On an inverse problem for a linear combination of orthogonal polynomials

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Abstract. This paper deals with the analysis of the orthogonality of a monic polynomial sequence $\{Q_n\}_{n\geq 0}$ defined as a linear combination of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with

$$Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + r_n P_{n-3}(x)\ ,\ n \geq 0,$$

where $r_n \neq 0$ for $n \geq 3$. Moreover, we obtain the relation between the corresponding linear functionals as well as an explicit expression for the sequence of monic orthogonal polynomials $\{Q_n\}_{n\geq 0}$. We obtain the connection between the Jacobi matrices associated with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively, by using a LU factorization. Some special cases of the above type relation are analyzed.

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1. Introduction

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the action of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A sequence of monic polynomials $\{P_n\}_{n\geq 0}$ is called orthogonal with respect to the linear functional u if the following orthogonality conditions hold

$$\begin{cases} \langle u, P_n P_m\rangle = 0\ , & n \neq m, \\ \langle u, P_n^2\rangle \neq 0\ , & n = 0, 1, ..., \end{cases}$$

where $\deg P_n = n$ for every $n = 0, 1, ...$ (see [1]).

The linear functional u is called quasi-definite (regular) if the leading principal submatrices H_n of the Hankel matrix $H = (u_{i+j})_{i,j\geq 0}$ related to the moments $u_k = \langle u, x^k\rangle$, $k \geq 0$, are nonsingular for each $n \geq 0$ (see [1]).

The following theorem gives a well-known characterization for the sequence of monic orthogonal polynomials (SMOP) $\{P_n\}_{n\geq 0}$ associated with the quasi-definite linear functional u.

Theorem 1.1. *Let u be a quasi-definite linear functional and $\{P_n\}_{n\geq 0}$ be the corresponding SMOP. Then, $\{P_n\}_{n\geq 0}$ satisfies the following three-term recurrence relation*

$$\begin{aligned} P_{n+1}(x) &= (x-\beta_n) P_n(x) - \gamma_n P_{n-1}(x) \quad , \quad n \geq 0, \\ P_{-1}(x) &= 0 \quad , \quad P_0(x) = 1, \end{aligned} \tag{1.1}$$

where $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ are sequences of complex numbers with $\gamma_n \neq 0$ for each $n \geq 1$.

Conversely, if a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ satisfies a three-term recurrence relation like (1.1), then there exists a unique quasi-definite linear functional u such that $\{P_n\}_{n\geq 0}$ is the corresponding SMOP . This result is known in the literature as Favard's Theorem (see [1]).

Let $\{P_n^{(1)}\}_{n\geq 0}$ be the associated SMOP of the first kind related to a SMOP $\{P_n\}_{n\geq 0}$ with respect to u. Then, $\{P_n^{(1)}\}_{n\geq 0}$ is defined by the three-term recurrence relation

$$\begin{aligned} P_{n+1}^{(1)}(x) &= (x-\beta_{n+1}) P_n^{(1)}(x) - \gamma_{n+1} P_{n-1}^{(1)}(x) \quad , \quad n \geq 0, \\ P_{-1}^{(1)}(x) &= 0 \quad , \quad P_0^{(1)}(x) = 1 \quad . \end{aligned}$$

Furthermore, $P_n^{(1)}(x) = \frac{1}{u_0}\left\langle u, \frac{P_{n+1}(x)-P_{n+1}(t)}{x-t}\right\rangle$ where the linear functional u acts on the variable t (see [1]). Also, we will denote by $\{P_n(.,\alpha)\}_{n\geq 0}$ the co-recursive SMOP defined by

$$\begin{aligned} P_{n+1}(x,\alpha) &= (x-\beta_n) P_n(x,\alpha) - \gamma_n P_{n-1}(x,\alpha) \quad , \quad n \geq 1, \\ P_1(x,\alpha) &= P_1(x) - \alpha \quad , \quad P_0(x,\alpha) = 1 \quad . \end{aligned}$$

The following connection formula is widely known in the literature (see [2])

$$P_n(x,\alpha) = P_n(x) - \alpha P_{n-1}^{(1)}(x) \quad , \quad n \geq 1 \quad .$$

In the theory of orthogonal polynomials, the well-known basic canonical spectral transformations of a quasi-definite linear functional u are (see [3])

(i) Christoffel Transformation

$$v_1 = (x-a)\, u \quad ,$$

(ii) Uvarov Transformation

$$v_2 = u + M\delta_a \quad ,$$

(iii) Geronimus Transformation

$$v_3 = (x-a)^{-1} u + M\delta_a \quad ,$$

where left-multiplication of a linear functional $u \in \mathcal{P}'$ by a polynomial $f \in \mathcal{P}$ is defined by

$$\langle fu, p\rangle = \langle u, fp\rangle \quad , \quad p \in \mathcal{P},$$

division of u by a first degree polynomial

$$\left\langle (x-a)^{-1} u, p \right\rangle = \left\langle u, \frac{p(x) - p(a)}{x-a} \right\rangle \quad , \quad p \in \mathcal{P} \quad .$$

There, δ_a is the Dirac functional at the mass point a given by

$$\langle \delta_a, p \rangle = p(a) \quad , \quad p \in \mathcal{P},$$

and $M \in \mathbb{R}$.

Necessary and sufficient conditions under which the linear functionals v_1, v_2, and v_3 are quasi-definite and the algebraic relations between the corresponding SMOP's are deeply investigated in the literature (see $[2-6]$).

Let u be a quasi-definite linear functional and $\{P_n\}_{n\geq 0}$ be the corresponding SMOP. When there is an algebraic relation between $\{P_n\}_{n\geq 0}$ and a given monic polynomial sequence $\{Q_n\}_{n\geq 0}$ as

$$Q_n(x) + \sum_{i=1}^{m-1} a_{i,n} Q_{n-i}(x) = P_n(x) + \sum_{i=1}^{k-1} b_{i,n} P_{n-i}(x) \quad , \quad n \geq 0 \quad , \tag{1.2}$$

to find necessary and sufficient conditions such that $\{Q_n\}_{n\geq 0}$ is also orthogonal and the relation between the corresponding quasi-definite linear functionals is said to be an inverse problem. Here m, $k \in \mathbb{Z}^+$ and $(a_{i,n})_{n\geq 0}$ and $(b_{i,n})_{n\geq 0}$ are sequences of complex numbers. The cases $m=1$, $k=2$ and $m=2$, $k=1$ have been studied in [7]. For the case $m=2$, $k=2$, one can find a detailed investigation in [8]. Later on, the situation $m=1$, $k=N$, with constant coefficients has been considered in [9]. In this contribution, necessary and sufficient conditions in order to $\{Q_n\}_{n\geq 0}$ be a SMOP are obtained and a matrix method by using Jacobi matrices associated with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ is derived. Moreover, for the case $N=3$, the authors described the recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ corresponding to $\{P_n\}_{n\geq 0}$ such that $\{Q_n\}_{n\geq 0}$ is also orthogonal. Then, in [10] the authors dealt with the case $m=1$, $k=3$. The characterization of the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$, the relation between the quasi-definite linear functionals such that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are the corresponding SMOP's and a matrix interpretation, where Jacobi matrices and Christoffel formula ([11]) play central role, are remarkably deduced. First time, this type of inverse problem was discussed in [12]. Recently, similar analysis has been done in [13] for the case $m=2$, $k=3$. On the other hand, the general linear structure relations given by (1.2) have been analyzed in [14] with an additional assumption about the orthogonality of $\{Q_n\}_{n\geq 0}$.

In this paper, our aim is to focus our attention on the special case of the general linear structure relations (1.2) with $m=1$, $k=4$ i. e. for $n \geq 0$

$$Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x) + r_n P_{n-3}(x) \quad , \tag{1.3}$$

where s_n, t_n, and r_n are complex numbers with the initial conditions $s_0 = t_0 = t_1 = r_0 = r_1 = r_2 = 0$ and $r_n \neq 0$ when $n \geq 3$. We obtain necessary and sufficient conditions for the orthogonality of the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$. In addition, we get the relation between the linear functionals u and v, respectively, corresponding to the SMOP's $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ by $q(x)\,v = ku$, where $q(x) = x^3 + ax^2 + bx + c$ is a monic cubic polynomial and $k \in \mathbb{C}\setminus\{0\}$. It is worthy to note that for the special case $q(x) = x^3$, this problem was solved in [15]. Moreover, an explicit expression of the SMOP $\{Q_n\}_{n\geq 0}$ is derived. A connection between the corresponding monic Jacobi matrices is presented and some special cases of the relation (1.3) are finally discussed.

2. Basic Results

In this section, $\{P_n\}_{n\geq 0}$ will denote a SMOP with respect to a quasi-definite linear functional u. An immediate consequence for $\{P_n\}_{n\geq 0}$ is to satisfy the three-term recurrence relation (1.1) with recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$. First of all, we characterize the orthogonality of the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ which is related to $\{P_n\}_{n\geq 0}$ by (1.3).

Theorem 2.1. *Let $\{P_n\}_{n\geq 0}$ be a given SMOP and let $\{Q_n\}_{n\geq 0}$ be a sequence of polynomials given by (1.3), with $r_n \neq 0$ for all $n \geq 3$. Then $\{Q_n\}_{n\geq 0}$ is a SMOP with recurrence coefficients $\{\tilde{\beta}_n\}_{n\geq 0}$ and $\{\tilde{\gamma}_n\}_{n\geq 1}$ given by (2.5)-(2.6) if and only if the following conditions hold*

$$\tilde{\gamma}_n = \gamma_n + t_n - t_{n+1} + s_n\left(\beta_{n-1} - \beta_n - s_n + s_{n+1}\right) \neq 0,\ n \geq 1, \tag{2.1}$$

as well as

$$s_{n-1}\tilde{\gamma}_n = s_n\gamma_{n-1} + r_n - r_{n+1} + t_n\left(\beta_{n-2} - \beta_n - s_n + s_{n+1}\right),\ n \geq 2, \tag{2.2}$$

$$t_{n-1}\tilde{\gamma}_n = t_n\gamma_{n-2} + r_n\left(\beta_{n-3} - \beta_n - s_n + s_{n+1}\right)\ ,\ n \geq 3, \tag{2.3}$$

$$r_{n-1}\tilde{\gamma}_n = r_n\gamma_{n-3}\ ,\quad n \geq 4\ , \tag{2.4}$$

together with $\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 \neq 0$. Furthermore, the coefficients of the three-term recurrence relation for $\{Q_n\}_{n\geq 0}$ are

$$\tilde{\beta}_n = \beta_n + s_n - s_{n+1}\ ,\quad n \geq 0, \tag{2.5}$$

$$\tilde{\gamma}_n = \gamma_n + t_n - t_{n+1} + s_n\left(\beta_{n-1} - \beta_n - s_n + s_{n+1}\right)\ ,\ n \geq 1\ . \tag{2.6}$$

Proof. Multiplying each hand side of the relation (1.3) by x, we obtain

$$xQ_n(x) = xP_n(x) + s_n xP_{n-1}(x) + t_n xP_{n-2}(x) + r_n xP_{n-3}(x)\ ,\ n \geq 0\ . \tag{2.7}$$

Substituting the recurrence relation (1.1) into (2.7) for $xP_n(x)$, $xP_{n-1}(x)$, $xP_{n-2}(x)$, and $xP_{n-3}(x)$, then using relation (1.3) for $P_{n+1}(x)$, we get for $n \geq 0$

$$\begin{aligned} xQ_n(x) = {} & Q_{n+1}(x) + \left(\beta_n + s_n - s_{n+1}\right)P_n(x) \\ & + \left[\gamma_n + t_n - t_{n+1} + s_n\beta_{n-1}\right]P_{n-1}(x) \\ & + \left[s_n\gamma_{n-1} + r_n - r_{n+1} + t_n\beta_{n-2}\right]P_{n-2}(x) \\ & + \left[t_n\gamma_{n-2} + r_n\beta_{n-3}\right]P_{n-3}(x) + r_n\gamma_{n-3}P_{n-4}(x) \end{aligned}$$

with the convention $P_{-n}(x) = 0$, $n \geq 1$. By using relation (1.3) for $P_n(x)$, the last equality becomes for $n \geq 0$

$$\begin{aligned} xQ_n(x) &= Q_{n+1}(x) + \tilde{\beta}_n Q_n(x) \\ &+ [\gamma_n + t_n - t_{n+1} + s_n(\beta_{n-1} - \beta_n - s_n + s_{n+1})] P_{n-1}(x) \\ &+ [s_n\gamma_{n-1} + r_n - r_{n+1} + t_n(\beta_{n-2} - \beta_n - s_n + s_{n+1})] P_{n-2}(x) \\ &+ [t_n\gamma_{n-2} + r_n(\beta_{n-3} - \beta_n - s_n + s_{n+1})] P_{n-3}(x) \\ &+ r_n\gamma_{n-3} P_{n-4}(x) \end{aligned}$$

where $\tilde{\beta}_n$ is exactly given by (2.5). Then, according to relation (1.3) for $P_{n-1}(x)$, we have for $n \geq 0$

$$\begin{aligned} xQ_n(x) &= Q_{n+1}(x) + \tilde{\beta}_n Q_n(x) + \tilde{\gamma}_n Q_{n-1}(x) \\ &+ [s_n\gamma_{n-1} + r_n - r_{n+1} + t_n(\beta_{n-2} - \beta_n - s_n + s_{n+1}) - s_{n-1}\tilde{\gamma}_n] P_{n-2}(x) \\ &+ [t_n\gamma_{n-2} + r_n(\beta_{n-3} - \beta_n - s_n + s_{n+1}) - t_{n-1}\tilde{\gamma}_n] P_{n-3}(x) \\ &+ [r_n\gamma_{n-3} - r_{n-1}\tilde{\gamma}_n] P_{n-4}(x) \end{aligned} \tag{2.8}$$

where $\tilde{\gamma}_n$ is given by (2.6). Thus, from (2.8), $\{Q_n\}_{n\geq 0}$ is a SMOP if and only if the conditions (2.1) – (2.4) are satisfied. □

Remark 2.2. Taking into account Theorem 2.1, given a SMOP $\{P_n\}_{n\geq 0}$ in order to $\{Q_n\}_{n\geq 0}$ be a SMOP, the coefficients s_{n+1}, t_{n+1}, and r_{n+1} can be generated from the previous ones as follows from (2.3) and (2.4)

$$s_{n+1} = s_n + \beta_n - \beta_{n-3} + \frac{t_{n-1}}{r_{n-1}}\gamma_{n-3} - \frac{t_n}{r_n}\gamma_{n-2} \ ,$$

from (2.6) and (2.4)

$$t_{n+1} = t_n + s_n(\beta_{n-1} - \beta_{n-3}) + \gamma_n + s_n\left(\frac{t_{n-1}}{r_{n-1}}\gamma_{n-3} - \frac{t_n}{r_n}\gamma_{n-2}\right) - \frac{r_n}{r_{n-1}}\gamma_{n-3} \ ,$$

from (2.2) and (2.4)

$$\begin{aligned} r_{n+1} &= r_n + t_n(\beta_{n-2} - \beta_{n-3}) + s_n\gamma_{n-1} - s_{n-1}\frac{r_n}{r_{n-1}}\gamma_{n-3} \\ &+ t_n\left(\frac{t_{n-1}}{r_{n-1}}\gamma_{n-3} - \frac{t_n}{r_n}\gamma_{n-2}\right) \ , \end{aligned}$$

for $n \geq 4$. Note that s_1, s_2, s_3, t_2, t_3, and r_3 are free parameters.

On the other hand, the first previous relation (equivalent to (2.1)) can be written as

$$s_{n+1} - \beta_n - \beta_{n-1} - \beta_{n-2} + \frac{t_n}{r_n}\gamma_{n-2} = s_n - \beta_{n-1} - \beta_{n-2} - \beta_{n-3} + \frac{t_{n-1}}{r_{n-1}}\gamma_{n-3}, \ \ n \geq 4,$$

and so there exists a constant A such that

$$s_{n+1} - \beta_n - \beta_{n-1} - \beta_{n-2} + \frac{t_n}{r_n}\gamma_{n-2} = A, \ \ n \geq 3,$$

Next, we state that there is a relation between quasi-definite linear functionals when $\{Q_n\}_{n\geq 0}$ is a SMOP with respect to a quasi-definite linear functional v. If the sequence of linear functionals $\{\omega_n\}_{n\geq 0}$ according to the SMOP $\{P_n\}_{n\geq 0}$ satisfies the condition $\langle \omega_n, P_m\rangle = \delta_{nm}$, then $\{\omega_n\}_{n\geq 0}$ is called the dual basis of $\{P_n\}_{n\geq 0}$ where δ_{nm} is the Kronecker delta. It is well known that

$$\omega_n = \frac{P_n u}{\langle u, P_n^2\rangle} \quad .$$

Theorem 2.3. *(see* [16]*) Let* $\{P_n\}_{n\geq 0}$ *be a SMOP with respect to a quasi-definite linear functional* u *and the sequence of monic polynomials* $\{Q_n\}_{n\geq 0}$ *be given by the relation* (1.3)*. If* $\{Q_n\}_{n\geq 0}$ *is a SMOP with respect to a quasi-definite linear functional* v*, then*

$$q(x)\, v = ku$$

where $q(x) = x^3 + ax^2 + bx + c$, $k \in \mathbb{C}\backslash\{0\}$ *and the normalizations for these linear functionals* $\langle u, 1\rangle = \langle v, 1\rangle = 1$.

Proof. Applying the quasi-definite linear functional u corresponding to the SMOP $\{P_n\}_{n\geq 0}$ in (1.3), we get for $n \geq 4$

$$\langle u, Q_n\rangle = 0 \quad .$$

Then, according to [6] and taking into account (1.3), we expand the linear functional u in terms of the dual basis $\left\{\frac{Q_j v}{\langle v, Q_j^2\rangle}\right\}_{j\geq 0}$ of the SMOP $\{Q_n\}_{n\geq 0}$ as

$$\begin{aligned} u &= \sum_{j=0}^{3} \frac{\langle u, Q_j\rangle}{\langle v, Q_j^2\rangle} Q_j v \\ &= \left[1 + \frac{s_1}{\langle v, Q_1^2\rangle} Q_1 + \frac{t_2}{\langle v, Q_2^2\rangle} Q_2 + \frac{r_3}{\langle v, Q_3^2\rangle} Q_3\right] v \end{aligned} \tag{2.9}$$

where $\langle u, Q_0\rangle = 1$, $\langle u, Q_1\rangle = s_1$, $\langle u, Q_2\rangle = t_2$, and $\langle u, Q_3\rangle = r_3$.

Since $\{Q_n\}_{n\geq 0}$ is the SMOP with respect to v, the recurrence coefficients $\left\{\tilde{\beta}_n\right\}_{n\geq 0}$ and $\{\tilde{\gamma}_n\}_{n\geq 1}$ are given as in (2.5) and (2.6). Furthermore,

$$\tilde{\gamma}_n = \frac{\langle v, Q_n^2\rangle}{\langle v, Q_{n-1}^2\rangle} \neq 0 \ , \ n \geq 1 \ . \tag{2.10}$$

Combining (2.10) and the relation (1.3) with (2.9), we get the desired result with $k = \frac{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}{r_3}$ where $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\gamma}_3$ are given by (2.6).
On the other hand, notice that $a = \frac{q''(0)}{2} = s_3 - \beta_0 - \beta_1 - \beta_2 + t_2\frac{\tilde{\gamma}_3}{r_3}$. According to relation (2.3), we get $a = s_4 - \beta_1 - \beta_2 - \beta_3 + t_3\frac{\gamma_1}{r_3} = A$. □

Now, we will find the explicit expression of the SMOP $\{Q_n\}_{n\geq 0}$ given by relation (1.3).

Theorem 2.4. *Let $\{P_n\}_{n\geq 0}$ be a SMOP with respect to a quasi-definite linear functional u, $\{Q_n\}_{n\geq 0}$ be given by the relation* (1.3) *and let assume the cubic polynomial $q(x)$ has simple zeros α_1, α_2, α_3. If $\{Q_n\}_{n\geq 0}$ is a SMOP with respect to a quasi-definite linear functional v, then for $n \geq 3$*

$$Q_n(x) = \frac{1}{\Delta_n}\begin{vmatrix} P_n(x) & P_{n-1}(x) & P_{n-2}(x) & P_{n-3}(x) \\ R_n(\alpha_1;c_1) & R_{n-1}(\alpha_1;c_1) & R_{n-2}(\alpha_1;c_1) & R_{n-3}(\alpha_1;c_1) \\ R_n(\alpha_2;c_2) & R_{n-1}(\alpha_2;c_2) & R_{n-2}(\alpha_2;c_2) & R_{n-3}(\alpha_2;c_2) \\ R_n(\alpha_3;c_3) & R_{n-1}(\alpha_3;c_3) & R_{n-2}(\alpha_3;c_3) & R_{n-3}(\alpha_3;c_3) \end{vmatrix}$$

where

$$\Delta_n = \begin{vmatrix} R_{n-1}(\alpha_1;c_1) & R_{n-2}(\alpha_1;c_1) & R_{n-3}(\alpha_1;c_1) \\ R_{n-1}(\alpha_2;c_2) & R_{n-2}(\alpha_2;c_2) & R_{n-3}(\alpha_2;c_2) \\ R_{n-1}(\alpha_3;c_3) & R_{n-2}(\alpha_3;c_3) & R_{n-3}(\alpha_3;c_3) \end{vmatrix} \neq 0 \quad ,$$

$R_n(x;\eta) = \eta P_n(x) + P_{n-1}^{(1)}(x)$, $\eta \in \mathbb{C}$, $k \in \mathbb{C}\setminus\{0\}$, $\langle u, 1\rangle = \langle v, 1\rangle = 1$, *and*

$$\begin{aligned} c_1 &= \frac{1}{k}\langle v, (x-\alpha_2)(x-\alpha_3)\rangle \ , \\ c_2 &= \frac{1}{k}\langle v, (x-\alpha_1)(x-\alpha_3)\rangle \ , \\ c_3 &= \frac{1}{k}\langle v, (x-\alpha_1)(x-\alpha_2)\rangle \quad . \end{aligned}$$

Proof. Assume that $\{Q_n\}_{n\geq 0}$ is a SMOP with respect to v. Then, according to Theorem 2.3,

$$q(x)\, v = ku \quad . \tag{2.11}$$

On one hand, if we take $x = \alpha_1$ at relation (1.3), then we obtain

$$Q_n(\alpha_1) = P_n(\alpha_1) + s_n P_{n-1}(\alpha_1) + t_n P_{n-2}(\alpha_1) + r_n P_{n-3}(\alpha_1) \quad , \ n \geq 3 \ .$$

Substracting the last equality from relation (1.3), dividing each hand side by $x - \alpha_1$ and after applying the linear functional u, we get

$$\begin{aligned} \left\langle u, \frac{Q_n(x) - Q_n(\alpha_1)}{x-\alpha_1} \right\rangle &= P_{n-1}^{(1)}(\alpha_1) + s_n P_{n-2}^{(1)}(\alpha_1) + t_n P_{n-3}^{(1)}(\alpha_1) \\ &\quad + r_n P_{n-4}^{(1)}(\alpha_1) \quad . \end{aligned} \tag{2.12}$$

At the left-hand side of (2.12), we get by using (2.11) for $n \geq 3$

$$\begin{aligned} \left\langle u, \frac{Q_n(x) - Q_n(\alpha_1)}{x-\alpha_1} \right\rangle &= \frac{1}{k}\langle v, (x-\alpha_2)(x-\alpha_3)[Q_n(x) - Q_n(\alpha_1)]\rangle \\ &= -c_1 Q_n(\alpha_1) \end{aligned}$$

where

$$c_1 = \frac{1}{k}\langle v, (x-\alpha_2)(x-\alpha_3)\rangle \quad .$$

Thus, for $n \geq 3$

$$-c_1 Q_n(\alpha_1) = P_{n-1}^{(1)}(\alpha_1) + s_n P_{n-2}^{(1)}(\alpha_1) + t_n P_{n-3}^{(1)}(\alpha_1) + r_n P_{n-4}^{(1)}(\alpha_1) \quad .$$

Taking into account relation (1.3) and the definition of the polynomials $R_n(x;\eta)$ in the last equality, we have for $n \geq 3$

$$-R_n(\alpha_1;c_1) = s_n R_{n-1}(\alpha_1;c_1) + t_n R_{n-2}(\alpha_1;c_1) + r_n R_{n-3}(\alpha_1;c_1) \quad . \tag{2.13}$$

After similar computations, it follows for $n \geq 3$

$$-R_n(\alpha_2;c_2) = s_n R_{n-1}(\alpha_2;c_2) + t_n R_{n-2}(\alpha_2;c_2) + r_n R_{n-3}(\alpha_2;c_2) \tag{2.14}$$

$$-R_n(\alpha_3;c_3) = s_n R_{n-1}(\alpha_3;c_3) + t_n R_{n-2}(\alpha_3;c_3) + r_n R_{n-3}(\alpha_3;c_3) \tag{2.15}$$

where

$$\begin{aligned} c_2 &= \frac{1}{k}\langle v, (x-\alpha_1)(x-\alpha_3)\rangle \quad , \\ c_3 &= \frac{1}{k}\langle v, (x-\alpha_1)(x-\alpha_2)\rangle \quad . \end{aligned}$$

Hence, we reach the desired result from (2.13) – (2.15) and the relation (1.3). □

Remark 2.5. Let α_1 be a simple zero and α_2 be a zero with multiplicity 2 of the cubic polynomial $q(x)$. Then, taking into account Theorem 2.4, substracting (2.15) from (2.14) and after dividing each hand side by $\alpha_2 - \alpha_3$, we obtain when α_3 tends to α_2

$$-R_n^{'}(\alpha_2;c_3) = s_n R_{n-1}^{'}(\alpha_2;c_3) + t_n R_{n-2}^{'}(\alpha_2;c_3) + r_n R_{n-3}^{'}(\alpha_2;c_3) \tag{2.16}$$

where we take $\beta_0 - s_1 = \alpha_1$ and $\frac{\beta_0+\beta_1-s_2}{2} \neq \alpha_2$. Notice that $k = \langle v, q(x)\rangle \in \mathbb{C}\setminus\{0\}$. In a similar way, we get the explicit expression of the SMOP $\{Q_n\}_{n\geq 0}$ from (2.13), (2.14), and (2.16) by

$$Q_n(x) = \frac{1}{\Lambda_n}\begin{vmatrix} P_n(x) & P_{n-1}(x) & P_{n-2}(x) & P_{n-3}(x) \\ R_n(\alpha_1;c_1) & R_{n-1}(\alpha_1;c_1) & R_{n-2}(\alpha_1;c_1) & R_{n-3}(\alpha_1;c_1) \\ R_n(\alpha_2;c_2) & R_{n-1}(\alpha_2;c_2) & R_{n-2}(\alpha_2;c_2) & R_{n-3}(\alpha_2;c_2) \\ R_n^{'}(\alpha_2;c_3) & R_{n-1}^{'}(\alpha_2;c_3) & R_{n-2}^{'}(\alpha_2;c_3) & R_{n-3}^{'}(\alpha_2;c_3) \end{vmatrix}$$

where $n \geq 3$, $\beta_0 - s_1 = \alpha_1$, $\frac{\beta_0+\beta_1-s_2}{2} \neq \alpha_2$, and

$$\Lambda_n = \begin{vmatrix} R_{n-1}(\alpha_1;c_1) & R_{n-2}(\alpha_1;c_1) & R_{n-3}(\alpha_1;c_1) \\ R_{n-1}(\alpha_2;c_2) & R_{n-2}(\alpha_2;c_2) & R_{n-3}(\alpha_2;c_2) \\ R_{n-1}^{'}(\alpha_2;c_3) & R_{n-2}^{'}(\alpha_2;c_3) & R_{n-3}^{'}(\alpha_2;c_3) \end{vmatrix} \neq 0 \quad .$$

In this situation, notice that

$$\begin{aligned} c_1 &= \frac{1}{k}\left\langle v, (x-\alpha_2)^2\right\rangle \quad , \\ c_2 &= c_3 = \frac{1}{k}\langle v, (x-\alpha_1)(x-\alpha_2)\rangle \quad . \end{aligned}$$

Additionally, if the cubic polynomial $q(x)$ has a zero with multiplicity 3, then one can find a detailed study in [15].

2.1. Jacobi matrices and inverse cubic polynomial transformations. Let $\mathbf{P} = (P_0, P_1, ...)^T$ and $\mathbf{Q} = (Q_0, Q_1, ...)^T$ be the column vectors associated with the SMOP's $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively. From Theorem 1.1, $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ satisfy the three-term recurrence relation of the form (1.1) which can be read

$$x\mathbf{P} = \mathbf{J}_P\mathbf{P} \quad , \quad x\mathbf{Q} = \mathbf{J}_Q\mathbf{Q} \tag{2.17}$$

where $\mathbf{J}_P$ and $\mathbf{J}_Q$ are the corresponding monic Jacobi matrices. Now, we explain a method in order to find the matrix $\mathbf{J}_Q$ with the help of the matrix $\mathbf{J}_P$, the polynomial $q(x) = x^3 + ax^2 + bx + c$ and the relation (1.3).

Taking into account the relation (1.3), we have

$$\mathbf{Q} = \mathbf{A}\mathbf{P} \tag{2.18}$$

where $\mathbf{A} = (a_{i,j})_{i,j\geq 1}$ is a banded lower triangular matrix with $a_{i,i} = 1$ and $a_{i,j} = 0$ for $i - j > 3$. Then, according to (2.17) and (2.18), we obtain

$$\begin{aligned} x\mathbf{A}\mathbf{P} &= \mathbf{J}_Q\mathbf{A}\mathbf{P} \quad , \quad \text{i. e.} \\ \mathbf{A}\mathbf{J}_P &= \mathbf{J}_Q\mathbf{A} \end{aligned}$$

and, as a consequence,

$$\mathbf{J}_Q = \mathbf{A}\mathbf{J}_P\mathbf{A}^{-1} \quad . \tag{2.19}$$

Besides, it follows from Christoffel formula [11] that

$$q(x)\,\mathbf{P} = \mathbf{B}\mathbf{Q} \tag{2.20}$$

where $\mathbf{B} = (b_{i,j})_{i,j\geq 1}$ is a banded upper triangular matrix with $b_{i,i+3} = 1$ and $b_{i,j} = 0$ for $j - i > 3$. Combining (2.17) and (2.18) with (2.20) provides

$$q(\mathbf{J}_P) = \mathbf{B}\mathbf{A} \quad . \tag{2.21}$$

Notice that $q(\mathbf{J}_P)$ is a seventh-diagonal matrix. Consequently, from the equality (2.21), we can find the matrix $\mathbf{B}$. On the other hand, in view of the equalities (2.19) and (2.21), we get

$$\begin{aligned} q(\mathbf{J}_Q) &= \mathbf{A}q(\mathbf{J}_P)\,\mathbf{A}^{-1} \\ &= \mathbf{A}\mathbf{B} \quad . \end{aligned}$$

Finally, we find the matrix $\mathbf{J}_Q$ from the last equality. Notice that we have extended the standard Geronimus transformation to a cubic case. Thus (2.21) is the UL factorization of the matrix $q(\mathbf{J}_P)$ and we have three free parameters which are the masses of the linear functional v at the zeros of the polynomial $q(x)$.

Next, we describe a method to state the connection between the truncated monic Jacobi matrices. Indeed, for $(\mathbf{P})_n = (P_0, P_1, ..., P_n)^T$ and $(\mathbf{Q})_n = (Q_0, Q_1, ..., Q_n)^T$, the truncated form of the three-term recurrence relations (2.17) and the relation (2.18) are

$$x(\mathbf{P})_n = (\mathbf{J}_P)_{n+1}(\mathbf{P})_n + P_{n+1}e_{n+1} \tag{2.22}$$

$$x(\mathbf{Q})_n = (\mathbf{J}_Q)_{n+1}(\mathbf{Q})_n + Q_{n+1}e_{n+1} \tag{2.23}$$

$$(\mathbf{Q})_n = (\mathbf{A})_{n+1}(\mathbf{P})_n \tag{2.24}$$

where the symbol $(.)_n$ denotes the truncation of any infinite matrix at level n and $e_{n+1} = (0, ..., 0, 1)^T \in \mathbb{R}^{n+1}$. Considering the relation (2.24) in (2.23) leads us to

$$\begin{aligned} x\left(\mathbf{A}\right)_{n+1}\left(\mathbf{P}\right)_n &= \left[\left(\mathbf{J}_Q\right)_{n+1}\left(\mathbf{A}\right)_{n+1} + e_{n+1}\left(s_{n+1}e_{n+1}^T + t_{n+1}e_n^T + r_{n+1}e_{n-1}^T\right)\right]\left(\mathbf{P}\right)_n \\ &\quad + P_{n+1}e_{n+1} \end{aligned}$$

where $e_n = (0, ..., 0, 1, 0)^T$ and $e_{n-1} = (0, ..., 0, 1, 0, 0)^T$ are elements of $\mathbb{R}^{n+1}$. Replacing (2.22) in the last equality gives

$$\begin{aligned} \left(\mathbf{A}\right)_{n+1}\left(\mathbf{J}_P\right)_{n+1} &= \left(\mathbf{J}_Q\right)_{n+1}\left(\mathbf{A}\right)_{n+1} \\ &\quad + \left(\mathbf{A}\right)_{n+1} e_{n+1}\left(s_{n+1}e_{n+1}^T + t_{n+1}e_n^T + r_{n+1}e_{n-1}^T\right) \quad . \end{aligned}$$

Thus, we find the truncated form of $\mathbf{J}_Q$ by a rank-one perturbation of the corresponding truncated form of $\mathbf{J}_P$ with

$$\left(\mathbf{J}_Q\right)_{n+1} = \left(\mathbf{A}\right)_{n+1}\left[\left(\mathbf{J}_P\right)_{n+1} - e_{n+1}\left(s_{n+1}e_{n+1}^T + t_{n+1}e_n^T + r_{n+1}e_{n-1}^T\right)\right]\left(\mathbf{A}\right)_{n+1}^{-1} \quad ,$$

that is to say, $\left(\mathbf{J}_Q\right)_{n+1}$ is a rank-one perturbation of the matrix $\left(\mathbf{J}_P\right)_{n+1}$.

3. Special Cases

In this section, we will discuss some special cases of relation (1.3).

In [17], the author pointed out that Bernstein-Szegö polynomials can be represented as a linear combination of Chebyshev polynomials with constant coefficients, i. e.,

$$Q_{n,i}(x) = P_{n,i}(x) + a_{1,i}P_{n-1,i}(x) + ... + a_{k,i}P_{n-k,i}(x) \quad , \quad n > k \quad ,$$

where $\{P_{n,i}\}_{n\geq 0}$ is the sequence of Chebyshev polynomials of i-th kind $(i = 1, 2, 3, 4)$ and $a_{k,i} \neq 0$. Then, each kind of Bernstein-Szegö polynomials $\{Q_{n,i}\}_{n\geq 0}$ $(i = 1, 2, 3, 4)$ is a sequence of orthogonal polynomials with respect to weight functions $\omega_i(x)$ if and only if

$$\omega_i(x) = \frac{\mu_i(x)}{\sigma_k(x)} \quad , \quad (i = 1, 2, 3, 4) \quad ,$$

$\mu_i(x)$ is the Chebyshev weight function of each kind $(i = 1, 2, 3, 4)$ and $\sigma_k(x)$ is a positive polynomial of degree k on $(-1, 1)$.

In the relation (1.3), let us assume that the coefficients are constant

$$Q_n(x) = P_n(x) + h_1P_{n-1}(x) + h_2P_{n-2}(x) + h_3P_{n-3}(x) \quad , \quad n \geq 4 \quad , \quad (3.1)$$

where h_1, h_2, $h_3 \in \mathbb{R}$ and $h_3 \neq 0$. This means that the sequence $\{Q_n\}_{n\geq 0}$ is quasi-orthogonal of order 3 with respect to the linear functional u (see [1]). Under this hypothesis, from Theorem 2.1, we obtain necessary and sufficient conditions for the orthogonality of $\{Q_n\}_{n\geq 0}$ given by (3.1). Indeed,

$$\tilde{\gamma}_n = \gamma_n + h_1(\beta_{n-1} - \beta_n) \neq 0 \quad , \quad n \geq 4 \quad ,$$

and for $n \geq 5$

$$h_2(\beta_n - \beta_{n-2}) = h_1(\gamma_{n-1} - \gamma_{n-3}) \quad , \tag{3.2}$$
$$h_3(\beta_n - \beta_{n-3}) = h_2(\gamma_{n-2} - \gamma_{n-3}) \quad , \tag{3.3}$$
$$h_1(\beta_n - \beta_{n-1}) = \gamma_n - \gamma_{n-3} \quad , \tag{3.4}$$

where the recurrence coefficients for the SMOP $\{Q_n\}_{n\geq 0}$ are

$$\tilde{\gamma}_n = \gamma_n + h_1(\beta_{n-1} - \beta_n) \quad , \quad n \geq 4 \quad ,$$
$$\tilde{\beta}_n = \beta_n \quad , \quad n \geq 4 \quad .$$

These orthogonality conditions are also obtained in [9]. Now, our aim is to define all SMOP $\{P_n\}_{n\geq 0}$ such that $\{Q_n\}_{n\geq 0}$ given by (3.1) is also orthogonal.

(i) Case $h_1 = 0$.

In this case, we will analyze two subcases.

(a) $h_2 = 0$.

In such a situation, it is clear from (3.2) – (3.4) that for $n \geq 5$

$$\beta_n = \beta_{n-3} \quad ,$$
$$\gamma_n = \gamma_{n-3} \quad .$$

Hence, $\{P_n\}_{n\geq 0}$ is a 3-periodic SMOP up to the initial conditions $\beta_0, \beta_1, \gamma_1$. Then, $\{Q_n\}_{n\geq 0}$ is also a 3-periodic SMOP with the recurrence coefficients

$$\tilde{\gamma}_n = \gamma_n \quad , \quad \tilde{\beta}_n = \beta_n \quad , \quad n \geq 4 \quad .$$

Sequences of monic orthogonal polynomials with recurrence relations with periodic coefficients were firstly studied by Ya. L. Geronimus (see [18] and [19]) as well as in [20]. Such SMOP can be expressed in terms of a polynomial mapping on the Chebyshev polynomials of the second kind (suitably shifted and rescaled) as was proved in Theorem 5.1 [21]. For a general theory of orthogonality and polynomial mappings see [22].

(b) $h_2 \neq 0$.

From (3.2) – (3.4), we get for $n \geq 5$

$$\beta_n = \beta_{n-2} \quad , \tag{3.5}$$
$$\gamma_n = \gamma_{n-3} \quad , \tag{3.6}$$
$$h_3(\beta_n - \beta_{n-3}) = h_2(\gamma_{n-2} - \gamma_{n-3}) \quad . \tag{3.7}$$

By combining (3.5) and (3.7), we obtain

$$h_3(\beta_{n-2} - \beta_{n-3}) = h_2(\gamma_{n-2} - \gamma_{n-3}) \quad , \quad n \geq 5 \ , \tag{3.8}$$

that is to say,

$$h_2\gamma_n = h_3\beta_n + C \quad , \quad n \geq 3 \ ,$$

where $C = h_2\gamma_2 - h_3\beta_2$ is a fixed constant. According to (3.5) and the last equalities, we have

$$\gamma_n = \gamma_{n-2} \quad , \quad n \geq 5 \ ,$$

and together with (3.6)

$$\gamma_n = \gamma_2 \quad , \quad n \geq 2 \ .$$

Furthermore, from (3.8)

$$\begin{aligned} \beta_{n-2} &= \beta_{n-3} \quad , \quad n \geq 5 \ , \quad \text{i. e. ,} \\ \beta_n &= \beta_2 \quad , \quad n \geq 2 \quad , \end{aligned}$$

with $\beta_2 = \frac{1}{h_3}(h_2\gamma_2 - C)$. Then, $\{P_n\}_{n\geq 0}$ is a SMOP with constant recurrence coefficients and $\beta_2 = \frac{1}{h_3}(h_2\gamma_2 - C)$ but $\beta_0, \beta_1, \gamma_1$ are free parameters. These yields a perturbation in the initial conditions of Chebyshev polynomials of second kind, together with a shift in the variable. On the other hand, $\{Q_n\}_{n\geq 0}$ is also a SMOP with constant recurrence coefficients

$$\tilde{\gamma}_n = \gamma_n = \gamma_2 \quad , \quad \tilde{\beta}_n = \beta_n = \beta_2 \quad , \quad n \geq 4 \quad .$$

(ii) Case $h_1 \neq 0$.

In this case, the following two subcases will appear.

(a) $h_2 = 0$.

The conditions (3.2) – (3.4) become for $n \geq 5$

$$\gamma_{n-1} = \gamma_{n-3} \quad , \tag{3.9}$$

$$\beta_n = \beta_{n-3} \quad , \tag{3.10}$$

$$\beta_n - \beta_{n-1} = \frac{1}{h_1}(\gamma_n - \gamma_{n-3}) \quad . \tag{3.11}$$

By using (3.9) – (3.11), we find

$$\begin{aligned} \beta_n - \beta_{n-1} &= \frac{1}{h_1}(\gamma_n - \gamma_{n-1}) \quad , \quad n \geq 5 \quad , \ \text{i. e. ,} \\ \beta_n &= \frac{1}{h_1}\gamma_n + C^* \quad , \quad n \geq 5 \quad , \end{aligned}$$

where $C^* = \beta_4 - \frac{1}{h_1}\gamma_4$ is a fixed constant. Similar computations lead us to

$$\begin{aligned} \beta_n &= \beta_2 \quad , \quad n \geq 2 \quad , \\ \gamma_n &= \gamma_2 \quad , \quad n \geq 2 \quad , \end{aligned}$$

with $\beta_2 = \frac{1}{h_1}\gamma_2 + C^*$. Thus, $\{P_n\}_{n\geq 0}$ is a SMOP with constant recurrence coefficients defined by the above equalities and $\beta_2 = \frac{1}{h_1}\gamma_2 + C^*$ while $\beta_0, \beta_1, \gamma_1$ are free parameters. Then, we have a similar situation to the analyzed in (b) of (i). Furthermore, the recurrence coefficients of the SMOP $\{Q_n\}_{n\geq 0}$ are

$$\begin{aligned} \tilde{\gamma}_n &= \gamma_n + h_1(\beta_{n-1} - \beta_n) = \gamma_2 \quad , \quad n \geq 4 \quad , \\ \tilde{\beta}_n &= \beta_n = \beta_2 \quad , \quad n \geq 4 \quad . \end{aligned}$$

(b) $h_2 \neq 0$.

Taking into account (3.2) – (3.4), we deduce that β_n and γ_n satisfy the following difference equation

$$y_n + \left(1 - \frac{h_2^2}{h_1 h_3}\right) y_{n-1} - \left(1 - \frac{h_2^2}{h_1 h_3}\right) y_{n-3} - y_{n-4} = 0 \quad , \quad n \geq 6 \quad . \tag{3.12}$$

The characteristic equation of the above difference equation is

$$\left(\lambda^2-1\right)\left[\lambda^2+\left(1-\frac{h_2^2}{h_1h_3}\right)\lambda+1\right]=0 \ . \qquad (3.13)$$

The solution of the characteristic equation (3.13) depends on the behavior of the constant $\frac{h_2^2}{h_1h_3}$ (see [23]).

(b_1) If $h_2^2=-h_1h_3$, then, from (3.13), $\lambda=-1$ is a zero with multiplicity 3. Thus,

$$\begin{aligned}\beta_n &= f_{11}+\left[f_{12}+f_{13}n+f_{14}n^2\right](-1)^n \quad , \quad n\geq 2 \ ,\\ \gamma_n &= g_{11}+\left[g_{12}+g_{13}n+g_{14}n^2\right](-1)^n \quad , \quad n\geq 2 \ .\end{aligned}$$

Moreover, f_{1j} and g_{1j} are related according to (3.2) – (3.4).

(b_2) If $h_2^2=3h_1h_3$, then, from (3.13), $\lambda=1$ is a zero with multiplicity 3. Therefore,

$$\begin{aligned}\beta_n &= f_{21}+f_{22}n+f_{23}n^2+f_{24}\left(-1\right)^n \quad , \quad n\geq 2 \ ,\\ \gamma_n &= g_{21}+g_{22}n+g_{23}n^2+g_{24}\left(-1\right)^n \quad , \quad n\geq 2 \ .\end{aligned}$$

(b_3) If $\frac{h_2^2}{h_1h_3}\in\mathbb{R}\backslash\left[-1,3\right]$, then from (3.13)

$$\begin{aligned}\beta_n &= f_{31}+f_{32}\left(-1\right)^n+f_{33}\lambda^n+f_{34}\lambda^{-n} \quad , \quad n\geq 2 \ ,\\ \gamma_n &= g_{31}+g_{32}\left(-1\right)^n+g_{33}\lambda^n+g_{34}\lambda^{-n} \quad , \quad n\geq 2 \ .\end{aligned}$$

Here, λ is the unique solution of the equation (3.13) such that $\lambda\in(-1,1)$.

(b_4) If $\frac{h_2^2}{h_1h_3}\in(-1,3)$, then from (3.13)

$$\begin{aligned}\beta_n &= f_{41}+f_{42}\left(-1\right)^n+f_{43}e^{in\theta}+f_{44}e^{-in\theta} \quad , \quad n\geq 2 \ ,\\ \gamma_n &= g_{41}+g_{42}\left(-1\right)^n+g_{43}e^{in\theta}+g_{44}e^{-in\theta} \quad , \quad n\geq 2 \ ,\end{aligned}$$

where $\lambda=e^{i\theta}$ is the unique solution of the equation (3.13) with $\theta\in(0,\pi)$.

Remark 3.1. It is worthy to notice that each $\{P_n\}_{n\geq 0}$ given by the recurrence coefficients β_n and γ_n explicitly obtained in $(b_1)-(b_4)$ is a SMOP but β_0,β_1,γ_1 are free parameters.

Remark 3.2. In each situation $(b_1)-(b_4)$, the recurrence coefficients of the SMOP $\{Q_n\}_{n\geq 0}$ are

$$\begin{aligned}\tilde{\gamma}_n &= \gamma_n+h_1\left(\beta_{n-1}-\beta_n\right) \quad , \quad n\geq 4 \ ,\\ \tilde{\beta}_n &= \beta_n \quad , \quad n\geq 4 \ ,\end{aligned}$$

with the condition $\gamma_n+h_1\left(\beta_{n-1}-\beta_n\right)\neq 0$, $n\geq 4$, where β_n and γ_n are obtained in $(b_1)-(b_4)$.

Remark 3.3. The comparison with the results given in [9] and [10] shows that the periodic character of the coefficients of the three term recurrence relation pointed out in **(i)** and **(ii)** (a) can be expected but the coefficients of the three term recurrence relation obtained in **(ii)** (b) are apparently unexpected. An interesting problem is the analysis of the integral representation of the linear functionals associated with the SMOP such that the coefficients of the three term recurrence relation are described in **(ii)** (b). As far as we know they are not yet studied in the literature of orthogonal polynomials (see [1] and [24]). Notice that those described in **(i)** and **(ii)**(a) are very well known. This constitutes an added value of the problem analyzed in our contribution.

Remark 3.4. If relation (3.1) holds for $n \geq 3$, then we also have

$$\tilde{\gamma}_3 = \gamma_3 + h_1(\beta_2 - \beta_3) \quad , \quad \tilde{\beta}_3 = \beta_3 \quad .$$

Notice that in order to have orthogonality, the condition $\tilde{\gamma}_3 \neq 0$ must hold. On the other hand,

$$s_2 = \frac{1}{\tilde{\gamma}_3}(h_1\gamma_2 + h_2(\beta_1 - \beta_3))$$

as well as

$$t_2 = \frac{1}{\tilde{\gamma}_3}(h_2\gamma_1 + h_3(\beta_0 - \beta_3)).$$

Thus, we get the conditions about these two parameters which give the explicit expression for $Q_2(x)$. On the other hand, from (2.5) and (2.6)

$$\tilde{\gamma}_2 = \gamma_2 + t_2 - h_2 + s_2(\beta_1 - \beta_2 + h_1 - s_2) \quad , \quad \tilde{\beta}_2 = \beta_2 - h_1 + s_2 \quad .$$

Notice that in order to have orthogonality, the condition $\tilde{\gamma}_2 \neq 0$ must be satisfied. On the other hand,

$$s_1 = \frac{1}{\tilde{\gamma}_2}(s_2\gamma_1 - h_3 + t_2(\beta_0 - \beta_2 + h_1 - s_2)).$$

Finally,

$$\tilde{\gamma}_1 = \gamma_1 - t_2 + s_1(\beta_0 - \beta_1 - s_1 + s_2) \quad , \quad \tilde{\beta}_1 = \beta_1 + s_1 - s_2 \quad .$$

Notice that in order to have orthogonality, the condition $\tilde{\gamma}_1 \neq 0$ must be satisfied. As an immediate consequence, you have $\tilde{\beta}_0 = \beta_0 - s_1$. Thus, all the coefficients of the three term recurrence relation for the sequence $\{Q_n\}_{n\geq 0}$ are completely determined.
As an application of the above results, we will focus our attention in the case when the sequence $\{P_n\}_{n\geq 0}$ is symmetric, i.e. $\beta_n = 0$ for every $n \geq 0$. The above conditions yield

$$\tilde{\gamma}_n = \gamma_n, n \geq 3, \tilde{\beta}_n = \beta_n, n \geq 3 \quad .$$

Furthermore,

$$s_2 = h_1\frac{\gamma_2}{\gamma_3},$$

as well as

$$t_2 = h_2\frac{\gamma_1}{\gamma_3}.$$

On the other hand,

$$\tilde{\gamma}_2 = \gamma_2 - h_1^2\frac{\gamma_2}{\gamma_3}(\frac{\gamma_2}{\gamma_3} - 1) + h_2(\frac{\gamma_1}{\gamma_3} - 1)$$

$$\tilde{\beta}_2 = h_1(\frac{\gamma_2}{\gamma_3} - 1).$$

Notice that if $\{P_n\}_{n\geq 0}$ is the sequence of monic Chebyshev polynomials of second kind, the above conditions become

$$\tilde{\gamma}_n = \frac{1}{4}, \tilde{\beta}_n = 0, n \geq 2, s_2 = h_1, t_2 = h_2,$$

as well as

$$s_1 = h_1 - 4h_3.$$

In this case,

$$\tilde{\gamma}_1 = \frac{1}{4} - h_2 + 4h_3(h_1 - 4h_3), \tilde{\beta}_1 = -4h_3, \tilde{\beta}_0 = 4h_3 - h_1.$$

In order to have orthogonality, you can fix any h_1 and $h_3 \neq 0$ in such a way $h_2 \neq \frac{1}{4} + 4h_3(4h_3 - h_1)$.
If $\{P_n\}_{n\geq 0}$ is the sequence of monic Hermite polynomials, the above conditions become

$$\tilde{\gamma}_n = \frac{n}{2}, \tilde{\beta}_n = 0, n \geq 3, s_2 = \frac{2h_1}{3}, t_2 = \frac{h_2}{3}.$$

Thus

$$\tilde{\gamma}_2 = 1 - \frac{2h_2}{3} + \frac{2h_1^2}{9}, \tilde{\beta}_2 = -\frac{h_1}{3}.$$

The orthogonality condition means that $h_2 \neq \frac{2h_1^2+9}{6}$. On the other hand,

$$s_1 = \frac{1}{\tilde{\gamma}_2}(\frac{h_1}{3} - h_3 - \frac{h_1h_2}{9}).$$

Finally,

$$\tilde{\gamma}_1 = \frac{1}{2} - \frac{h_2}{3} + s_1(\frac{2h_1}{3} - s_1), \tilde{\beta}_1 = s_1 - \frac{2h_1}{3}, \tilde{\beta}_0 = -s_1. .$$

Thus, the orthogonality condition yields another constraint about the possible choices of h_3. This Hermite case shows that, for a fixed h_1, you can choose h_2 and $h_3 \neq 0$ in order to have orthogonality if and only if you have the above constraints.

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