On a direct Uvarov-Chihara problem and some extensions

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Abstract In this contribution, we analyze a perturbation of a nontrivial probability measure $d\mu$ supported on an infinite subset on the real line, which consists on the addition of a time dependent mass point. For the associated sequence of monic orthogonal polynomials, we study its dynamics with respect to the time parameter. In particular, we determine the time evolution of their zeros in the special case when the measure is semiclassical. We also study the dynamics of the Verblunsky coefficients, i.e. the recurrence relation coefficients of a polynomial sequence, orthogonal with respect to a nontrivial probability measure supported on the unit circle, induced from $d\mu$ through the Szegő transformation.

1 Introduction

Let consider the classical mechanical problem of a 1-dimensional chain of particles with neighbor interactions. Assume that the system is homogeneous (contains no impurities) and that the mass of each particle is m. We denote by y_n the displacement of the n-th particle, and by $\varphi(y_{n+1}-y_n)$ the interaction potential between neighboring particles. We can consider this system as a chain of infinitely many particles joined together with non-linear springs. Therefore, if

$$F(r) = -\varphi'(r)$$

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is the force of the spring when it is stretched by the amount r, and $r_n = y_{n+1} - y_n$ is the mutual displacement. Then, according to the Newton's law, the equation that governs the evolution is

$$m\ddot{y}_n = \varphi'(y_{n+1} - y_n) - \varphi'(y_n - y_{n-1}),$$

where, as usual, $\dot{y}$ denotes the derivative with respect to the time. If $F(r)$ is proportional to r, that is, when $F(r)$ obeys the Hooke's law, the spring is linear and the potential can be written as $\varphi(r) = \frac{\kappa}{2}r^2$. Thus, the equation of motion is

$$m\ddot{y}_n = \kappa(y_{n-1} - 2y_n + y_{n+1}),$$

and the solutions $y_n^{(l)}$, $l \in \mathbb{N}$, are given by a linear superposition of the normal modes. In particular, when the particles located at y_0 and y_{N+1} are fixed,

$$y_n^{(l)} = C_n \sin\left(\frac{\pi l}{N+1}\right)\cos(\omega_l t + \delta_l), \quad l = 1, 2, \ldots, N,$$

where $\omega_l = 2\sqrt{\kappa/m}\sin(\pi l/(2N+2))$, the amplitude C_n of each mode is a constant determined by the initial conditions. In this case there is no transfer of energy between the modes. Therefore, the linear lattice is non-ergodic, and cannot be an object of statistical mechanics unless some modification is made. In the early 1950s, the general belief was that if a non-linearity is introduced in the model, then the energy flows between the different modes, eventually leading to a stable state of statistical equilibrium [5]. This phenomenon was explained by the connection to solitons [1].

There are non-linear lattices which admit periodic behavior at least when the energy is not too high. Lattices with exponential interaction have the desired properties. The Toda lattice [18] is given by setting

$$\varphi(r) = e^{-r} + r - 1.$$

Flaschka [6] (see also [15, 14]) proved the complete integrability for the Toda lattice by recasting it as a Lax equation for Jacobi matrices. Later, Van Moerbeke [19], following a similar work [13] on Hill's equation [10], used the Jacobi matrices to define the Toda hierarchy for the periodic Toda lattices, and to find the corresponding Lax pairs.

Flaschka's change of variable is given by

$$a_n = \frac{1}{2}e^{-(y_{n+1}-y_n)/2}, \quad b_n = \frac{1}{2}\dot{y}_n.$$

Hence the new variables obey the evolution equations

[1] In mathematics and physics, a soliton is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. Solitons are caused by a cancellation of non-linear and dispersive effects in the medium.

$$\dot{a}_n = a_n(b_{n+1} - b_n), \tag{1}$$
$$\dot{b}_n = 2(a_n^2 - a_{n-1}^2), \quad a_{-1} = 0, \quad n \geqslant 0, \tag{2}$$

with initial data $b_n^0 = b_n(0) = \overline{b_n(0)}$, $a_n^0 = a_n(0) > 0$, which we suppose uniformly bounded.

Let $\mathbf{J}_t$ be the semi-infinite Jacobi matrix associated with the system (1)-(2), that is

$$\mathbf{J}_t = \begin{bmatrix} b_0(t) & a_0(t) & 0 & 0 & \cdots \\ a_0(t) & b_1(t) & a_1(t) & 0 & \cdots \\ 0 & a_1(t) & b_2(t) & a_2(t) & \ddots \\ 0 & 0 & a_2(t) & b_3(t) & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

If μ is a nontrivial probability measure supported on some interval $E \subset \mathbb{R}$, then it is very well known that there exists a unique sequence of polynomials $\{p_n\}_{n\geqslant 0}$, assuming the leading coefficient of p_n is positive, satisfying

$$\int_E p_n(x)p_m(x)d\mu(x) = \delta_{n,m}, \quad n,m \geqslant 0.$$

$\{p_n\}_{n\geqslant 0}$ is then said to be the sequence of orthonormal polynomials with respect to μ. $\{p_n\}_{n\geqslant 0}$ satisfies the three term recurrence relation

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x), \quad n \geqslant 0,$$

with the initial condition $p_{-1} = 0$, $p_0(x) = 1$. Notice that the matrix representation of the recurrence relation is the Jacobi matrix defined above. We use the notation $\mathbf{J}_\mu = \mathbf{J}_0$, i.e. with entries $a_n(0) = a_n^0$ and $b_n(0) = b_n^0$. Favard's theorem says that, given any Jacobi matrix $\widetilde{\mathbf{J}}$, there exists a measure μ on the real line for which $\widetilde{\mathbf{J}} = \mathbf{J}_\mu$. In general, μ is not unique.

Flaschka's main observation is that the equations (1)-(2) can be reformulated in terms of the Jacobi matrix $\mathbf{J}_t$ as the Lax pair

$$\dot{\mathbf{J}}_t = [\mathbf{A}, \mathbf{J}_t] = \mathbf{A}\mathbf{J}_t - \mathbf{J}_t\mathbf{A},$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & a_0(t) & 0 & 0 & \cdots \\ -a_0(t) & 0 & a_1(t) & 0 & \cdots \\ 0 & -a_1(t) & 0 & a_2(t) & \ddots \\ 0 & 0 & -a_2(t) & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} = (\mathbf{J}_t)_+ - (\mathbf{J}_t)_-,$$

where we use the standard notation $(\mathbf{J}_t)_+$ (resp. $(\mathbf{J}_t)_-$) for the upper-triangular (resp. lower-triangular) projection of the matrix $\mathbf{J}_t$, and $[\cdot,\cdot]$ denotes the commutator. At the same time, the corresponding orthogonality measure $d\mu(\cdot,t)$ goes through a simple spectral transformation,

$$d\mu(x,t) = e^{-tx}d\mu(x,0), \quad t > 0. \tag{3}$$

Notice that spectral transformations of orthogonal polynomials on the real line play a central role in the solution of the problem. Indeed, the solution of Toda lattice is a combination of the inverse spectral problem from $\{a_n^0\}_{n\geqslant 0}$, $\{b_n^0\}_{n\geqslant 0}$ associated with the measure $d\mu = d\mu(\cdot,0)$, the spectral transformation (3), and the direct spectral problem from $\{a_n(t)\}_{n\geqslant 0}$, $\{b_n(t)\}_{n\geqslant 0}$ associated with the measure $d\mu(\cdot,t)$. A generalization of the perturbation (3) has been analyzed in [9], where the authors also describe the time evolution of the zeros of such polynomials.

In this contribution, we are interested in the analysis of the dynamical properties of the family of orthogonal polynomials $P_n(x,t)$ with respect to the measure

$$d\tilde{\mu}(x) = (1 - J(t))d\mu(x) + J(t)\delta(x), \tag{4}$$

where μ is a symmetric (i.e, $d\mu(x) = \omega(x)dx$ with $\omega(x) = \omega(-x)$ and $supp(d\mu(x))$ symmetric) nontrivial probability measure supported on the real line, and $J : \mathbb{R}_+ \to [0,1]$ is a positive C^1 function. In other words, a time-dependent mass $J(t)$ is added to μ, in such a way that the new measure $\tilde{\mu}$ is also normalized. This problem has been analyzed in [21], where the authors describe the dynamics of the corresponding orthogonal polynomials and the recurrence relation coefficients, and the connection of this problem with the Darboux transformation. This kind of perturbations are particular examples of the so-called Uvarov perturbations. They have been extensively studied in [4], where end mass points are considered, and in [1], [11] in a more general framework. In [8], the author deals with an electrostatic interpretation of the zeros of the orthogonal polynomials associated to the perturbed measure, when it is assumed that μ is a measure satisfying some extra conditions.

The manuscript is organized as follows. In Section 2, we extend the results in [21] for non symmetric measures, using a symmetrization process. In Section 3, we analyze the dynamical behavior of the zeros of $P_n(x,t)$, when the orthogonality measure is semiclassical. Some representative examples of such dynamics are shown, when μ is a symmetric classical measure. Finally, in Section 4 we deal with a similar transformation for orthogonal polynomials with respect to measures supported on the unit circle.

2 Time dependence of orthogonal polynomials and symmetrization problems

Let $\{P_n\}_{n\geqslant 0}$ be the sequence of monic orthogonal polynomials with respect to a symmetric measure μ supported on a symmetric infinite subset of the real line. If we denote by $\{P_n(x,t)\}_{n\geqslant 0}$ the sequence of monic orthogonal polynomials associated with $\tilde{\mu}$ defined in (4), then (see [1], [11])

$$P_n(x,t) = P_n(x) - \frac{J(t)P_n(0)}{1 - J(t) + J(t)K_{n-2}(0,0)} K_{n-2}(x,0), \tag{5}$$

where $K_n(x,y)$ is the n-th reproducing kernel defined by

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\|P_k\|^2} = \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1)}(y)}{\|P_n\|^2(x-y)},$$

where the expression in the right hand side is known as the Christoffel-Darboux formula and it is valid if $x \neq y$. Notice that $P_n(x) = P_n(x,0)$, i.e., the perturbed polynomials at zero time. Since μ is symmetric, we have $P_{2n+1}(0) = 0$, so that

$$\begin{aligned} P_{2n+1}(x,t) &= P_{2n+1}(x), \quad n \geqslant 0, \\ P_{2n}(x,t) &= P_{2n}(x) - \frac{J(t)P_{2n}(0)}{1 - J(t) + J(t)K_{2n-2}(0,0)} K_{2n-2}(x,0), \quad n \geqslant 0, t > 0. \end{aligned}$$

In other words, the odd degree polynomials are invariant under time. Our interest is to find the differential equation satisfied by $P_n(x,t)$ with respect to the time parameter. Obviously, $\dot{P}_{2n+1}(x,t) = 0$. Differentiating $P_{2n}(x,t)$ with respect to the time we have

$$\dot{P}_{2n}(x,t) = -\frac{\dot{J}(t)P_{2n}(0)}{[1 - J(t) + J(t)K_{2n-2}(0,0)]^2} K_{2n-2}(x,0),$$

and using the Christoffel-Darboux formula, we get

$$\dot{P}_{2n}(x,t) = r_n \frac{P_{2n-1}(x)}{x}, \tag{6}$$

with

$$r_n = -\frac{\dot{J}(t)P_{2n}(0)P_{2n-2}(0)}{\|P_{2n-2}\|^2[1 - J(t) + J(t)K_{2n-1}(0,0)]^2}. \tag{7}$$

Furthermore, since

$$K_{2n-1}(x,0) = \frac{P_{2n-1}(x)P_{2n-2}(0)}{\|P_{2n-2}\|^2 x},$$

we have

$$K_{2n-1}(0,0) = \frac{P'_{2n-1}(0)P_{2n}(0)}{\|P_{2n-2}\|^2},$$

so that

$$r_n = -\frac{\dot{J}(t)P_{2n}(0)P_{2n-2}(0)\|P_{2n-2}\|^2}{[(1 - J(t))\|P_{2n-2}\|^2 + J(t)P'_{2n-1}(0)P_{2n}(0)]^2}. \tag{8}$$

In [21], the authors show that in this case, the dynamics of the coefficients of the recurrence relation is given by

$$\dot{d}_{2n} = r_n, \quad \dot{d}_{2n+1} = -r_{n+1},$$

where $d_n = a_n^2$. This represents a non-local integrable chain with continuous time and discrete space variable. It is related to the so-called Uvarov-Chihara problem in the theory of orthogonal polynomials (see [20]).

The dynamics of the sequence of polynomials P_n with respect to the time can be easily obtained for the general (non symmetric) case using a symmetrization process. Given a measure μ, we can define a linear functional u in the linear space of polynomials with real coefficients $\mathbb{P}$ such that

$$u[q(x)] = \int_E q(x)d\mu(x), \quad q \in \mathbb{P}.$$

If μ is a probability measure, then u is said to be positive definite. In a more general framework, it is enough for u to be quasi definite (i.e, the principal leading submatrices of its Gram matrix with respect to the canonical basis $\{x^n\}_{n\geqslant 0}$ are nonsingular) for the existence of a sequence monic polynomials with respect to u to be guaranteed. Let denote such a sequence by $\{P_n\}_{n\geqslant 0}$, and define the linear functional u_s as

$$u_s[x^{2n}] := u[x^n], \quad u_s[x^{2n+1}] := 0, \quad n \geqslant 0.$$

This is, the linear functional u_s is symmetric. Thus, it is well known ([3]) that, if we denote by $\{Q_n\}_{n\geqslant 0}$ the sequence of monic polynomials orthogonal with respect to u_s, then

$$Q_{2n}(x) = P_n(x^2), \quad Q_{2n+1}(x) = x\tilde{P}_n(x^2), \quad n \geqslant 0,$$

where $\{\tilde{P}_n\}_{n\geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to the linear functional $\tilde{u} = xu$ (i.e $\tilde{u}[q] = u[xq]$) for any $q \in \mathbb{P}$. $\{\tilde{P}_n\}_{n\geqslant 0}$ is the sequence of *kernel* polynomials of parameter 0 (see [3]), and they can be expressed in terms of $\{P_n(x)\}_{n\geqslant 0}$ by

$$\tilde{P}_n(x) = \frac{1}{x}\left(P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)}P_n(x)\right), \quad n \geqslant 0.$$

A necessary and sufficient condition for their existence is that $P_n(0) \neq 0$, $n \geqslant 0$ (in the positive definite case, that $0 \notin supp(\mu)$).

Therefore, if u is a (non necessarily symmetric) positive definite linear functional, then let $\{P_n(x,t)\}_{n\geqslant 0}$ be the sequence of monic polynomials orthogonal with respect to the linear functional $u_t := (1 - J(t))u + J(t)\delta(x)$. Thus,

$$P_n(x^2,t) = Q_{2n}(x,t), \quad n \geqslant 0,$$

where $\{Q_n(x,t)\}_{n\geqslant 0}$ are symmetric polynomials orthogonal with respect to the linear functional u_s obtained from the symmetrization of u_t and, therefore,

$$\dot{P}_n(x^2,t) = \dot{Q}_{2n}(x,t) = r_n\frac{Q_{2n-1}(x,t)}{x} = r_n\frac{x\tilde{P}_{n-1}(x^2,t)}{x},$$

where r_n is computed using the polynomials Q_n and the polynomials $\tilde{P}_n(x,t)$ are orthogonal with respect to the linear functional xu_t, provided $P_n(0,t) \neq 0$, $n \geqslant 1$. Then,

$$\dot{P}_n(x,t) = r_n \tilde{P}_{n-1}(x,t), \quad n \geqslant 1.$$

Furthermore, from $Q_{2n-1}(x,t) = x\tilde{P}_n(x^2,t)$, we get

$$\dot{Q}_{2n-1}(x,t) = x\dot{\tilde{P}}_n(x^2,t) = 0,$$

we get $\dot{\tilde{P}}_n(x,t) = 0$, $n \geqslant 0$. As a consequence

Proposition 1. *Let $\{P_n(x)\}_{n\geqslant 0}$ be the sequence of monic polynomials with respect a nontrivial probability measure $d\mu$. Let $d\tilde{\mu}$ be defined as in* (4) *and denote by $\{P_n(x,t)\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Then,*

$$\dot{P}_n(x,t) = \frac{r_n}{x}\left(P_n(x,t) - \frac{P_n(0,t)}{P_{n-1}(0,t)}P_{n-1}(x,t)\right), \quad n \geqslant 1.$$

3 Time evolution of zeros of semiclassical orthogonal polynomials

Let us consider a symmetric positive definite linear functional u which is semiclassical, i.e.,

$$\mathcal{D}(\phi(x)u) = \Psi(x)u,$$

for some polynomials ϕ and Ψ, which are even and odd functions, respectively, with $\deg \Psi \geqslant 1$, and let us define the linear functional

$$\tilde{u} = (1 - J(t))u + J(t)\delta(x). \tag{9}$$

Here, as above, $J : \mathbb{R}_+ \to [0,1]$ is a positive C^1 function. Then, we have

$$x^2\phi(x)\tilde{u} = (1 - J(t))x^2\phi(x)u.$$

Applying the derivative operator in both sides we get

$$\begin{aligned}
\mathcal{D}[x^2\phi(x)\tilde{u}] &= (1 - J(t))\mathcal{D}[x^2\phi(x)u] \\
&= (1 - J(t))[2x\phi(x)u + x^2\mathcal{D}(\phi u)] \\
&= 2x\phi\tilde{u} + (1 - J(t))x^2\Psi u \\
&= (2x\phi + x^2\Psi)\tilde{u}.
\end{aligned}$$

Thus, $\tilde{u}$ is also semiclassical, and then its corresponding sequence of monic orthogonal polynomials, $\{P_n(x,t)\}_{n\geqslant 0}$, satisfies the structure relation ([11], [12])

$$x^2\phi(x)\frac{\partial}{\partial x}P_n(x;t) = A_n(x;t)P_n(x;t) + B_n(x;t)P_{n-1}(x;t), \tag{10}$$

where the functions $A_n(x;t), B_n(x;t)$ can be calculated explicitly using the measure associated with u and its corresponding sequence or orthogonal polynomials (see

[2], [9], [12]). Let $x_{n,k}(t)$ be the k-th zero of $P_n(x;t)$, i.e.,

$$P_n(x_{n,k}(t),t)=0.$$

Following [9], differentiating the last equation with respect t, we obtain

$$\frac{\partial}{\partial x}P_n(x;t)\Big|_{x=x_{n,k}}\dot{x}_{n,k}+\dot{P}_n(x_{n,k},t)=0.$$

Thus, evaluating (10) with $n=2m$ at $x=x_{2m,k}(t)$ we get

$$x^2_{2m,k}(t)\phi(x_{2m,k}(t))\frac{\partial}{\partial x}P_{2m}(x_{2m,k}(t);t)=B_n(x_{2m,k}(t);t)P_{2m-1}(x_{2m,k}(t);t),$$

and, as a consequence, from (6) we obtain

$$\dot{x}_{2m,k}(t)=-r_m\frac{x_{2m,k}(t)\phi(x_{2m,k}(t))}{B_{2m}(x_{2m,k}(t))}. \tag{11}$$

Next, we consider two examples of classical families (semiclassical of class zero) of orthogonal polynomials that are symmetric, namely the Gegenbauer (with parameter $\alpha=\beta=1$) and Hermite polynomials. In both cases, since their structure relations are known, $A_n(x,t)$ and $B_n(x,t)$ can be easily obtained directly from the structure and recurrence relations.

First, notice that from (5), we have

$$P'_{2n}(x,t)=P'_{2n}(x)-\frac{J(t)P_{2n}(0)P_{2n-2}(0)}{1-J(t)+J(t)K_{n-2}(0,0)}\frac{xP'_{2n-1}(x)-P_{2n-1}(x)}{||P_{2n-2}||^2x^2}, \tag{12}$$

where P' denotes the derivative with respect to x. Thus,

$$x^2\phi(x)P'_{2n}(x,t)=x^2\phi(x)P'_{2n}(x)-M(t)\phi(x)[xP'_{2n-1}(x)-P_{2n-1}(x)], \tag{13}$$

where

$$M(t)=\frac{J(t)P_{2n}(0)P_{2n-2}(0)}{[1-J(t)+J(t)K_{n-2}(0,0)]||P_{2n-2}||^2}.$$

1. For the Gegenbauer polynomials with $\alpha=\beta=1$, we have $\phi(x)=1-x^2$ and (see [12])

$$\phi(x)P'_n(x) = a_nP_{n+1}+c_nP_{n-1}(x), \tag{14}$$
$$xP_n(x) = P_{n+1}(x)-\gamma_nP_{n-1}(x), \tag{15}$$

where a_n,c_n,γ_n are given by

$$
\begin{aligned}
a_n &= -n,\\
c_n &= \frac{4n(n+1)^2(n+2)(n+3)}{(2n+1)(2n+2)^2(2n+3)},\\
\gamma_n &= \frac{4n(n+1)^2(n+2)}{(2n+1)(2n+2)^2(2n+3)}.
\end{aligned}
$$

As a consequence,

$$
(1-x^2)P_n'(x) = a_n x P_n(x) - (a_n\gamma_n - c_n)P_{n-1}(x). \tag{16}
$$

Thus, from (13) and (16), it is straightforward to show that

$$
x^2(1-x^2)P_{2n}'(x,t) = A_n(x,t)P_{2n}(x,t) + B_n(x,t)P_{2n-1}(x,t),
$$

with

$$
\begin{aligned}
A_n(x,t) &= a_{2n}x^3 - M(t)\frac{a_{2n-1}\gamma_{2n-1} - c_{2n-1}}{\gamma_{2n-1}}x,\\
B_n(x,t) &= -(a_{2n-1}\gamma_{2n-1} - c_{2n-1} + M(t))x^2 - M(t)\left(a_{2n-1}x^2 - \phi(x) - \frac{A_n(x,t)}{x}\right),
\end{aligned}
$$

which can be reduced after some calculations to

$$
\begin{aligned}
A_n(x,t) &= a_{2n}x^3 - M(t)\frac{(4n-1)(2n-1)^2}{n^2}x,\\
B_n(x,t) &= \left[\frac{(4n^2-1)(4n-1)^2}{4n+1} - a_{2n} - (2+a_{2n-1})M(t)\right]x^2\\
&+ \frac{(4n-1)(2n-1)^2}{n^2}M^2(t) + M(t).
\end{aligned}
$$

Notice that in this case, $A_n(x,t)$ and $B_n(x,t)$ are polynomials in x. Thus, from (11), the dynamics of the zeros of $P_n(x,t)$ can be described as

$$
\dot{x}_{2m,k}(t) = -r_m \frac{x_{2m,k}(t)(1-x_{2m,k}^2(t))}{B_{2m}(x_{2m,k}(t))}.
$$

2. Now, we consider the Hermite polynomials H_n. In this case, we have $\phi(x) = 1$, $H_n'(x) = nH_{n-1}(x)$, and

$$
H_{n+1}(x) = xH_n(x) - \frac{1}{2}nH_{n-1}(x).
$$

Thus, proceeding as above, we get

$$
x^2H_{2n}'(x,t) = A_n(x,t)H_{2n}(x,t) + B_n(x,t)H_{2n-1}(x,t),
$$

with

$$A_n(x,t) = \frac{2(2n-1)}{n}M(t)x,$$
$$B_n(x,t) = 2\left(n - \frac{2n-1}{n}M(t)\right)x^2 + \left(1 - \frac{2(2n-1)}{n}M(t)\right)M(t).$$

Again, since we have a classical family, $A_n(x,t)$ and $B_n(x,t)$ are polynomials in x. As a consequence, the behavior of the zeros of $H_{2m}(x,t)$ can be described as

$$\dot{x}_{2m,k}(t) = -r_m \frac{x_{2m,k}(t)}{B_{2m}(x_{2m,k}(t))}.$$

4 Time dependence of Verblunsky coefficients for OPUC

Given a nontrivial probability measure σ supported on the unit circle $\mathbb{T}$, there exists a sequence of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ which is orthogonal with respect to σ, i.e.

$$\int_{\mathbb{T}} \Phi_n(z)\overline{\Phi_m(z)}d\sigma(z) = \kappa_n \delta_{n,m}, \quad \kappa_n > 0, \quad n,m \geqslant 0.$$

They are called orthogonal polynomials on the unit circle (OPUC). These polynomials satisfy the recurrence relation (see [16], [17])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \geqslant 1,$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ is called the reversed polynomial, and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ satisfy $|\Phi_n(0)| < 1$. They are called Verblunsky (reflection, Schur, Szegő) coefficients.

On the other hand, if μ is a nontrivial probability measure supported on $[-1,1]$, then it is very well know ([17]) that it induces a nontrivial positive measure σ supported on the unit circle. This process is called the Szegő transformation. On the other hand, if σ is induced through the Szegő transformation, then their corresponding orthogonal polynomials Φ_n have real coefficients, and the Verblunsky coefficients are also real. In this case, consider the perturbation

$$d\tilde{\sigma}(z) = (1 - J(t))d\sigma(z) + J(t)\delta(z-1),$$

i.e., a time dependent mass is added at the point $z = 1$, where $J : \mathbb{R}_+ \to [0,1]$ is a positive C^1 function. Notice that this is the same perturbation defined in the previous sections for orthogonal polynomials on the real line, although the symmetry requirement has been removed. As before, if $\Phi_n(z;t)$ is the MOPS with respect to $\tilde{\sigma}$, then

$$\Phi_n(z;t) = \Phi_n(z) - \frac{J(t)\Phi_n(1)}{1 - J(t) + J(t)K_{n-1}(1,1)}K_{n-1}(z,1), \tag{17}$$

where $K_n(z,y)$, the reproducing kernel, is now defined as (see [16], [17])

$$K_n(z,y)=\sum_{k=0}^{n}\frac{\Phi_k(z)\overline{\Phi_k(y)}}{\|\Phi_k\|^2}=\frac{\Phi^*_{n+1}(z)\overline{\Phi^*_{n+1}(y)}-\Phi_{n+1}(z)\overline{\Phi_{n+1)}(y)}}{\|\Phi_{n+1}\|^2(1-z\bar{y})},$$

provided $z\bar{y}\neq 1$. Therefore,

$$\Phi_n(0;t)=\Phi_n(0)-\frac{J(t)\Phi_n(1)}{1-J(t)+J(t)K_{n-1}(1,1)}K_{n-1}(0,1), \tag{18}$$

and since we have real coefficients and $\Phi^*_n(0)=1$,

$$\Phi_n(0;t)=\Phi_n(0)-\frac{J(t)\Phi_n^2(1)(1-\Phi_n(0))}{\|\Phi_n\|^2[1-J(t)+J(t)K_{n-1}(1,1)]}. \tag{19}$$

Thus,

$$\dot{\Phi}_n(0;t)=-\frac{\dot{J}(t)\Phi_n^2(1)(1-\Phi_n(0))}{\|\Phi_n\|^2[1-J(t)+J(t)K_{n-1}(1,1)]^2},$$

which describes the dynamic behavior of the Verblunsky coefficients of the perturbed measure with respect to the time. We will show that $\Phi_n(1)$ and $K_{n-1}(1,1)$ can be expressed in terms of the previous Verblunsky coefficients. Notice that, from the recurrence relation, we have

$$\Phi_n(1)=\Phi_{n-1}(1)+\Phi_n(0)\Phi^*_{n-1}(1),$$

but since Φ_{n-1} has real coefficients, we get

$$\Phi_n(1)=[1+\Phi_n(0)]\Phi_{n-1}(1),$$

and, recursively,

$$\Phi_n(1)=\prod_{k=1}^{n}(1+\Phi_k(0)).$$

On the other hand,

$$K_{n-1}(1,1)=\sum_{k=0}^{n-1}\frac{\Phi_k^2(1)}{\|\Phi_k\|^2}=\sum_{k=0}^{n-1}\frac{\prod_{j=1}^{k}(1+\Phi_j(0))^2}{\prod_{j=1}^{k}(1-\Phi_j^2(0))}=\sum_{k=0}^{n-1}\frac{\prod_{j=1}^{k}(1+\Phi_j(0))}{\prod_{j=1}^{k}(1-\Phi_j(0))}.$$

As a consequence, in order to describe the dynamics of $\Phi_n(0;t)$, the values of $\{\Phi_k(0)\}_{k=1}^{n}$ are required. The situation can be simplified if symmetric measures are considered. As an example, consider the perturbation of the Lebesgue measure on the real line defined by

$$d\tilde{\mu}(x,t)=dx+\frac{1}{J(t)}\delta(x+1)+\frac{1}{J(t)}\delta(x-1).$$

Notice that $d\tilde{\mu}(x,t)$ is symmetric. Applying the Szegő transformation to $d\tilde{\mu}(x)$ will induce a measure $d\sigma(z,t)$ on the unit circle which is also symmetric. It was shown in [7] that in such a case, the Verblunsky coefficients associated with $d\sigma$ are

$$\begin{aligned} \Phi_{2n}(0,t) &= \frac{-1}{2n+1}\frac{3n^2(n+1)^2+2n(n+1)J(t)-J^2(t)}{n^2(n+1)^2+2n(n+1)J(t)+J^2(t)}, \\ \Phi_{2n+1}(0,t) &= 0. \end{aligned}$$

In other words, the dynamics of $\Phi_n(0)$ can be obtained easily only in terms of $J(t)$.

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