$(1,1)$-D_ω-Coherent Pairs

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Abstract

In this work, we introduce the notion of $(1,1)$-D_ω-coherent pair of weakly quasi-definite linear functionals $(\mathcal{U},\mathcal{V})$ as the D_ω-analogue to the generalized coherent pair studied by A. Delgado and F. Marcellán in [8]. This means that their corresponding families of monic orthogonal polynomials $\{P_n(x)\}_{n=0}^{M_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$ satisfy

$$\frac{D_\omega P_{n+1}(x)}{n+1}+a_n\frac{D_\omega P_n(x)}{n}=R_n(x)+b_nR_{n-1}(x),$$
$$a_n\neq 0,\quad 1\leq n\leq \min\{M_0-1,M_1\}.$$

We prove that $(1,1)$-D_ω-coherence is a sufficient condition for the weakly quasi-definite linear functionals to be D_ω-semiclassical, one of them of class at most 1 and the another of class at most 5, and they are related by a expression of rational type. Additionally, a matrix interpretation of $(1,1)$-D_ω-coherence in terms of the corresponding monic Jacobi matrices is given. The particular case when $\mathcal{U}$ is D_ω-classical linear functional is studied.

Keywords: Linear functionals, discrete orthogonal polynomials, D_ω-coherent pairs.

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1 Introduction

A pair of quasi-definite linear functionals $(\mathcal{U},\mathcal{V})$ is said to be a $(1,1)$-*coherent pair* if their corresponding sequences of monic orthogonal polynomials (SMOP), $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$\frac{P'_{n+1}(x)}{n+1}+a_n\frac{P'_n(x)}{n}=R_n(x)+b_nR_{n-1}(x),\quad a_n\neq 0,\quad n\geq 1. \tag{1.1}$$

When $b_n=0$ for all $n\geq 1$, the pair of linear functionals is called either a $(1,0)$-*coherent pair*, or a *coherent pair*. Coherent pairs have been introduced in [13]) in the framework of weighted Sobolev inner products with respect to a vector of two measures supported on the real line. The corresponding sequences of orthogonal polynomials can be easily computed in terms of the sequence $\{P_n(x)\}_{n\geq 0}$ and thus the study of their analytic properties can be done in a friendly way. On the other hand, they are very useful in the analysis of Sobolev-Fourier expansions which are more competitive in terms of speed of convergence than the standard Fourier expansions (see [12]).

In [8], A. Delgado and F. Marcellán stated that the $(1,1)$-coherence (for them, *generalized coherence*) of a pair of positive Borel measures (μ_0, μ_1) on the real line is a necessary and sufficient condition for

$$Q_{n+1}(x;\lambda) + c_n(\lambda) Q_n(x;\lambda) = P_{n+1}(x) + \frac{n+1}{n}\, a_n P_n(x), \quad n \geq 1, \tag{1.2}$$

where $\{c_n(\lambda)\}_{n\geq 1}$ are rational functions in $\lambda > 0$ and $\{Q_n(x;\lambda)\}_{n\geq 0}$ is the SMOP associated with the Sobolev inner product

$$\langle p(x), q(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1, \quad \lambda > 0,\ p, q \in \mathbb{P},$$

where $\mathbb{P}$ denotes the linear space of polynomials with complex coefficients. In the sequel, $\mathbb{P}_n$ will denote the linear subspace of polynomials of degree at most n.

They determined all $(1,1)$-coherent pairs of quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$ proving that at least one of them must be semiclassical of class at most 1 and they are related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \leq 3, \deg(\rho(x)) = 1$. This is a generalization of the results obtained by H. G. Meijer in [20] for $(1,0)$-coherence. There it was shown that at least one of the quasi-definite linear functionals either $\mathcal{U}$ or $\mathcal{V}$ must be classical (Laguerre or Jacobi) and they are related by a expression of rational type as above with $\deg(\sigma(x)) \leq 2$. But, A. Iserles, et al., in [13] were the first ones who introduced the concept of coherent pair (for us, $(1,0)$-coherent pair) of positive Borel measures (μ_0, μ_1) on the real line which arose as a sufficient condition for (1.2).

On the other hand, Marcellán and N. C. Pinzón-Cortés in [15] extended the notion of $(1,1)$-coherent pair of quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$ to $(1,1)$-*q-coherent pair* as follows. The corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$\frac{(D_q P_{n+1})(x)}{[n+1]_q} + a_n \frac{(D_q P_n)(x)}{[n]_q} = R_n(x) + b_n R_{n-1}(x)\,, \quad a_n \neq 0,\ n \geq 1. \tag{1.3}$$

where $0 < q < 1$, $[n]_q = \frac{q^n - 1}{q-1}$, $n \geq 1$, and D_q is the q-difference operator defined by $(D_q p)(x) = \frac{p(qx) - p(x)}{(q-1)x}$ for $x \neq 0$, and by continuity $(D_q p)(0) = p'(0)$, $p \in \mathbb{P}$. When $b_n = 0$ for all $n \geq 1$, $(\mathcal{U}, \mathcal{V})$ is said to be $(1,0)$-*q-coherent pair*. This problem is motivated by the discretization of a Sobolev inner product in the geometric q-lattice. They proved that $(1,1)$-q-coherence of a pair of quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$ is a sufficient condition for at least one of them to be q-semiclassical of class at most 1 and they to be related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \leq 3, \deg(\rho(x)) = 1$, and as a consequence, the companion linear functional must be q-semiclassical of class at most 5. Besides, they analyzed the case when $\mathcal{U}$ is q-classical. This is a generalization of the results obtained by I. Area, et al., in [3, 5] for $(1,0)$-q-coherent pairs. They showed that if $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-q-coherent pair of quasi-definite linear functionals then at least one of them must be q-classical and one is a rational modification of the other as above with $\deg(\sigma(x)) \leq 2$. Also, they determined all q-coherent pairs of positive-definite linear functionals when $\mathcal{U}$ or $\mathcal{V}$ is some specifical q-classical linear functional. Notice that from the study of q-coherent pairs it is possible to recover the properties of coherent pairs in the continuous case, for $(1,0)$-coherence and $(1,1)$-coherence, taking limits when $q \uparrow 1$.

Finally, a pair of weakly quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$, of order $M_0 \geq 2$ and $M_1 \geq 1$, respectively, is called a $(1,1)$-D_ω-*coherent pair* if their corresponding families of MOP, $\{P_n(x)\}_{n=0}^{M_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$ satisfy

$$\frac{D_\omega P_{n+1}(x)}{n+1} + a_n \frac{D_\omega P_n(x)}{n} = R_n(x) + b_n R_{n-1}(x), \qquad (1.4)$$
$$a_n \neq 0, \quad 1 \leq n \leq \min\{M_0 - 1, M_1\},$$

where D_ω is the difference operator defined by $(D_\omega p)(x) = \frac{p(x+\omega)-p(x)}{\omega}$, $p \in \mathbb{P}$. When $b_n = 0$ for $1 \leq n \leq \min\{M_0 - 1, M_1\}$, the pair is said to be a $(1,0)$-D_ω-*coherent pair.*

I. Area, et al, in [3, 4, 6] studied the $(1,0)$-D_ω-coherent pairs in the framework of the discretizations of Sobolev inner products when you consider uniform lattices. In other words, the measures involved in the inner product are discrete and supported on a uniform lattice of length ω in each step. They proved that if $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-D_ω-coherent pair of weakly quasi-definite linear functionals then at least one of them must be D_ω-classical as well as they are related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \leq 2, \deg(\rho(x)) = 1$. Also, they determined all $(1,0)$-D_1-coherent pairs of nonnegative-definite linear functionals and by using a limit process when $\omega \to 0$, they recovered the classification given by Meijer in [20].

The aim of this work is to generalize these results obtained by I. Area, et al., for $(1,0)$-D_ω-coherent pairs of weakly quasi-definite linear functionals and to get the D_ω-analogue results obtained by A. Delgado and F. Marcellán in [8] for $(1,1)$-coherent pairs of quasi-definite linear functionals.

The structure of this paper is as follows. In Section 2 we give the definitions and present the basic results which will be used in the forthcoming sections. In Section 3 we prove that $(1,1)$-D_ω-coherence is a necessary and sufficient condition for (1.2) which establishes a relationship between D_ω-Sobolev orthogonal polynomials and $(1,1)$-D_ω-coherent pairs. In Section 4 we study $(1,1)$-D_ω-coherent pairs of weakly quasi-definite linear functionals. We show that if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ω-coherent pair then at least one of them must be D_ω-semiclassical of class at most 1 and they are related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \leq 3, \deg(\rho(x)) = 1$, and thus the companion linear functional is D_ω-semiclassical of class at most 5. Also, we analyze the case of $(1,0)$-D_ω-coherent pairs and we recover the results obtained by I. Area, et al. In Section 5 we study the case when $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ω-coherent pair of weakly quasi-definite linear functionals and $\mathcal{U}$ is D_ω-classical. Finally, in Section 6, we state a matrix interpretation of $(1,1)$-D_ω-coherence of a pair of quasi-definite linear functionals $(\mathcal{U},\mathcal{V})$, in terms of the corresponding monic Jacobi matrices. Indeed, we obtain $[\mathcal{M}_p,\mathcal{M}_r]=(\mathcal{M}_p-\mathcal{M}_r)(\mathcal{M}_p-\mathcal{M}_r-\omega)$, where $[\mathcal{M}_p,\mathcal{M}_r]$ is the commutator of $\mathcal{M}_p$ and $\mathcal{M}_r$, and $\mathcal{M}_p$ (resp. $\mathcal{M}_r$) is a similar matrix to the monic Jacobi matrix associated with $\mathcal{U}$ (resp. $\mathcal{V}$). Furthermore, when $\mathcal{U}$ is D_ω-classical, $\mathcal{M}_{\tilde{p}}=\mathcal{M}_r$, where $\mathcal{M}_{\tilde{p}}$ is a similar matrix to the monic Jacobi matrix associated with the SMOP $\{\frac{D_\omega P_{n+1}(x)}{n+1}\}_{n\geq 0}$.

2 Preliminaries

2.1 Linear Functionals and Orthogonal Polynomials

$\mathbb{P}^*$ will denote the dual space of the linear space of polynomials with complex coefficients $\mathbb{P}$. For $\mathcal{U}\in\mathbb{P}^*$, $\{u_n=\langle\mathcal{U},x^n\rangle\}_{n\geq 0}$ is called the *sequence of moments of* $\mathcal{U}$, where $\langle\mathcal{U},p(x)\rangle\in\mathbb{C}$ denotes the image of polynomial $p(x)$ by $\mathcal{U}$. Also, for a nonzero polynomial $q(x)$ we define the linear functionals

$$\langle q(x)\mathcal{U},p(x)\rangle=\langle\mathcal{U},q(x)p(x)\rangle\,,\ \langle(q(x))^{-1}\mathcal{U},p(x)\rangle=\left\langle\mathcal{U},\frac{p(x)-L_q(x;p)}{q(x)}\right\rangle,$$

where $p\in\mathbb{P}$ and $L_q(x;p)$ denotes the interpolation polynomial of $p(x)$ at the zeros of $q(x)$ taking into account their multiplicity. Notice that, for $a\in\mathbb{C}$, $(x-a)(x-a)^{-1}\mathcal{U}=\mathcal{U}$ but $(x-a)^{-1}(x-a)\mathcal{U}=\mathcal{U}-\langle\mathcal{U},1\rangle\delta_a$, where δ_a is the *Dirac Delta linear functional at* a, defined by $\langle\delta_a,p(x)\rangle=p(a)$, $\forall p\in\mathbb{P}$.

From now, we assume that ω is a nonzero complex number. Then, the *difference operator* D_ω is defined by

$$\left(D_\omega p\right)(x)=\frac{p(x+\omega)-p(x)}{\omega},\quad p\in\mathbb{P}.$$

When $\omega=1$, D_1 is the well-known *forward difference operator* Δ, and when $\omega=-1$, D_{-1} is the *backward difference operator* ∇. Also, for $\mathcal{U}\in\mathbb{P}^*$, we can define the linear functional $D_\omega\mathcal{U}$ by

$$\langle D_\omega\mathcal{U},p(x)\rangle=-\langle\mathcal{U},D_\omega p(x)\rangle\,,\quad p\in\mathbb{P}.$$

Notice that in [1] and [18] the authors have introduced another notation for the left hand side of the above expression. Indeed, using the transposition operator, you must write $D_{-\omega}\mathcal{U}$. Nevertheless, we prefer to use the new notation to be consistent with [6] and the results therein.

It is easy to check the following properties. Let $p,r\in\mathbb{P}$

$$\Big(D_\omega\left[p(x+a)\right]\Big)(x)=\Big(D_\omega\left[p(x)\right]\Big)(x+a),\quad a\in\mathbb{C},\tag{2.1}$$

$$\left(D_{\omega}\left[p\,r\right]\right)(x) = r(x)\left(D_{\omega}p\right)(x) + p(x+\omega)\left(D_{\omega}r\right)(x), \tag{2.2}$$

$$\left(D_{-\omega}p\right)(x+\omega) = \left(D_{\omega}p\right)(x), \quad D_{\omega}D_{-\omega} = D_{-\omega}D_{\omega}, \quad D_{\omega} = D_{-\omega} + \omega D_{\omega}D_{-\omega},$$

$$D_{\omega}\left[p(x)\mathcal{U}\right] = p(x-\omega)D_{\omega}\mathcal{U} + \left(D_{\omega}p\right)(x-\omega)\mathcal{U}. \tag{2.3}$$

Notice that the difference operator D_ω becomes the usual derivative operator $D = \frac{d}{dx}$ when $\omega \to 0$. Indeed, when $\omega \to 0$, $(D_\omega p)(x) \to p'(x)$ in $\mathbb{P}$ and $D_\omega \mathcal{U} \to D\mathcal{U}$ in $\mathbb{P}^*$, where $D\mathcal{U}$ is defined by $\langle D\mathcal{U}, p(x)\rangle = -\langle \mathcal{U}, p'(x)\rangle, \forall p \in \mathbb{P}$.

$\mathcal{U} \in \mathbb{P}^*$ is said to be a *weakly quasi-definite linear functional of order* M, $M \in \mathbb{N} \cup \{\infty\}$, if the leading principal submatrices of the Hankel matrix associated with the moments of the functional $H_n = (u_{i+j})_{i,j=0}^{n}$ are nonsingular for $0 \leq n \leq M$ and, if $M < \infty$, H_{M+1} is a singular matrix. As a consequence, there exists a countable family $\{P_n(x)\}_{n=0}^{M}$ called the *family of monic orthogonal polynomials (MOP)* with respect to $\mathcal{U}$, such that $\deg(P_n(x)) = n$, $\langle \mathcal{U}, P_n(x)P_m(x)\rangle = k_n^P \delta_{n,m}$, $k_n^P \neq 0$, $0 \leq n, m \leq M$. Besides, this family of MOP satisfies the following *three-term recurrence relation (TTRR)*

$$\begin{gathered} P_n(x) = \left(x - \alpha_n^P\right) P_{n-1}(x) - \beta_n^P P_{n-2}(x), \quad \beta_n^P \neq 0, \quad 1 \leq n \leq M, \\ P_0(x) = 1, \quad P_{-1}(x) = 0. \end{gathered} \tag{2.4}$$

Conversely, if a family of monic polynomials $\{P_n(x)\}_{n=0}^{M}$ satisfies (2.4), then $\{P_n(x)\}_{n=0}^{M-1}$ is orthogonal with respect to some weakly quasi-definite linear functional.

Notice that if $M = \infty$, the concept of weakly quasi-definite linear functional coincides with the notion of *quasi-definite or regular* linear functional ([7]). In this case, the TTRR (2.4) can be written in matrix form as

$$x\mathfrak{p}(x) = \mathcal{J}_p \mathfrak{p}(x), \tag{2.5}$$

$$\mathfrak{p}(x) = \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \end{bmatrix}, \quad \mathcal{J}_p = \begin{bmatrix} \alpha_1^P & 1 & 0 & 0 & \cdots \\ \beta_2^P & \alpha_2^P & 1 & 0 & \ddots \\ 0 & \beta_3^P & \alpha_3^P & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where the semi-infinite tridiagonal matrix $\mathcal{J}_p$ is said to be the *monic Jacobi matrix* associated with the quasi-definite linear functional $\mathcal{U}$.

If $\mathcal{U}$ is a weakly quasi-definite linear functional of order M with $M < \infty$, then there exists a unique family of monic polynomials $\{P_n(x)\}_{n=0}^{M+1}$ such that $\langle \mathcal{U}, x^m P_n(x)\rangle = 0$ for $0 \leq m \leq n-1$ and $1 \leq n \leq M+1$, $\langle \mathcal{U}, x^n P_n(x)\rangle \neq 0$ for $0 \leq n \leq M$, and $\langle \mathcal{U}, x^{M+1} P_{M+1}(x)\rangle = 0$. Therefore, $\{P_n(x)\}_{n=0}^{M}$ is the family of MOP associated with $\mathcal{U}$.

A linear functional $\mathcal{U}$ is said to be *positive definite* ([7]) if $\langle \mathcal{U}, p(x)\rangle > 0$ for every nonzero polynomial $p(x)$ such that $p(x) \geq 0$, $\forall x \in \mathbb{R}$, or, equivalently, if its moments are all real and $\det(H_n) > 0$, $n \in \mathbb{N}$, or, equivalently, there exists a nondecreasing and bounded function $\varrho(x)$ with an infinite set of points of increase such that $\langle \mathcal{U}, p(x)\rangle = \int_{\mathbb{R}} p(x) d\varrho(x)$, $p \in \mathbb{P}$.

Given a family of monic polynomials $\{P_n(x)\}_{n=0}^M$ with $\deg(P_n(x)) = n, 0 \leq n \leq M$, and $M \in \mathbb{N}\cup\{\infty\}$, we can associate with it a family of linear functionals $\{\wp_n\}_{n=0}^M$ called the *dual family* of $\{P_n(x)\}_{n=0}^M$ such that $\langle \wp_n, P_m(x)\rangle = \delta_{n,m}$ for $0 \leq n, m \leq M$. When $M = \infty$, $\{\wp_n\}_{n\geq 0} \subset \mathbb{P}^*$ is said to be the *dual basis* of $\{P_n(x)\}_{n\geq 0}$.

Furthermore, if $\{P_n(x)\}_{n=0}^M$ is the family of MOP associated with a weakly quasi-definite linear functional $\mathcal{U}$ of order M, then

$$\wp_n = \frac{P_n(x)}{\langle \mathcal{U}, P_n^2(x)\rangle}\mathcal{U}, \quad 0 \leq n \leq M, \tag{2.6}$$

and, as a consequence,

$$D_\omega \wp_n^{[1]} = -(n+1)\wp_{n+1}\,, \quad 0 \leq n \leq M-1, \tag{2.7}$$

where $\{\wp_n^{[1]}\}_{n=0}^{M-1}$ is the dual family of the monic polynomials $\{\frac{D_\omega P_{n+1}(x)}{n+1}\}_{n=0}^{M-1}$.

2.2 D_ω-Semiclassical and D_ω-Classical Linear Functionals

$\mathcal{U} \in \mathbb{P}^*$ is said to be a *D_ω-semiclassical* linear functional if it is weakly quasi-definite and there exist polynomials $\sigma(x)$ and $\tau(x)$ such that $\mathcal{U}$ satisfies the distributional equation (D_ω-Pearson equation)

$$D_\omega(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U},\ ^1 \tag{2.8}$$

with $\sigma(x)$ a monic polynomial and $\deg(\tau(x)) \geq 1$. In these conditions, the *class of $\mathcal{U}$* is defined by the non-negative integer $s := \min\max\{\deg(\sigma(x)) - 2, \deg(\tau(x)) - 1\}$, where the minimum is taken among all pairs of polynomials $(\sigma(x), \tau(x))$ such that (2.8) holds[2]. In this case, we also say that the family of MOP associated with $\mathcal{U}$ is a *D_ω-semiclassical family of MOP of class s*.

The following result provides a criterion for determining the class of a D_ω-semiclassical linear functional.

Theorem 1 ([3, 18])**.** *If $\mathcal{U}$ is a D_ω-semiclassical linear functional satisfying* (2.8) *then, the class of $\mathcal{U}$ is s if and only if*

$$\prod_{\{c\in\mathbb{C}:\sigma(c)=0\}} \Big[\big|(\theta_c\sigma)(c+\omega) - \tau(c+\omega)\big| + \big|\langle \mathcal{U}, \theta_{c+\omega}\left(\theta_c\sigma(x) - \tau(x)\right)\rangle\big|\Big] > 0,$$

holds, where $\theta_c p(x) = \frac{p(x)-p(c)}{x-c}$, for $p \in \mathbb{P}$, $c \in \mathbb{C}$. If there exists $c \in \mathbb{C}$ such that $\sigma(c) = 0$ and $(\theta_c\sigma)(c+\omega) - \tau(c+\omega) = \langle \mathcal{U}, \theta_{c+\omega}(\theta_c\sigma(x) - \tau(x)\rangle = 0$, (2.8) *becomes $D_\omega(\theta_c\sigma(x)\mathcal{U}) = -\left[\theta_{c+\omega}\left(\theta_c\sigma(x) - \tau(x)\right)\right]\mathcal{U}$.*

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[1] This definition implies that $\sigma(x)$ can not be zero and $\tau(x)$ can not be a constant, otherwise, $u_0 = 0$.

[2] This class is defined as a minimum because if $(\sigma(x), \tau(x))$ satisfies (2.8), then so does $(p(x+\omega)\sigma(x), (D_\omega p)(x)\sigma(x) + p(x)\tau(x))$, for all $p \in \mathbb{P}\setminus\{0\}$.

Notice that this relation appears in [18] in a different way taking into account our definition of the linear functional $D_\omega\mathcal{U}$. Indeed, they are the same replacing ω by $-\omega$.

Proposition 2. *Let $\mathcal{U}, \mathcal{V}$ be two weakly quasi-definite linear functionals such that $p(x)\mathcal{U} = r(x)\mathcal{V}$, for some nonzero polynomials $p(x), r(x)$, i.e., $\mathcal{U}$ and $\mathcal{V}$ are related by an expression of rational type. Then, $\mathcal{U}$ is D_ω-semiclassical if and only if $\mathcal{V}$ is D_ω-semiclassical. Moreover, if the class of $\mathcal{U}$ is s, then the class of $\mathcal{V}$ is at most $s + \deg(p(x)) + \deg(r(x))$.*

Proof. It is easy to check that if $D_\omega\left[\sigma_u(x)\mathcal{U}\right] = \tau_u(x)\mathcal{U}$ holds, with $\deg(\tau_u(x)) \geq 1$, then $\mathcal{V}$ satisfies $D_\omega[p(x+\omega)r(x)\sigma_u(x)\mathcal{V}] = [\frac{p(x+\omega)-p(x-\omega)}{\omega}\sigma_u(x) + p(x-\omega)\tau_u(x)]r(x)\mathcal{V}$. The proof of the class is also easy. □

A D_ω-semiclassical linear functional $\mathcal{U}$ of class $s = 0$ is said to be D_ω-*classical*, i.e., it is weakly quasi-definite and satisfies

$$D_\omega\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U}, \quad \text{with } \deg(\sigma(x)) \leq 2,\ \deg(\tau(x)) = 1. \tag{2.9}$$

Its corresponding family of MOP is said to be a *D_ω-classical family of MOP*. A characterization of these polynomials is the following.

Theorem 3 ([1]). *Let $\mathcal{U}$ be a weakly quasi-definite lineal functional of order M and let $\{P_n(x)\}_{n=0}^{M}$ be its corresponding MOP. The following statements are equivalent*

- **i)** $\{P_n(x)\}_{n=0}^{M}$ *is a D_ω-classical family of MOP and $\mathcal{U}$ satisfies* (2.9).
- **ii)** $\{\frac{D_\omega P_{n+1}(x)}{n+1}\}_{n=0}^{M-1}$ *is a family of MOP with respect to $\mathcal{U}^{[1]} \in \mathbb{P}^*$.*

Moreover, $\mathcal{U}^{[1]} = \sigma(x)\mathcal{U}$ and $\{\frac{D_\omega P_{n+1}(x)}{n+1}\}_{n=0}^{M-1}$ is also a D_ω-classical family of MOP of the same type as $\{P_n(x)\}_{n=0}^{M}$ because $\mathcal{U}^{[1]}$ satisfies

$$D_\omega\left[\sigma(x+w)\mathcal{U}^{[1]}\right] = \left[\tau(x) + (D_\omega\sigma)(x)\right]\mathcal{U}^{[1]}.$$

When $\omega = 1$, Kravchuk, Hahn, Charlier, and Meixner are all the D_1-classical families of MOP ([10]). The linear functionals associated with Kravchuk and Hahn family of MOP are weakly quasi-definite because they have a finite set as support and their families of MOP satisfy a finite orthogonality relation. However, Charlier and Meixner linear functionals are quasi-definite ([7]). In Table 1 and Table 2, we give the polynomials $\sigma(x)$ and $\tau(x)$ which appear in (2.9), the weight function $w(x)$ such that the D_1-classical functional can be represented as $\langle\mathcal{U}, p(x)\rangle = \sum_{x_k=a}^{b-1} p(x_k)w(x_k)$, $x_{k+1} = x_k + 1$, for all $p \in \mathbb{P}$, with $a, b \in \mathbb{N} \cup \{\infty\}$, the coefficients α_n^P and β_n^P of the TTRR (2.4), and the monic polynomial $\frac{D_1 P_{n+1}(x)}{n+1}$.

For characterizations of the D_ω-semiclassical and D_ω-classical linear functionals see [1, 3, 9, 10, 11, 14, 16, 17, 18, 19, 21, 22].

Table 1: D_1-Classical Families of MOP (Weakly Quasi-Definite L.F.).

	Kravchuk	Hahn
$P_n(x)$	$K_n^{(p)}(x;N)$	$H_n^{(\alpha,\beta)}(x;N)$
$\sigma(x)$	$N-x$	$(N-x-1)(x+\beta+1)$
$\tau(x)$	$\frac{Np-x}{p}$	$(N-1)(\beta+1)-x(\alpha+\beta+2)$
x	$\{0,1,\cdots,N\}$	$\{0,1,\cdots,N-1\}$
$w(x)$	$\binom{N}{x}p^x(1-p)^{N-x}$	$\frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(\alpha+N-x)\Gamma(\beta+x+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+N+1)\Gamma(N-x)\Gamma(x+1)}$
Restriction	$p\in(0,1), N\in\mathbb{Z}^+$	$\alpha,\beta>-1, N\in\mathbb{Z}^+$
α_{n+1}^P	$n+p(N-2n)$	$\frac{\alpha-\beta+2N-2}{4}+\frac{(\beta^2-\alpha^2)(\alpha+\beta+2N)}{4(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$
β_{n+1}^P	$pn(1-p)(N-n+1)$	$\frac{n(N-n)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}$
$\frac{D_1P_{n+1}(x)}{n+1}$	$K_n^{(p)}(x;N-1)$	$H_n^{(\alpha+1,\beta+1)}(x;N-1)$

Table 2: D_1-Classical Sequences of MOP (Quasi-Definite L.F.).

	Charlier	Meixner
$P_n(x)$	$C_n^{(\mu)}(x)$	$M_n^{(\gamma,\mu)}(x)$
$\sigma(x)$	μ	$\mu(\gamma+x)$
$\tau(x)$	$\mu-x$	$\mu\gamma-x(1-\mu)$
x	$\mathbb{N}$	$\mathbb{N}$
$w(x)$	$\frac{e^{-\mu}\mu^x}{\Gamma(x+1)}$	$\frac{\mu^x(1-\mu)^\gamma\Gamma(x+\gamma)}{\Gamma(x+1)\Gamma(\gamma)}$
Restriction	$\mu>0$	$\gamma>0, \mu\in(0,1)$
α_{n+1}^P	$n+\mu$	$\frac{\gamma\mu+n(1+\mu)}{1-\mu}$
β_{n+1}^P	$n\mu$	$\frac{\mu n(\gamma+n-1)}{(1-\mu)^2}$
$\frac{D_1P_{n+1}(x)}{n+1}$	$C_n^{(\mu)}(x)$	$M_n^{(\gamma+1,\mu)}(x)$

3 D_ω-Sobolev Orthogonal Polynomials and D_ω-Coherent Pairs

In the sequel, we will denote $M := \min\{M_0-1, M_1\}$.

A pair of weakly quasi-definite linear functionals $(\mathcal{U},\mathcal{V})$ is said to be a $(1,1)$-D_ω-*coherent pair* if their corresponding families of MOP, $\{P_n(x)\}_{n=0}^{M_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$, with $M_0\geq 2$ and $M_1\geq 1$, satisfy

$$\begin{gathered}\frac{D_\omega P_{n+1}(x)}{n+1}+a_n\frac{D_\omega P_n(x)}{n}=R_n(x)+b_nR_{n-1}(x),\\ a_n\neq 0,\quad 1\leq n\leq M.\end{gathered}\tag{3.1}$$

If $b_n=0$ for $1\leq n\leq M$, the pair of linear functionals is said to be a $(1,0)$-D_ω-*coherent pair*.

In this context, we can consider the following Sobolev inner product, where the weakly quasi-definite linear functionals that determine this product constitute a $(1,1)$ or $(1,0)$ -D_ω-coherent pair,

$$\langle p(x), r(x)\rangle_{\lambda,\omega} = \langle \mathcal{U}, p(x)r(x)\rangle + \lambda \langle \mathcal{V}, (D_\omega p)(x)(D_\omega r)(x)\rangle\,, \quad \lambda > 0, \tag{3.2}$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients. Thus, there is a close relationship between $(1,1)$-D_ω-coherent pairs and D_ω-Sobolev orthogonal polynomials.

Proposition 4. *If* $(\mathcal{U}, \mathcal{V})$ *is a* $(1,1)$-D_ω*-coherent pair given by* (3.1)*, then*

$$\begin{gathered} Q_{n+1}(x;\lambda,\omega) + c_n(\lambda,\omega)Q_n(x;\lambda,\omega) = P_{n+1}(x) + a_n\frac{n+1}{n}P_n(x)\,, \\ a_n \neq 0, \quad 1 \le n \le M, \end{gathered} \tag{3.3}$$

holds, where $\{c_n(\lambda,\omega)\}_{n=1}^{M}$ *are rational functions in* $\lambda > 0$ *given by*

$$c_n(\lambda,\omega) = \frac{a_n\frac{n+1}{n}\left\langle \mathcal{U}, P_n^2(x)\right\rangle + b_n n(n+1)\lambda\left\langle \mathcal{V}, R_{n-1}^2(x)\right\rangle}{\langle Q_n(x;\lambda,\omega), Q_n(x;\lambda,\omega)\rangle_{\lambda,\omega}}, \tag{3.4}$$

and $\{Q_n(x;\lambda,\omega)\}$ *is the family of MOP associated with the* D_ω*-Sobolev inner product* (3.2)*.*

Conversely, if there are constants $a_n \neq 0$ *and* $c_n(\lambda,\omega)$*,* $1 \le n \le M$*, such that* (3.3) *holds, then there exist constants* b_n *with*

$$b_n = \frac{\left\langle \mathcal{V}, \frac{D_\omega P_{n+1}(x)}{n+1}R_{n-1}(x)\right\rangle}{\left\langle \mathcal{V}, R_{n-1}^2(x)\right\rangle} + a_n, \quad 1 \le n \le M, \tag{3.5}$$

such that (3.1) *holds, i.e.,* $(\mathcal{U}, \mathcal{V})$ *is a* $(1,1)$-D_ω*-coherent pair.*

Proof. For $1 \le n \le M$, we have the following Fourier series expansion

$$P_{n+1}(x) + a_n\frac{n+1}{n}P_n(x) = Q_{n+1}(x;\lambda,\omega) + \sum_{k=0}^{n} c_{k,n+1}(\lambda,\omega)Q_k(x;\lambda,\omega),$$

where $c_{k,n+1}(\lambda,\omega) = \frac{\langle P_{n+1}(x)+a_n\frac{n+1}{n}P_n(x), Q_k(x;\lambda,\omega)\rangle_{\lambda,\omega}}{\langle Q_k(x;\lambda,\omega), Q_k(x;\lambda,\omega)\rangle_{\lambda,\omega}}$. Then using (3.1), (3.2), and the orthogonality of $\{P_n(x)\}_{n=0}^{M_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$ with respect $\mathcal{U}$ and $\mathcal{V}$, respectively, we get $c_{k,n+1}(\lambda,\omega) = 0$, $k = 0, \ldots, n-1$, and $c_n(\lambda,\omega) := c_{n,n+1}(\lambda,\omega)$ is given by (3.4), for $1 \le n \le M$. Therefore (3.3) holds.

Conversely, let $r(x)$ be a polynomial with $\deg(r(x)) \le n-1$. If we apply $\langle\,\cdot\,, r(x)\rangle_{\lambda,\omega}$ to both sides of (3.3), then from (3.2) and (3.3), we obtain $\lambda(n+1)\left\langle \mathcal{V}, \left(\frac{D_\omega P_{n+1}(x)}{n+1} + a_n\frac{D_\omega P_n(x)}{n}\right)D_\omega r(x)\right\rangle = 0$, for $1 \le n \le M$. Since for $k \in \mathbb{N}$ every polynomial of degree k is the D_ω-derivative of some polynomial of degree $k+1$, from the previous equation it follows that for every polynomial $p(x)$

with real coefficients of degree at most $n-2$, $2 \le n \le M$, $\left\langle \mathcal{V}, \left(\frac{D_\omega P_{n+1}(x)}{n+1} + a_n \frac{D_\omega P_n(x)}{n}\right) p(x) \right\rangle = 0$ holds. On other hand,

$$\frac{D_\omega P_{n+1}(x)}{n+1} + a_n \frac{D_\omega P_n(x)}{n} = R_n(x) + \sum_{k=0}^{n-1} b_{k,n} R_k(x), \quad 1 \le n \le M,$$

where $b_{k,n} = \frac{\left\langle \mathcal{V}, \left(\frac{D_\omega P_{n+1}(x)}{n+1} + a_n \frac{D_\omega P_n(x)}{n}\right) R_k(x) \right\rangle}{\langle \mathcal{V}, R_k^2(x) \rangle}$. Thus, for $1 \le n \le M$, $b_n := b_{n-1,n}$ is given by (3.5), and $b_{k,n} = 0$ for $k = 0, \ldots, n-2$. □

The family $\{c_n(\lambda,\omega)\}_{n=1}^{M}$ can be characterized in the following way.

Corollary 5. *If* $(\mathcal{U}, \mathcal{V})$ *is a* $(1,1)$-D_ω*-coherent pair given by* (3.1)*, then the family* $\{c_n(\lambda,\omega)\}_{n=1}^{M}$ *in* (3.3) *satisfies*

$$\begin{aligned} c_n(\lambda,\omega) &= \frac{A_n(\lambda,\omega)}{B_n(\lambda,\omega) - c_{n-1}(\lambda,\omega) E_n(\lambda,\omega)}, \quad 2 \le n \le M, \\ c_1(\lambda,\omega) &= \frac{A_1(\lambda,\omega)}{\langle \mathcal{U}, P_1^2(x) \rangle + \lambda \langle \mathcal{V}, R_0^2(x) \rangle}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} A_n(\lambda,\omega) &= a_n \frac{n+1}{n} \left\langle \mathcal{U}, P_n^2(x) \right\rangle + \lambda\, b_n n(n+1) \left\langle \mathcal{V}, R_{n-1}^2(x) \right\rangle, \\ B_n(\lambda,\omega) &= \left\langle \mathcal{U}, P_n^2(x) \right\rangle + \left(a_{n-1} \frac{n}{n-1}\right)^2 \left\langle \mathcal{U}, P_{n-1}^2(x) \right\rangle \\ &\qquad + \lambda n^2 \left[\left\langle \mathcal{V}, R_{n-1}^2(x) \right\rangle + b_{n-1}^2 \left\langle \mathcal{V}, R_{n-2}^2(x) \right\rangle \right], \\ E_n(\lambda,\omega) &= a_{n-1} \frac{n}{n-1} \left\langle \mathcal{U}, P_{n-1}^2(x) \right\rangle + \lambda (n-1) n b_{n-1} \left\langle \mathcal{V}, R_{n-2}^2(x) \right\rangle. \end{aligned}$$

Proof. Using (3.1), (3.2), and (3.3), we get $\langle Q_n(x;\lambda,\omega), Q_n(x;\lambda,\omega) \rangle_{\lambda,\omega} = B_n(\lambda,\omega) - c_{n-1}(\lambda,\omega) E_n(\lambda,\omega)$, for $2 \le n \le M+1$. Besides, since $Q_1(x;\lambda,\omega) = P_1(x)$, then from (3.4) it follows (3.6). □

Under the conditions of Corollary 5 we get

Corollary 6. *The family* $\{c_n(\lambda,\omega)\}_{n=1}^{M}$ *satisfies*

$$c_n(\lambda,\omega) = \frac{g_n(\lambda,\omega)}{h_n(\lambda,\omega)}, \quad 1 \le n \le M,$$

where $g_n(\lambda,\omega)$ *and* $h_n(\lambda,\omega)$ *are polynomials on* λ *of degree at most* n*.*

Proof. This is a straightforward consequence of (3.6) and induction on n. □

Notice that if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ω-coherent pair, then from (3.6) we get the family $\{c_n(\lambda,\omega)\}_{n=1}^{M}$. Thus, from (3.3) and $Q_1(x;\lambda,\omega) = P_1(x)$, we can obtain recursively the D_ω-Sobolev polynomials $\{Q_n(x;\lambda,\omega)\}_{n=0}^{M+1}$.

4 $(1,1)$-D_ω-Coherent Pairs of Linear Functionals

In this section, we assume that $\mathcal{U}$ and $\mathcal{V}$ are two weakly quasi-definite linear functionals with corresponding family of MOP $\{P_n(x)\}_{n=0}^{M_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$, $M_0 \geq 2$ and $M_1 \geq 1$.

Lemma 7. *Let* $(\mathcal{U},\mathcal{V})$ *be a* $(1,1)$-D_ω*-coherent pair as in* (3.1)*. Then*

a. $a_1 \neq b_1$ *if and only if* $\frac{D_\omega P_{n+1}(x)}{n+1} \neq R_n(x)$, $1 \leq n \leq M$.

b. For $1 \leq n \leq M$,

$$\frac{D_\omega P_{n+1}(x)}{n+1} = R_n(x) + (b_n - a_n)R_{n-1}(x)$$
$$+ \sum_{k=2}^{n}(-1)^{k-1} a_n a_{n-1} \cdots a_{n-(k-2)} \left(b_{n-(k-1)} - a_{n-(k-1)}\right) R_{n-k}(x). \quad (4.1)$$

Proof. From (3.1) is easy to prove (4.1) as well as, $a_1 = b_1$ if and only if $\frac{D_\omega P_{N+1}(x)}{N+1} = R_N(x)$ for some $1 \leq N \leq M$. ☐

In the remainder of this section we assume that $a_1 \neq b_1$.

Lemma 8. *Let* $(\mathcal{U},\mathcal{V})$ *be a* $(1,1)$-q*-coherent pair given by* (3.1)*. Then there exists a monic polynomial* $\gamma_n(x)$ *of degree* $1 \leq n \leq M-1$ *such that*

$$\left\langle \gamma_n(x)\mathcal{V}, \frac{D_\omega P_{m+1}(x)}{m+1} \right\rangle = 0, \quad 2 \leq n+1 \leq m \leq M-1, \quad (4.2)$$

$$\textit{and} \left\langle \mathcal{V}, \gamma_n(x)\frac{D_\omega P_{n+2}(x)}{n+2} \right\rangle = 0, \quad \textit{for } n = M-1.$$

Proof. Let $\gamma_n(x) = R_n(x) + \sum_{j=0}^{n-1} A_{j,n}R_j(x)$ with $0 \leq n \leq M$. Then, for $0 \leq n \leq M-1$,

$$\left\langle \mathcal{V}, \gamma_n(x)\frac{D_\omega P_{n+2}(x)}{n+2} \right\rangle \overset{(4.1)}{=} (b_{n+1} - a_{n+1})\langle \mathcal{V}, R_n^2(x)\rangle + \sum_{k=2}^{n+1}(-1)^{k-1}$$
$$a_{n+1} \cdots a_{n+1-(k-2)} \left(b_{n+1-(k-1)} - a_{n+1-(k-1)}\right) A_{n+1-k,n}\langle \mathcal{V}, R_{n+1-k}^2(x)\rangle. \quad (4.3)$$

Hence, for $1 \leq n \leq M-1$, we can choose real numbers $A_{0,n}, \ldots, A_{n-1,n}$, not all zero, such that (4.3) is zero, because $a_1 \neq b_1$. On the other hand, for $0 \leq n \leq M$ and $0 \leq m \leq M-1$, if we apply $\langle \gamma_n(x)\mathcal{V}, \cdot\rangle$ to (3.1), then we obtain $\langle \gamma_n(x)\mathcal{V}, \frac{D_\omega P_{m+2}(x)}{m+2}\rangle = -a_{m+1}\langle \gamma_n(x)\mathcal{V}, \frac{D_\omega P_{m+1}(x)}{m+1}\rangle$ for $n < m$. Thus, the proof is complete. ☐

Notice that in the previous lemma we can choose $A_{1,n} = \cdots = A_{n-1,n} = 0$. Hence, for $1 \leq n \leq M-1$,

$$\gamma_n(x) = R_n(x) + A_{0,n} = R_n(x) + \frac{(-1)^{n+1}(b_{n+1} - a_{n+1})\langle \mathcal{V}, R_n^2(x)\rangle}{a_{n+1}a_n \cdots a_3 a_2 (b_1 - a_1)\langle \mathcal{V}, 1\rangle}. \quad (4.4)$$

Lemma 9. *Let* $(\mathcal{U},\mathcal{V})$ *be a* $(1,1)$-D_ω*-coherent pair given by* (3.1) *and let* $\gamma_n(x)$ *be the monic polynomial introduced in Lemma 8,* $\deg(\gamma_n(x))=n$. *Then there exists a polynomial* $\varphi_{n+1}(x)$ *with* $\deg(\varphi_{n+1}(x))\le n+1$ *such that*

$$D_\omega[\gamma_n(x)\mathcal{V}] = -\varphi_{n+1}(x)\mathcal{U}, \quad 1\le n\le M-1, \tag{4.5}$$

holds. Moreover, for $1\le n\le M-1$,

$$\varphi_{n+1}(x)=\sum_{k=0}^{n}\frac{(k+1)\left\langle \gamma_n(x)\mathcal{V}, \frac{D_\omega P_{k+1}(x)}{k+1}\right\rangle}{\langle \mathcal{U}, P_{k+1}^2(x)\rangle}P_{k+1}(x). \tag{4.6}$$

Proof. Let $\{\wp_k\}_{k=0}^{M_0}$ and $\{\wp_k^{[1]}\}_{k=0}^{M_0-1}$ be the dual families of $\{P_k(x)\}_{k=0}^{M_0}$ and $\{\frac{D_\omega P_{k+1}(x)}{k+1}\}_{k=0}^{M_0-1}$, respectively, and let $1\le n\le M-1$. Since $\{\wp_k^{[1]}\}_{k=0}^{M-1}$ is a basis of the algebraic dual space of the space of polynomials of degree at most $M-1$, then $\gamma_n(x)\mathcal{V}=\sum_{k=0}^{M-1}\lambda_{k,n}\wp_k^{[1]}$ where $\lambda_{k,n}=\langle\gamma_n(x)\mathcal{V},\frac{D_\omega P_{k+1}(x)}{k+1}\rangle$. Hence, from Lemma 8 it follows that $\lambda_{k,n}=0$ for $2\le n+1\le k\le M-1$. Thus $\gamma_n(x)\mathcal{V}=\sum_{k=0}^{n}\lambda_{k,n}\wp_k^{[1]}$, for $1\le n\le M-1$, and, as a consequence, using (2.7) and (2.6), (4.5) holds. ☐

Corollary 10. *If* $(\mathcal{U},\mathcal{V})$ *is a* $(1,1)$-D_ω*-coherent pair given by* (3.1) *with* $M_0\ge 4$ *and* $M_1\ge 3$, *then there exist polynomials* $\alpha(x)$ *and* $\phi(x)$, *and a monic polynomial* $\beta(x)$, *with* $\deg(\alpha(x))\le 4$, $\deg(\phi(x))\le 3$, *and* $\deg(\beta(x))=2$, *such that*

$$\alpha(x)\mathcal{U}=\beta(x)\mathcal{V}, \tag{4.7}$$

$$\alpha(x)D_\omega\mathcal{V}=\phi(x)\mathcal{V}, \tag{4.8}$$

$$\phi(x)\mathcal{U}=\beta(x)D_\omega\mathcal{V}, \tag{4.9}$$

where

$$\alpha(x)=\gamma_2(x-\omega)\varphi_2(x)-\gamma_1(x-\omega)\varphi_3(x), \tag{4.10}$$

$$\beta(x)=\gamma_1(x-\omega)\left(D_\omega\gamma_2\right)(x-\omega)-\gamma_2(x-\omega), \tag{4.11}$$

$$\phi(x)=\varphi_3(x)-\left(D_\omega\gamma_2\right)(x-\omega)\varphi_2(x), \tag{4.12}$$

Besides, for $1\le n\le M-1$,

$$\phi(x)\gamma_n(x-\omega)+\alpha(x)\left(D_\omega\gamma_n\right)(x-\omega)=-\varphi_{n+1}(x)\beta(x), \tag{4.13}$$

where $\gamma_n(x)$ *and* $\varphi_{n+1}(x)$ *are the polynomials given in Lemma 9.*

Proof. From (4.5) for $n=1$ and $n=2$ and from (2.3) we get

$$\gamma_1(x-\omega)D_\omega\mathcal{V}+\mathcal{V}=-\varphi_2(x)\mathcal{U}, \tag{4.14}$$

$$\gamma_2(x-\omega)D_\omega\mathcal{V}+\left(D_\omega\gamma_2\right)(x-\omega)\mathcal{V}=-\varphi_3(x)\mathcal{U}. \tag{4.15}$$

Then, the elimination of $D_\omega\mathcal{V}$, $\mathcal{U}$, and $\mathcal{V}$ yields (4.7)-(4.9), respectively. Furthermore, from Lemma 9 it is immediate to check the degrees of these polynomials. On the other hand, from (4.7), (4.5), (2.3) and (4.8), $-\varphi_{n+1}(x)\beta(x)\mathcal{V}=[\gamma_n(x-\omega)\phi(x)+\alpha(x)(D_\omega\gamma_n)(x-\omega)]\mathcal{V}$ follows, for $1\le n\le M-1$. ☐

Notice that if $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ω-coherent pair with $M_0 \geq 4$ and $M_1 \geq 3$, then from (4.6), (4.4) and (4.1), the leading coefficients of $\varphi_2(x)$ and $\varphi_3(x)$ are, respectively,

$$\frac{2b_2\left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_2\langle \mathcal{U}, P_2^2(x)\rangle} \quad \text{and} \quad \frac{3b_3\langle \mathcal{V}, R_2^2(x)\rangle}{a_3\langle \mathcal{U}, P_3^2(x)\rangle}. \tag{4.16}$$

Hence, the leading coefficients of $\beta(x)$, $\alpha(x)$, and $\phi(x)$ are, respectively, 1,

$$\frac{2b_2\left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_2\langle \mathcal{U}, P_2^2(x)\rangle} - \frac{3b_3\langle \mathcal{V}, R_2^2(x)\rangle}{a_3\langle \mathcal{U}, P_3^2(x)\rangle}, \ \frac{3b_3\langle \mathcal{V}, R_2^2(x)\rangle}{a_3\langle \mathcal{U}, P_3^2(x)\rangle} - \frac{4b_2\left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_2\langle \mathcal{U}, P_2^2(x)\rangle}. \tag{4.17}$$

To prove that the $(1,1)$-D_ω-coherence is a sufficient condition for $\mathcal{U}$ and $\mathcal{V}$ to be D_ω-semiclassical linear functionals, we consider the zeros of the monic polynomial $\beta(x)$ given by (4.11). Indeed, if ξ_1 and ξ_2 are the zeros of $\beta(x)$, then

$$\beta(x) = (x-\xi_1)(x-\xi_2), \quad \left(D_\omega\beta\right)(x) = 2\left[x - \frac{\xi_1+\xi_2-\omega}{2}\right] \underset{(2.2)}{\overset{(4.11)}{=}} 2\gamma_1(x). \tag{4.18}$$

Therefore, the possible cases to analyze are the following:

i. ξ and $\xi-\omega$ are the zeros of $\beta(x)$, equivalently, ξ is a zero of $\beta(x)$ such that $\xi-\omega$ is the zero of $\left(D_\omega\beta\right)(x)$, (Theorem 11).

ii. ξ_1 and ξ_2 are the zeros of $\beta(x)$ such that $\xi_1 \neq \xi_2$, $\xi_2 \neq \xi_1-\omega$ and $\xi_1 \neq \xi_2-\omega$, equivalently, ξ_1 and ξ_2 are the zeros of $\beta(x)$ such that $\xi_1 \neq \xi_2$, $\left(D_\omega\beta\right)(\xi_1-\omega) \neq 0$ and $\left(D_\omega\beta\right)(\xi_2-\omega) \neq 0$, (Theorem 15).

iii. ξ is a double zero of $\beta(x)$, (Theorem 16).

Theorem 11. *If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-D_ω-coherent pair given by* (3.1) *with $M_0 \geq 4$ and $M_1 \geq 3$, and if ξ and $\xi-\omega$ are the zeros of $\beta(x)$, then there exist polynomials $\tilde{\alpha}_3(x)$, $\varphi_2(x)$, and $\gamma_1(x)$ of degrees $\leq 3, \leq 2$, and 1, respectively, such that*

$$D_\omega\left[\tilde{\alpha}_3(x)\mathcal{U}\right] = -\varphi_2(x)\mathcal{U}, \tag{4.19}$$

$$\tilde{\alpha}_3(x)\mathcal{U} = \gamma_1(x)\mathcal{V}. \tag{4.20}$$

Hence, $\mathcal{U}$ and $\mathcal{V}$ are D_ω-semiclassical linear functionals of class at most 1 *and* 5*, respectively.*

Proof. From (4.18), $\beta(x) = (x-\xi)(x-\xi+\omega) = (x-\xi)\gamma_1(x)$. Then from (4.11), $\gamma_2(\xi-\omega) = 0$ and thus $\gamma_2(x) = \gamma_1(x)\nu_1(x)$, where $\nu_1(x)$ is a monic polynomial of degree 1. Also, from (4.10) we obtain $\alpha(\xi) = 0$ and, thus, $\alpha(x) = (x-\xi)\tilde{\alpha}_3(x)$, where $\tilde{\alpha}_3(x) = \nu_1(x-\omega)\varphi_2(x) - \varphi_3(x)$. Hence, $(D_\omega\gamma_2)(x) = \gamma_1(x+\omega) + \nu_1(x)$ and, therefore, (4.14) and (4.15) become

$$\gamma_2(x-\omega)D_\omega\mathcal{V} + \nu_1(x-\omega)\mathcal{V} = -\nu_1(x-\omega)\varphi_2(x)\mathcal{U},$$
$$\gamma_2(x-\omega)D_\omega\mathcal{V} + \left[\gamma_1(x) + \nu_1(x-\omega)\right]\mathcal{V} = -\varphi_3(x)\mathcal{U}.$$

As a consequence, (4.20) follows by elimination of $\gamma_2(x-\omega)D_\omega\mathcal{V}$. Besides, taking D_ω in (4.20) and using (4.5), (4.19) holds. Furthermore, from Proposition 2, we obtain the desired result. □

For the second case we need some previous results which will be stated as lemmas.

Lemma 12. *Let* $(\mathcal{U},\mathcal{V})$ *be a* $(1,1)$*-*D_ω*-coherent pair given by* (3.1) *with* $M_0 \geq 4$ *and* $M_1 \geq 3$*, and let* $\alpha(x)$*,* $\beta(x)$*, and* $\phi(x)$ *be the polynomials introduced in Corollary 10. If* ξ *is a zero of* $\beta(x)$ *such that* $\beta(\xi-\omega) \neq 0$ *and* $\alpha(\xi) = 0$*, then* $\gamma_1(\xi-\omega) \neq 0$ *and* $\phi(\xi) = 0$*.*

Proof. From (4.18) we get $\gamma_1(\xi-\omega) \neq 0$. Thus, from (4.13) for $n = 1$, $\phi(\xi) = 0$ holds. □

Lemma 13. *Let* $(\mathcal{U},\mathcal{V})$ *be a* $(1,1)$*-*D_ω*-coherent pair given by* (3.1) *with* $M_0 \geq 4$ *and* $M_1 \geq 3$*, and let* $\alpha(x)$*,* $\beta(x)$*,* $\phi(x)$*, and* $\gamma_n(x)$ *be the polynomials introduced in Corollary 10. If* ξ *is a zero of* $\beta(x)$ *such that* $\alpha(\xi) \neq 0$*, then there exists a constant* $C \neq 0$*, independent on* n*, such that*

$$\gamma_n(\xi-\omega) + C\,(D_\omega\gamma_n)(\xi-\omega) = 0, \quad 1 \leq n \leq M-1.$$

Proof. $\phi(\xi) \neq 0$ follows from (4.13) for $n = 1$. Hence, if $C = \alpha(\xi)/\phi(\xi)$ and using (4.13), the proof is complete. □

Lemma 14. *Let* $(\mathcal{U},\mathcal{V})$ *be a* $(1,1)$*-*D_ω*-coherent pair given by* (3.1) *with* $M_0 \geq 4$ *and* $M_1 \geq 3$*, and let* $\gamma_n(x)$ *be given by* (4.4)*. If there exist constants* ξ_1, ξ_2, C_1, C_2 *independent on* n*, such that* $\xi_2 \neq \xi_1 - \omega$*,* $\xi_1 \neq \xi_2 - \omega$*, and*

$$\gamma_n(\xi_k-\omega) + C_k\,(D_\omega\gamma_n)(\xi_k-\omega) = 0, \quad k = 1,2, \tag{4.21}$$

for $1 \leq n \leq M-1$*, then* $\xi_1 = \xi_2$ *and* $C_1 = C_2$*.*

Proof. As a consequence of (4.4) and (4.21), for $1 \leq n \leq M-1$, we get

$$R_n(\xi_1-\omega) + C_1\,(D_\omega R_n)(\xi_1-\omega) = R_n(\xi_2-\omega) + C_2\,(D_\omega R_n)(\xi_2-\omega).$$

Besides, since this equation also holds for $n = 0$ and $\{R_n(x)\}_{n=0}^{M-1}$ is a basis of $\mathbb{P}_{M-1}$, then for every $p \in \mathbb{P}_{M-1}$,

$$p(\xi_1-\omega) + C_1\,(D_\omega p)(\xi_1-\omega) = p(\xi_2-\omega) + C_2\,(D_\omega p)(\xi_2-\omega) \tag{4.22}$$

holds. In particular, (4.22) is true for $p(x) = (x-\xi_2)^n(x-\xi_2+\omega)^n$ with $1 \leq n \leq M-1$. Therefore, $(\xi_1-\xi_2)^n\big[(\xi_1-\xi_2-\omega)^n + C_1\frac{(\xi_1-\xi_2+\omega)^n-(\xi_1-\xi_2-\omega)^n}{\omega}\big] = 0$ follows for $1 \leq n \leq M-1$. If $\xi_1 \neq \xi_2$, then when $n = 1$ we can conclude that $C_1 = (\xi_2-\xi_1+\omega)/2$. If we replace this value when $n = 2$, we obtain $(\xi_2-\xi_1+\omega)(\xi_2-\xi_1-\omega) = 0$, which yields a contradiction. So $\xi_1 = \xi_2$ and thus, $C_1 = C_2$ follows from (4.22) for $p(x) = x$. □

Theorem 15. *Let* $(\mathcal{U},\mathcal{V})$ *be a* $(1,1)$*-*D_ω*-coherent pair given by* (3.1) *with* $M_0 \geq 4$ *and* $M_1 \geq 3$*, and let* $\beta(x)$ *be the monic polynomial given by* (4.11)*. If* ξ_1 *and* ξ_2 *are the zeros of* $\beta(x)$ *such that* $\xi_1 \neq \xi_2, \xi_2 \neq \xi_1-\omega, \xi_1 \neq \xi_2-\omega$*, then there* *exist polynomials* $\tilde{\alpha}(x)$ *and* $\tilde{\phi}(x)$*, with* $\deg(\tilde{\alpha}(x)) \leq 3$ *and* $\deg(\tilde{\phi}(x)) \leq 2$*, such that*

$$\tilde{\alpha}(x)\mathcal{U} = \tilde{\beta}(x)\mathcal{V}, \tag{4.23}$$

$$\tilde{\alpha}(x)D_\omega\mathcal{V} = \tilde{\phi}(x)\mathcal{V}, \tag{4.24}$$

$$\tilde{\phi}(x)\mathcal{U} = \tilde{\beta}(x)D_\omega\mathcal{V}, \tag{4.25}$$

where $\tilde{\beta}(x) = x - \xi$ *for some* $\xi \in \{\xi_1, \xi_2\}$*. Moreover,*

$$D_\omega\left[\tilde{\alpha}(x)\mathcal{V}\right] = \Big(\tilde{\phi}(x-\omega) + \left(D_\omega\tilde{\alpha}\right)(x-\omega)\Big)\,\mathcal{V}. \tag{4.26}$$

Thus, $\mathcal{V}$ *and* $\mathcal{U}$ *are* D_ω*-semiclassical linear functionals of class at most* 1 *and* 5*, respectively.*

Proof. Let $\alpha(x), \beta(x)$, and $\phi(x)$ be the polynomials introduced in Corollary 10 and let $\beta(x) = (x-\xi_1)\tilde{\beta}(x)$ with $\tilde{\beta}(x) = x - \xi_2$. Since $\xi_1 \neq \xi_2, \xi_2 \neq \xi_1 - \omega, \xi_1 \neq \xi_2 - \omega$, then from Lemmas 13 and 14 we get either $\alpha(\xi_1) = 0$ or $\alpha(\xi_2) = 0$. If $\alpha(\xi_1) = 0$, i.e., $\alpha(x) = (x-\xi_1)\tilde{\alpha}(x)$, then from Lemma 12, $\gamma_1(\xi_1 - \omega) \neq 0$ and $\phi(\xi_1) = 0$, i.e., $\phi(x) = (x-\xi_1)\tilde{\phi}(x)$. Thus, (4.7)-(4.9) and (4.13) become

$$\tilde{\alpha}(x)\mathcal{U} = \tilde{\beta}(x)\mathcal{V} + \eta_1\delta_{\xi_1}, \tag{4.27}$$

$$\tilde{\alpha}(x)D_\omega\mathcal{V} = \tilde{\phi}(x)\mathcal{V} + \eta_2\delta_{\xi_1}, \tag{4.28}$$

$$\tilde{\phi}(x)\mathcal{U} = \tilde{\beta}(x)D_\omega\mathcal{V} + \eta_3\delta_{\xi_1}, \tag{4.29}$$

$$\tilde{\phi}(x)\gamma_n(x-\omega) + \tilde{\alpha}(x)\left(D_\omega\gamma_n\right)(x-\omega) = -\varphi_{n+1}(x)\tilde{\beta}(x), \tag{4.30}$$

for $1 \leq n \leq M-1$. Hence,

$$\begin{aligned}\Big(\tilde{\phi}(x)\gamma_n(x-\omega) + \tilde{\alpha}(x)\left(D_\omega\gamma_n\right)(x-\omega)\Big)\mathcal{U} &\underset{(4.5)}{\overset{(4.30)}{=}} \tilde{\beta}(x)D_\omega[\gamma_n(x)\mathcal{V}] \\ &\underset{(4.27)}{\overset{(4.29)}{=}} \gamma_n(x-\omega)\Big(\tilde{\phi}(x)\mathcal{U} - \eta_3\delta_{\xi_1}\Big) + \left(D_\omega\gamma_n\right)(x-\omega)\Big(\tilde{\alpha}(x)\mathcal{U} - \eta_1\delta_{\xi_1}\Big),\end{aligned}$$

for $1 \leq n \leq M-1$, and, as a consequence,

$$\eta_3\gamma_n(\xi_1-\omega) = -\eta_1\left(D_\omega\gamma_n\right)(\xi_1-\omega), \quad 1 \leq n \leq M-1. \tag{4.31}$$

Since $\left(D_\omega\gamma_1\right)(\xi_1-\omega) = 1$ and $\gamma_1(\xi_1-\omega) \neq 0$, then, $\eta_1 = 0$ if and only if $\eta_3 = 0$. If $\eta_3 = 0$, (4.23) and (4.25) follow. If $\eta_3 \neq 0$ and $\tilde{\alpha}(\xi_2) \neq 0$, then $\alpha(\xi_2) \neq 0$ and hence, from Lemma 13, there exists $C \neq 0$, which is independent on n, such that $\gamma_n(\xi_2-\omega) + C\left(D_\omega\gamma_n\right)(\xi_2-\omega) = 0$, for $1 \leq n \leq M-1$. But, since $\xi_1 \neq \xi_2, \xi_2 \neq \xi_1-\omega, \xi_1 \neq \xi_2-\omega$, from Lemma 14 we obtain that neither the previous equation nor (4.31) hold, which is a contradiction. On the other hand, if $\eta_3 \neq 0$ and $\tilde{\alpha}(\xi_2) = 0$, then $\alpha(\xi_2) = 0$ and we can do the same analysis as for ξ_1 and we get

$$\tilde{\eta}_3\gamma_n(\xi_2-\omega) = -\tilde{\eta}_1\left(D_\omega\gamma_n\right)(\xi_2-\omega), \quad 1 \leq n \leq M-1. \tag{4.32}$$

Therefore, $\tilde{\eta}_1 = 0$ if and only if $\tilde{\eta}_3 = 0$. If $\tilde{\eta}_3 = 0$, (4.23) and (4.25) follow. If $\tilde{\eta}_3 \neq 0$, then from Lemma 14 either (4.31) or (4.32) can not be hold, which is a contradiction. So $\tilde{\eta}_3 = 0$.

Assume that $\eta_3 = 0$ (otherwise $\tilde{\eta}_3 = 0$ and the following holds for ξ_2). From (4.23), (4.5), (4.28) and (4.30), we obtain $-\varphi_2(x)\tilde{\beta}(x)\mathcal{V} = \gamma_1(x-\omega)\eta_2\delta_{\xi_1} - \varphi_2(x)\tilde{\beta}(x)\mathcal{V}$. Since $\gamma_1(\xi_1-\omega) \neq 0$, then $\eta_2 = 0$ and (4.24) follows. As a consequence, (4.26) holds. Finally, from Proposition 2 we get our result. $\square$

Theorem 16. *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-D_ω-coherent pair given by* (3.1) *with $M_0 \geq 5$ and $M_1 \geq 4$, and let $\beta(x)$ be the monic polynomial given by* (4.11). *If ξ is a double zero of $\beta(x)$ and $\gamma_N(\xi-\omega) + \frac{\omega}{2}(D_\omega\gamma_N)(\xi-\omega) \neq 0$ for some $3 \leq N \leq M-1$, with $\gamma_N(x)$ the monic polynomial introduced in Lemma 8, then there exist polynomials $\tilde{\alpha}(x)$ and $\tilde{\phi}(x)$, with* $\deg(\tilde{\alpha}(x)) \leq 3$, *and* $\deg(\tilde{\phi}(x)) \leq 2$, *such that*

$$\tilde{\alpha}(x)\mathcal{U} = \tilde{\beta}(x)\mathcal{V}, \tag{4.33}$$

$$\tilde{\alpha}(x)D_\omega\mathcal{V} = \tilde{\phi}(x)\mathcal{V}, \tag{4.34}$$

$$\tilde{\phi}(x)\mathcal{U} = \tilde{\beta}(x)D_\omega\mathcal{V}, \tag{4.35}$$

where $\tilde{\beta}(x) = x - \xi$. Moreover,

$$D_\omega\left[\tilde{\alpha}(x)\mathcal{V}\right] = \Big(\tilde{\phi}(x-\omega) + (D_\omega\tilde{\alpha})(x-\omega)\Big)\mathcal{V}. \tag{4.36}$$

Thus, $\mathcal{V}$ and $\mathcal{U}$ are D_ω-semiclassical linear functionals of class at most 1 *and* 5, *respectively.*

Proof. Let $\alpha(x)$ and $\phi(x)$ be the polynomials introduced in Corollary 10. Since $\gamma_1(\xi-\omega) = -\frac{\omega}{2} \neq 0$ follows from (4.18), then $\gamma_1(\xi-\omega) + C\,(D_\omega\gamma_1)(\xi-\omega) \neq 0$ for $C \neq \frac{\omega}{2}$. But for $C = \frac{\omega}{2}$, $\gamma_N(\xi-\omega) + \frac{\omega}{2}(D_\omega\gamma_N)(\xi-\omega) \neq 0$ for some $3 \leq N \leq M-1$, by hypothesis. Thus, from Lemma 13 it follows that $\alpha(\xi) = 0$, and then from Lemma 12, $\phi(\xi) = 0$. Hence, $\beta(x) = (x-\xi)\tilde{\beta}(x)$, $\alpha(x) = (x-\xi)\tilde{\alpha}(x)$, and $\phi(x) = (x-\xi)\tilde{\phi}(x)$. Therefore (4.7) - (4.9) and (4.13) become

$$\tilde{\alpha}(x)\mathcal{U} = \tilde{\beta}(x)\mathcal{V} + \tilde{\eta}_1\delta_\xi, \tag{4.37}$$

$$\tilde{\alpha}(x)D_\omega\mathcal{V} = \tilde{\phi}(x)\mathcal{V} + \tilde{\eta}_2\delta_\xi, \tag{4.38}$$

$$\tilde{\phi}(x)\mathcal{U} = \tilde{\beta}(x)D_\omega\mathcal{V} + \tilde{\eta}_3\delta_\xi, \tag{4.39}$$

$$\tilde{\phi}(x)\gamma_n(x-\omega) + \tilde{\alpha}(x)\,(D_\omega\gamma_n)(x-\omega) = -\varphi_{n+1}(x)\tilde{\beta}(x), \tag{4.40}$$

for $1 \leq n \leq M-1$. Then,

$$\begin{aligned}\Big(\tilde{\phi}(x)\gamma_n(x-\omega) + \tilde{\alpha}(x)\,(D_\omega\gamma_n)(x-\omega)\Big)\mathcal{U} &\underset{(4.5)}{\overset{(4.40)}{=}} \tilde{\beta}(x)D_\omega[\gamma_n(x)\mathcal{V}] \\ &\underset{(4.37)}{\overset{(4.39)}{=}} \gamma_n(x-\omega)\Big(\tilde{\phi}(x)\mathcal{U} - \tilde{\eta}_3\delta_\xi\Big) + (D_\omega\gamma_n)(x-\omega)\Big(\tilde{\alpha}(x)\mathcal{U} - \tilde{\eta}_1\delta_\xi\Big),\end{aligned}$$

for $1 \leq n \leq M-1$, and thus

$$\tilde{\eta}_3 \gamma_n(\xi-\omega) + \tilde{\eta}_1 \left(D_\omega \gamma_n\right)(\xi-\omega) = 0, \quad 1 \leq n \leq M-1. \tag{4.41}$$

Since $\gamma_1(\xi-\omega) \neq 0$ and $\left(D_\omega \gamma_1\right)(\xi-\omega) = 1$, then, $\tilde{\eta}_1 = 0$ if and only if $\tilde{\eta}_3 = 0$. If $\tilde{\eta}_3 \neq 0$, from (4.41) for $n=1$, we get $\tilde{\eta}_1/\tilde{\eta}_3 = -\gamma_1(\xi-\omega)$ and, as a consequence, $\gamma_n(\xi-\omega) + \frac{\omega}{2}\left(D_\omega \gamma_n\right)(\xi-\omega) = 0$ for all $1 \leq n \leq M-1$, which yields a contradiction. So $\tilde{\eta}_3 = 0$ and hence, (4.33) and (4.35) follow. Furthermore, from (4.33), (4.5), (4.38), and (4.40) we obtain $-\varphi_2(x)\tilde{\beta}(x)\mathcal{V} = \gamma_1(x-\omega)\tilde{\eta}_2\delta_\xi - \varphi_2(x)\tilde{\beta}(x)\mathcal{V}$. Thus, $\tilde{\eta}_2 = 0$ and then (4.34) follows. As a consequence, (4.36) holds. Finally, from Proposition 2 we deduce our desired result. ☐

5 The Case When $\mathcal{U}$ is D_ω-Classical

Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-D_ω-coherent pair of weakly quasi-definite linear functionals of order $M_0 \geq 2$ and $M_1 \geq 1$, respectively. In this section, we will analyze the case when $\mathcal{U}$ is a D_ω-classical linear functional given by (2.9), i.e.,

$$D_\omega\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U}, \quad \deg(\sigma(x)) \leq 2, \deg(\tau(x)) = 1.$$

The following theorem is proved for the continuous case in [2, p. 314], but its proof is similar to the D_ω-case.

Theorem 17. *Let $\{T_n(x)\}_{n=0}^{\hat{M}_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$ be two families of MOP with respect to the weakly quasi-definite linear functionals $\hat{\mathcal{U}}$ and $\mathcal{V}$ of order $\hat{M}_0 \geq 2$ and $M_1 \geq 2$, respectively, and $\langle\hat{\mathcal{U}},1\rangle = 1 = \langle\mathcal{V},1\rangle$. Then, the following statements are equivalent*

- ***i)*** *There exist complex numbers $\{a_n\}_{n=1}^{\min\{\hat{M}_0,M_1\}}, \{b_n\}_{n=1}^{\min\{\hat{M}_0,M_1\}}$, with $a_1 \neq b_1$, $a_n b_n \neq 0$, $1 \leq n \leq \min\{\hat{M}_0,M_1\}$, such that*

$$T_n(x) + a_n T_{n-1}(x) = R_n(x) + b_n R_{n-1}(x), \quad 1 \leq n \leq \min\{\hat{M}_0,M_1\}. \tag{5.1}$$

- ***ii)*** *$T_n(x) \neq R_n(x)$, for $1 \leq n \leq \min\{\hat{M}_0,M_1\}$, and there exist constants C^T, C^R, and η such that*

$$\left(x - C^T\right)\hat{\mathcal{U}} = \eta\left(x - C^R\right)\mathcal{V}. \tag{5.2}$$

Remark 18. *If $\{P_n(x)\}_{n=0}^{M_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$ are families of MOP with respect to the weakly quasi-definite linear functionals $\mathcal{U}$ and $\mathcal{V}$ of order $M_0 \geq 3$ and $M_1 \geq 2$, respectively, $\mathcal{U}$ is D_ω-classical given by* (2.9) *(this is, $\{P_n^{[1]}(x) = \frac{D_\omega P_{n+1}(x)}{n+1}\}_{n=0}^{M_0-1}$ is a family of MOP with respect to $\mathcal{U}^{[1]} = \sigma(x)\mathcal{U}$), and corresponding TTRR given as in* (2.4)*, then from the proof of Theorem 17 we obtain the following results:*

- *From proof of (i) $\Longrightarrow$ (ii), the condition $b_n \neq 0$, $1 \leq n \leq M$, can be replaced by $b_2 \neq 0$. Besides,*

$$C^{P^{[1]}} = \alpha_1^{P^{[1]}} - \frac{\beta_2^{P^{[1]}}(a_2 - b_2)}{b_2(a_1 - b_1)}, \quad C^R = \alpha_1^R - \frac{\beta_2^R(a_2 - b_2)}{a_2(a_1 - b_1)},$$
$$\eta = \frac{\beta_2^{P^{[1]}} a_2}{\beta_2^R b_2} \frac{\langle \mathcal{U}, \sigma(x) \rangle}{\langle \mathcal{V}, 1 \rangle}.$$

- *From proof of (ii) $\Longrightarrow$ (i) we get $P_1^{[1]}(x) - R_1(x) = b_1 - a_1 \neq 0$, $a_1 b_1 \neq 0$ and for $2 \leq n \leq M$,*

$$a_n = -\frac{\langle \mathcal{V}, P_n^{[1]}(x) \rangle}{\langle \mathcal{V}, P_{n-1}^{[1]}(x) \rangle} \neq 0, \quad b_n = -\frac{\langle \sigma(x)\mathcal{U}, R_n(x) \rangle}{\langle \sigma(x)\mathcal{U}, R_{n-1}(x) \rangle} \neq 0.$$

Finally, the next result it is a straightforward consequence of Theorem 17, Theorem 3, and Proposition 2.

Corollary 19. *Let $\mathcal{U}$ be a D_ω-classical linear functional given by (2.9), let $\mathcal{V}$ be a weakly quasi-definite linear functional, and let $\{P_n(x)\}_{n=0}^{M_0}$ and $\{R_n(x)\}_{n=0}^{M_1}$ be their corresponding families of MOP, with $M_0 \geq 3$ and $M_1 \geq 2$. The following statements are equivalent*

i) *$(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ω-coherent pair given by (3.1), with $a_1 \neq b_1$ and $a_n b_n \neq 0$, for $1 \leq n \leq M$.*

ii) *$\frac{D_\omega P_{n+1}(x)}{n+1} \neq R_n(x)$, for $1 \leq n \leq M$, and there exist constants $C^{P^{[1]}}$, C^R, and η (see Remark 18) such that*

$$\left(x - C^{P^{[1]}}\right) \sigma(x)\mathcal{U} = \eta \left(x - C^R\right) \mathcal{V}.$$

In this case, $\mathcal{V}$ is a D_ω-semiclassical linear functional of class at most 2.

Remark 20. *From the previous Corollary and Remark 18 it follows that if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-D_ω-coherent pair given by (3.1) with $a_1 \neq b_1$ and $b_2 \neq 0$, and $\mathcal{U}$ is a D_ω-classical linear functional given by (2.9), then*

$$\mathcal{V} = \frac{1}{\eta} \left(x - C^R\right)^{-1} \left(x - C^{P^{[1,D_\omega]}}\right) \sigma(x)\mathcal{U} + \langle \mathcal{V}, 1 \rangle \, \delta_{C^R}.$$

In particular, this equation holds when $\mathcal{U}$ is any of the D_1-classical linear functionals given in the Table 1 and Table 2.

6 A Matrix Interpretation of $(1,1)$-D_ω-Coherence

In this section, we assume that $\mathcal{U}$ and $\mathcal{V}$ are two quasi-definite linear functionals, i.e., $M = N = \infty$. We will denote by $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ their corresponding SMOP, and we will assume that they are a $(1,1)$-D_ω-coherent pair given by (3.1), i.e.,

$$\frac{D_\omega P_{n+1}(x)}{n+1} + a_n \frac{D_\omega P_n(x)}{n} = R_n(x) + b_n R_{n-1}(x), \quad a_n \neq 0, \quad n \geq 1.$$

We can write this algebraic relation in a matrix form as

$$\mathcal{A} D_\omega \mathfrak{p}(x) = \mathcal{B} \mathfrak{r}(x), \tag{6.1}$$

where

$$\mathfrak{p}(x) = \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \end{bmatrix}, \quad \mathfrak{r}(x) = \begin{bmatrix} R_0(x) \\ R_1(x) \\ \vdots \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} 1 & 1/1 & 0 & 0 & \cdots \\ 0 & a_1/1 & 1/2 & 0 & \ddots \\ 0 & 0 & a_2/2 & 1/3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ b_1 & 1 & 0 & 0 & \ddots \\ 0 & b_2 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Notice that $\mathcal{A}$ (an upper bidiagonal matrix) and $\mathcal{B}$ (a lower bidiagonal matrix) are nonsingular because $a_n \neq 0$ for $n \geq 1$. Besides, from (2.5) we have that

$$x\mathfrak{p}(x) = \mathcal{J}_p \mathfrak{p}(x), \quad x\mathfrak{r}(x) = \mathcal{J}_r \mathfrak{r}(x),$$

where $\mathcal{J}_p$ and $\mathcal{J}_r$ are the monic Jacobi matrices associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Then

$$\begin{aligned} \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r\mathfrak{r}(x) + \mathfrak{p}(x) &\overset{(2.5)}{=} x\mathcal{A}^{-1}\mathcal{B}\mathfrak{r}(x) + \mathfrak{p}(x) \overset{(6.1)}{=} xD_\omega\mathfrak{p}(x) + \mathfrak{p}(x) \\ &\overset{(2.2)}{=} D_\omega\left[(x-\omega)\mathfrak{p}(x)\right] \overset{(2.5)}{=} \left(\mathcal{J}_p - \omega\mathcal{I}\right) D_\omega \mathfrak{p}(x) \overset{(6.1)}{=} \left(\mathcal{J}_p - \omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B}\mathfrak{r}(x), \end{aligned}$$

where $\mathcal{I}$ is the infinite identity matrix. As a consequence,

$$\mathfrak{p}(x) = \left[\left(\mathcal{J}_p - \omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B} - \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r\right]\mathfrak{r}(x). \tag{6.2}$$

Hence,

$$\begin{aligned} \mathcal{J}_p\left[\left(\mathcal{J}_p - \omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B} - \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r\right]\mathfrak{r}(x) &\overset{(6.2)}{=} \mathcal{J}_p\mathfrak{p}(x) \\ &\overset{(2.5)}{=} x\mathfrak{p}(x) \overset{(6.2)}{=} x\left[\left(\mathcal{J}_p - \omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B} - \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r\right]\mathfrak{r}(x) \\ &\overset{(2.5)}{=} \left[\left(\mathcal{J}_p - \omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B} - \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r\right]\mathcal{J}_r\mathfrak{r}(x). \end{aligned}$$

In other words,

$$\mathcal{J}_p\left(\mathcal{J}_p - \omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B} - \mathcal{J}_p\mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r = \left(\mathcal{J}_p - \omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r - \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r^2.$$

Multiplying on the left by $\mathcal{A}$ and on the right by $\mathcal{B}^{-1}$ we get

$$\mathcal{A}\mathcal{J}_p\left(\mathcal{J}_p-\omega\mathcal{I}\right)\mathcal{A}^{-1}-\mathcal{A}\mathcal{J}_p\mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r\mathcal{B}^{-1}=\mathcal{A}\left(\mathcal{J}_p-\omega\mathcal{I}\right)\mathcal{A}^{-1}\mathcal{B}\mathcal{J}_r\mathcal{B}^{-1}-\mathcal{B}\mathcal{J}_r^2\mathcal{B}^{-1}. \tag{6.3}$$

Let

$$\mathcal{M}_p=\mathcal{A}\mathcal{J}_p\mathcal{A}^{-1} \quad \textit{and} \quad \mathcal{M}_r=\mathcal{B}\mathcal{J}_r\mathcal{B}^{-1}, \tag{6.4}$$

i.e., $\mathcal{M}_p$ (resp. $\mathcal{M}_r$) and $\mathcal{J}_p$ (resp. $\mathcal{J}_r$) are similar matrices. Then, (6.3) becomes

$$\begin{aligned} 0 &= \mathcal{M}_p^2-\omega\mathcal{M}_p-2\mathcal{M}_p\mathcal{M}_r+\omega\mathcal{M}_r+\mathcal{M}_r^2 \\ &= \left(\mathcal{M}_p-\mathcal{M}_r\right)^2+\mathcal{M}_r\mathcal{M}_p-\mathcal{M}_p\mathcal{M}_r-\omega\left(\mathcal{M}_p-\mathcal{M}_r\right) \\ &= \left(\mathcal{M}_p-\mathcal{M}_r\right)\left(\mathcal{M}_p-\mathcal{M}_r-\omega\right)-[\mathcal{M}_p,\mathcal{M}_r], \end{aligned}$$

where $[\mathcal{S},\mathcal{T}]$ is the *commutator* of the matrices $\mathcal{S}$ and T, defined by $[\mathcal{S},\mathcal{T}]=\mathcal{S}\mathcal{T}-\mathcal{T}\mathcal{S}$. Therefore, we have proved the following result.

Proposition 21. *If* $(\mathcal{U},\mathcal{V})$ *is a* $(1,1)$*-*D_ω*-coherent pair given by* (6.1)*, then*

$$[\mathcal{M}_p,\mathcal{M}_r]=\left(\mathcal{M}_p-\mathcal{M}_r\right)\left(\mathcal{M}_p-\mathcal{M}_r-\omega\right),$$

where $[\mathcal{M}_p,\mathcal{M}_r]$ *is the commutator of* $\mathcal{M}_p$ *and* $\mathcal{M}_r$*, and* $\mathcal{M}_p$ *and* $\mathcal{M}_r$ *are the matrices given by* (6.4)*.*

Furthermore, when $\mathcal{U}$ is a D_ω-classical linear functional we have the following result.

Proposition 22. *If* $(\mathcal{U},\mathcal{V})$ *is a* $(1,1)$*-*D_ω*-coherent pair given by* (6.1) *and* $\mathcal{U}$ *is a* D_ω*-classical linear functional, then*

$$\tilde{\mathcal{A}}\mathcal{J}_{\tilde{p}}\tilde{\mathcal{A}}^{-1}=\mathcal{M}_{\tilde{p}}=\mathcal{M}_r=\mathcal{B}\mathcal{J}_r\mathcal{B}^{-1}.$$

Therefore, $\mathcal{J}_{\tilde{p}}$ *and* $\mathcal{J}_r$*, the monic Jacobi matrices associated with the SMOP* $\{\frac{D_\omega P_{n+1}(x)}{n+1}\}_{n\geq 0}$ *and* $\{R_n(x)\}_{n\geq 0}$ *respectively, are similar matrices.*

Proof. Since $\{P_n(x)\}_{n\geq 0}$ is a D_ω-classical SMOP, so is $\{\frac{D_\omega P_{n+1}(x)}{n+1}\}_{n\geq 0}$ (see Theorem 3). Thus (6.1) becomes

$$\tilde{\mathcal{A}}\tilde{\mathfrak{p}}(x)=\mathcal{B}\mathfrak{r}(x), \tag{6.5}$$

where

$$\tilde{\mathfrak{p}}(x)=\begin{bmatrix} \frac{D_\omega P_1(x)}{1} \\ \frac{D_\omega P_2(x)}{2} \\ \vdots \end{bmatrix}, \quad \tilde{\mathcal{A}}=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ a_1 & 1 & 0 & 0 & \ddots \\ 0 & a_2 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Notice that $\tilde{\mathcal{A}}$ is a nonsingular lower bidiagonal matrix as $\mathcal{B}$. Hence,

$$\tilde{\mathcal{A}}\mathcal{J}_{\tilde{p}}\tilde{\mathcal{A}}^{-1}\mathcal{B}\mathfrak{r}(x)\overset{(6.5)}{=}\tilde{\mathcal{A}}\mathcal{J}_{\tilde{p}}\tilde{\mathfrak{p}}(x)\overset{(2.5)}{=}x\tilde{\mathcal{A}}\tilde{\mathfrak{p}}(x)\overset{(6.5)}{=}x\mathcal{B}\mathfrak{r}(x)\overset{(2.5)}{=}\mathcal{B}\mathcal{J}_r\mathfrak{r}(x),$$

where $\mathcal{J}_{\tilde{p}}$ and $\mathcal{J}_r$ are the monic Jacobi matrices associated with the SMOP $\{\frac{D_\omega P_{n+1}(x)}{n+1}\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$, respectively. Therefore, $\tilde{\mathcal{A}}\mathcal{J}_{\tilde{p}}\tilde{\mathcal{A}}^{-1} = \mathcal{B}\mathcal{J}_r\mathcal{B}^{-1}$. Finally, if $\mathcal{M}_{\tilde{p}} = \tilde{\mathcal{A}}\mathcal{J}_{\tilde{p}}\tilde{\mathcal{A}}^{-1}$ and $\mathcal{M}_r$ is as in (6.4), then the proof is complete. $\square$

For example, when $\omega = 1$, the Proposition 22 holds for the Charlier and Meixner D_1-classical SMOP, $\{C_n^{(\mu)}(x)\}_{n\geq 0}$ and $\{M_n^{(\gamma,\mu)}(x)\}_{n\geq 0}$. In these cases,

$$\frac{D_1 C_{n+1}^{(\mu)}(x)}{n+1} = C_n^{(\mu)}(x), \quad \frac{D_1 M_{n+1}^{(\gamma,\mu)}(x)}{n+1} = M_n^{(\gamma+1,\mu)}(x), \quad n \geq 0,$$

and the entries of the monic Jacobi matrix $\mathcal{J}_{\tilde{p}}$ associated with the SMOP $\{\frac{D_1 P_{n+1}(x)}{n+1}\}_{n\geq 0}$ are given in Table 2.

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