(M, N)-Coherent Pairs of linear functionals and Jacobi matrices

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Abstract

A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ in the linear space of polynomials with complex coefficients is said to be a (M, N)-*coherent pair of order* m if their corresponding sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy a structure relation

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0,$$

where M, N, and m are non-negative integers, $\{a_{i,n}\}_{n\geq 0}$, $0 \leq i \leq M$, and $\{b_{i,n}\}_{n\geq 0}$, $0 \leq i \leq N$, are sequences of complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ if $i > n$. When $m = 1$, $(\mathcal{U}, \mathcal{V})$ is called a (M, N)-*coherent pair* .

In this work, we give a matrix interpretation of (M, N)-coherent pairs of linear functionals. Indeed, an algebraic relation between the corresponding monic tridiagonal (Jacobi) matrices associated with such linear functionals is stated. As a particular situation, we analyze the case when one the linear functionals is classical. Finally, the relation between the Jacobi matrices associated with (M, N)-coherent pairs of linear functionals of order m and the Hessenberg matrix associated with the multiplication operator in terms of the basis of monic polynomials orthogonal with respect to the Sobolev inner product defined by the pair $(\mathcal{U}, \mathcal{V})$ is deduced.

Keywords: coherent pairs, structure relations, regular linear functionals, orthogonal polynomials, classical orthogonal polynomials, Sobolev orthogonal polynomials, monic Jacobi matrix.

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1 Introduction

The more general notion of coherent pair was introduced and studied by M. N. de Jesus, F. Marcellán, J. Petronilho, and N. C. Pinzón-Cortés in [9], which includes, as a particular case, the concept of (M, N)-coherent pair of order m.

A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ in the linear space of polynomials with complex coefficients is said to be a (M, N)-*coherent pair of order* m, M, N, and m fixed non-negative integers, if their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy the algebraic relation

$$P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x) = Q_n(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0, \tag{1.1}$$

where $\{a_{i,n}\}_{n\geq 0}, \{b_{i,n}\}_{n\geq 0} \subset \mathbb{C}$, $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, $a_{i,n} = b_{i,n} = 0$ when $i > n$, and $P_n^{[m]}(x)$ denotes the monic polynomial of degree n

$$P_n^{[m]}(x) = \frac{P_{n+m}^{(m)}(x)}{(n+1)_m}, \quad m, n \geq 0,$$

where $(n+1)_m$ is the *Pochhammer symbol* defined by $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, $n \geq 1$, and $(\alpha)_0 = 1$. Besides, when $m = 1$, $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-*coherent pair*.

They unify the generalizations given in the literature of the concept of coherent pair (in our terminology, $(1,0)$-coherent pair) introduced by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna in [8] (see also the work of H. G. Meijer [19]). In fact, A. Delgado and F. Marcellán ([5]) introduced $(1,1)$-coherent pairs, K. H. Kwon, J. H. Lee, and F. Marcellán ([12]) considered $(2,0)$-coherent pairs, P. Maroni and R. Sfaxi ([17, 18]) looked at the $(M+N, 2M+1)$-coherence relation, F. Marcellán, A. Martínez-Finkelshtein, and J. Moreno-Balcázar ([14]) presented the $(M,0)$-coherence relation, M. Alfaro, F. Marcellán, A. Peña, and M. L. Rezola ([1, 2]) studied $(1,1)$-coherent pairs of order 0, J. Petronilho ([20]) considered the (M,N)-coherence relation of order 0, M. N. de Jesus and J. Petronilho ([10, 11]) analyzed (M,N)-coherent pairs, A. Branquinho and M. N. Rebocho ([3]) looked at the $(1,0)$-coherence relation of order 2, and, F. Marcellán and N. C. Pinzón-Cortés ([15]) studied $(1,1)$-coherent pairs of order m. For a review about these and other works, see for instance, the introductory sections in the papers [15] and [11].

Additionally, there is a close relation between Sobolev orthogonal polynomials and (M,N)-coherent pairs of order m. Indeed, when the linear functionals $\mathcal{U}$ and $\mathcal{V}$ are positive definite and μ_0 and μ_1 are the positive Borel measures associated with them, respectively, M. N. de Jesus, F. Marcellán, J. Petronilho, and N. C. Pinzón-Cortés ([9]) showed that the (M,N)-coherence relation of order m (1.1), $m \geq 1$, implies

$$\begin{gathered} P_{n+m}(x) + \sum_{i=1}^{M} \frac{(n+1)_m a_{i,n}}{(n-i+1)_m} P_{n-i+m}(x) = S_{n+m}(x;\lambda) + \sum_{j=1}^{K} c_{j,n}(\lambda) S_{n-j+m}(x;\lambda), \; n \geq 0, \\ S_n(x;\lambda) = P_n(x), \quad n \leq m, \end{gathered} \tag{1.2}$$

where $K = \max\{M,N\}$, $c_{j,n}(\lambda)$, $1 \leq j \leq K$, $n \geq 0$, are rational functions in $\lambda > 0$ such that $c_{j,n}(\lambda) = 0$ for $n < j \leq K$, and $\{S_n(x;\lambda)\}_{n\geq 0}$ is the SMOP with respect to the Sobolev inner product

$$\langle p(x), r(x) \rangle_\lambda = \int_{\mathbb{R}} p(x) r(x) d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x) r^{(m)}(x) d\mu_1, \quad \lambda > 0, \, m \in \mathbb{Z}^+, \tag{1.3}$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients. The algebraic relations in (1.2) allow to study analytic properties of Sobolev orthogonal polynomials (see [9]).

In this paper, we will study the (M,N)-coherence relations from a matrix point of view. In Section 2, we will state notations, basic definitions, and the background which will be helpful in the following sections. In Section 3, we will analyze (M,N)-coherent pairs of regular linear functionals focusing our attention on the monic Jacobi matrices associated with the linear functionals which constitute such a coherent pair. Besides, the classical case will be analyzed in more detail. Finally, we will obtain the matrix representation of the multiplication operator by x with respect to the basis of monic Sobolev orthogonal polynomials $\{S_n(x;\lambda)\}_{n\geq 0}$.

2 Basic Background

2.1 Orthogonal Polynomials

A linear functional $\mathcal{U}:\mathbb{P}\to\mathbb{C}$ defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients is said to be *quasi-definite or regular* ([4]) if the leading principal submatrices

$$H_n=\left[u_{i+j}\right]_{i,j=0}^{n},\quad u_n=\langle\mathcal{U},x^n\rangle,\quad n\geq 0,$$

of the Hankel matrix associated with the *moments* of the linear functional are nonsingular. Equivalently, there exists a unique sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ such that

$$\deg(P_n(x))=n\quad\text{and}\quad\langle\mathcal{U},P_n(x)P_m(x)\rangle=k_n\delta_{n,m},\ \text{with } k_n\neq 0,\ n,m\geq 0.$$

$\{P_n(x)\}_{n\geq 0}$ is said to be the *sequence of monic orthogonal polynomials (SMOP)* with respect to $\mathcal{U}$. When $\det(H_n)>0$, $n\geq 0$, then $\mathcal{U}$ is said to be *positive definite*. In this case, there exists a positive Borel measure μ supported on an infinite subset E of the real line such that $\langle\mathcal{U},p(x)\rangle=\int_E p(x)d\mu(x)$, $p\in\mathbb{P}$.

An important characterization of orthogonal polynomials is given by the *Favard Theorem* ([4]) as follows. A sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ is a SMOP with respect to a regular linear functional $\mathcal{U}$ if and only if there exist $\{\alpha_n^P\}_{n\geq 0},\{\beta_n^P\}_{n\geq 0}\subset\mathbb{C}$, $\beta_n^P\neq 0$, $n\geq 2$, such that they satisfy a *three-term recurrence relation (TTRR)*

$$P_n(x)=\left(x-\alpha_n^P\right)P_{n-1}(x)-\beta_n^P P_{n-2}(x),\ n\geq 1,\quad P_0(x)=1,\ P_{-1}(x)=0. \tag{2.1}$$

Moreover, $\mathcal{U}$ is positive definite if and only if α_n^P is real and $\beta_{n+1}^P>0$, for $n\geq 1$.

This TTRR can be written in a matrix form as

$$x\mathbf{p}(x)=\mathcal{J}_P\mathbf{p}(x), \tag{2.2}$$

where

$$\mathbf{p}(x)=\left[\begin{array}{ccc} P_0(x), & P_1(x), & \cdots \end{array}\right]^T,\quad \mathcal{J}_P=\left[\begin{array}{ccccc} \alpha_1^P & 1 & 0 & 0 & \cdots \\ \beta_2^P & \alpha_2^P & 1 & 0 & \ddots \\ 0 & \beta_3^P & \alpha_3^P & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{array}\right],$$

where the semi-infinite tridiagonal matrix $\mathcal{J}_P$ is said to be the *monic Jacobi matrix* associated with the regular linear functional $\mathcal{U}$.

On the other hand, for a linear functional $\mathcal{U}$ and a nonzero polynomial $r(x)$, $r(x)\mathcal{U}$ is the linear functional defined by

$$\big\langle r(x)\mathcal{U}, p(x)\big\rangle = \big\langle \mathcal{U}, r(x)p(x)\big\rangle, \quad p \in \mathbb{P},$$

and the (distributional) *derivative of* $\mathcal{U}$, $D\mathcal{U}$, is defined by

$$\langle D\mathcal{U}, p(x)\rangle = -\langle \mathcal{U}, p'(x)\rangle, \quad p \in \mathbb{P}.$$

2.2 Classical Linear Functionals

When a linear functional $\mathcal{U}$ is regular and there exist $\sigma(x), \tau(x) \in \mathbb{P} \setminus \{0\}$, $\sigma(x)$ monic, such that

$$D\big[\sigma(x)\mathcal{U}\big] = \tau(x)\mathcal{U} \quad \text{with } \deg(\sigma(x)) \leq 2 \text{ and } \deg(\tau(x)) = 1, \tag{2.3}$$

holds, then $\mathcal{U}$ is said to be a *classical* linear functional, and its associated SMOP is called *classical SMOP*. For characterizations of the classical SMOP and its classification see for example [4, 13, 16, 21]. One of these is as follows.

Theorem 2.1 ([7]). *Let $\mathcal{U}$ be a regular linear functional and let $\{P_n(x)\}_{n\geq 0}$ be its corresponding SMOP. Then $\{P_n(x)\}_{n\geq 0}$ is a classical SMOP if and only if for fixed $m \geq 1$, $\{P_n^{[m]}(x)\}_{n\geq 0}$ is a SMOP with respect to some (regular) linear functional $\mathcal{U}^{[m]}$.*

Moreover, if $\mathcal{U}$ satisfies (2.3), then $\mathcal{U}^{[m]} = \sigma^m(x)\mathcal{U}$ and $\{P_n^{[m]}(x)\}_{n\geq 0}$ is also a classical SMOP of the same type as $\{P_n(x)\}_{n\geq 0}$ since $\mathcal{U}^{[m]}$ satisfies

$$D\Big[\sigma(x)\mathcal{U}^{[m]}\Big] = \big[\tau(x) + m\sigma'(x)\big]\mathcal{U}^{[m]}.$$

Under linear transformations of the variable and some conditions on the parameters, the Hermite, Laguerre and Jacobi SMOP are the classical SMOP with respect to definite positive linear functionals. In Table 2.1 we give the polynomials $\sigma(x)$ and $\tau(x)$ satisfying (2.3), the weight function $w(x)$ (positive, integrable and supported on a infinite set of the real line) such that the classical linear functional can be represented as $\langle \mathcal{U}, p(x)\rangle = \int_a^b p(x)w(x)dx$, $p \in \mathbb{P}$, the coefficients α_n^P and β_n^P, $n \geq 1$, appearing in the TTRR (2.1), and the monic orthogonal polynomials $P_n^{[m]}(x)$, $n, m \geq 0$.

3 A Matrix Interpretation of (M, N)-Coherence

Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be the SMOP associated with regular linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively. Then, let us consider a relation such as

$$\mathbf{p}'(x) = \mathcal{M}\mathbf{q}(x), \tag{3.1}$$

where $\mathcal{M}$ is a infinite matrix and

$$\mathbf{p}(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad \mathbf{q}(x) = \begin{bmatrix} Q_0(x), & Q_1(x), & \cdots \end{bmatrix}^T.$$

Notice that the entries of the 0th row of $\mathcal{M}$ are all 0's (since $P_0'(x) = 0$). On the other hand, from the TTRR (2.2) satisfied by $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, it follows that

$$x\mathbf{p}(x) = \mathcal{J}_P\mathbf{p}(x) \quad \text{and} \quad x\mathbf{q}(x) = \mathcal{J}_Q\mathbf{q}(x) \tag{3.2}$$

Table 2.1: Classical SMOP: Hermite, Laguerre and Jacobi.

	Hermite	**Laguerre**	**Jacobi**
$P_n(x)$	$H_n(x)$	$L_n^{(\alpha)}(x)$	$P_n^{(\alpha,\beta)}(x)$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$-x+\alpha+1$	$-(\alpha+\beta+2)x+\beta-\alpha$
$(a,b)\subset\mathbb{R}$	$(-\infty,\infty)$	$(0,\infty)$	$(-1,1)$
$w(x)$	e^{-x^2}	$x^\alpha e^{-x}$	$(1-x)^\alpha(1+x)^\beta$
Restriction	$-$	$\alpha>-1$	$\alpha>-1, \beta>-1$
α_{n+1}^P	0	$2n+\alpha+1$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$
β_{n+1}^P	$\frac{1}{2}n$	$n(n+\alpha)$	$\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}$
$P_n^{[m]}(x)$	$H_n(x)$	$L_n^{(\alpha+m)}(x)$	$P_n^{(\alpha+m,\beta+m)}(x)$

hold, where $\mathcal{J}_P$ and $\mathcal{J}_Q$ are their corresponding Jacobi Matrices. Thus,

$$\mathcal{M}\mathcal{J}_Q\mathbf{q}(x)+\mathbf{p}(x) \underset{(3.2)}{\overset{(3.1)}{=}} x\mathbf{p}'(x)+\mathbf{p}(x) = \big(x\mathbf{p}(x)\big)' \overset{(3.2)}{=} \mathcal{J}_P\mathbf{p}'(x) \overset{(3.1)}{=} \mathcal{J}_P\mathcal{M}\mathbf{q}(x),$$

i.e.,

$$\mathbf{p}(x) = (\mathcal{J}_P\mathcal{M}-\mathcal{M}\mathcal{J}_Q)\,\mathbf{q}(x). \tag{3.3}$$

As a consequence,

$$\mathcal{J}_P\,(\mathcal{J}_P\mathcal{M}-\mathcal{M}\mathcal{J}_Q)\,\mathbf{q}(x) \overset{(3.3)}{=} \mathcal{J}_P\mathbf{p}(x) \underset{(3.2)}{\overset{(3.3)}{=}} (\mathcal{J}_P\mathcal{M}-\mathcal{M}\mathcal{J}_Q)\,\mathcal{J}_Q\mathbf{q}(x),$$

Therefore, we have proved the following result

Lemma 3.1. *If two SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are related by an expression such as (3.1), i.e.,*

$$\mathbf{p}'(x)=\mathcal{M}\mathbf{q}(x),$$

then

$$\mathcal{J}_P^2\mathcal{M}-2\mathcal{J}_P\mathcal{M}\mathcal{J}_Q+\mathcal{M}\mathcal{J}_Q^2=0, \tag{3.4}$$

where $\mathcal{J}_P$ and $\mathcal{J}_Q$ are the Jacobi matrices associated with $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, respectively.

Now, let us consider a (M,N)-coherent pair of regular linear functionals $(\mathcal{U},\mathcal{V})$ given by

$$\frac{P'_{n+1}(x)}{n+1}+\sum_{i=1}^{M}a_{i,n}\frac{P'_{n-i+1}(x)}{n-i+1} = Q_n(x)+\sum_{i=1}^{N}b_{i,n}Q_{n-i}(x), \quad n\geq 0, \tag{3.5}$$

where $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are the SMOP associated with $\mathcal{U}$ and $\mathcal{V}$, respectively, $a_{M,n}\neq 0$ if $n\geq M$, $b_{N,n}\neq 0$ if $n\geq N$, and $a_{i,n}=b_{i,n}=0$ if $i>n$. In order to use Lemma 3.1, the (M,N)-coherence relation (3.5) can be written in two matrix forms as

$$\mathcal{A}_1\mathbf{p}'_1(x)=\mathcal{B}\mathbf{q}(x), \tag{3.6}$$

$$\mathcal{A}\mathbf{p}'(x) = \mathcal{B}\mathbf{q}(x), \tag{3.7}$$

where $\mathbf{p}(x)$ and $\mathbf{q}(x)$ are given as in (3.1),

$$\mathbf{p}_1(x) = \begin{bmatrix} P_1(x), & P_2(x), & \cdots \end{bmatrix}^T,$$

$\mathcal{A}$ is a lower Hessenberg matrix with $M+1$ nonzero diagonals (such that the entries of its superdiagonal are $\frac{1}{n+1}$, $n \geq 0$, and the entries of its main diagonal are 1 and $\frac{a_{1,n}}{n}$, $n \geq 1$), and, $\mathcal{A}_1$ and $\mathcal{B}$ are nonsingular lower triangular matrices with $M+1$ and $N+1$ nonzero diagonals, respectively, (whose entries of their main diagonals are $\frac{1}{n+1}$, $n \geq 0$, and 1's, respectively). These infinite matrices $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{B}$ are such that

$$\begin{array}{c} \left[\begin{array}{cccccccc} \underbrace{0 \ \cdots \ 0}_{n-M+1 \text{ zeros}} & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{2,n}}{n-1} & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots \end{array} \right], \\ \uparrow \\ n\text{th position} \\ \downarrow \\ \left[\begin{array}{ccccccc} \underbrace{0 \ \cdots \ 0}_{n-M \text{ zeros}} & \frac{a_{M,n}}{n-M+1} & \cdots & \frac{a_{1,n}}{n} & \frac{1}{n+1} & 0 & \cdots \end{array} \right], \\ \uparrow \\ n\text{th position} \\ \downarrow \\ \left[\begin{array}{ccccccc} \underbrace{0 \ \cdots \ 0}_{n-N \text{ zeros}} & b_{N,n} & \cdots & b_{1,n} & 1 & 0 & \cdots \end{array} \right] \end{array}$$

are their corresponding nth rows, for $n \geq 1$, and, $[1\ 1\ 0\ \cdots]$, $[1\ 0\ \cdots]$, and $[1\ 0\ \cdots]$ are their 0th rows, respectively, (counting the rows from zero).

Proposition 3.2. *Let $(\mathcal{U}, \mathcal{V})$ be a (M,N)-coherent pair given by (3.6) and let $\mathcal{J}_P$ and $\mathcal{J}_Q$ be the Jacobi matrices associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Then*

$$\mathcal{J}_P^2 \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} - 2\mathcal{J}_P \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} \mathcal{J}_Q + \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} \mathcal{J}_Q^2 = 0$$

holds, where $\mathbf{0}$ is the zero row, i.e., $\mathbf{0} = \begin{bmatrix} 0, & 0, & \cdots \end{bmatrix}$.

Proof. (3.6) can be read as (3.1) as follows

$$\mathbf{p}_1'(x) = \mathcal{A}_1^{-1}\mathcal{B}\mathbf{q}(x) \quad \Longrightarrow \quad \mathbf{p}'(x) = \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} \mathbf{q}(x),$$

i.e., the matrix $\mathcal{M}$ in (3.1) is the matrix obtained from $\mathcal{A}_1^{-1}\mathcal{B}$ by shifting the matrix one position downward, adding a zero row to top. As a consequence, Lemma 3.1 holds and, in particular, (3.4) holds replacing $\mathcal{M}$ by $\begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix}$. □

Proposition 3.3. *If $(\mathcal{U}, \mathcal{V})$ is a (M,N)-coherent pair given by (3.7) such that $\mathcal{A}$ is a nonsingular matrix (for example, when $M = 1$ and $N \geq 0$, $\mathcal{A}$ is a nonsingular upper bidiagonal matrix, since $a_{1,n} \neq 0$ for $n \geq 1$), then*

$$(\mathcal{M}_P - \mathcal{M}_Q)^2 = [\mathcal{M}_P, \mathcal{M}_Q],$$

where $[\mathcal{M}_P, \mathcal{M}_Q]$ is the commutator of the matrices $\mathcal{M}_P$ and $\mathcal{M}_Q$ defined by

$$[\mathcal{M}_P, \mathcal{M}_Q] = \mathcal{M}_P\mathcal{M}_Q - \mathcal{M}_Q\mathcal{M}_P,$$

and, $\mathcal{M}_P$ and $\mathcal{M}_Q$ are given by

$$\mathcal{M}_P = \mathcal{A}\mathcal{J}_P\mathcal{A}^{-1} \quad \text{and} \quad \mathcal{M}_Q = \mathcal{B}\mathcal{J}_Q\mathcal{B}^{-1},$$

i.e., $\mathcal{M}_P$ and $\mathcal{M}_Q$ are similar matrices to the monic Jacobi matrices $\mathcal{J}_P$ and $\mathcal{J}_Q$ associated with $\mathcal{U}$ and $\mathcal{V}$, respectively.

Proof. Let $\mathcal{M} = \mathcal{A}^{-1}\mathcal{B}$. Then, from (3.7) and Lemma 3.1,

$$\mathcal{J}_P^2\mathcal{A}^{-1}\mathcal{B} - 2\mathcal{J}_P\mathcal{A}^{-1}\mathcal{B}\mathcal{J}_Q + \mathcal{A}^{-1}\mathcal{B}\mathcal{J}_Q^2 = 0$$

holds. Therefore, if we multiply in the left by $\mathcal{A}$ and in the right by $\mathcal{B}^{-1}$ in both sides of the previous equation, we obtain

$$0 = \mathcal{M}_P^2 - 2\mathcal{M}_P\mathcal{M}_Q + \mathcal{M}_Q^2 = (\mathcal{M}_P - \mathcal{M}_Q)^2 - [\mathcal{M}_P, \mathcal{M}_Q],$$

which is the desired result. □

As a particular case, let us consider the situation when $\mathcal{U}$ is a classical linear functional, since $\{P_n^{[m]}(x)\}_{n\geq 0}$, $m \geq 0$, is again a classical SMOP from Theorem 2.1. Hence, (3.5) reads

$$P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n}P_{n-i}^{[m]}(x) = Q_n(x) + \sum_{i=1}^{N} b_{i,n}Q_{n-i}(x), \quad n \geq 0, \tag{3.8}$$

where $M, N, m \in \mathbb{Z}^+ \cup \{0\}$, $\{a_{i,n}\}_{n\geq 0}, \{b_{i,n}\}_{n\geq 0} \subset \mathbb{C}$, $a_{M,n} \neq 0$, $n \geq M$, $b_{N,n} \neq 0$, $n \geq N$, $a_{i,n} = b_{i,n} = 0$, $i > n$. Writing (3.8) in a matrix form, we get

$$\widehat{\mathcal{A}}\widehat{\mathbf{p}}(x) = \mathcal{B}\mathbf{q}(x), \tag{3.9}$$

where $\mathcal{B}$ and $\mathbf{q}(x)$ are given as in (3.7),

$$\widehat{\mathbf{p}}(x) = \begin{bmatrix} P_0^{[m]}(x), & P_1^{[m]}(x), & \cdots \end{bmatrix}^T,$$

and $\widehat{\mathcal{A}}$ is the lower triangular matrix with $M+1$ nonzero diagonals, such that its nth row for $n \geq 1$ (counting the rows from zero), is

$$[\quad \underbrace{0 \quad \cdots \quad 0}_{n-M \text{ zeros}} \quad a_{M,n} \quad \cdots \quad a_{1,n} \quad 1 \quad 0 \quad \cdots \quad],$$

and the entries of its main diagonal are all 1's.

A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ satisfying (3.8) is said to be a (M,N)-*coherent pair of order* m. Notice that a (M,N)-coherent pair is a (M,N)-coherent pair of order 1.

Proposition 3.4. *Let $(\mathcal{U}, \mathcal{V})$ be a (M,N)-coherent pair of order m, $m \geq 0$, given by (3.9) and such that $\mathcal{U}$ is a classical linear functional. Then $\mathcal{J}_{P^{[m]}}$ and $\mathcal{J}_Q$, the monic Jacobi matrices associated with the SMOP $\{P_n^{[m]}(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ respectively, are similar matrices satisfying*

$$\widehat{\mathcal{A}}\mathcal{J}_{P^{[m]}}\widehat{\mathcal{A}}^{-1} = \mathcal{M}_{P^{[m]}} = \mathcal{M}_Q = \mathcal{B}\mathcal{J}_Q\mathcal{B}^{-1}.$$

Proof. Since $\{P_n^{[m]}(x)\}_{n\geq 0}$ is a classical SMOP, then from (2.2), $\widehat{\mathbf{p}}(x)$ in (3.9) satisfies

$$x\widehat{\mathbf{p}}(x) = \mathcal{J}_{P^{[m]}}\widehat{\mathbf{p}}(x). \tag{3.10}$$

Thus, multiplying (3.10) by $\widehat{\mathcal{A}}$ and taking into account that $\widehat{\mathcal{A}}$ and $\mathcal{B}$ are nonsingular matrices, we get

$$\widehat{\mathcal{A}}\mathcal{J}_{P^{[m]}}\widehat{\mathcal{A}}^{-1}\mathcal{B}\mathbf{q}(x) \overset{(3.9)}{=} \widehat{\mathcal{A}}\mathcal{J}_{P^{[m]}}\widehat{\mathbf{p}}(x) \overset{(3.10)}{=} x\widehat{\mathcal{A}}\widehat{\mathbf{p}}(x) \overset{(3.9)}{=} x\mathcal{B}\mathbf{q}(x) \overset{(3.2)}{=} \mathcal{B}\mathcal{J}_Q\mathbf{q}(x).$$

Since the basis are the same, the proof is complete. □

Remark 3.5. Proposition 3.4 can be used for Hermite, Laguerre, and Jacobi SMOP taking into account the information presented in Table 2.1.

Remark 3.6. The well known Hahn's condition characterizing classical orthogonal polynomials (see [6]) has been considered from a matrix approach in [22] where nice and elegant proofs of some well known characterizations of classical orthogonal polynomials are given. Therein, the connection between the Jacobi matrices associated with the SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{\frac{P_n'(x)}{n+1}\}_{n\geq 0}$ is stated in formula (4.2) and, as a consequence, it is possible to get the general expression of the coefficients of the TTRR for classical orthogonal polynomials by identifying the corresponding entries. Notice that the statement of Lemma 3.1 generalizes (4.2) for families of orthogonal polynomials related by a relation like $\mathbf{p}'(x) = \mathcal{M}\mathbf{q}(x)$. On the other hand, (4.2) is a straightforward consequence of Proposition 3.2 when $\mathcal{A}_1 = diag(1, 1/2, 1/3, ...)$ and $\mathcal{B} = I$.

3.1 A Matrix Interpretation of Sobolev Orthogonal Polynomials and (M, N)-Coherence of Order m

Let us consider the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$ given in the introductory section, which is orthogonal with respect the inner product (1.3), this is,

$$\langle p(x), r(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)r(x)d\mu_0 + \lambda\int_{\mathbb{R}} p^{(m)}(x)r^{(m)}(x)d\mu_1, \quad \lambda > 0,\ m\in\mathbb{Z}^+,$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients and μ_0 and μ_1 are positive Borel measures supported on an infinite subset of the real line.

In [9], it was proved that if $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order m, $m \geq 1$, of positive definite linear functionals such that μ_0 and μ_1 are their corresponding positive Borel measures, then the SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{S_n(x;\lambda)\}_{n\geq 0}$ with respect to $\mathcal{U}$ and $\langle\cdot,\cdot\rangle_\lambda$, respectively, satisfy the algebraic relation (1.2). So, using the notation

$$\widetilde{a}_{i,n} = \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n}, \quad n \geq 0,$$

where $\widetilde{a}_{i,n} = 0$ when $i > n$, and $\widetilde{a}_{0,n} = 1$ for $n \geq 0$, (since $a_{i,n} = 0$, $i > n$, $a_{0,n} = 1$, $n \geq 0$), (1.2) reads

$$P_{n+m}(x) + \sum_{i=1}^{M}\widetilde{a}_{i,n}P_{n-i+m}(x) = S_{n+m}(x;\lambda) + \sum_{j=1}^{K}c_{j,n}(\lambda)S_{n-j+m}(x;\lambda), \quad n \geq 0,$$

$$S_n(x;\lambda) = P_n(x), \quad n \leq m,$$

where $c_{j,n}(\lambda) = 0$, $n < j \leq K$, and $K = \max\{M, N\}$. Hence, we can express these relations as

$$\widetilde{\mathcal{A}}\mathbf{p}(x) = \mathcal{C}\mathbf{s}(x;\lambda), \tag{3.11}$$

where

$$\mathbf{p}(x) = \begin{bmatrix} P_0(x), & P_1(x), & \cdots \end{bmatrix}^T, \quad \mathbf{s}(x;\lambda) = \begin{bmatrix} S_0(x;\lambda), & S_1(x;\lambda), & \cdots \end{bmatrix}^T,$$

and the matrices $\widetilde{\mathcal{A}}$ and $\mathcal{C}$ are banded matrices. More precisely, they are lower triangular matrices with $M+1$ and $K+1$ nonzero diagonals, respectively, whose entries of their main diagonal are all 1's, their first m rows (counting from zero) are $[0\ \cdots\ 0\ 1\ 0\ \cdots]$, and their $(n+m)$th rows, $n \geq 0$, are, respectively,

$$\begin{bmatrix} \underbrace{0 \quad \cdots \quad 0}_{n-M+m \text{ zeros}} & \widetilde{a}_{M,n} & \cdots & \widetilde{a}_{1,n} & 1 & 0 & \cdots \end{bmatrix},$$

$$\begin{array}{c} \uparrow \\ (n+m)\text{th place} \\ \downarrow \end{array}$$

$$\begin{bmatrix} \underbrace{0 \quad \cdots \quad 0}_{n-K+m \text{ zeros}} & c_{K,n}(\lambda) & \cdots & c_{1,n}(\lambda) & 1 & 0 & \cdots \end{bmatrix}.$$

Therefore,

$$x\mathbf{s}(x;\lambda) \overset{(3.11)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}x\mathbf{p}(x) \overset{(2.2)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{J}_P\mathbf{p}(x) \overset{(3.11)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{J}_P\widetilde{\mathcal{A}}^{-1}\mathcal{C}\mathbf{s}(x;\lambda).$$

In other words, the matrix representation of the multiplication operator by x in terms of the basis $\{S_n(x;\lambda)\}_{n\geq 0}$ is

$$x\mathbf{s}(x;\lambda) = \mathcal{H}_{P,\lambda}\mathbf{s}(x;\lambda),$$

where $\mathcal{H}_{P,\lambda}$ is the infinite lower Hessenberg matrix

$$\mathcal{H}_{P,\lambda} = \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{J}_P\widetilde{\mathcal{A}}^{-1}\mathcal{C},$$

similar to the monic Jacobi matrix $\mathcal{J}_P$ associated with the SMOP $\{P_n(x)\}_{n\geq 0}$.

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