ON $W^{1,p}$-CONVERGENCE OF FOURIER-SOBOLEV EXPANSIONS

FRANCISCO MARCELLÁN[(1)], YAMILET QUINTANA[(2)], AND ALEJANDRO URIELES

Abstract. Let $\{q_n\}_{n\geq 0}$ be the sequence of polynomials orthonormal with respect to the Sobolev inner product

$$\langle f, g\rangle_S := \int_{-1}^{1} f(x)g(x)w_0(x)dx + \int_{-1}^{1} f'(x)g'(x)w_1(x)dx,$$

where $w_0 \in L^\infty([-1,1])$ and w_1 is a weight of Kufner-Opic type. We study necessary and/or sufficient conditions for the convergence in the $W^{1,p}([-1,1],(w_0,w_1))$ norm of the Fourier expansion in terms of $\{q_n\}_{n\geq 0}$, with $1 < p < \infty$.

Key words and phrases: Sobolev orthogonal polynomials; weighted Sobolev spaces; Fourier expansions; Sobolev-Fourier expansions.

1. Introduction

Given $1 \leq p < \infty$, let $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ be the following weighted Sobolev space

$$\mathbb{W}^{1,p}([-1,1],(w_0,w_1)) := \{f : [-1,1] \to \mathbb{R} : f \in L^p([-1,1],w_0),\ f' \in L^p([-1,1],w_1)\},$$

with the norm

$$\|f\|^p_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} := \|f\|^p_{L^p([-1,1],w_0)} + \|f'\|^p_{L^p([-1,1],w_1)},$$

where $w_0 \in L^\infty([-1,1])$ and w_1 is a Kufner-Opic type weight (see definition 2.1).

For $f, g \in \mathbb{W}^{1,2}([-1,1],(w_0,w_1))$ we introduce the weighted Sobolev inner product

$$\langle f, g\rangle_S := \int_{-1}^{1} f(x)g(x)w_0(x)dx + \int_{-1}^{1} f'(x)g'(x)w_1(x)dx. \tag{1.1}$$

Let $\mathbb{P}$ be the space of the polynomials with real coefficients. In general it is not true that $\mathbb{P} \subseteq \mathbb{W}^{1,2}([-1,1],(w_0,w_1))$, but when it holds we can consider the sequence $\{q_n\}_{n\geq 0}$ of orthonormal polynomials with respect to (1.1) and for $f \in \mathbb{W}^{1,2}([-1,1],(w_0,w_1))$ its Fourier (or Fourier-Sobolev) expansion in terms of $\{q_n\}_{n\geq 0}$

$$\mathcal{S}f \sim \sum_{k=0}^{\infty} \tilde{f}(k) q_k, \tag{1.2}$$

where $\tilde{f}(k) = \langle f, q_k\rangle_S$, for $k \geq 0$.

Date: September 10, 2012.
2010 AMS Subject Classification. 42C10, 33C45.
(1) Supported in part by a grant from Ministerio de Ciencia e Innovación (MTM 2009-12740-C03-01), Spain.
(2) Supported in part by the Research Sabbatical Fellowship Program (2011-2012) from Universidad Simón Bolívar, Venezuela.

This definition of the Fourier-Sobolev expansion of f is purely formal and it is not obvious whether it converges to f. In fact, the solution of this problem can be very hard, or relatively easy, depending either of the sense of the convergence, or in terms of additional restrictions on f and the pair of weights (w_0, w_1), (see for instance, [10, 11, 17, 18, 20]).

The main goal of this work is to study necessary and/or sufficient conditions for the $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$-norm convergence of the Fourier-Sobolev expansion (1.2).

The structure of the paper it is as follows. Section 2 contains the notation as well as some basic background to be needed in the sequel. Section 3 is focused on the study of necessary and sufficient conditions for the $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$-norm convergence of the Fourier-Sobolev expansion (1.2) (see Theorems 3.1 and 3.2).

2. Previous definitions and notations

Following the ideas of [14], the authors in [33] and [34] introduced general classes of Sobolev spaces appearing in the context of orthogonal polynomials on the real line. We will use the approach given in these papers to establish the Kufner-Opic type property as follows.

Definition 2.1. *Let $1 \le p \le \infty$. A weight function w on $[a,b]$ is said to satisfy the Kufner-Opic type property (or belongs to $B_p([a,b])$ if and only if*

$$\begin{aligned} w^{-1} &\in L^{1/(p-1)}([a,b])\,, && \text{for } 1 \le p < \infty\,,\\ w^{-1} &\in L^{1}([a,b])\,, && \text{for } p = \infty\,. \end{aligned}$$

Also, if J is any interval we say that $w \in B_p(J)$ if $w \in B_p(I)$ for every compact interval $I \subseteq J$. We say that a weight belongs to $B_p(J)$, where J is a union of disjoint intervals $\cup_{i\in A} J_i$, if it belongs to $B_p(J_i)$, for $i \in A$.

Notice if $v \ge w$ in J and $w \in B_p(J)$, then $v \in B_p(J)$.

This class contains the classical Muckenhoupt A_p weights appearing in Harmonic Analysis (see [8]). Other properties of the class of weights of the Kufner-Opic type we will need in the sequel are contained in the following result.

Lemma 2.1. [33, Lemma 3.1] *Let us consider $1 \le p \le \infty$ and $w \in B_p((a,b))$. For any compact interval $I \subseteq (a,b)$, there is a positive constant c_1, which only depends on p, w, and I, such that*

$$\|g\|_{L^1(I)} \le c_1 \|g\|_{L^p(I,w)} \le c_1 \|g\|_{L^p([a,b],w)}\,, \qquad \text{for any } g \in L^p([a,b],w).$$

Furthermore, if $w \in B_p([a,b])$, then there is a positive constant c_2, which only depends on p and w, such that

$$\|g\|_{L^1([a,b])} \le c_2 \|g\|_{L^p([a,b],w)}\,, \qquad \text{for any } g \in L^p([a,b],w).$$

As a consequence, if $w \in B_p([a,b])$ and $f' \in L^p([a,b],w)$, then $f \in AC([a,b])$.

The proof of this Lemma will be not included here, however the reader can check it in [33].

Definition 2.2. *We denote by $AC([a,b])$ the set of absolutely continuous functions in $[a,b]$, i.e. the functions $f \in C([a,b])$ such that $f(x) - f(a) = \int_a^x f'(t)\,dt$ for every $x \in [a,b]$. If J is any interval, $AC_{loc}(J)$ denotes the set of absolutely continuous functions in every compact subinterval of J.*

For $1 \le p < \infty$, let us consider the weighted Sobolev space $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, given by

$$\mathbb{W}^{1,p}([-1,1],(w_0,w_1)) := \{f : [-1,1] \to \mathbb{R} : f \in L^p([-1,1],w_0),\ f' \in L^p([-1,1],w_1)\}$$

where $w_0 \in L^\infty([-1,1])$ and $w_1 \in B_p([-1,1])$. In [33] is showed that $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ with the norm

$$\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} := \left(\|f\|^p_{L^p([-1,1],w_0)} + \|f'\|^p_{L^p([-1,1],w_1)}\right)^{1/p}$$

is a Banach space.

Let $\mathbb{P}$ be the space of polynomials with real coefficients. In general it is not true that $\mathbb{P} \subset \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, however if we denote by $\mathbb{P}^{1,p}([-1,1],(w_0,w_1))$ the subset $\mathbb{P} \cap \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, then as a consequence of [34, Theorem 4.1] $\mathbb{P}^{1,p}([-1,1],(w_0,w_1))$ is dense in $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ and from Lemma 2.1 it follows in a straightforward way that $\mathbb{W}^{1,p}([-1,1],(w_0,w_1)) \subseteq AC([-1,1])$.

Notice that if $\mathbb{P} \subset L^p([-1,1],w_1)$, since $w_1 \in B_p([-1,1])$, then $w_1 \in A_p([-1,1])$. Also, when $p = 2$ and $w_1 \in L^1([-1,1])$, $\mathbb{W}^{1,2}([-1,1],(w_0,w_1))$ is a Hilbert space and we can consider the sequence of orthonormal polynomials $\{q_n\}_{n\ge 0}$ associated with the inner Sobolev inner product

$$\langle f,g\rangle_S := \int_{-1}^{1} f(x)g(x)w_0(x)dx + \int_{-1}^{1} f'(x)g'(x)w_1(x)dx. \tag{2.3}$$

With these remarks in mind, we can give the following definition.

Definition 2.3. *Let $\{q_n\}_{n\ge 0}$ be the sequence of orthonormal polynomials with respect to Sobolev inner product* (1.1)*. For $1 < p < \infty$ let us consider (w_0,w_1) a vector of weights such that $w_0 \in L^\infty([-1,1])$ and $w_1 \in A_p([-1,1])$. Let $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ and $x \in [-1,1]$, for each $n \ge 0$, we define the n-th Fourier-Sobolev partial sum*

$$\mathcal{S}_n(f,x) = \sum_{k=0}^{n} \hat{f}(k)q_k(x), \qquad \text{where } \hat{f}(k) = \langle f,q_k\rangle_S, \tag{2.4}$$

as well as the Fourier-Sobolev expansion of f by means the formal expression

$$\mathcal{S}f \sim \sum_{n=0}^{\infty} \tilde{f}(n)q_n, \tag{2.5}$$

In a similar way to the classical case, for each $n \ge 0$ the n-th Fourier-Sobolev partial sum (2.4) induces a linear operator $\mathcal{S}_n : \mathbb{W}^{1,p}([-1,1],(w_0,w_1)) \to \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, given by

$$(\mathcal{S}_n f)(x) := \mathcal{S}_n(f,x), \quad \text{for } x \in [-1,1].$$

3. $W^{1,p}$-convergence of Fourier-Sobolev expansions

The following result shows that under the conditions of the definition 2.3, the convergence in $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$-norm of the Fourier-Sobolev expansion (2.5) is equivalent to the uniform boundedness of the operator $\mathcal{S}_n$, for each n.

Theorem 3.1. *Let $\{q_n\}_{n\ge 0}$ be the sequence of orthonormal polynomials with respect to* (1.1)*. Let (w_0,w_1) be a pair of weight functions such that $w_0 \in L^\infty([-1,1])$ and $w_1 \in A_p([-1,1])$ for $1 < p < \infty$. Then the following conditions are equivalent.*

i) *$\mathcal{S}_n f \to f$ in $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, for all $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$.*

ii) *There exists $C>0$, independent of n, such that*

$$\|\mathcal{S}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq C\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}, \quad \forall f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1)).$$

Proof. i)⇒ii) Using the Hölder inequality,

$$\begin{aligned} |\hat{f}(k)| &= \left|\int_{-1}^{1} f(x)q_k(x)w_0(x)dx + \int_{-1}^{1} f'(x)q_k'(x)w_1(x)dx\right| \\ &\leq \|f\|_{L^p([-1,1],w_0)}\|q_k\|_{L^q([-1,1],w_0)} + \|f'\|_{L^p([-1,1],w_1)}\|q_k'\|_{L^q([-1,1],w_1)}. \end{aligned}$$

Then for each n, we have

$$\|\mathcal{S}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq \max(A_n, B_n)\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))},$$

where $A_n = \sum_{k=0}^{n} \|q_k\|_{L^q([-1,1],w_0)}$ and $B_n = \sum_{k=0}^{n} \|q_k'\|_{L^q([-1,1],w_1)}$. Consequently, $\mathcal{S}_n$ is a continuous operator for each n. Furthermore,

$$\|\mathcal{S}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq \|\mathcal{S}_n f - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} + \|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq \tilde{C}(f),$$

where $\tilde{C}(f)$ is a constant independent of n. Thus, $\sup_{n\in\mathbb{N}} \|\mathcal{S}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} < \infty$ and from Banach-Steinhaus theorem we obtain ii).

ii)⇒i) Since $w_0 \in L^\infty([-1,1])$ and $w_1 \in A_p([-1,1])$, $1<p<\infty$, then as a consequence of [34, Theorem 4.1] the linear space $\mathbb{P}$ is dense in $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$. Then, given $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ and $\varepsilon>0$, let $p(x) = \sum_{k=0}^{m} a_k q_k(x)$ such that $\|p-f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} < \varepsilon$.

Using that $\mathcal{S}_n p = p$, whenever $n \geq m$, we have

$$\begin{aligned} \|\mathcal{S}_n f - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} &\leq \|\mathcal{S}_n f - \mathcal{S}_n p\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} + \|\mathcal{S}_n p - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \\ &= \|\mathcal{S}_n (f-p)\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} + \|p-f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \\ &\leq (C+1)\|p-f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq (C+1)\varepsilon, \end{aligned}$$

and from these last inequalities we can deduce i).

□

The advantage of the previous result is that it allows us to work as in the case of $L^p[-1,1]$, where a similar condition to ii) is stated for studying necessary conditions for the mean convergence of the Fourier expansions in terms of classical orthogonal polynomials (see [3, 23, 25, 27, 28, 29, 30, 31], and more recently [16]).

When $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ is a Banach space, some of their properties can be easily deduced taking into account that $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ is a closed subspace of the cartesian product $L^p([-1,1],w_0) \times L^p([-1,1],w_1)$ with the norm

$$\begin{aligned} \|u\|_{L^p([-1,1],w_0)\times L^p([1,1],w_1)} &= \|(u_1,u_2)\|_{L^p([-1,1],w_0)\times L^p([-1,1],w_1)} \\ &:= \begin{cases} \left(\|u_1\|^p_{L^p([-1,1],w_0)} + \|u_2\|^p_{L^p([-1,1],w_1)}\right)^{1/p}, & 1 \leq p < \infty, \\ \max\left\{\|u_1\|_{L^\infty([-1,1],w_0)}, \|u_2\|_{L^\infty([-1,1],w_1)}\right\}, & p = \infty. \end{cases} \end{aligned}$$

LEMMA 3.1. *Let (w_0, w_1) be a pair of weights on $[-1,1]$ such that $w_j \in L^1([-1,1])$ and $1 \le p < \infty$. If q is the conjugate of p, i.e. $\frac{1}{p}+\frac{1}{q}=1$, then we can associate with every continuous linear functional*
$L \in (L^p([-1,1], w_0) \times L^p([1-,1], w_1))'$ a unique $v = (v_1, v_2) \in L^q([-1,1], w_0) \times L^q([-1,1], w_1)$ such that for every $u = (u_1, u_2) \in L^p([-1,1], w_0) \times L^p([-1,1], w_1)$

$$\begin{aligned} L(u) &= \langle u_1, v_1\rangle_{w_0} + \langle u_2, v_2\rangle_{w_1} \\ &= \int_{-1}^{1} u_1(x)v_1(x)w_0(x)dx + \int_{-1}^{1} u_2(x)v_2(x)w_1(x)dx. \end{aligned} \tag{3.6}$$

Moreover,

$$\|L\| = \|v\|_{L^q([-1,1],w_0)\times L^q([-1,1],w_1)} = \left(\|v_1\|^p_{L^p([-1,1]),w_0} + \|v_2\|^p_{L^p([-1,1]),w_1}\right)^{1/p}.$$

Thus $L \in (L^p([-1,1], w_0) \times L^p([1,1], w_1))' \cong L^q([-1,1], w_0) \times L^q([1,1], w_1)$.

Proof. It suffices to follow the proof given in [1, Lemma 3.7]. □

PROPOSITION 3.1. *If (w_0, w_1) is a pair of weights on $[-1,1]$ such that $w_j \in L^1([-1,1])$, $j = 0,1$, $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, $\mathbb{W}^{1,q}([-1,1],(w_0,w_1))$ are Banach spaces, with q the conjugate of p, $1 \le p < \infty$, then $(\mathbb{W}^{1,p}([-1,1],(w_0,w_1)))' = \mathbb{W}^{1,q}([-1,1],(w_0,w_1))$ and*

$$\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} = \sup\{|\langle f, g\rangle_S| : \|g\|_{\mathbb{W}^{1,q}([-1,1],(w_0,w_1))} = 1\}. \tag{3.7}$$

Proof. It suffices to consider the projection

$$\mathcal{P} : \mathbb{W}^{1,p}([-1,1],(w_0,w_1)) \longrightarrow L^p([-1,1], w_0) \times L^p([1,1], w_1),$$

given by $\mathcal{P}f = (f, f')$ and to follow the arguments of the proof presented in [1, Theorem 3.8]. □

THEOREM 3.2. *Let $\{q_n\}_{n\ge 0}$ be the sequence of orthonormal polynomials with respect to* (1.1)*, (w_0, w_1) be a pair of weights such that $w_0 \in L^\infty([-1,1])$ and $w_1 \in A_p([-1,1])$ for $1 < p < \infty$. If there exists $C > 0$, independent of n, such that*

$$\|\mathcal{S}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \le C\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}, \tag{3.8}$$

for all $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, then

$$\|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\|q_n\|_{\mathbb{W}^{1,q}([-1,1],(w_0,w_1))} \le C, \quad \frac{1}{p}+\frac{1}{q}=1.$$

Proof. We apply the same argument as in [27] (see also [11, 37]). Assume that (3.8) holds, then

$$\|\langle f, q_n\rangle_S\, q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} = \|\mathcal{S}_n f - \mathcal{S}_{n-1} f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \le 2C,$$

with $\mathcal{S}_{-1} \equiv 0$. Now, we consider the functionals L_n on $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ given by

$$L_n f := \langle f, q_n\rangle_S \|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}.$$

Hence, for every $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ we have $\sup_n\{|L_n f|\} < \infty$ and from the Banach-Steinhaus theorem we obtain that $\sup_n\{\|L_n\|\} < \infty$.

On the other hand, taking into account Proposition 3.1 we get

$$\|L_n\| = \|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\|q_n\|_{\mathbb{W}^{1,q}([-1,1],(w_0,w_1))},$$

where q is the conjugate of p. Therefore

$$\|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\|q_n\|_{\mathbb{W}^{1,q}([-1,1],(w_0,w_1))} < \infty.$$

From the above inequality our statement follows.

□

3.1. **Máté-Nevai-Totik theorems vs. Opic-Kufner type weights.** In this section we include a well-known result of [16], which allows to find necessary conditions for the convergence of the Fourier expansions in terms of orthogonal polynomials in $L^p([-1,1],d\mu)$-norm.

THEOREM 3.3. [16, Theorem 2] *Let $\{p_n\}_{n\geq 0}$ be a orthonormal system with respect to a non-trivial probability measure $d\mu$ in $[-1,1]$, $\mu' > 0$ a.e. in $[-1,1]$ and $0 < r < \infty$. If g is a measurable function in $[-1,1]$, then*

$$\int_{-1}^{1} |g(x)|^r (1-x^2)^{-r/4}\mu'(x)^{-r/2}dx \leq \pi^{r/2}2^{\max\{1-r/2,0\}} \liminf_{n\to\infty} \int_{-1}^{1} |g(x)p_n(x)|^r dx.$$

In particular, if the above inferior limit is 0, then $g = 0$ a.e.

As an immediate consequence of the above theorem we get

COROLLARY 3.1. *Let $\{p_n\}_{n\geq 0}$ be a orthonormal system with respect to $d\mu$ supported in $[-1,1]$, such that $\mu' > 0$ a.e. in $[-1,1]$, and $1 < p < \infty$. If there exists a constant C, independent of n, such that*

$$\|S_n f\|_{L^p([-1,1],d\mu)} \leq C\|f\|_{L^p([-1,1],d\mu)},$$

for all $f \in L^p([-1,1],d\mu)$. Then

i) $\int_{-1}^{1} d\mu(x) < \infty$.

ii) $\int_{-1}^{1}(1-x^2)^{-p/4}(\mu'(x))^{1-p/2}dx < \infty$.

From Theorems 3.2 and 3.3 we get

THEOREM 3.4. *Let $1 < p < \infty$, $\{p_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ the sequences of orthonormal polynomials with respect to $w_0(x)dx$ and $w_1(x)dx$, respectively. If there exists a constant C such that condition (ii) of Theorem 3.1 holds, then*

i) $w_j \in L^1([-1,1])$, $j = 0,1$.

ii) $\int_{-1}^{1}(1-x^2)^{-p/4}(w_0(x))^{1-p/2}dx < \infty$.

iii) $\liminf_{n\to\infty} \frac{1}{(n+1)\|p_n\|_S} \left(\int_{-1}^{1} |p_n(x)|^p w_0(x)dx\right)^{1/p} < \infty$.

iv) $\liminf_{n\to\infty} \frac{1}{(n+1)\|t_n\|_S} \left(\int_{-1}^{1} |t_n'(x)|^p w_1(x)dx\right)^{1/p} < \infty$.

Proof. From Theorem 3.2 we deduce that

$$\left(\int_{-1}^{1} |q_n(x)|^p w_0(x)dx\right)^{1/p} \left(\int_{-1}^{1} |q_n(x)|^q w_1(x)dx\right)^{1/q} \leq C.$$

Therefore, when $n = 0$ i) follows in a straightforward way.

Let us consider the function $g_k(x) = q_k(x)w_0^{1/p}(x)$, $k \geq 0$. Then, by Theorem 3.3 we have

$$\int_{-1}^{1} |q_k(x)|^p (1-x^2)^{-p/4} (w_0(x))^{1-p/2} dx \leq \pi^{p/2} 2^{\max\{1-p/2,0\}} \liminf_{n\to\infty} \int_{-1}^{1} |q_k(x) p_n(x)|^p w_0(x) dx,$$

for each $k \geq 0$. In particular, when $k = 0$ the above equation becomes condition ii).

Finally, we only need to prove the condition iii), taking into account similar arguments yield condition iv).

For $x \in [-1,1]$, we have that $p_n(x) = \sum_{k=0}^{n} \hat{p_n}(k) q_k(x)$ and by the Cauchy-Schwarz inequality

$$|p_n(x)|^p \leq \|p_n\|_S^p \left(\sum_{k=0}^{n} |q_k(x)| \right)^p .$$

On the other hand, using the Hölder inequality for finite sums we have

$$\left(\sum_{k=0}^{n} |q_k(x)| \right)^p \leq (n+1)^{p-1} \sum_{k=0}^{n} |q_k(x)|^p, \text{ for every } x \in [-1,1].$$

Consequently,

$$\left| \frac{p_n(x)}{(n+1)\|p_n\|_S} \right|^p w_0(x) \leq \frac{1}{n+1} \sum_{k=0}^{n} |q_k(x)|^p w_0(x) \text{ a. e.} \tag{3.9}$$

From Theorem 3.2, we have

$$\left(\int_{-1}^{1} |q_k(x)|^p w_0(x) dx \right)^{1/p} \leq C, \quad \text{for each } k \geq 0.$$

Therefore,

$$\frac{1}{n+1} \left(\int_{-1}^{1} \left(\sum_{k=0}^{n} |q_n(x)|^p \right) w_0(x) dx \right)^{1/p} \leq C. \tag{3.10}$$

Condition iii) is deduced from (3.9) and (3.10).

□

Acknowledgments.

The authors wish to thank B. Xh. Fejzullahu for his useful comments, which have contributed to improve the presentation of the manuscript.

References

[1] R. Adams, *Sobolev spaces*, Academic Press Inc. New York, 1978.

[2] V. Alvarez, D. Pestana, J. M. Rodríguez, and E. Romera, Weighted Sobolev spaces on curves, *J. Approx. Theory* **119** (2002) 41–85.

[3] V. M. Badkov, The uniform convergence of Fourier series in orthogonal polynomials, *Math. Notes*, **5** (1969), 174–179.

[4] R. C. Brown and B. Opic, Embeddings of weighted Sobolev spaces into spaces of continuous functions, *Proc. Royal Soc. London A* **439** (1992), 279–296.

[5] W. N. Everitt and L. L. Littlejohn, The density of polynomials in a weighted Sobolev space, *Rend. Mat.*, Serie VII, **10** (1990), 835–852.

[6] W. N. Everitt, L. L. Littlejohn and S. C. Williams, Orthogonal polynomials in weighted Sobolev spaces, Lecture Notes in Pure and Appl. Math. **117** (1989), Marcel Dekker Inc., New York, 53–72.

[7] W. N. Everitt, L. L. Littlejohn and S. C. Williams, Orthogonal polynomials and approximation in Sobolev spaces, *J. Comput. Appl. Math.* **48** (1993), 69–90.

[8] J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*. North Holland Math. Studies **116**, Amsterdam, 1985.

[9] A. Iserles, P. E. Koch, S. P. Nørsett and J. M. Sanz-Serna, Orthogonality and approximation in a Sobolev space in *Algorithms for Approximations*, J. C. Mason and M. G. Cox Editors, Chapman & Hall, London, 1990. 117–124.

[10] B. Xh. Fejzullahu, Divergent Legendre-Sobolev polynomial series, *Novi. Sad J. Math.* **38** (1) (2008), 35–41.

[11] _________, Asymptotic properties and Fourier expansions of orthogonal polynomials with a non-discrete Gegenbauer-Sobolev inner product, *J. Approx. Theory* **162** (2010), 397-406.

[12] T. Kilpeläinen, Weighted Sobolev spaces and capacity, *Ann. Acad. Sci. Fennicae*, Series A. I. Math. **19** (1994), 95–113.

[13] A. Kufner, *Weighted Sobolev Spaces*, Teubner Verlagsgesellschaft, Teubner-Texte zur Mathematik (Band 31). Leipzig, 1980. Also published by John Wiley & Sons. New York, 1985.

[14] A. Kufner and B. Opic, How to define reasonably Weighted Sobolev Spaces, *Comment. Math. Univ. Carolin.* **25**(3) (1984), 537–554.

[15] A. Kufner and A. M. Sändig, *Some Applications of Weighted Sobolev Spaces*, Teubner Verlagsgesellschaft, Teubner-Texte zur Mathematik (Band 100), Leipzig, 1987.

[16] A. Máté, P. Nevai, and V. Totik, Necessary conditions for weighted mean convergence of Fourier series in orthogonal polynomials, *J. Approx. Theory*, **46** (1986), 314–322.

[17] F. Marcellán and B. P. Osilenker, Estimates of polynomials orthogonal with respect to the Legendre-Sobolev inner product, *Math. Zamet.* **62** (1997), 871–880.

[18] F. Marcellán, B. P. Osilenker and I. A. Rocha, Problemas abiertos en series de Fourier de Jacobi-Sobolev. in *Margarita Mathematica en memoria de José Javier (Chicho) Guadalupe Hernández*. L. Español and J. L. Varona Editors, Servicio de Publicaciones, Universidad de La Rioja (2001), 183–189. Logroño, Spain. In Spanish.

[19] _________, On Fourier series of Jacobi-Sobolev orthogonal polynomials, *J. of Inequal. & Appl.* **7** (5) (2002), 673–699.

[20] F. Marcellán, I. A. Rocha and L. Salto, Relative asymptotics and Fourier series of orthogonal polynomials with a discrete Sobolev inner product, *J. Approx. Theory*, **121** (2003), 336–356.

[21] F. Marcellán and Y. Quintana, *Polinomios ortogonales no-estándar. Propiedades algebraicas y analíticas*. XXII Escuela Venezolana de Matemáticas. Ediciones IVIC, Mérida, Venezuela. 2009. In Spanish.

[22] F. Marcellán and J. M. Rodríguez: *Markov type inequalities in weighted Sobolev spaces*, manuscript.

[23] C. Meaney, Divergent Jacobi polynomials series, *Proc. Amer. Math. Soc.* **87** (1983), 459–462.

[24] V. G. Maz'ja, *Sobolev spaces*. Springer-Verlag, Berlin, 1985.

[25] B. Muckenhoupth, Mean convergence of Jacobi series, *Proc. Amer. Math. Soc.* **23** (1) (1969), 306–310.

[26] B. Muckenhoupt, Hardy's inequality with weights, *Studia Math.* **44** (1972), 31–38.

[27] J. Newman and W. Rudin, Mean convergence of orthogonal series, *Proc. Amer. Math. Soc.* **3** (1952) 219–222.

[28] H. Pollard, The mean convergence of orthogonal series I, *Trans. Amer. Math. Soc.* **62** (1947), 387–403.

[29] _________, The mean convergence of orthogonal series II, *Trans. Amer. Math. Soc.* **63** (1948), 355–367.

[30] _________, The mean convergence of orthogonal series III, *Duke Math. J.* **16** (1949), 189–191.

[31] _________, The convergence almost everywhere of Legendre series, *Proc. Amer. Math. Soc.* **35** (1972), 442–444.
[32] A. Portilla, Y. Quintana, J. M. Rodríguez and E. Tourís, Weighted Weierstrass' Theorem with first derivatives, *J. Math. Anal. Appl.* **334** (2) (2007), 1167–1198.
[33] J. M. Rodríguez, V. Alvarez, E. Romera and D. Pestana, Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials I, *Acta Appl. Math.* **80** (2004), 273-308.
[34] J. M. Rodríguez, V. Alvarez, E. Romera and D. Pestana, Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials II, *Approx. Theory and its Appl.* **18** (2002), 1–32.
[35] J. M. Rodríguez, Approximation by polynomials and smooth functions in Sobolev spaces with respect to measures, *J. Approx. Theory* **120** (2003) 185–216.
[36] W. Rudin, *Real and Complex Analysis*, (2nd. ed.), McGraw Hill, New York, 1974.
[37] K. Stempak, On convergence and divergence of Fourier-Bessel series, *Electron. Trans. Numer. Anal.* **14** (2002), 223-235.
[38] G. Szegő, *Orthogonal Polynomials*. Amer. Math. Soc. Coll. Publ. **23**, (4th ed.), Providence, R.I., 1975.
[39] J. L. Varona, *Convergencia en L^p con pesos de la serie de Fourier respecto de algunos sistemas ortogonales*. Doctoral Dissertation. Universidad de Cantabria, Spain. 1988. In Spanish.
[40] _________, Condiciones necesarias para la convergencia de series de Fourier, *Mathemàtiques* **3** (2005), 135–153. In Spanish.
[41] G. M. Wing, The mean convergence of orthogonal series, *Amer. J. Math.* **72** (1950), 792–808.
[42] A. Zygmund, Notes on the history of Fourier series, Studies in Harmonic Analysis, Proc. Conf. De Paul Univ., *MAA Stud. Math.* **13** Math. Assoc. Amer. Washington DC (1976) 1–19.

Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
E-mail address: `pacomarc@ing.uc3m.es`

Departamento de Matemáticas Puras y Aplicadas, Edificio Matemáticas y Sistemas (MYS), Apartado Postal: 89000, Caracas 1080 A, Universidad Simón Bolívar, Venezuela
E-mail address: `yquintana@usb.ve`

Departamento de Matemáticas Puras y Aplicadas, Edificio Matemáticas y Sistemas (MYS), Apartado Postal: 89000, Caracas 1080 A, Universidad Simón Bolívar, Venezuela
E-mail address: `aurielesg@gmail.com`