambda(1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha))}. \tag{18}$$

Another way to obtain the expression for $\varphi_n^{(j)}(\alpha)$ is to take derivatives in (12) and evaluate at $z = \alpha$. In such a way that

$$\varphi_n^{(j)}(\alpha) = \frac{(\beta_n/\alpha_n)\phi_n^{(j)}(\alpha)}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}. \tag{19}$$

Thus, (18) and (19) yield

$$\left(\frac{\beta_n}{\alpha_n}\right)^2 = \frac{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)},$$

which is the first equation in (13). Inserting it either in (18) or in (19), we obtain the second equation in (13).

Conversely, define $\{\varphi_n\}_{n\geqslant 0}$ as in (12). To show the orthogonality, let $k < n$ and put

$$\langle \varphi_n(z), \varphi_k(z)\rangle_{s'} = \left\langle \frac{\beta_n}{\alpha_n}\phi_n(z) - \lambda\varphi_n^{(j)}(\alpha)K_{n-1}^{(0,j)}(z,\alpha), \varphi_k(\varphi)\right\rangle_{\mu} + \lambda\varphi_n^{(j)}(\alpha)\overline{\varphi_k^{(j)}(\alpha)}$$
$$= -\lambda\varphi_n^{(j)}(\alpha)\overline{\varphi_k^{(j)}(\alpha)} + \lambda\varphi_n^{(j)}(\alpha)\overline{\varphi_k^{(j)}(\alpha)} = 0.$$

On the other hand,

$$\langle \varphi_n(z), \varphi_n(z)\rangle_{s'} = \frac{\beta_n}{\alpha_n}\langle \phi_n(z), \varphi_n(z)\rangle_{\mu} - \lambda\varphi_n^{(j)}(\alpha)\langle K_{n-1}^{(0,j)}(z,\alpha), \varphi_n(z)\rangle_{\mu} + \lambda\varphi_n^{(j)}(\alpha)\overline{\varphi_n^{(j)}(\alpha)}$$
$$= \frac{\beta_n^2}{\alpha_n^2} + \lambda^2 K_{n-1}^{(j,j)}(\alpha,\alpha)|\varphi_n^{(j)}(\alpha)|^2 + \lambda|\varphi_n^{(j)}(\alpha)|^2 = 1. \qquad \blacksquare$$

Now, let $\{c_{m,n}\}_{m,n\geqslant 0}$, with $c_{m,n} = c_{m-n} = \int_{\mathbb{T}} z^{m-n}\,\mathrm{d}\mu(z)$, be the sequence of moments associated with μ and let $\mathbf{T} = (c_{m,n})_{m,n\geqslant 0}$ be its corresponding Gram matrix, which is a Toeplitz matrix. If $\{\tilde{c}_{m,n}\}_{m,n\geqslant 0}$ are the moments associated with (3), then $\tilde{c}_{m,n} = c_{m-n}$ for $m < j$ or $n < j$. On the other hand, if $m, n \geqslant j$, then we have

$$\tilde{c}_{m,n} = c_{m-n} + \lambda\frac{m!}{(m-j)!}\alpha^{m-j}\frac{n!}{(n-j)!}\bar{\alpha}^{n-j}.$$

Note that when $|\alpha| = 1$, the previous expression becomes

$$\tilde{c}_{m,n} = c_{m-n} + \lambda\frac{m!}{(m-j)!}\frac{n!}{(n-j)!}\alpha^{m-n}.$$

It is straightforward to deduce that $\widetilde{\mathbf{T}} = (\tilde{c}_{m,n})_{m,n\geqslant 0}$, the infinite moment matrix associated with (3), can be expressed as

$$\widetilde{\mathbf{T}} = \mathbf{T} + \lambda\mathbf{Z}^j\mathbf{D}_j\mathbf{D}_\alpha\mathbf{D}_j\left(\mathbf{Z}^{\mathrm{t}}\right)^j,$$

where $\mathbf{Z}$ is the shift matrix with 1's on the first lower-diagonal and 0's on the remaining entries, $\mathbf{Z}^{\mathrm{t}}$ is its transpose, $\mathbf{D}_j = \operatorname{diag}\{j!/0!, (j+1)!/1!, (j+2)!/2!, \ldots\}$, and $\mathbf{D}_\alpha$ is the Toeplitz matrix

$$\mathbf{D}_\alpha = \begin{pmatrix} 1 & \bar{\alpha} & \bar{\alpha}^2 & \ldots \\ \alpha & 1 & \bar{\alpha} & \\ \alpha^2 & \alpha & 1 & \\ \vdots & & & \ddots \end{pmatrix}.$$

In other words, $\tilde{\mathbf{T}}$ is a perturbation of $\mathbf{T}$ consisting on the addition of a shifted α-dependent Toeplitz matrix. If $|\alpha| \neq 1$, then

$$\tilde{\mathbf{T}} = \mathbf{T} + \lambda\mathbf{Z}^j\overline{\mathbf{V}}_\alpha^{\mathrm{t}}\mathbf{V}_\alpha(\mathbf{Z}^{\mathrm{t}})^j,$$

where $\mathbf{V}_\alpha = (j!/0!, ((j+1)!/1!)\bar{\alpha}^1, ((j+2)!/2!)\bar{\alpha}^2, \ldots)$.

4. Asymptotics

4.1. *Relative asymptotics*

4.1.1. *Case* $\alpha \in \mathbb{C}\backslash\overline{\mathbb{D}}$

The following lemma gives necessary conditions for μ to belong to the $\mathcal{N}$ class in terms of the limit of the ratio of the leading coefficients of $\{\varphi_n\}_{n\geqslant 0}$ and $\{\phi_n\}_{n\geqslant 0}$.

LEMMA 4.1 *Let* $\alpha \in \mathbb{C}\backslash\overline{\mathbb{D}}$. *If* $\mu \in \mathcal{N}$, *then* $\lim_{n\to\infty} \beta_n/\alpha_n = 1/|\alpha|$.

Proof Assume $\mu \in \mathcal{N}$. Using (13) and Proposition 5,

$$\lim_{n\to\infty} \frac{\beta_n^2}{\alpha_n^2} = \lim_{n\to\infty} \frac{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)} = \lim_{n\to\infty} \frac{|\phi_{n-1}^{(j)}(\alpha)|^2}{|\phi_n^{(j)}(\alpha)|^2} = \frac{1}{|\alpha|^2},$$

and the lemma is proved. ∎

As a straightforward consequence, we have the following:

COROLLARY 4.2 *If* $\mu \in \mathcal{N}$, $\Psi_n = \varphi_n/\beta_n$ *and* $\alpha \in \mathbb{C}\setminus\mathbb{D}$, *then* $\lim_{n\to\infty}(\|\Psi_n\|_S/\|\Phi_n\|_\mu) = |\alpha|$.

Using the previous lemma, we will prove the relative asymptotics on $\mathbb{C}\backslash\overline{\mathbb{D}}$.

PROPOSITION 4.3 *If* $\mu \in \mathcal{N}$ *and* $\alpha \in \mathbb{C}\backslash\overline{\mathbb{D}}$, *then*

$$\lim_{n\to\infty} \left(\frac{\varphi_n^{(k)}}{\phi_n^{(k)}}\right)(z) = \frac{\bar{\alpha}(z-\alpha)}{|\alpha|(\bar{\alpha}z-1)}, \quad k=0,1,\ldots \tag{20}$$

uniformly for z *on every compact subset of* $\mathbb{C}\backslash\overline{\mathbb{D}}$.

Proof From (12), we have

$$\frac{\varphi_n^{(k)}}{\phi_n^{(k)}}(z) = \frac{\beta_n}{\alpha_n} - \lambda\varphi_n^{(j)}(\alpha)\overline{\phi_n^{(j)}(\alpha)}\frac{K_{n-1}^{(k,j)}(z,\alpha)}{\phi_n^{(k)}(z)\overline{\phi_n^{(j)}(\alpha)}}. \tag{21}$$

In Proposition 7, we proved that

$$\lim_{n\to\infty} \frac{K_{n-1}^{(k,j)}(z,\alpha)}{\phi_n^{(k)}(\zeta)\overline{\phi_n^{(j)}(\alpha)}} = \frac{1}{z\overline{\alpha}-1}, \tag{22}$$

and using (19) we obtain

$$\lim_{n\to\infty} \lambda\varphi_n^{(j)}(\alpha)\overline{\phi_n^{(j)}(\alpha)} = \left(|\alpha| - \frac{1}{|\alpha|}\right). \tag{23}$$

The outer relative asymptotics (20) follows letting n tends to infinity in (21), using Lemma 4.1, (22) and (23). ∎

Remark 2 Note that the two previous results are special cases of Theorem 2 in [6].

4.1.2. *Case* $\alpha \in \mathbb{T}$

To obtain the previous relative asymptotics for $\alpha \in \mathbb{T}$, we need the following:

LEMMA 4.4 *If* $\mu \in \mathcal{N}$ *and* $\alpha \in \mathbb{T}$, *then* $\lim_{n\to\infty}(\beta_n/\alpha_n) = 1$.

Proof Note that

$$\frac{\lambda|\phi_n^{(j)}(\alpha)|^2}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)} = 1 - \frac{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)}. \tag{24}$$

Since

$$0 \leqslant \frac{\lambda|\phi_n^{(j)}(\alpha)|^2}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} \leqslant \frac{|\phi_n^{(j)}(\alpha)|^2}{K_n^{(j,j)}(\alpha,\alpha)},$$

then taking limit when n tends to infinity in (24) and using Proposition 10, the result follows. ∎

PROPOSITION 4.5 *If $\mu \in \mathcal{N}$ and $\alpha \in \mathbb{T}$, then*

$$\lim_{n\to\infty} \left(\frac{\varphi_n^{(k)}}{\phi_n^{(k)}}\right)(z) = 1, \quad k = 0, 1 \ldots \tag{25}$$

uniformly in compact subsets of $\mathbb{C}\backslash\overline{\mathbb{D}}$.

Proof Since $K_{n-1}^{(k,k)}(z,z)/|\phi_n^{(k)}(z)|^2$ is bounded, the Cauchy–Schwarz inequality yields $|K_{n-1}^{(k,j)}(\alpha,z)|^2 \leq K_{n-1}^{(k,k)}(\alpha,\alpha)K_{n-1}^{(j,j)}(z,z)$ and, thus,

$$\frac{|K_{n-1}^{(k,j)}(\alpha,z)|^2}{|\phi_n^{(k)}(\alpha)|^2|\phi_n^{(j)}(z)|^2} \leq C\frac{K_{n-1}^{(j,j)}(\alpha,\alpha)}{|\phi_n^{(j)}(\alpha)|^2}.$$

Taking limit when n tends to infinity in the previous inequality, we obtain

$$\lim_{n\to\infty} \frac{|K_{n-1}^{(k,j)}(\alpha,z)|^2}{|\phi_n^{(k)}(\alpha)|^2|\phi_n^{(j)}(z)|^2} = 0.$$

On the other hand, $\lim_{n\to\infty} \varphi_n^{(j)}(\alpha)\overline{\phi_n^{(j)}(\alpha)} = 0$. Therefore, using (21), $\lim_{n\to\infty}(\varphi_n^{(k)}/\phi_n^{(k)})(z) = \lim_{n\to\infty}(\beta_n/\alpha_n)$ and the result follows. ∎

PROPOSITION 4.6 *Let $\mu \in \mathcal{N}$ and $\alpha \in \mathbb{T}$. If $C_1 \leq |\phi_n(\alpha)| \leq C_2$ with $C_1, C_2 > 0$, $n \geq 0$, and*

$$\lim_{n\to\infty} \frac{|(\phi_n^*)^{(i)}(\alpha)|}{n^i} = 0, \quad i = 1, 2, \ldots. \tag{26}$$

Then,

$$D_1 \leq \liminf_{n\to\infty} \frac{|(\varphi_n^*)'(z)|}{n} \leq \limsup_{n\to\infty} \frac{|(\varphi_n^*)'(z)|}{n} \leq D_2.$$

Proof Using (3.1) and (13), we have

$$\begin{aligned}\frac{\alpha_n}{\beta_n}\varphi_n(z) &= \phi_n(z) - \lambda\frac{\alpha_n}{\beta_n}\varphi_n^{(j)}(\alpha)K_{n-1}^{(0,j)}(z,\alpha)\\ &= \phi_n(z) - \frac{\alpha_n}{\beta_n}\frac{\lambda\phi_n^{(j)}(\alpha)}{\sqrt{(1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha))(1+\lambda K_n^{(j,j)}(\alpha,\alpha))}}K_{n-1}^{(0,j)}(z,\alpha)\\ &= \phi_n(z) - \frac{\lambda\phi_n^{(j)}(\alpha)}{1+K_{n-1}^{(j,j)}(\alpha,\alpha)}K_{n-1}^{(0,j)}(z,\alpha).\end{aligned}$$

Applying in the previous equality the reversed operator $*$ and taking i derivatives, we obtain

$$r_n(\varphi_n^*)^{(i)}(z) = (\phi_n^*)^{(i)}(z) - \frac{\lambda\phi_n^{(j)}(\alpha)}{1+K_n^{(j,j)}(\alpha,\alpha)}\sum_{k=0}^{n-1}(z^{n-k}\phi_k^*(z))^{(i)}\phi_k^{(j)}(\alpha).$$

According to the Leibniz rule, we obtain

$$r_n(\varphi_n^*)^{(i)}(z) = (\phi_n^*)^{(i)}(z) + \sum_{l=0}^{i} a_l(z,\alpha),$$

where

$$a_l(z,\alpha) = -\frac{\lambda\phi_n^{(j)}(\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)}\sum_{k=0}^{n-1}\frac{i!}{l!(i-l)!}\frac{(n-k)!}{(n-k-l+1)!}z^{n-k-l}(\phi_k^*)^{(i-l)}(z)\phi_k^{(j)}(\alpha).$$

Using (26), we obtain

$$\frac{\lambda C_1}{1+n\lambda C_2^2} \le \frac{\lambda|\phi_n^{(j)}(\alpha)|}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)} \le \frac{\lambda C_2}{1+n\lambda C_1^2}. \tag{27}$$

Therefore, from (27), for $l < i$, we have

$$\frac{|a_l(\alpha,\alpha)|}{n^l} \le \frac{k\lambda C_2^2}{n^l(1+n\lambda C_1^2)}\frac{i!}{l!(i-l)!}\frac{(n-k)!}{(n-k-l+1)!}\sum_{k=0}^{n-1}\left|\frac{(\phi_k^*)^{(i-l)}(\alpha)}{k}\right|,$$

and thus

$$\lim_{n\to\infty}\frac{|a_i(z,\alpha)|}{n^l} = 0.$$

On the other hand, for $i = l$, we have

$$C_1^2 l\sum_{k=0}^{n-1}k^j\frac{(n-k)!}{(n-k-l+1)!} \le \sum_{k=0}^{n-1}\frac{(n-k)!}{(n-k-l+1)!}|\phi_k^*(\alpha)\phi_k^{(j)}(\alpha)| \le C_2^2 u\sum_{k=0}^{n-1}k^j\frac{(n-k)!}{(n-k-l+1)!}.$$

Thus,

$$D_1 \le \liminf_{n\to\infty}\frac{|a_l(z,\alpha)|}{n^l} \le \limsup_{n\to\infty}\frac{|a_l(z,\alpha)|}{n^l} \le D_2,$$

and since

$$\limsup_{n\to\infty}\frac{|(\varphi_n^*)'(z)|}{n^l} \le \limsup_{n\to\infty}\frac{|a_l(z,\alpha)|}{n^l},$$

the result follows. ■

COROLLARY 4.7 $\{\varphi_n\}_{n\geqslant 0}$ *does not have normal derivative behaviour.*

4.1.3. *Case* $\alpha \in \mathbb{D}$

If $\mu \in \mathcal{N}$, the behaviour of the sequence β_n/α_n with $\alpha \in \mathbb{D}$ should tend to infinity. For this reason, we will consider in this subsection $\mu \in \mathcal{S}$.

LEMMA 4.8 *If* $\mu \in \mathcal{S}$ *and* $\alpha \in \mathbb{D}$, *then* $\lim_{n\to\infty}(\beta_n/\alpha_n) = 1$.

Proof It is a straightforward consequence of (13). ■

4.2. *Inner strong asymptotic*

4.2.1. *Case* $\alpha \in \mathbb{C}\backslash\mathbb{D}$

PROPOSITION 4.9 *If* $\mu \in \mathcal{S}$ *and* $\alpha \in \mathbb{C}\backslash\mathbb{D}$, *then*

$$\lim_{n\to\infty} \frac{\varphi_n^{(k)}(z)}{n^k z^n} = \overline{D\left(\frac{1}{\overline{z}}\right)} \lim_{n\to\infty} \left(\frac{\varphi_n^{(k)}}{\phi_n^{(k)}}\right)(z)$$

uniformly for z *on every compact subset of* $\mathbb{C}\backslash(\mathbb{D}\cup\{\alpha\})$.

Proof For $k,j = 0,1,\ldots,$ we have

$$\begin{aligned} K_n^{(k,j)}(z,\zeta) &= \overline{\frac{\partial^j}{\partial \zeta^j}\overline{\frac{\partial^k}{\partial z^k}\left(\frac{\phi_n^*(z)\overline{\phi_n^*(\zeta)} - \phi_n(z)\overline{\phi_n(\zeta)}}{1 - z\overline{\zeta}}\right)}} \\ &= \sum_{l=0}^{k}\sum_{r=0}^{j} C_k^j C_j^r \overline{\frac{\partial^{j-r}}{\partial \zeta^{j-r}}\overline{\frac{\partial^{k-l}}{\partial z^{k-l}}\frac{1}{1 - z\overline{\zeta}}}}(\phi_n^{*(l)}(z)\overline{\phi_n^{*(r)}(\zeta)} - \phi_n^{(l)}(z)\overline{\phi_n^{(r)}(\zeta)}). \end{aligned}$$

Therefore,

$$\frac{K_n^{(k,j)}(z,\alpha)}{z^n\overline{\phi_n^{(j)}(\alpha)}} = \sum_{l=0}^{k}\sum_{r=0}^{j} C_k^j C_j^r \overline{\frac{\partial^{j-r}}{\partial \zeta^{j-r}}\overline{\frac{\partial^{k-l}}{\partial z^{k-l}}\frac{1}{1 - z\overline{\zeta}}}}\left(\frac{\phi_n^{*(l)}(z)}{z^n}\frac{\overline{\phi_n^{*(r)}(\alpha)}}{\phi_n^{(j)}(\alpha)} - \frac{\phi_n^{(l)}(z)}{z^n}\frac{\overline{\phi_n^{(r)}(\alpha)}}{\phi_n^{(j)}(\alpha)}\right)$$

and the result follows using Lemma 13, Proposition 14, (12) and (23). ■

5. Zeros

In this section, we analyse the behaviour of the zeros of the orthogonal polynomials with respect to (3), when λ tends to infinity, for $j = 0$ (Uvarov's case) and $j = 1$. Denote by Ψ_n, $\Phi_n(z, \mathrm{d}\mu_1)$, and $\Phi_n(z, \mathrm{d}\mu_2)$ the nth monic orthogonal polynomial with respect to (3), $\mathrm{d}\mu_1$ and $\mathrm{d}\mu_2$, respectively, where $\mathrm{d}\mu_2 = |z-\alpha|^4\mathrm{d}\mu$, i.e. the product of two Christoffel transformations of $\mathrm{d}\mu$. Note that the monic expressions of (3.1) and (7) are, respectively,

$$\Psi_n(z) = \Phi_n(z) - \frac{\lambda \Phi_n^{(j)}(\alpha)}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} K_{n-1}^{(0,j)}(z,\alpha), \tag{28}$$

$$(z-\alpha)\Phi_{n-1}(z, \mathrm{d}\mu_1) = \Phi_n(z) - \frac{\Phi_n(\alpha)}{K_{n-1}(\alpha,\alpha)} K_{n-1}(z,\alpha). \tag{29}$$

Now, if $j = 0$, using the last two equations, we obtain

$$\Psi_n(z) = \frac{1}{\lambda K_{n-1}(\alpha,\alpha)} \Phi_n(z) + \frac{\lambda K_{n-1}(\alpha,\alpha)}{1+\lambda K_{n-1}(\alpha,\alpha)} (z-\alpha)\Phi_{n-1}(z, \mathrm{d}\mu_1).$$

Thus,

$$\lim_{\lambda\to\infty} \Psi_n(z) = (z-\alpha)\Phi_{n-1}(z, \mathrm{d}\mu_1).$$

This means that, for a fixed n and $\lambda \to \infty$, $n-1$ zeros of Ψ_n tend to the zeros of $\Phi_{n-1}(z, \mathrm{d}\mu_1)$, and the remaining zero tends to $z = \alpha$. Note that this is the scalar case of Theorem 4.6 in [10].

On the other hand, for the monic polynomials orthogonal with respect to $\mathrm{d}\mu_2$, we have (see [10])

$$(z-\alpha)^2\Phi_{n-2}(z, \mathrm{d}\mu_2) = \frac{1}{\Delta_{1,1}} \begin{vmatrix} \Phi_n(z) & K_{n-1}(z,\alpha) & K_{n-1}^{(0,1)}(z,\alpha) \\ \Phi_n(\alpha) & K_{n-1}(\alpha,\alpha) & K_{n-1}^{(0,1)}(\alpha,\alpha) \\ \Phi_n'(\alpha) & K_{n-1}^{(1,0)}(\alpha,\alpha) & K_{n-1}^{(1,1)}(\alpha,\alpha) \end{vmatrix}, \tag{30}$$

where $\Delta_{1,j}$, $j = 1,2,3$, denotes the minor of the $1,j$ entry of the above 3×3 matrix. Thus,

$$K_{n-1}^{(0,1)}(z,\alpha) = \frac{\Delta_{1,1}}{\Delta_{1,3}}[(z-\alpha)^2\Phi_{n-2}(z, \mathrm{d}\mu_2) - \Phi_n(z)] + \frac{\Delta_{1,2}}{\Delta_{1,3}} K_{n-1}(z,\alpha),$$

and, from (29),

$$K_{n-1}(z,\alpha) = \frac{K_{n-1}(\alpha,\alpha)}{\Phi_n(\alpha)}(\Phi_n(z) - (z-\alpha)\Phi_{n-1}(z, \mathrm{d}\mu_1)).$$

Therefore, if $j = 1$, then using the last two equations, (28) becomes

$$\begin{aligned} \Psi_n(z) = {} & \left[1 + \frac{\lambda\Phi_n'(\alpha)}{1+\lambda K_{n-1}^{(1,1)}(\alpha,\alpha)} \left(\frac{\Delta_{1,1}}{\Delta_{1,3}} - \frac{\Delta_{1,2}}{\Delta_{1,3}} \frac{K_{n-1}(\alpha,\alpha)}{\Phi_n(\alpha)}\right)\right]\Phi_n(z) \\ & + \frac{\lambda\Phi_n'(\alpha)}{1+\lambda K_{n-1}^{(1,1)}(\alpha,\alpha)} \frac{\Delta_{1,2}}{\Delta_{1,3}} \frac{K_{n-1}(\alpha,\alpha)}{\Phi_n(\alpha)} (z-\alpha)\Phi_{n-1}(z, \mathrm{d}\mu_1) \\ & - \frac{\lambda\Phi_n'(\alpha)}{1+\lambda K_{n-1}^{(1,1)}(\alpha,\alpha)} \frac{\Delta_{1,1}}{\Delta_{1,3}} (z-\alpha)^2\Phi_{n-2}(z, \mathrm{d}\mu_2). \end{aligned}$$

Simplifying and taking limit when $\lambda \to \infty$,

$$\begin{aligned} \lim_{\lambda\to\infty} \Psi_n(z) = {} & \frac{\Delta_{1,2}}{K_{n-1}^{(1,1)}(\alpha,\alpha)\Phi_n(\alpha)}\Phi_n(z) + \frac{\Phi_n'(\alpha)K_{n-1}(\alpha,\alpha)\Delta_{1,2}}{\Phi_n(\alpha)K_{n-1}^{(1,1)}(\alpha,\alpha)\Delta_{1,3}}(z-\alpha)\Phi_{n-1}(z, \mathrm{d}\mu_1) \\ & - \frac{\Phi_n'(\alpha)\Delta_{1,1}}{K_{n-1}^{(1,1)}(\alpha,\alpha)\Delta_{1,3}}(z-\alpha)^2\Phi_{n-2}(z, \mathrm{d}\mu_2), \end{aligned}$$

and, as a consequence, the zeros of $\Psi_n(z)$ tend to the zeros of a linear combination of $\Phi_n(z)$, $(z-\alpha)\Phi_{n-1}(z, \mathrm{d}\mu_1)$ and $(z-\alpha)^2\Phi_{n-2}(z, \mathrm{d}\mu_2)$ when $\lambda \to \infty$.

Remark 3 Note that, according to Theorem 4.6 in [10], $n-2$ zeros of the polynomial Ψ_n orthogonal with respect to the Sobolev inner product

$$\langle f,g\rangle_{S_1} := \int_{\mathbb{T}} f(z)\overline{g(z)}\mathrm{d}\mu(z) + \lambda_0 f(\alpha)\overline{g(\alpha)} + \lambda_1 f'(\alpha)\overline{g'(\alpha)}, \quad \alpha \in \mathbb{C},\ \lambda \in \mathbb{R}^+\backslash\{0\},\ j \in \mathbb{N},$$

tend to the zeros of the polynomial orthogonal $\Phi_{n-2}(x, \mathrm{d}\mu_2)$ when both λ_0 and λ_1 tend to ∞ and the other two zeros of Ψ_n tend to α. This is not the case for the orthogonal polynomials with respect to (3), as we have just proved.

5.1. *Examples*

We show some basic examples to exhibit the behaviour of the zeros of the orthogonal polynomials associated with perturbations of the form (3) for two special examples of probability measures on the unit circle: the Lebesgue and Bernstein–Szegő measures. Note that, in contrast with the real line case [4], there is not a well-developed theory on the unit circle. For the first one, it is very

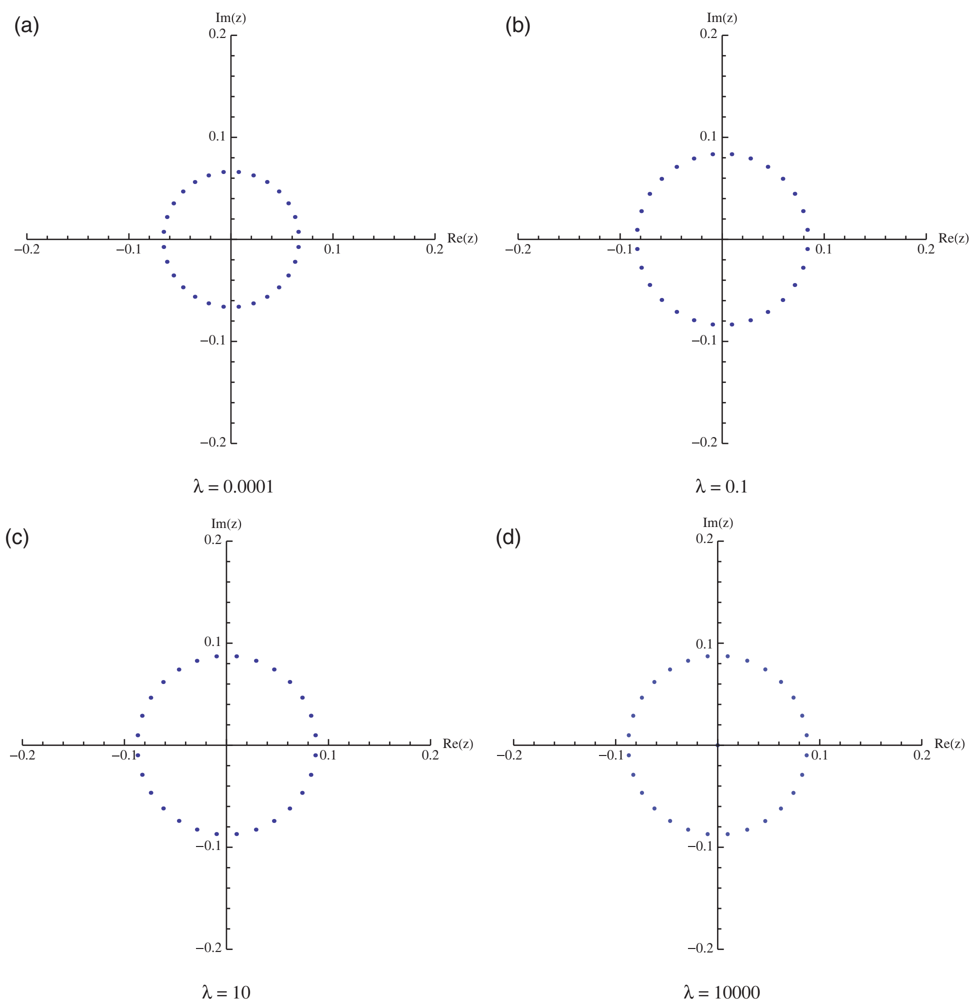

Figure 1. Lebesgue case, $n = 30, j = 2, \alpha = 0.05 + 0.05\mathrm{i}$.

well known that its corresponding monic orthogonal polynomial sequence is $\phi_n(z) = z^n$, $n \geqslant 0$. Thus, $\phi_n^{(j)}(\alpha) = (n!/(n-j)!)\alpha^{n-j}$,

$$K_{n-1}^{(0,j)}(z,\alpha) = \sum_{k=j}^{n-1} \frac{k!}{(k-j)!} z^k \bar{\alpha}^{k-j}, \quad K_n^{(j,j)}(\alpha,\alpha) = \sum_{k=j}^{n} \left(\frac{k!}{(k-j)!}\right)^2 |\alpha|^{2(k-j)},$$

and we can calculate explicitly $\{\varphi_n\}_{n\geqslant 0}$ from (12). We obtained some numerical estimates about the behaviour of the zeros of φ_n when λ varies. Figure 1 shows the zeros of φ_{30} for fixed values of n, j and α and different values of λ. For $\lambda \geqslant 10$, the zeros have almost no variation. Thus, for fixed n, j and α, the zeros of φ_n converge to the zeros of a fixed polynomial when λ tends to infinity.

For the Bernstein–Szegő measure, we have $\phi_n(z) = z^{n-1}(z - \bar{b})$, $n \geqslant 1$, where $|b| < 1$ is the parameter of the measure. Thus, we obtain

$$\phi_n^{(j)}(\alpha) = \frac{n!}{(n-j)!}\alpha^{n-j} - \bar{b}\frac{(n-1)!}{(n-j-1)!}\alpha^{n-j-1},$$

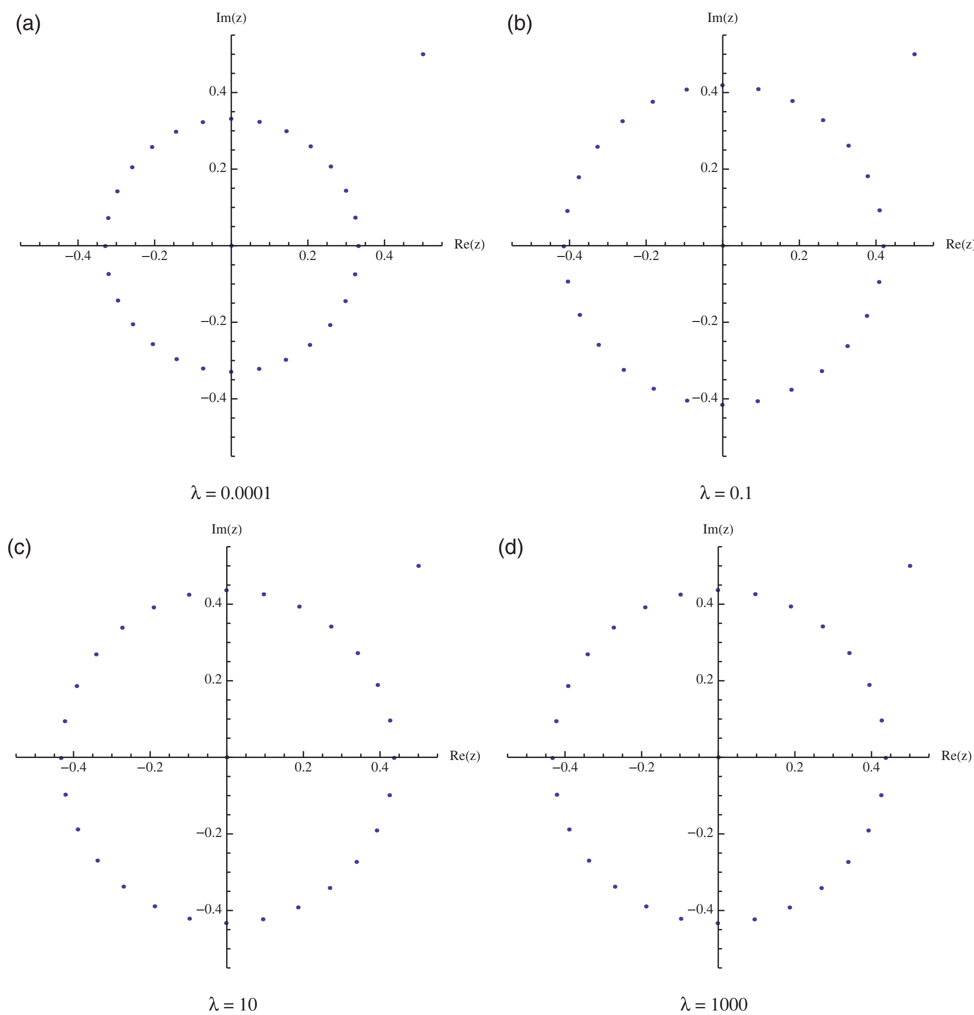

Figure 2. Bernstein–Szegő case, $n = 30$, $j = 2$, $\alpha = 0.25 + 0.25\mathrm{i}$, $\beta = 0.5 - 0.5\mathrm{i}$.

$$K_{n-1}^{(0,j)}(z,\alpha) = \sum_{k=j}^{n-1}(z^k - \bar{b}z^{k-1})\left(\frac{k!}{(k-j)!}\bar{\alpha}^{k-j} - \bar{b}\frac{(k-1)!}{(k-j-1)!}\bar{\alpha}^{k-j-1}\right),$$

$$K_n^{(j,j)}(\alpha,\alpha) = \sum_{k=j}^{n}\left|\frac{k!}{(k-j)!}\bar{\alpha}^{k-j} - \bar{b}\frac{(k-1)!}{(k-j-1)!}\bar{\alpha}^{k-j-1}\right|^2,$$

and, again, we obtain the expression of $\varphi_n(z)$ using (12). We performed a similar numerical analysis of the zeros of φ_n as a function of λ as in the Lebesgue case. Figure 2 shows the behaviour of the zeros of φ_{30} for fixed j, α and β.

For a general measure, the behaviour of the zeros of the polynomials orthogonal with respect to (3) as a function of λ constitutes an interesting open problem.

Acknowledgements

The work of the K.C. and F.M. was supported by Dirección General de Investigación, Ministerio de Ciencia e Inovación of Spain, grant MTM2009-12740-C03-01. The work of L.E.G. was supported by Consejo Nacional de Ciencia y Tecnología of México, grant 156668. K.C. thanks the Katholieke Universiteit Leuven for the hospitality during the visit in which part of this work was done.

References

[1] M. Alfaro, G. López, and M.L. Rezola, *Some properties of zeros of Sobolev-type orthogonal polynomials*, J. Comput. Appl. Math. 69 (1996), pp. 171–179.
[2] A. Branquinho, A. Foulquié, and F. Marcellán, *Asymptotic behavior of Sobolev-type orthogonal polynomials on a rectifiable Jordan curve or arc*, Constr. Approx. 18 (2002), pp. 161–182.
[3] K. Castillo, *A new approach to relative asymptotic behavior to Sobolev-type inner product on the unit circle*, Appl. Math. Lett. (2011), DOI:10.1016/j.aml.2011.11.015.
[4] K. Castillo, M.V. Mello, and F.R. Rafaeli, *Monotonicity and asymptotics of zeros of Sobolev type orthogonal polynomials: A general case*, 2011, submitted for publication.
[5] L. Daruis, J. Hernández, and F. Marcellán, *Spectral transformations for Hermitian Toeplitz matrices*, J. Comput. Appl. Math. 202 (2007), pp. 155–176.
[6] A. Foulquié, F. Marcellán, and K. Pan, *Asymptotic behavior of Sobolev-type orthogonal polynomials on the unit circle*, J. Approx. Theory 100 (1999), pp. 345–363.
[7] A. Foulquié Moreno, F. Marcellán, F. Peherstorfer, and R. Steinbauer, *Strong asymptotics on the support of the measure of orthogonality for polynomials orthogonal with respect to a discrete Sobolev inner product on the unit circle*, Rendi. Circ. Matem. Palermo 52 (1998), pp. 411–425.
[8] Y.L. Geronimus, *Orthogonal Polynomials*, Consultants Bureau, New York, 1961.
[9] A.A. Gonchar, *On the convergence of Padé approximants for some classes of meromorphic functions*, Math. USSR Sbornik 26 (1975), pp. 555–575.
[10] F. Marcellán and L. Moral, *Sobolev-type orthogonal polynomials on the unit circle*, Appl. Math. Comput. 128 (2002), pp. 329–363.
[11] F. Marcellán and F.R. Rafaeli, *Monotonicity and asymptotic of zeros of Laguerre–Sobolev-type orthogonal polynomials of higher derivatives*, Proc. Amer. Math. Soc. 139 (2011), pp. 3929–3936.
[12] A. Martínez-Finkelshtein and B. Simon, *Asymptotics of the L^2 Norm of Derivatives of OPUC*, J. Approx. Theory 163 (2011), pp. 747–773.
[13] A. Maté and P. Nevai, *Remarks on E.A. Rahmanov's paper on the asymptotics of the ratio of orthogonal polynomials*, J. Approx. Theory 36 (1982), pp. 64–72.
[14] A. Maté, P. Nevai, and V. Totik, *Extensions of Szegő's theory of orthogonal polynomials II*, Constr. Approx 3 (1987), pp. 51–72.
[15] E.A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polinomials II*, USSR Sbornik 53 (1983), pp. 105–117.
[16] B. Simon, *Orthogonal Polynomials on the Unit Circle* (2 volumes), American Mathematical Society Colloquium Publication Series, Vol. 54, American Mathematical Society, Providence, RI, 2005.
[17] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publication Series, Vol. 23, 4th ed., American Mathematical Society, Providence, RI, 1975.