(M, N)-Coherent Pairs of Order (m, k) and Sobolev Orthogonal Polynomials

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Abstract

A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-coherent pair of order (m, k) if their corresponding sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy a structure relation such as

$$\sum_{i=0}^{M} a_{i,n} P^{(m)}_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n} Q^{(k)}_{n+k-i}(x), \quad n \geq 0,$$

where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$. In the first part of this work we focus our attention in the algebraic properties of an (M, N)-coherent pair of order (m, k). To be more precise, let us assume that $m \geq k$. If $m = k$ then $\mathcal{U}$ and $\mathcal{V}$ are related by a rational factor (in the distributional sense); if $m > k$ then $\mathcal{U}$ and $\mathcal{V}$ are semiclassical and they are again related by a rational factor. In the second part of this work we deal with a Sobolev type inner product defined in the linear space of polynomials with real coefficients, $\mathbb{P}$, as

$$\langle p(x), q(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1(x), \quad p, q \in \mathbb{P},$$

where λ is a positive real number, m is a positive integer number and (μ_0, μ_1) is a (M, N)-coherent pair of order m of positive Borel measures supported on an infinite subset of the real line, meaning that the sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ with respect to μ_0 and μ_1, respectively, satisfy a structure relation as above with $k = 0$, $a_{i,n}$ and $b_{i,n}$ being real numbers fulfilling the above mentioned conditions. We generalize several recent results known in the literature in the framework of Sobolev orthogonal polynomials and their connections with coherent pairs (introduced in [A. Iserles et al., J. Approx. Theory **65** (2), 151-175 (1991)]) and their extensions. In particular, we show how to compute the coefficients of the Fourier expansion of functions on an appropriate Sobolev space (defined by the above inner product) in terms of the sequence of Sobolev orthogonal polynomials $\{S_n(x; \lambda)\}_{n\geq 0}$.

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1. Introduction

In this work we deal with sequences of monic polynomials, $\{S_n(x;\lambda)\}_{n\geq 0}$, orthogonal with respect to the Sobolev inner product

$$\langle p(x), q(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1(x), \quad p,q \in \mathbb{P}, \tag{1.1}$$

where λ is a positive real number, m is a positive integer number (it indicates a derivative) and (μ_0, μ_1) is a (M,N)-coherent pair of order m of positive Borel measures supported on an infinite subset of the real line, i.e., if $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are the sequences of monic orthogonal polynomials (SMOPs) with respect to μ_0 and μ_1, respectively, then

$$\sum_{i=0}^{M} a_{i,n} P^{(m)}_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0, \tag{1.2}$$

where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$.

The case $(M,N) = (1,0)$ and $m = 1$ has a special historical importance. Such a pair of measures is said to be a coherent pair and it has been introduced and analyzed by A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna [14]. Later on, F. Marcellán and J. Petronilho [19] described all the coherent pairs of measures when one of the measures is a classical one. Finally, H. G. Meijer [23] proved that there are no other coherent pairs, showing that, indeed, in a coherent pair one of the measures must be a classical one (Jacobi or Laguerre) and the other one is a rational perturbation of it. Meijer's paper had a great influence in the subsequent developments of the theory of coherent pairs of orthogonal polynomials. Indeed, after these works, several other ones appeared in the literature, dealing with generalizations of the notion of coherence, including in a more general framework of quasi-definite linear functionals. For instance, among others, we mention here the works by K. H. Kwon, J. H. Lee, and F. Marcellán [17], F. Marcellán, A. Martínez-Finkelshtein and J. J. Moreno-Balcázar [18], M. de Bruin and H. G. Meijer [24], M. Alfaro, F. Marcellán, A. Peña, and M. L. Rezola [1, 2, 3, 4], A. M. Delgado and F. Marcellán [9, 10], J. Petronilho [25], M. N. de Jesus and J. Petronilho [15, 16], A. Branquinho and M. N. Rebocho [6], F. Marcellán and N. C. Pinzón-Cortés [20], and M. Alfaro, A. Peña, J. Petronilho, and M. L. Rezola [5]. For a review about these and other contributions, see e.g. the introductory sections in the recent papers [16] and [20].

All these generalizations of the notion of coherence may be regarded as special cases of the notion of (M,N)-coherence of order (m,k) to be considered in the present paper. Indeed, given two SMOPs $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, and four nonnegative integer

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numbers M, N, m, k, we say these two SMOPs form a (M, N)-coherent pair of order (m, k) if a relation such as

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

holds, where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$. The above structure relation has been already considered in [15], where it has been proved that, under some natural conditions, and assuming, without lost of generality, that $0 \leq k \leq m$, the regular moment linear functionals $\mathcal{U}$ and $\mathcal{V}$ associated with the SMOPs $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ (respectively) fulfill a distributional differential equation

$$D^{m-k}(\phi(x)\mathcal{V}) = \psi(x)\mathcal{U},$$

where $\phi(x)$ and $\psi(x)$ are some polynomials. Furthermore in [15] the authors also proved that if $m = k$ then $\mathcal{U}$ and $\mathcal{V}$ are related by a rational factor and, if $m = k + 1$, then both $\mathcal{U}$ and $\mathcal{V}$ must be semiclassical, being also related by a rational factor. For a survey about the theory of semi-classical linear functionals, the basic reference is P. Maroni [21].

When $m > k + 1$ the problem of determining whether $\mathcal{U}$ and $\mathcal{V}$ are semiclassical (for arbitrary M and N) remained open. In the present work we fill this gap by proving that even when $m > k + 1$ both $\mathcal{U}$ and $\mathcal{V}$ are semiclassical and they are related by a rational factor. This will be stated in Section 3. On the other hand, when the above linear functionals are associated with positive Borel measures, then a useful algebraic relation between the sequences $\{S_n(x; \lambda)\}_{n\geq 0}$ and $\{P_n(x)\}_{n\geq 0}$ will be deduced, provided that the measures form an (M, N)-coherent pair of order m in the sense of (1.2) and $\{S_n(x; \lambda)\}_{n\geq 0}$ is an SMOP with respect to the inner product (1.1). This will be the topic to be analyzed in Section 4. Notice that an inner product of this type, involving higher order derivatives, was already considered in [20] in a situation corresponding to $(1, 1)$-coherence of order m. In Section 5 we built and implement an efficient algorithm for the computation of the Fourier-Sobolev coefficients, i.e. the coefficients of the Fourier expansion of functions of the Sobolev space $W^{m,2}(I, \mu_0, \mu_1)$ in terms of the SMOP $\{S_n(x; \lambda)\}_{n\geq 0}$, thus extending to the more general framework of (M, N)-coherence of order m the previous algorithms known in the literature for coherence [14], generalized coherence [17], and (M, N)-coherence (of order 1) [16]. Notice that from such an algorithm the evaluation of the Fourier-Sobolev coefficients does not need the explicit expressions of the Sobolev orthogonal polynomials. This is an extension of a remarkable fact pointed out by Iserles et. al. in [13] for coherent pairs. In such a paper the authors point out that when we wish to approximate a function by its projection into the linear space of polynomials and, simultaneously, to approximate its derivative by the derivative of the polynomial approximant in the linear space $L^2([-1, 1]; dx)$ the Fourier-Sobolev projector in the Sobolev space $W^{1,2}(([-1, 1]; dx, dx)$ is more valuable than the standard Fourier projector in such a space. Given that the derivative of the function is steep, it is only expected that the quality of the projection in the conventional L^2 norm deteriorates. In general, the Fourier Legendre projector is poor near the end points whereas the Fourier-Sobolev projector displays a reasonably good behaviour throughout the interval $[-1, 1]$. At the end of Section 5 an illustrative example of a Fourier-Sobolev expansion is presented for a particular situation involving a $(2,1)$-coherent pair of order 3. Thus, a comparison of the remainder errors for the Fourier and Fourier-Sobolev projectors for higher derivatives is analyzed, from a computational point of view, in a more general framework than [13]. Before going to the main part of this work in the next Section 2 we recall some basic background from the general theory of orthogonal polynomials needed in the sequel.

2. Basic Tools

For each nonnegative integer number n, $\mathbb{P}_n$ will denote the linear space of the polynomials with complex coefficients of degree less than or equal to n, and $\mathbb{P} = \cup_{n=0}^{\infty}\mathbb{P}_n$. $\langle \mathcal{U}, p(x)\rangle$ will denote the image of the polynomial $p \in \mathbb{P}$ by the linear functional $\mathcal{U}$. If $P_n(x)$ is a monic polynomial, then $P_n^{[m]}(x)$ denotes the monic polynomial of degree n defined by

$$P_n^{[m]}(x) := \frac{P_{n+m}^{(m)}(x)}{(n+1)_m}, \quad n = 0, 1, 2, \cdots,$$

where $(\alpha)_n$ is the *Pochhammer symbol*: $(\alpha)_0 = 1$; $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ if $n \geq 1$.

A linear functional $\mathcal{U}$ is said to be *quasi-definite* or *regular* if $\det\left([u_{i+j}]_{i,j=0}^{n}\right) \neq 0$ for every $n \geq 0$, where $u_n = \langle \mathcal{U}, x^n\rangle$ is its *moment of order* n. In this way, there exists a sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ such that $\deg(P_n(x)) = n$ and $\langle \mathcal{U}, P_n(x)P_m(x)\rangle = \kappa_n\delta_{n,m}$, with $\kappa_n \neq 0$, for $n, m \geq 0$. $\{P_n(x)\}_{n\geq 0}$ is called the *sequence of monic orthogonal polynomials (SMOP)* with respect to $\mathcal{U}$. In this case, if $\{\mathfrak{p}_n\}_{n\geq 0}$ is the *dual basis* associated with $\{P_n(x)\}_{n\geq 0}$, which is defined by $\langle \mathfrak{p}_m, P_n(x)\rangle = \delta_{m,n}$ for $n, m \geq 0$, then

$$\mathfrak{p}_n = \frac{P_n(x)}{\langle \mathcal{U}, P_n^2(x)\rangle}\mathcal{U}, \quad \forall n \geq 0. \tag{2.1}$$

Besides, if $\{\mathfrak{e}_n\}_{n\geq 0}$ is the dual basis of the sequence $\{P_n^{[m]}(x)\}_{n\geq 0}$ (for fixed $m \geq 0$), then

$$D^m\mathfrak{e}_n = (-1)^m(n+1)_m\mathfrak{p}_{n+m}, \quad \forall n \geq 0, \tag{2.2}$$

where, for a linear functional $\mathcal{V}$, $D\mathcal{V}$ denotes its (distributional) *derivative*, which is defined as the linear functional such that

$$\langle D\mathcal{V}, p(x)\rangle = -\langle \mathcal{V}, p'(x)\rangle, \quad p \in \mathbb{P}.$$

When $\det\left([u_{i+j}]_{i,j=0}^{n}\right) > 0$ for all $n \geq 0$, $\mathcal{U}$ is said to be a *positive definite* linear functional. In this case, there exists a positive Borel measure μ supported on the real line such that $\langle \mathcal{U}, p(x)\rangle = \int_{\mathbb{R}} p(x)d\mu(x)$, $\forall p \in \mathbb{P}$. Besides, $\int_{\mathbb{R}} P_n^2(x)d\mu(x) < \int_{\mathbb{R}} p^2(x)d\mu(x)$ holds for every monic polynomial of degree n, $p(x) \neq P_n(x)$, which is called the *extremal property* of the SMOP $\{P_n(x)\}_{n\geq 0}$ (see e.g. [26]).

The linear functionals *Dirac Delta at* a, $\varphi(x)\mathcal{U}$ and $(x-a)^{-1}\mathcal{U}$, for $a \in \mathbb{C}$ and $\varphi \in \mathbb{P}$, are defined by $\langle \delta_a, p(x)\rangle = p(a)$, $\langle \varphi(x)\mathcal{U}, p(x)\rangle = \langle \mathcal{U}, \varphi(x)p(x)\rangle$, and $\left\langle (x-a)^{-1}\mathcal{U}, p(x)\right\rangle = \left\langle \mathcal{U}, \frac{p(x)-p(a)}{x-a}\right\rangle$, for $p \in \mathbb{P}$.

Lemma 2.1. *Let $\mathcal{U}$ be a linear functional, and let $\varphi(x)$ be a polynomial of degree n whose zeros $x_i \in \mathbb{C}$, $1 \leq i \leq n$, are simple. Then*

$$\left\langle \varphi^{-1}(x)\mathcal{U}, p(x)\right\rangle = \left\langle \mathcal{U}, \frac{p(x) - L_{\varphi}(x;p)}{\varphi(x)}\right\rangle, \quad p \in \mathbb{P}, \tag{2.3}$$

$$\varphi^{-1}(x)\varphi(x)\mathcal{U} = \mathcal{U} - \sum_{i=1}^{n} \frac{1}{\varphi'(x_i)} \left\langle \mathcal{U}, \frac{\varphi(x)}{x - x_i} \right\rangle \delta_{x_i}, \tag{2.4}$$

where $L_{\varphi}(x;p)$ denotes the interpolatory polynomial of $p(x)$ at the zeros of $\varphi(x)$ given by

$$L_{\varphi}(x;p) = \sum_{i=1}^{n} p(x_i) \frac{\varphi(x)}{(x - x_i)\varphi'(x_i)}.$$

Proof. The proof of (2.3) uses induction on n, and (2.4) follows from (2.3). □

Proposition 2.2. *Let $\mathcal{U}$ and $\mathcal{V}$ be two positive-definite linear functionals related by the expression of rational type*

$$\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}, \tag{2.5}$$

where $\varphi(x)$ and $\rho(x)$ are polynomials of degree r and t, respectively, and let μ_0 and μ_1 be their corresponding positive Borel measures supported on the real line. Assume that μ_1 has compact support and that all the zeros of $\varphi(x)$ are real and simple, and they lie out the convex-hull of the support of μ_1, i.e., $x_i \in \mathbb{R} \setminus \mathrm{co}(\mathrm{supp}(\mu_1))$ for all $1 \le i \le r$. For each $\ell = 1, \cdots, r$, define

$$\begin{aligned}\eta_{\ell} = \frac{1}{\varphi'(x_{\ell})} \int_{\mathbb{R}} \frac{\varphi(x)}{x - x_{\ell}} d\mu_0 - \frac{1}{r\varphi'(x_{\ell})} \sum_{i=1}^{r} \Bigg\{ \rho(x_i) F(x_{\ell}, \mu_1) \\ + \sum_{j=0}^{t-1} \frac{\left(\theta_{x_i}\rho(x)\right)^{(j)}(0)}{j!} \left[v_j + (x_{\ell} - x_i) \sum_{k=0}^{j-1} x_{\ell}^{j-1-k} v_k + x_{\ell}^{j}(x_{\ell} - x_i) F(x_{\ell}, \mu_1) \right] \Bigg\},\end{aligned}$$

where $\theta_{x_i}\rho(x) = \frac{\rho(x) - \rho(x_i)}{x - x_i}$ and $F(\,\cdot\,, \mu_1)$ is the Cauchy transform of the measure μ_1 defined by

$$F(z, \mu_1) = \int_{\mathbb{R}} \frac{d\mu_1(x)}{x - z}, \qquad z \in \mathbb{C} \setminus \mathrm{co}(\mathrm{supp}(\mu_1)).$$

Then the measures μ_0 and μ_1 are related by

$$d\mu_0(x) = \frac{\rho(x)}{\varphi(x)} d\mu_1(x) + \sum_{\ell=1}^{r} \eta_{\ell} \delta_{x_{\ell}}, \tag{2.6}$$

provided $\eta_{\ell} \ge 0$ for all $\ell = 1, \cdots, r$ and the right-hand side of (2.6) defines a positive Borel measure.

Proof. From (2.5) we have that $\varphi^{-1}(x)\varphi(x)\mathcal{U} = \varphi^{-1}(x)\rho(x)\mathcal{V}$. Hence, from (2.3) we get, for every fixed $p \in \mathbb{P}$,

$$\left\langle \mathcal{U}, \frac{p(x) - L_{\varphi}(x;p)}{\varphi(x)} \varphi(x) \right\rangle = \left\langle \mathcal{V}, \frac{p(x) - L_{\varphi}(x;p)}{\varphi(x)} \rho(x) \right\rangle.$$

Consequently,

$$\int_{\mathbb{R}} p(x) d\mu_0 - \int_{\mathbb{R}} L_{\varphi}(x;p) d\mu_0 = \int_{\mathbb{R}} p(x) \frac{\rho(x)}{\varphi(x)} d\mu_1 - \int_{\mathbb{R}} L_{\varphi}(x;p) \frac{\rho(x)}{\varphi(x)} d\mu_1,$$

where, since $L_\varphi(x;p) = \sum_{i=1}^r p(x_i)\frac{\varphi(x)}{(x-x_i)\varphi'(x_i)}$, it follows that

$$\int_{\mathbb{R}} L_\varphi(x;p)d\mu_0 - \int_{\mathbb{R}} L_\varphi(x;p)\frac{\rho(x)}{\varphi(x)}d\mu_1 = \sum_{\ell=1}^{r}\frac{p(x_\ell)}{\varphi'(x_\ell)}\left[\int_{\mathbb{R}}\frac{\varphi(x)}{x-x_\ell}d\mu_0 - \int_{\mathbb{R}}\frac{\rho(x)}{x-x_\ell}d\mu_1\right].$$

On the other hand, since $\theta_{x_i}\rho(x)$ is a polynomial of degree $t-1$, then from its definition and its Taylor polynomial, we obtain

$$\rho(x) = \frac{1}{r}\sum_{i=1}^{r}\rho(x) = \frac{1}{r}\sum_{i=1}^{r}\left[\left(\sum_{j=0}^{t-1}\frac{(\theta_{x_i}\rho(x))^{(j)}(0)}{j!}x^j\right)(x-x_i)+\rho(x_i)\right].$$

Thus, the proof is complete taking in account that

$$\frac{x^j(x-x_i)}{x-x_\ell} = x^j + (x_\ell - x_i)\sum_{k=0}^{j-1}x_\ell^{j-1-k}x^k + \frac{x_\ell^j(x_\ell-x_i)}{x-x_\ell}.$$

□

Remark 2.3. Under the remaining hypothesis of Proposition 2.2, the right-hand side of (2.6) defines a positive Borel measure if, for instance, the polynomials $\rho(x)$ and $\varphi(x)$ have the same sign in the interval $\mathrm{co}(\mathrm{supp}(\mu_1))$.

An important characterization of OPs is given by the *Favard Theorem* ([8]): $\{P_n(x)\}_{n\geq 0}$ is the SMOP with respect to a regular linear functional $\mathcal{U}$ if and only if there exist complex numbers $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$, $\beta_n \neq 0$, $n \geq 1$, such that they satisfy the *three-term recurrence relation (TTRR)*

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x),\ n\geq 0, \quad P_0(x) = 1, \quad P_{-1}(x) = 0.$$

Moreover, $\mathcal{U}$ is positive definite if and only if $\alpha_n \in \mathbb{R}$ and $\beta_{n+1} > 0$, for $n \geq 0$.

A linear functional $\mathcal{U}$ is said to be *semiclassical* if it is quasi-definite and there exist $\sigma, \tau \in \mathbb{P}\setminus\{0\}$, $\deg(\tau(x)) \geq 1$, such that $D(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}$ holds. In this case, the *class of* $\mathcal{U}$ is the nonnegative integer $s := \min\max\{\deg(\sigma(x))-2, \deg(\tau(x))-1\}$, where the minimum is taken among all pairs $(\sigma(x), \tau(x))$ such that $D(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}$ holds.

Proposition 2.4 ([22])**.** *If $\mathcal{U}$ and $\mathcal{V}$ are quasi-definite linear functionals and they are related by $p(x)\mathcal{U} = q(x)\mathcal{V}$, $p, q \in \mathbb{P}\setminus\{0\}$, then, $\mathcal{U}$ is semiclassical if and only if so is $\mathcal{V}$. Moreover, if the class of $\mathcal{U}$ is s, then the class of $\mathcal{V}$ is at most $s + \deg(p(x)) + \deg(q(x))$.*

A semiclassical linear functional $\mathcal{U}$ (resp. SMOP $\{P_n(x)\}_{n\geq 0}$) of class $s = 0$ is said to be a *classical linear functional* (resp. *classical SMOP*).

Theorem 2.5 ([11, 12, 21])**.** *A regular linear functional $\mathcal{U}$ is classical satisfying $D(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}$ if and only if, for $m \geq 1$ fixed, $\{P_n^{[m]}(x)\}_{n\geq 0}$ is a SMOP associated to $\widehat{\mathcal{U}}_m = \sigma^m(x)\mathcal{U}$. Moreover, $D(\sigma(x)\widehat{\mathcal{U}}_m) = [\tau(x) + m\sigma'(x)]\widehat{\mathcal{U}}_m$ holds.*

Finally, the *formal Stieltjes series* of a linear functional $\mathcal{U}$ is defined by

$$S_{\mathcal{U}}(z) := -\sum_{n\geq 0}\frac{u_n}{z^{n+1}}.$$

3. (M,N)-Coherent Pairs of Order (m,k)

Definition 3.1. A pair of regular linear functionals $(\mathcal{U},\mathcal{V})$ is said to be a $(M,N)-$*coherent pair of order* (m,k), with M,N,m,k fixed nonnegative integer numbers, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ fulfill the following linear algebraic structure relation

$$P_n^{[m]}(x)+\sum_{i=1}^{M}a_{i,n}P_{n-i}^{[m]}(x)=Q_n^{[k]}(x)+\sum_{i=1}^{N}b_{i,n}Q_{n-i}^{[k]}(x),\quad n\geq 0, \tag{3.1}$$

where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n}\neq 0$ if $n\geq M$, $b_{N,n}\neq 0$ if $n\geq N$, and $a_{i,n}=b_{i,n}=0$ if $i>n$. Furthermore, $(\mathcal{U},\mathcal{V})$ is said to be a (M,N)-*coherent pair of order* m if it is a (M,N)-coherent pair of order $(m,0)$.

Remark 3.2. When $(\mathcal{U},\mathcal{V})$ is a (M,N)-coherent pair of order (m,k) and $\mathcal{U}$ or $\mathcal{V}$ is a classical linear functional, then $(\mathcal{U},\mathcal{V})$ can be regarded as a (M,N)-coherent pair of order $(0,k)$ or $(m,0)$, respectively, and thus it can be seen as a (N,M)-coherent pair of order k or a (M,N)-coherent pair of order m, respectively.

The next theorem improves several results stated in [15, 16, 20, 25] by giving a complete description of the semiclassical case in the framework of (M,N)-coherence of order (m,k).

Theorem 3.3. *Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-coherent pair of order (m,k) given by (3.1), with $m\geq k$, and $\det(\mathcal{L}_{M+N})\neq 0$, where $\mathcal{L}_{M+N}=[l_{i,j}]_{i,j=0}^{M+N-1}$ is the matrix of order $M+N$ with entries*

$$l_{i,j}=\begin{cases} a_{j-i,j} & \text{if}\quad 0\leq i\leq N-1\quad \text{and}\quad i\leq j\leq M+i,\\ b_{j-i+N,j} & \text{if}\quad N\leq i\leq M+N-1\quad \text{and}\quad i-N\leq j\leq i,\\ 0 & \text{otherwise,}\end{cases} \tag{3.2}$$

and the convention $a_{0,j_1}=b_{0,j_2}=1$ for $0\leq j_1\leq N-1$ and $0\leq j_2\leq M-1$. Then, there exist polynomials $\phi_{M+k+n}(x)$ and $\psi_{N+m+n}(x)$ of degrees $M+k+n$ and $N+m+n$, respectively, such that

$$D^{m-k}[\phi_{M+k+n}(x)\mathcal{V}]=\psi_{N+m+n}(x)\mathcal{U},\quad n\geq 0, \tag{3.3}$$

and each one of the functionals $\mathcal{U}$ and $\mathcal{V}$ is a rational modification of the other one, i.e., there exist polynomials $\varphi(x)$ and $\rho(x)$ such that

$$\varphi(x)\mathcal{U}=\rho(x)\mathcal{V}. \tag{3.4}$$

Moreover,

- ***(i)*** *If $m=k$, then $\mathcal{U}$ is a semiclassical linear functional if and only if so is $\mathcal{V}$.*
- ***(ii)*** *If $m>k$, then $\mathcal{U}$ and $\mathcal{V}$ are semiclassical linear functionals.*

Proof. According to (3.1), set

$$R_n(x)=\sum_{i=0}^{M}a_{i,n}P_{n-i}^{[m]}(x)=\sum_{i=0}^{N}b_{i,n}Q_{n-i}^{[k]}(x),\quad n\geq 0, \tag{3.5}$$

where $a_{0,n} = b_{0,n} = 1$. Let $\{\mathfrak{p}_n\}_{n\geq 0}$, $\{\mathfrak{q}_n\}_{n\geq 0}$, $\{\mathfrak{r}_n\}_{n\geq 0}$, $\{\mathfrak{e}_n\}_{n\geq 0}$ and $\{\mathfrak{h}_n\}_{n\geq 0}$ be the dual bases associated with the SMOP $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and the sequences $\{R_n(x)\}_{n\geq 0}$, $\{P_n^{[m]}(x)\}_{n\geq 0}$ and $\{Q_n^{[k]}(x)\}_{n\geq 0}$, respectively. Since

$$\langle \mathfrak{e}_n, R_j(x)\rangle \overset{(3.5)}{=} \sum_{i=0}^{M} \langle \mathfrak{e}_n, a_{i,j} P_{j-i}^{[m]}(x)\rangle = \begin{cases} a_{j-n,j} & \text{if} \quad n \leq j \leq n+M, \\ 0 & \text{otherwise}, \end{cases}$$

$$\langle \mathfrak{h}_n, R_j(x)\rangle \overset{(3.5)}{=} \sum_{i=0}^{N} \langle \mathfrak{h}_n, b_{i,j} Q_{j-i}^{[k]}(x)\rangle = \begin{cases} b_{j-n,j} & \text{if} \quad n \leq j \leq n+N, \\ 0 & \text{otherwise}, \end{cases}$$

we get

$$\mathfrak{e}_n = \sum_{j\geq 0} \langle \mathfrak{e}_n, R_j(x)\rangle \mathfrak{r}_j = \sum_{j=n}^{n+M} a_{j-n,j}\mathfrak{r}_j, \quad n \geq 0, \tag{3.6}$$

$$\mathfrak{h}_n = \sum_{j\geq 0} \langle \mathfrak{h}_n, R_j(x)\rangle \mathfrak{r}_j = \sum_{j=n}^{n+N} b_{j-n,j}\mathfrak{r}_j, \quad n \geq 0. \tag{3.7}$$

These equations for $0 \leq n \leq N-1$ and $0 \leq n \leq M-1$, respectively, yield the following system of linear equations

$$\mathcal{L}_{M+N} \begin{bmatrix} \mathfrak{r}_0 \\ \vdots \\ \mathfrak{r}_{N-1} \\ \mathfrak{r}_N \\ \vdots \\ \mathfrak{r}_{N+M-1} \end{bmatrix} = \begin{bmatrix} \mathfrak{e}_0 \\ \vdots \\ \mathfrak{e}_{N-1} \\ \mathfrak{h}_0 \\ \vdots \\ \mathfrak{h}_{M-1} \end{bmatrix},$$

where the matrix $\mathcal{L}_{M+N} = [l_{i,j}]_{i,j=0}^{M+N-1}$ is given by (3.2). Since $\det(\mathcal{L}_{M+N}) \neq 0$, we can solve this linear system and get

$$\mathfrak{r}_i = \alpha_{i,0}\mathfrak{e}_0 + \cdots + \alpha_{i,N-1}\mathfrak{e}_{N-1} + \alpha_{i,N}\mathfrak{h}_0 + \cdots + \alpha_{i,N+M-1}\mathfrak{h}_{M-1}, \quad 0 \leq i \leq M+N-1, \tag{3.8}$$

where $\alpha_{i,j}$, $0 \leq j \leq N+M-1$, are some constants. On the other hand, for every $i \geq 0$, if we multiply (3.6) for $n = N+i$ by $b_{N,M+N+i}$, and (3.7) for $n = M+i$ by $a_{M,M+N+i}$, and then we subtract the resulting equations, we obtain

$$\begin{aligned} &b_{N,M+N+i}\mathfrak{e}_{N+i} - a_{M,M+N+i}\mathfrak{h}_{M+i} \\ &\qquad = \beta_{1,i}\mathfrak{r}_{\min\{M,N\}+i} + \cdots + \beta_{\max\{M,N\},i}\mathfrak{r}_{M+N+i-1}, \quad i \geq 0, \end{aligned} \tag{3.9}$$

where $\beta_{j,i}$, $1 \leq j \leq \max\{M,N\}$, $i \geq 0$, are constants. On the other hand, for $t \geq 0$ fixed, from (3.6) we can recursively get an expression for $\mathfrak{r}_{M+N+t}$ as a linear combination of $\mathfrak{r}_i$, $0 \leq i \leq M+N-1$, and $\mathfrak{e}_j$, $N \leq j \leq N+t$, (since $a_{M,M+j} \neq 0$, $N \leq j \leq N+t$). As a consequence and using (3.8), (3.9) becomes

$$\tilde{\alpha}_{i,0}\mathfrak{e}_0 + \cdots + \tilde{\alpha}_{i,N+i-1}\mathfrak{e}_{N+i-1} + b_{N,M+N+i}\mathfrak{e}_{N+i}$$

$$= \tilde{\beta}_{i,0}\mathfrak{h}_0 + \cdots + \tilde{\beta}_{i,M-1}\mathfrak{h}_{M-1} + a_{M,M+N+i}\mathfrak{h}_{M+i}, \quad i \geq 0,$$

where $\tilde{\alpha}_{i,j_1}$ and $\tilde{\beta}_{i,j_2}$, for $0 \leq j_1 \leq N+i-1$ and $0 \leq j_2 \leq M-1$, are constants. Taking the mth derivative in the above equation, since $m \geq k$, from (2.2) it follows that

$$\begin{aligned}\widehat{\alpha}_{i,0}\mathfrak{p}_m + \cdots + \widehat{\alpha}_{i,N+i-1}\mathfrak{p}_{N+i-1+m} + b_{N,M+N+i}(-1)^m(N+i+1)_m\mathfrak{p}_{N+i+m} = \\ D^{m-k}\left[\widehat{\beta}_{i,0}\mathfrak{q}_k + \cdots + \widehat{\beta}_{i,M-1}\mathfrak{q}_{M-1+k} + a_{M,M+N+i}(-1)^k(M+i+1)_k\mathfrak{q}_{M+i+k}\right],\end{aligned}$$

for $i \geq 0$. Therefore, from (2.1) we get (3.3) with

$$\phi_{M+k+n}(x) = (-1)^k \frac{(M+n+1)_k a_{M,M+N+n}}{\langle \mathcal{V}, Q^2_{M+k+n}(x)\rangle} x^{M+k+n} + \textit{lower degree terms}, \; n \geq 0,$$

$$\psi_{N+m+n}(x) = (-1)^m \frac{(N+n+1)_m b_{N,M+N+n}}{\langle \mathcal{U}, P^2_{N+m+n}(x)\rangle} x^{N+m+n} + \textit{lower degree terms}, \; n \geq 0.$$

Notice that when $m = k$, for every $n \geq 0$, (3.3) becomes (3.4) with $\rho(x) = \phi_{M+k+n}(x)$ and $\varphi(x) = \psi_{N+m+n}(x)$, and, as a consequence, the statement *(i)* follows from Proposition 2.4.

On the other hand, (3.3) becomes

$$\sum_{i=0}^{m-k} \binom{m-k}{i} \phi^{(i)}_{M+k+n}(x) D^{m-k-i}\mathcal{V} = \psi_{N+m+n}(x)\mathcal{U}, \quad n \geq 0, \tag{3.10}$$

with $\deg(\phi_{M+k+n}(x)) = M+k+n$ and $\deg(\psi_{N+m+n}(x)) = N+m+n$. Hence, let us consider the following linear system resulting from (3.10) for $n = 0, 1, \ldots m-k$,

$$\mathcal{T}_{m-k+1}(x) \begin{bmatrix} D^{m-k}\mathcal{V} \\ \vdots \\ D\mathcal{V} \\ \mathcal{V} \end{bmatrix} = \begin{bmatrix} \psi_{N+m}(x)\mathcal{U} \\ \psi_{N+m+1}(x)\mathcal{U} \\ \vdots \\ \psi_{N+m+(m-k)}(x)\mathcal{U} \end{bmatrix},$$

where $0 \neq \det(\mathcal{T}_{m-k+1}(x)) = \prod_{i=0}^{m-k}\binom{m-k}{i} W[\phi_{M+k}(x), \phi_{M+k+1}(x), \ldots, \phi_{M+k+(m-k)}(x)] = \rho(x)$ where $W[\cdot]$ denotes the Wronskian. If $m > k$ we can solve this system for $\mathcal{V}$ and $D\mathcal{V}$ and thus (3.4) follows as well as $\rho(x)D\mathcal{V} = \varsigma(x)\mathcal{U}$, where $\varphi(x)$ and $\varsigma(x)$ are some polynomials. As a consequence,

$$\begin{aligned} D\left[\varphi(x)\rho(x)\mathcal{V}\right] &= \left(\varphi(x)\rho(x)\right)'\mathcal{V} + \varphi(x)\varsigma(x)\mathcal{U} = \left[(\varphi(x)\rho(x))' + \varsigma(x)\rho(x)\right]\mathcal{V}, \\ D\left[\varphi(x)\rho(x)\mathcal{U}\right] &= D\left[\rho^2(x)\mathcal{V}\right] = 2\rho(x)\rho'(x)\mathcal{V} + \rho(x)\varsigma(x)\mathcal{U} = \left[2\varphi(x)\rho'(x) + \varsigma(x)\rho(x)\right]\mathcal{U}. \end{aligned}$$

Therefore, $\mathcal{V}$ and $\mathcal{U}$ are semiclassical linear functionals, which proves the statement *(ii)*. □

Remark 3.4. When $(\mathcal{U}, \mathcal{V})$ is a (M,N)-coherent pair of order (m,k) and $m = k$, it is not possible to conclude that $\mathcal{U}$ and $\mathcal{V}$ are semiclassical. Indeed, in [25, Section 4] it was proved that if $\mathcal{U}$ and $\mathcal{V}$ are related by $\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\varphi(x)) = N$ and $\deg(\rho(x)) = M$, then

$$\sum_{i=n-M}^{n+N} a_{i,n,1}P_i(x) = \sum_{i=n-N}^{n+N} b_{i,n,1}Q_i(x), \quad \text{and}$$
$$\sum_{i=n-M}^{n+M} a_{i,n,2}P_i(x) = \sum_{i=n-N}^{n+M} b_{i,n,2}Q_i(x), \quad \text{for } n \geq 0,$$

hold, where $\{a_{i,n,j}\}_{n\geq 0}$ and $\{b_{i,n,j}\}_{n\geq 0}$, $j = 1, 2$, are some constants. Thus, in this case, for any pair of nonzero polynomials $\varphi(x)$ and $\rho(x)$, we can choose either $\mathcal{U}$ or $\mathcal{V}$ being a non-semiclassical linear functional, and as a consequence, so is the other one.

Remark 3.5. When $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order (m, k) of positive-definite linear functionals satisfying the same conditions of Theorem 3.3, then there exist polynomials $\varphi(x)$ and $\rho(x)$ such that $\mathcal{U}$ and $\mathcal{V}$ are related by $\varphi(x)\mathcal{U} = \rho(x)\mathcal{V}$. Therefore, when the zeros of either $\varphi(x)$ or $\rho(x)$ satisfy certain conditions, Proposition 2.2 states the relation between the positive Borel measures μ_0 and μ_1 corresponding to $\mathcal{U}$ and $\mathcal{V}$, respectively. More precisely, it gives an expression for either μ_0 in terms of μ_1, or μ_1 in terms of μ_0.

In the following theorem we deduce some relations for the formal Stieltjes series associated with the linear functionals constituting a (M, N)-coherent pair of order (m, k). Thus, we generalize the results in [15, Section 4].

Theorem 3.6. *If $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order (m, k) given by (3.1), and assuming the same condition as in Theorem 3.3, then*

$$\psi_{N+m+n}(z)S_{\mathcal{U}}(z) - [\phi_{M+k+n}(z)S_{\mathcal{V}}(z)]^{(m-k)} = A_n(z), \quad n \geq 0, \tag{3.11}$$

where $A_n(z) = (\mathcal{V}\theta_0\phi_{M+k+n})^{(m-k)}(z) - (\mathcal{U}\theta_0\psi_{N+m+n})(z)$, $\deg(A_n(z)) \leq n-1+\max\{M+2k-m, N+m\}$, and

$$(\mathcal{U}\theta_0 p)(x) = \sum_{j=0}^{n-1} a_{j+1}\sum_{i=0}^{j} u_i x^{j-i} = \sum_{j=0}^{n-1}\left(\sum_{i=j}^{n-1} a_{i+1}u_{i-j}\right)x^j, \text{ for } p(x) = \sum_{j=0}^{n} a_j x^j.$$

Moreover, $S_{\mathcal{V}}(z)$ is the (formal) solution of the following non-homogeneous ordinary differential equations of order $m-k$

$$\sum_{i=0}^{m-k} B_{i,n}(z)S_{\mathcal{V}}^{(i)}(z) = C_n(z), \quad n \geq 0, \tag{3.12}$$

with polynomial coefficients

$$B_{i,n}(z) = \binom{m-k}{i}\left[\psi_{N+m+n+1}(z)\phi_{M+k+n}^{(m-k-i)}(z) - \psi_{N+m+n}(z)\phi_{M+k+n+1}^{(m-k-i)}(z)\right],$$
$$C_n(z) = \psi_{N+m+n}(z)A_{n+1}(z) - \psi_{N+m+n+1}(z)A_n(z),$$

and $\deg(B_{i,n}(z)) \leq M+N+2k+2n+i+1$, $\deg(C_n(z)) \leq 2n+N+\max\{M+2k, N+2m\}$.

Proof. From Theorem 3.3, for $n \geq 0$ there exist polynomials

$$\phi_{M+k+n}(x) = \sum_{j=0}^{M+k+n} r_{j,n} x^j \quad \text{and} \quad \psi_{N+m+n}(x) = \sum_{j=0}^{N+m+n} t_{j,n} x^j$$

such that $\langle D^{m-k}[\phi_{M+k+n}(x)\mathcal{V}], x^i\rangle = \langle \psi_{N+m+n}(x)\mathcal{U}, x^i\rangle$, for $i \geq 0$. So

$$(-1)^{m-k}(i-m+k+1)_{m-k} \sum_{j=0}^{M+k+n} r_{j,n} v_{i-m+k+j} = \sum_{j=0}^{N+m+n} t_{j,n} u_{i+j}, \quad i, n \geq 0,$$

where $v_{i-m+k+j} = 0$ if $i-m+k+j < 0$. Thus, multiplying the above expression by $z^{-(i+1)}$ and adding for $i = 0, 1, \ldots,$ we get in the left hand side

$$\begin{aligned}
&\sum_{i \geq m-k} (i-m+k+1)_{m-k} \sum_{j=0}^{M+k+n} r_{j,n} v_{i-m+k+j} z^{-(i+1)} \\
&\qquad = \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{\ell \geq 0} (\ell+1)_{m-k} \frac{v_{\ell+j}}{z^{\ell+j+1}} \\
&= \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{l=0}^{m-k} \binom{m-k}{l} (-j)_{m-k-l} z^l \sum_{\ell \geq 0} (\ell+1+j)_l \frac{v_{\ell+j}}{z^{\ell+j+1+l}} \\
&= \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{l=0}^{m-k} \binom{m-k}{l} (-j)_{m-k-l} z^l \left[(-1)^{l+1} S_{\mathcal{V}}^{(l)}(z) - \sum_{i=0}^{j-1} (i+1)_l \frac{v_i}{z^{i+1+l}} \right] \\
&= \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k+l} \sum_{l=0}^{m-k} \binom{m-k}{l} (-1)^{m-k-l} (j-m+k+l+1)_{m-k-l} (-1)^{l+1} S_{\mathcal{V}}^{(l)}(z) \\
&- \sum_{j=0}^{M+k+n} r_{j,n} z^{j-m+k} \sum_{i=0}^{j-1} (i+1-j)_{m-k} \frac{v_i}{z^{i+1}} = (-1)^{m-k+1} \sum_{l=0}^{m-k} \binom{m-k}{l} \phi_{M+k+n}^{(m-k-l)}(z) S_{\mathcal{V}}^{(l)}(z) \\
&\qquad - \sum_{j=0}^{M+k+n-1} r_{j+1,n} \sum_{i=0}^{j} (-1)^{m-k} \left((j-i)-m+k+1\right)_{m-k} v_i z^{j-i-m+k} \\
&\qquad = (-1)^{m-k-1} \left[\phi_{M+k+n}(z) S_{\mathcal{V}}(z)\right]^{(m-k)} + (-1)^{m-k-1} \left(\mathcal{V}\theta_0 \phi_{M+k+n}\right)^{(m-k)}(z),
\end{aligned}$$

(taking in account that $(a+b)_n = \sum_{k=0}^{n} \binom{n}{k} (a)_{n-k} (b)_k$ and $(-a)_n = (-1)^n (a-n+1)_n$). The right hand side becomes

$$\begin{aligned}
\sum_{i \geq 0} \left[\sum_{j=0}^{N+m+n} t_{j,n} u_{i+j} \right] z^{-(i+1)} &= \sum_{j=0}^{N+m+n} t_{j,n} z^j \left[-S_{\mathcal{U}}(z) - \sum_{i=0}^{j-1} \frac{u_i}{z^{i+1}} \right] \\
&= -\psi_{N+m+n}(z) S_{\mathcal{U}}(z) - \left(\mathcal{U}\theta_0 \psi_{N+m+n}\right)(z).
\end{aligned}$$

Therefore, (3.11) follows. On the other hand, from (3.11) for n and $n+1$, we can obtain

$$\psi_{N+m+n+1}(z)\left[\phi_{M+k+n}(z)S_{\mathcal{V}}(z)\right]^{(m-k)} - \psi_{N+m+n}(z)\left[\phi_{M+k+n+1}(z)S_{\mathcal{V}}(z)\right]^{(m-k)}$$
$$= \psi_{N+m+n}(z)A_{n+1}(z) - \psi_{N+m+n+1}(z)A_n(z), \quad n \geq 0.$$

Thus, using the Leibniz rule, (3.12) holds. □

Remark 3.7. We can get $S_{\mathcal{V}}$ if we solve (formally) any differential equation in (3.12). As a consequence, from (3.11) we can also obtain $S_{\mathcal{U}}$.

4. Sobolev OP's and (M, N)-Coherent Pairs of Order m of Measures

In the sequel, $\mathbb{P}$ will denote the linear space of polynomials with real coefficients. We also assume that $\mathcal{U}$ and $\mathcal{V}$ are positive definite linear functionals and, μ_0 and μ_1 are their respective positive Borel measures supported on the real line. Besides, we consider the Sobolev inner product

$$\langle p(x), q(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \lambda\int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1, \quad p,q \in \mathbb{P}, \lambda > 0, m \in \mathbb{Z}^+, \tag{4.1}$$

and its corresponding SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$. The completion of $\mathbb{P}$ with respect to the norm $\|\cdot\|_\lambda := \langle\cdot,\cdot\rangle_\lambda^{1/2}$ yields the appropriate Sobolev space of functions. Notice that (4.1) can be rewritten as

$$\langle p(x), q(x)\rangle_\lambda := \langle p(x), q(x)\rangle_{\mu_0} + \lambda\langle p^{(m)}(x), q^{(m)}(x)\rangle_{\mu_1},$$

where $\langle\cdot,\cdot\rangle_{\mu_i}$ is the inner product induced by $d\mu_i$, $i = 0, 1$.

Remark 4.1. If $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{S_n(x;\lambda)\}_{n\geq 0}$ are the SMOP with respect to μ_0, μ_1 and $\langle\cdot,\cdot\rangle_\lambda$, respectively, then

$$Q_n(x) = P_n^{[m]}(x) + \sum_{j=0}^{n-1}\frac{(j+1)_m}{(n+1)_m}\frac{\langle T_{n+m}(x), P_{j+m}(x)\rangle_{\mu_0}}{\|P_{j+m}\|^2_{\mu_0}}P_j^{[m]}(x), \quad n \geq 0, \tag{4.2}$$

$$S_n(x;\lambda) + \sum_{i=m}^{n-1}\frac{\langle T_n(x), S_i(x;\lambda)\rangle_{\mu_0}}{\|S_i\|^2_\lambda}S_i(x;\lambda) = P_n(x) + \sum_{i=m}^{n-1}\frac{\langle T_n(x), P_i(x)\rangle_{\mu_0}}{\|P_i\|^2_{\mu_0}}P_i(x), \tag{4.3}$$

for $n \geq m$, and $S_n(x;\lambda) = P_n(x)$ for $n \leq m$, hold, where

$$T_n(x) = \lim_{\lambda\longrightarrow\infty} S_n(x;\lambda), \quad n \geq 0. \tag{4.4}$$

Proof. From (4.1), $\langle P_n(x), x^i\rangle_\lambda = 0$, for $i < n < m$, and then $S_n(x;\lambda) = P_n(x)$ for $n < m$. Besides, the coefficients of the Sobolev MOP's $S_n(x;\lambda)$ are rational functions of λ, more precisely, their numerator and denominator are polynomials in λ of the same degree. Indeed, from the uniqueness of SMOP with respect to the bilinear functional $\mathcal{W}$ associated with the Sobolev inner product $\langle\cdot,\cdot\rangle_\lambda$, every $S_n(x;\lambda)$ can be written as

$$S_n(x;\lambda) = \frac{1}{\det\left([w_{i,j}]_{i,j=0}^{n-1}\right)}\begin{vmatrix} w_{0,0} & \cdots & w_{0,n-1} & w_{0,n} \\ \vdots & \ddots & \vdots & \vdots \\ w_{n-1,0} & \cdots & w_{n-1,n-1} & w_{n-1,n} \\ 1 & \cdots & x^{n-1} & x^n \end{vmatrix}, \; n \geq 1, \quad S_0(x;\lambda) = 1,$$

where $w_{i,j} = \langle x^i, x^j\rangle_\lambda = u_{i+j} + \lambda(i-m+1)_m(j-m+1)_m v_{(i-m)+(j-m)}$, for $i,j \ge 0$. Additionally, notice that $[w_{i,j}]_{i,j=0}^n$ is a symmetric matrix for $n \ge m$, and it is a Hankel matrix for $n < m$ (it is the Hankel matrix associated with $\mathcal{U}$). Thus, there exist the monic polynomials given by (4.4). From (4.4) and (4.1) it follows that, for $n \ge 0$,

$$\langle T_n(x), x^i\rangle_{\mu_0} = 0,\ i < \min\{n,m\}, \quad \langle T_n^{(m)}(x), x^j\rangle_{\mu_1} = 0,\ j < n-m. \tag{4.5}$$

Hence, from (4.5) we get

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{\langle T_n(x), P_i(x)\rangle_{\mu_0}}{\|P_i\|^2_{\mu_0}} P_i(x) = \sum_{j=0}^{n-m} \frac{\langle T_n(x), P_{j+m}(x)\rangle_{\mu_0}}{\|P_{j+m}\|^2_{\mu_0}} P_{j+m}(x), \quad n \ge m, \\ \frac{T_{n+m}^{(m)}(x)}{(n+1)_m} &= \sum_{i=0}^{n} \frac{\langle T_{n+m}^{(m)}(x)/(n+1)_m, Q_i(x)\rangle_{\mu_1}}{\|Q_i\|^2_{\mu_1}} Q_i(x) = Q_n(x), \quad n \ge 0, \end{aligned} \tag{4.6}$$

and therefore (4.2) follows. On the other hand, from (4.1) and (4.5) we obtain

$$T_n(x) = \sum_{i=0}^{n} \frac{\langle T_n(x), S_i(x;\lambda)\rangle_\lambda}{\|S_i\|^2_\lambda} S_i(x;\lambda) = S_n(x;\lambda) + \sum_{i=m}^{n-1} \frac{\langle T_n(x), S_i(x;\lambda)\rangle_{\mu_0}}{\|S_i\|^2_\lambda} S_i(x;\lambda),$$

for $n \ge 0$, and, as a consequence, (4.3) holds. □

Recall that the pair of measures (μ_0, μ_1) is said to be a (M,N)-coherent pair of order m if it is a (M,N)-coherent pair of order $(m,0)$, i.e, if their corresponding SMOP $\{P_n(x)\}_{n\ge0}$ and $\{Q_n(x)\}_{n\ge0}$ satisfy

$$P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x) = Q_n(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}(x), \quad n \ge 0, \tag{4.7}$$

where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n} \neq 0$ if $n \ge M$, $b_{N,n} \neq 0$ if $n \ge N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$.

The following Theorem extends a fundamental algebraic property known for $(1,0)$-coherent, $(2,0)$-coherent, $(k+1,0)$-coherent, $(1,1)$-coherent and (M,N)-coherent pairs of measures of order 1, (stated in [7, 9, 14, 16, 18]), to (M,N)-coherent pairs of order m.

Theorem 4.2. *Let (μ_0, μ_1) be a (M,N)-coherent pair of order m given by (4.7), and $K = \max\{M,N\}$. Then, $S_n(x;\lambda) = P_n(x)$ for $n < m$ and*

$$P_{n+m}(x) + \sum_{i=1}^{M} \frac{(n+1)_m a_{i,n}}{(n-i+1)_m} P_{n-i+m}(x) = S_{n+m}(x;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda} S_{n-j+m}(x;\lambda), \quad n \ge 0, \tag{4.8}$$

where $c_{j,n,\lambda} = 0$ for $n < j \le K$, and

$$\begin{aligned} c_{j,n,\lambda} = \frac{(n+1)_m}{\|S_{n-j+m}\|^2_\lambda} \Bigg[&\sum_{i=j}^{M} \frac{a_{i,n}}{(n-i+1)_m} \langle P_{n-i+m}(x), S_{n-j+m}(x;\lambda)\rangle_{\mu_0} \\ &+ \lambda \sum_{i=j}^{N} b_{i,n} \left\langle Q_{n-i}(x), S_{n-j+m}^{(m)}(x;\lambda)\right\rangle_{\mu_1} \Bigg], \quad 1 \le j \le K. \end{aligned} \tag{4.9}$$

Furthermore, for every $n \ge K$,

(i) *if $M > N$ and $a_{M,n} \neq 0$, then $c_{K,n,\lambda} \neq 0$,*

(ii) *if $M < N$ and $b_{N,n} \neq 0$, then $c_{K,n,\lambda} \neq 0$,*

(iii) *if $M = N(= K)$ and $a_{M,n} b_{N,n} \neq 0$ then,*

$$c_{K,n,\lambda} \neq 0 \quad \text{iff} \quad a_{K,n} \|P_{n-K+m}\|^2_{\mu_0} + \lambda(n-K+1)^2_m b_{K,n} \|Q_{n-K}\|^2_{\mu_1} \neq 0.$$

Conversely, if (4.8) holds for some constants $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, and $\{a_{i,n}\}_{n\geq 0}$, $1 \leq i \leq M$, such that $c_{j,n,\lambda} = 0$, when $n - j + m < 0$, and $a_{i,n} = 0$, when $n - i + m < 0$, then (μ_0, μ_1) is a (M, K)-coherent pair of order m given by

$$P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x) = Q_n(x) + \sum_{j=1}^{K} b_{j,n} Q_{n-j}(x), \quad n \geq 0, \tag{4.10}$$

where $b_{j,n} = 0$ for $n < j \leq K$, and

$$b_{j,n} = \frac{\left\langle P_n^{[m]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m]}(x), Q_{n-j}(x) \right\rangle_{\mu_1}}{\|Q_{n-j}\|^2_{\mu_1}}, \quad 1 \leq j \leq \min\{K, n\}, \ n \geq 0, \tag{4.11}$$

provided that the conditions $b_{K,n} \neq 0$ hold for all $n \geq K$.

Proof. Since $\langle P_n(x), x^i \rangle_\lambda = 0$ for $i < n < m$, then $S_n(x; \lambda) = P_n(x)$ for $n < m$. On the other hand, substituting (4.6) in (4.7), and integrating m times both sides of the resulting equation, we get

$$\frac{P_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{M} a_{i,n} \frac{P_{n-i+m}(x)}{(n-i+1)_m} = \frac{T_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x)}{(n-i+1)_m} + \sum_{j=0}^{m-1} \kappa_{n,j} x^j, \quad n \geq 0.$$

Applying $\langle \cdot, x^i \rangle_{\mu_0}$, $i < m$, and taking into account (4.5), we obtain for every fixed $n \geq 0$, the system of linear equations $\sum_{j=0}^{m-1} \kappa_{n,j} u_{j+i} = 0$ for $i = 0, \ldots, m-1$. Thus, taking into account that $\det\left([u_{i+j}]_{i,j=0}^{m-1}\right) \neq 0$, then $\kappa_{n,j} = 0$, for $j = 0, \ldots, m-1$ and $n \geq 0$. Therefore

$$\frac{P_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{M} a_{i,n} \frac{P_{n-i+m}(x)}{(n-i+1)_m} = \frac{T_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x)}{(n-i+1)_m}, \quad n \geq 0. \tag{4.12}$$

On the other hand,

$$\frac{T_{n+m}(x)}{(n+1)_m} + \sum_{i=1}^{N} b_{i,n} \frac{T_{n-i+m}(x)}{(n-i+1)_m} = \frac{S_{n+m}(x;\lambda)}{(n+1)_m} + \sum_{j=1}^{n+m} \frac{c_{j,n,\lambda}}{(n+1)_m} S_{n-j+m}(x;\lambda), \quad n \geq 0,$$

where from (4.1), (4.12) and (4.6), for $1 \leq j \leq n+m$,

$$\|S_{n-j+m}\|^2_\lambda \frac{c_{j,n,\lambda}}{(n+1)_m} = \sum_{i=1}^{M} \frac{a_{i,n}}{(n-i+1)_m} \langle P_{n-i+m}(x), S_{n-j+m}(x;\lambda) \rangle_{\mu_0}$$

$$+\lambda\sum_{i=1}^{N} b_{i,n}\left\langle Q_{n-i}(x), S^{(m)}_{n-j+m}(x;\lambda)\right\rangle_{\mu_1},$$

then $c_{j,n,\lambda}=0$ for $j>i$ or $j>\max\{M,N\}=K$. Therefore, (4.8) and (4.9) hold. Besides, for $n\geq K$,

$$\frac{c_{K,n,\lambda}}{(n+1)_m}=\frac{\frac{a_{M,n}}{(n-M+1)_m}\|P_{n-M+m}\|^2_{\mu_0}\delta_{M,K}+\lambda(n-N+1)_m b_{N,n}\|Q_{n-N}\|^2_{\mu_1}\delta_{N,K}}{\|S_{n-K+m}\|^2_{\lambda}},$$

from which *(i)*, *(ii)* and *(iii)* are deduced.

Finally, applying $\langle\cdot, p(x)\rangle_\lambda$ to both sides of (4.8), for any $p\in\mathbb{P}_{n-K+m-1}$, we get

$$0=\lambda\left\langle P^{(m)}_{n+m}(x)+\sum_{i=1}^{M}\frac{(n+1)_m a_{i,n}}{(n-i+1)_m}P^{(m)}_{n-i+m}(x), p^{(m)}(x)\right\rangle_{\mu_1},\quad\forall\, p\in\mathbb{P}_{n-K+m-1},$$

i.e.,

$$0=\left\langle P^{[m]}_{n}(x)+\sum_{i=1}^{M}a_{i,n}P^{[m]}_{n-i}(x), q(x)\right\rangle_{\mu_1},\quad\forall\, q\in\mathbb{P}_{n-K-1}.$$

Besides,

$$P^{[m]}_n(x)+\sum_{i=1}^{M}a_{i,n}P^{[m]}_{n-i}(x)=Q_n(x)+\sum_{j=1}^{n}b_{j,n}Q_{n-j}(x),\quad n\geq 0,$$

where $b_{j,n}$ for $1\leq j\leq n$, is given by (4.11). Therefore, (4.10) follows. □

Remark 4.3. Using Theorem 4.2, we can recursively compute the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$ and the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1\leq j\leq K$, when μ_0 and μ_1 form a (M,N)-coherent pair of order m and the coherence relation (4.7) is known. In the next section, we will prove the Algorithm 5.5 which allows to compute the Sobolev norms $\{\|S_n\|^2_\lambda\}_{n\geq 0}$ and the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1\leq j\leq K$, and, as a consequence, from (4.8) and $S_n(x;\lambda)=P_n(x)$ for $n<m$, we can get the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$.

5. Computation of the Fourier-Sobolev Coefficients for (M,N)-Coherent Pairs of Order m of Measures

Let I be a open interval of the real line and let $W^{m,2}[I,\mu_0,\mu_1]$ be the Sobolev space of smooth functions

$$W^{m,2}[I,\mu_0,\mu_1]=\left\{f: I\to\mathbb{R}\,|\, f\in L^2_{\mu_0}(I),\, f^{(m)}\in L^2_{\mu_1}(I)\right\}.$$

Every function $f\in W^{m,2}[I,\mu_0,\mu_1]$ generates the following Fourier-Sobolev series with respect to the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$,

$$f(x)\sim\sum_{n=0}^{\infty}\frac{f_n}{s_n}S_n(x;\lambda),\tag{5.1}$$

where

$$f_n \equiv f_n(\lambda) := \langle f(x), S_n(x;\lambda)\rangle_\lambda, \quad \text{and} \quad s_n \equiv s_n(\lambda) := \|S_n\|_\lambda^2, \quad n \geq 0. \tag{5.2}$$

In [14], [17] and [16] an efficient algorithm for computing the Fourier-Sobolev coefficients f_n/s_n, $n \geq 0$, when (μ_0, μ_1) is a $(1,0)$-coherent, $(2,0)$-coherent and (M,N)-coherent pair of order 1, respectively, is done. Here we extend these algorithms to the general case when (μ_0, μ_1) is a (M,N)-coherent pair of order m. For this purpose, first we show how to compute the sequences $\{f_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$, based on the algebraic property stated in Theorem 4.2. Finally, the algorithm will be a consequence of these results.

We use the following notation

$$\widetilde{a}_{i,n} = \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n}, \quad \text{and} \quad \widetilde{b}_{i,n} = (n+1)_m b_{i,n}, \tag{5.3}$$

where $\widetilde{a}_{i,n} = \widetilde{b}_{i,n} = 0$ when $i > n$, and $\widetilde{a}_{0,n} = 1$ and $\widetilde{b}_{0,n} = (n+1)_m$ for $n \geq 0$, (since $a_{i,n} = b_{i,n} = 0$ for $i > n$, and $a_{0,n} = b_{0,n} = 1$ for $n \geq 0$).

Theorem 5.1. *Let (μ_0, μ_1) be a (M,N)-coherent pair of order m given by (4.7), and $K = \max\{M,N\}$. Then the sequence $\{f_n\}_{n\geq 0}$, given by (5.2), satisfies the following non-homogeneous linear difference equation of order K*

$$f_{n+m} + \sum_{j=1}^{K} c_{j,n,\lambda} f_{n-j+m} = \varrho_n, \quad n \geq 0, \tag{5.4}$$

where $c_{j,n,\lambda}$ is given by (4.9) and $\varrho_n \equiv \varrho_n(\lambda; f)$ is defined by

$$\varrho_n = \left\langle f(x), \sum_{i=0}^{M} \widetilde{a}_{i,n} P_{n-i+m}(x) \right\rangle_{\mu_0} + \lambda \left\langle f^{(m)}(x), \sum_{i=0}^{N} \widetilde{b}_{i,n} Q_{n-i}(x) \right\rangle_{\mu_1}. \tag{5.5}$$

Proof. Applying $\langle f(x), \cdot\rangle_\lambda$ to both sides of (4.8) and using (4.1), (4.7), and (5.3), we get the desired result. □

Now, we will show that the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, together with the Sobolev norms $\{s_n\}_{n\geq 0}$ satisfy the system of $K+1$ difference equations given by (5.6) and (5.9), with initial conditions $s_n = \|P_n\|_{\mu_0}^2$, $0 \leq n < m$, and $c_{j,n,\lambda} = 0$, $0 \leq n < j \leq K$, from which they can be computed. Besides, since the sequence $\{\varrho_n\}_{n\geq 0}$ can be directly computed in terms of the data (the (M,N)-coherence relation (4.7), the parameter λ, and the function f), then using (5.4), we can recursively compute the sequence $\{f_n\}_{n\geq 0}$. Thus, the sequences $\{f_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ will be deduced and therefore we get the Fourier-Sobolev coefficients $\{f_n/s_n\}_{n\geq 0}$.

Theorem 5.2. *The following relations hold*

$$s_{n-K+\ell+m} c_{K-\ell,n,\lambda} + \sum_{i=1}^{\ell} c_{i,n-K+\ell,\lambda} c_{K-\ell+i,n,\lambda} s_{n-K+\ell-i+m} = \zeta_{\ell,n,\lambda}, \quad 0 \leq \ell \leq K, n \geq 0, \tag{5.6}$$

where $c_{j,n,\lambda}=0$ *for* $n<j\le K$*, and*

$$\zeta_{\ell,n,\lambda}=\sum_{i=K-\ell}^{M}\widetilde{a}_{i,n}\widetilde{a}_{i-K+\ell,n-K+\ell}\|P_{n-i+m}\|^2_{\mu_0}+\lambda\sum_{i=K-\ell}^{N}\widetilde{b}_{i,n}\widetilde{b}_{i-K+\ell,n-K+\ell}\|Q_{n-i}\|^2_{\mu_1}. \tag{5.7}$$

Proof. From (4.8), (4.7) and taking into account (5.3), for $j=K-\ell$ and $0\le\ell\le K$, $n\ge 0$, (4.9) becomes

$$\begin{aligned} s_{n-K+\ell+m}c_{K-\ell,n,\lambda}=&\sum_{i=K-\ell}^{M}\sum_{j=0}^{M}\widetilde{a}_{i,n}\widetilde{a}_{j,n-K+\ell}\left\langle P_{n-i+m}(x),P_{n-K+\ell-j+m}(x)\right\rangle_{\mu_0}\\ &-\sum_{i=K-\ell}^{M}\sum_{j=1}^{K}\widetilde{a}_{i,n}c_{j,n-K+\ell,\lambda}\left\langle P_{n-i+m}(x),S_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_0}\\ &+\lambda\sum_{i=K-\ell}^{N}\sum_{j=0}^{N}\widetilde{b}_{i,n}\widetilde{b}_{j,n-K+\ell}\left\langle Q_{n-i}(x),Q_{n-K+\ell-j}(x)\right\rangle_{\mu_1}\\ &-\lambda\sum_{i=K-\ell}^{N}\sum_{j=1}^{K}\widetilde{b}_{i,n}c_{j,n-K+\ell,\lambda}\left\langle Q_{n-i}(x),S^{(m)}_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_1}. \end{aligned} \tag{5.8}$$

Notice that, from orthogonality, the first and the third terms in the right-hand side of (5.8) are equal to

$$\sum_{i=K-\ell}^{M}\widetilde{a}_{i,n}\widetilde{a}_{i-K+\ell,n-K+\ell}\|P_{n-i+m}\|^2_{\mu_0}\quad\text{and}\quad\lambda\sum_{i=K-\ell}^{N}\widetilde{b}_{i,n}\widetilde{b}_{i-K+\ell,n-K+\ell}\|Q_{n-i}\|^2_{\mu_1},$$

respectively. On the other hand, the second and the fourth terms are equal to

$$-\sum_{j=1}^{\ell}c_{j,n-K+\ell,\lambda}\sum_{i=K-\ell+j}^{M}\widetilde{a}_{i,n}\langle P_{n-i+m}(x),S_{n-K+\ell-j+m}(x;\lambda)\rangle_{\mu_0}\quad\text{and}$$
$$-\lambda\sum_{j=1}^{\ell}c_{j,n-K+\ell,\lambda}\sum_{i=K-\ell+j}^{N}\widetilde{b}_{i,n}\langle Q_{n-i}(x),S^{(m)}_{n-K+\ell-j+m}(x;\lambda)\rangle_{\mu_1},$$

respectively. Indeed, since $\langle P_{n-i+m}(x),S_{n-K+\ell-j+m}(x;\lambda)\rangle_{\mu_0}=0$ if $i<K-\ell+j$ or if $K-\ell+j>M$ (because $i\le M$), then the second term in (5.8) is equal to

$$\begin{aligned} &\sum_{j=1}^{M-K+\ell}\sum_{i=K-\ell+j}^{M}\widetilde{a}_{i,n}c_{j,n-K+\ell,\lambda}\left\langle P_{n-i+m}(x),S_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_0}\\ &\qquad=\sum_{j=1}^{\ell}\sum_{i=K-\ell+j}^{M}\widetilde{a}_{i,n}c_{j,n-K+\ell,\lambda}\left\langle P_{n-i+m}(x),S_{n-K+\ell-j+m}(x;\lambda)\right\rangle_{\mu_0}, \end{aligned}$$

where the last equality follows from $\sum_{j=M-K+\ell+1}^{\ell}\sum_{i=K-\ell+j}^{M}(\,\cdot\,)=0$. In the same way we can obtain the fourth term. Furthermore, notice that from (4.9) the sum of the second and the fourth terms is $-\sum_{j=1}^{\ell} c_{j,n-K+\ell,\lambda} s_{n-K+\ell-j+m} c_{K-\ell+j,n,\lambda}$. Therefore, 5.6 follows. □

Corollary 5.3. *The sequence $\{s_n\}_{n\geq 0}$ satisfies the non-homogeneous linear difference equation of order K*

$$s_{n+m} + \sum_{i=1}^{K} c_{i,n,\lambda}^2 s_{n-i+m} = \zeta_{K,n,\lambda}, \quad n \geq 0, \tag{5.9}$$

where $c_{j,n,\lambda} = 0$ for $n < j \leq K$, and

$$\zeta_{K,n,\lambda} = \sum_{i=0}^{M} \widetilde{a}_{i,n}^2 \|P_{n-i+m}\|_{\mu_0}^2 + \lambda \sum_{i=0}^{N} \widetilde{b}_{i,n}^2 \|Q_{n-i}\|_{\mu_1}^2.$$

Proof. The proof is a straightforward consequence of (5.6) for $\ell = K$ (since (4.8) and (4.9) hold taking $c_{0,n,\lambda} = 1$ for $n \geq 0$). □

Additionally, from the (M, N)-coherence of order m, we can find bounds for $\{s_n\}_{n\geq 0}$, the norms of the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$, as follows

Corollary 5.4. *For $n \geq m$, the following inequalities hold*

$$\|P_n\|_{\mu_0}^2 + \lambda(n+1-m)_m^2 \|Q_{n-m}\|_{\mu_1}^2 \leq s_n \leq \sum_{i=0}^{M} \widetilde{a}_{i,n-m}^2 \|P_{n-i}\|_{\mu_0}^2 + \lambda \sum_{i=0}^{N} \widetilde{b}_{i,n-m}^2 \|Q_{n-m-i}\|_{\mu_1}^2. \tag{5.10}$$

Proof. From the extremal properties for monic Sobolev orthogonal and standard polynomials we get $s_n = \|S_n\|_{\mu_0}^2 + \lambda \|S_n^{(m)}\|_{\mu_1}^2 \geq \|P_n\|_{\mu_0}^2 + \lambda(n+1-m)_m^2 \|Q_{n-m}\|_{\mu_1}^2$, for $n \geq 0$. On the other hand, from (5.9) and since $\zeta_{K,n,\lambda} > 0$ for every $n \geq 0$, it follows that $s_{n+m} \leq \zeta_{K,n,\lambda}$, for $n \geq 0$. Substituting n by $n-m$, the proof is complete. □

Finally, substituting in (5.6) ℓ by $K-j$ and n by $n+j$, we get

$$s_{n+m} c_{j,n+j,\lambda} = \zeta_{K-j,n+j,\lambda} - \sum_{i=1}^{K-j} c_{i,n,\lambda} c_{j+i,n+j,\lambda} s_{n+m-i}, \quad 0 \leq j \leq K, n \geq 0. \tag{5.11}$$

These previous equations, $\|S_n\|_\lambda = \|P_n\|_{\mu_0}$ for $n < m$, and $c_{j,n,\lambda} = 0$ for $n < j \leq K$, allow us to compute all the Sobolev norms $\{s_n\}_{n\geq 0}$ as well as all the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, in the algebraic relation (4.8), as it is shown in the following algorithm. Additionally, using (5.4) and $f_n = \langle f(x), P_n(x)\rangle_{\mu_0}$ for $n < m$, we can compute the coefficients $\{f_n\}_{n\geq 0}$. Finally, as a consequence, it is possible to compute the Fourier-Sobolev coefficients $\{f_n/s_n\}_{n\geq 0}$ appearing in (5.1), for any function $f \in W^{m,2}[I, \mu_0, \mu_1]$.

Algorithm 5.5. This algorithm allows us to compute the Fourier-Sobolev coefficients $\{\hbar_{n,\lambda} = f_n/s_n\}_{n\geq 0}$ in (5.1) for a given function $f \in W^{m,2}[I, \mu_0, \mu_1]$, as well as the coefficients $\{c_{j,n,\lambda}\}_{n\geq 0}$, $1 \leq j \leq K$, in (4.8), when (μ_0, μ_1) is a (M, N)-coherent pair of order m given by (4.7).

- *Starting Data:* The initial conditions are $K = \max\{M, N\}$ and

$$c_{j,n,\lambda} = 0, \; j > K \text{ or } n < j \le K, n \ge 0; \quad c_{0,n,\lambda} = 1, \; n \ge 0;$$
$$s_n = \|P_n\|^2_{\mu_0}, \quad f_n = \langle f(x), P_n(x)\rangle_{\mu_0}, \quad \hbar_{n,\lambda} = f_n/s_n, \quad 0 \le n < m.$$

  Furthermore, we must take in account the expression for ϱ_n and $\zeta_{j,n,\lambda}$, $0 \le j \le K$, and $n \ge 0$, given by (5.5) and (5.7), respectively. (See also (5.3)).

- *Step n, for every $n \ge 0$ fixed:* From the Starting Data and the information obtained in the Steps 1 to $n-1$, we can compute

  (i) s_{m+n} from (5.11) taking $j = 0$, and the elements $c_{j,n+j,\lambda}$ for $j = 1, \cdots, K$,

  (ii) f_{m+n} from (5.4),

  (iii) and the Fourier-Sobolev coefficient $\hbar_{n,\lambda}$

  as follows

$$s_{m+n} = \zeta_{K,n,\lambda} - \sum_{i=1}^{\min\{K,n\}} c^2_{i,n,\lambda} s_{m+n-i};$$
$$c_{j,n+j,\lambda} = \left(\zeta_{K-j,n+j,\lambda} - \sum_{i=1}^{\min\{K-j,n\}} c_{i,n,\lambda} c_{i+j,n+j,\lambda} s_{m+n-i}\right) / s_{m+n}, \quad 1 \le j \le K;$$
$$f_{m+n} = \varrho_n - \sum_{i=1}^{\min\{K,n\}} c_{i,n,\lambda} f_{m+n-i};$$
$$\hbar_{m+n,\lambda} = f_{m+n}/s_{m+n}.$$

Remark 5.6. Notice that the computation of the Sobolev norms $\{s_{m+n}\}_{n\ge 0}$ and the coefficients $\{c_{j,n+j,\lambda}\}_{n\ge 0}$, $1 \le j \le K$, obeys the scheme illustrated by the following matrix with $K+1$ rows and infinitely many columns, where the computation must be done successively along the decreasing diagonals

$$\begin{bmatrix} s_m & s_{m+1} & s_{m+2} & \cdots & & & & \\ 0 & c_{1,1,\lambda} & c_{1,2,\lambda} & c_{1,3,\lambda} & \cdots & & & \\ 0 & 0 & c_{2,2,\lambda} & c_{2,3,\lambda} & c_{2,4,\lambda} & \cdots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & c_{K,K,\lambda} & c_{K,K+1,\lambda} & c_{K,K+2,\lambda} & \cdots \end{bmatrix}$$

Remark 5.7. As a consequence of the Algorithm 5.5, the computation of the Fourier-Sobolev coefficients does not need to know explicitly the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\ge 0}$, when (μ_0, μ_1) is a (M, N)-coherent pair of order m. However, to get the Fourier-Sobolev series, we can recursively compute the Sobolev SMOP using (4.8) and $S_n(x;\lambda) = P_n(x)$ for $n < m$, because the Sobolev norms $\{s_{m+n}\}_{n\ge 0}$ and the coefficients $\{c_{j,n+j,\lambda}\}_{n\ge 0}$, $1 \le j \le K$, already were obtained from the Algorithm 5.5.

5.1. Two Special Cases

Some sequences involved in Algorithm 5.5 satisfy additional properties when (μ_0, μ_1) is a $(1,1)$-coherent or $(1,0)$-coherent pair of order m.

Theorem 5.8. *Let (μ_0, μ_1) be a $(1,1)$-coherent pair of order m with corresponding pair of SMOP $(\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0})$, given by*

$$P_n^{[m]}(x) + a_{1,n} P_{n-1}^{[m]}(x) = Q_n(x) + b_{1,n} Q_{n-1}(x), \quad n \geq 0,$$

where $a_{1,0} = b_{1,0} = 0$. Then,

(i) *The Sobolev SMOP with respect to the inner product (4.1), $\{S_n(x;\lambda)\}_{n\geq 0}$, satisfies $S_n(x;\lambda) = P_n(x)$ for $n < m$, and*

$$P_{m+n}(x) + \frac{m+n}{n} a_{1,n} P_{m+n-1}(x) = S_{m+n}(x;\lambda) + c_{1,n,\lambda} S_{m+n-1}(x;\lambda), \quad n \geq 0, \tag{5.12}$$

where $c_{1,0,\lambda} = 0$ and

$$c_{1,n,\lambda} s_{m+n-1} = \frac{m+n}{n} a_{1,n} \|P_{m+n-1}\|^2_{\mu_0} + \lambda (n)_m (n+1)_m b_{1,n} \|Q_{n-1}\|^2_{\mu_1}.$$

(ii) *The sequences of Sobolev norms $\{s_n\}_{n\geq 0}$ with $s_n = \|S_n\|^2_\lambda$ and constants $\{c_{1,n,\lambda}\}_{n\geq 0}$ in (5.12) can be computed by*

$$s_{m+n+1} = \zeta_{1,n+1,\lambda} - \frac{\zeta^2_{0,n+1,\lambda}}{s_{m+n}} \quad \text{and} \quad \frac{\zeta_{0,n+1,\lambda}}{c_{1,n+1,\lambda}} = \zeta_{1,n,\lambda} - \zeta_{0,n,\lambda} c_{1,n,\lambda}, \quad n \geq 0, \tag{5.13}$$

(the above equation holds if $\zeta_{0,n+1,\lambda} \neq 0$ for $n \geq 0$), and $c_{1,n+1,\lambda} = \zeta_{0,n+1,\lambda}/s_{n+m}$, $n \geq 0$, holds, with initial conditions $c_{1,0,\lambda} = 0$, $s_m = \zeta_{1,0,\lambda}$, and $s_n = \|P_n\|_{\mu_0}$ for $n < m$, where

$$\zeta_{0,n,\lambda} = \frac{n+m}{n} a_{1,n} \|P_{m+n-1}\|^2_{\mu_0} + \lambda (n)_m (n+1)_m b_{1,n} \|Q_{n-1}\|^2_{\mu_1}, \tag{5.14}$$

$$\zeta_{1,n,\lambda} = \|P_{n+m}\|^2_{\mu_0} + \frac{(n+m)^2}{n^2} a^2_{1,n} \|P_{n+m-1}\|^2_{\mu_0} + \lambda (n+1)^2_m \left[\|Q_n\|^2_{\mu_1} + b^2_{1,n} \|Q_{n-1}\|^2_{\mu_1} \right].$$

Furthermore, lower and upper bounds for s_n, $n \geq 0$, are given in (5.10) by taking $M = N = 1$.

(iii) *If $\zeta_{0,n+1,\lambda} \neq 0$ for $n \geq 0$, then every Sobolev norm s_{m+n} for $n \geq 0$, and each constant $c_{1,n,\lambda}$ for $n \geq 1$, can be represented, respectively, by the continued fraction*

$$s_{m+n} = \frac{\zeta^2_{0,n+1,\lambda}}{|\zeta_{1,n+1,\lambda}} - \frac{\zeta^2_{0,n+2,\lambda}|}{|\zeta_{1,n+2,\lambda}} - \cdots, \quad n \geq 0,$$

$$c_{1,n,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}} - \frac{\frac{\zeta_{0,n+1,\lambda}}{\zeta_{0,n,\lambda}}}{\left|\frac{\zeta_{1,n+1,\lambda}}{\zeta_{0,n+1,\lambda}}\right.} - \frac{\left.\frac{\zeta_{0,n+2,\lambda}}{\zeta_{0,n+1,\lambda}}\right|}{\left|\frac{\zeta_{1,n+2,\lambda}}{\zeta_{0,n+2,\lambda}}\right.} - \cdots, \quad n \geq 1.$$

(iv) *If $\zeta_{0,n+1,\lambda} \neq 0$ for $n \geq 0$, then the Sobolev norms $\{s_n\}_{n\geq 0}$ and the constants $\{c_{1,n,\lambda}\}_{n\geq 0}$ in (5.12) are*

$$s_{m+n} = \frac{\varpi_{n+1}(0;\lambda)}{\varpi_n(0;\lambda)}, \quad c_{1,n+1,\lambda} = \zeta_{0,n+1,\lambda}\frac{\varpi_n(0;\lambda)}{\varpi_{n+1}(0;\lambda)}, \quad n \geq 0, \tag{5.15}$$

where $\{\varpi_n(x;\lambda)\}_{n\geq 0}$ is a SMOP with respect to some positive definite linear functional, satisfying the TTRR: $\varpi_0(x;\lambda) = 1$, $\varpi_{-1}(x;\lambda) = 0$,

$$\varpi_{n+1}(x;\lambda) = (x + \zeta_{1,n,\lambda})\varpi_n(x;\lambda) - \zeta^2_{0,n,\lambda}\varpi_{n-1}(x;\lambda), \quad n \geq 0. \tag{5.16}$$

(v) *Let $f \in W^{m,2}[I,\mu_0,\mu_1]$ and let $\sum_{n=0}^{\infty} \frac{f_n}{s_n} S_n(x;\lambda)$ be its Fourier-Sobolev series. Then, the Fourier-Sobolev coefficients $\{f_n/s_n\}_{n\geq 0}$ can be computed using (5.13) and $f_{m+n} = \varrho_n - c_{1,n,\lambda} f_{m+n-1}$, $n \geq 0$, where ϱ_n is given by (5.5) taking $M = N = 1$.*

Proof. ***(i)*** It is immediate from Theorem 4.2.

(ii) (5.11) becomes $s_{n+m} = \zeta_{1,n,\lambda} - c^2_{1,n,\lambda}s_{n+m-1}$ and $s_{n+m}c_{1,n+1,\lambda} = \zeta_{0,n+1,\lambda}$, for $n \geq 0$. As a consequence, (5.13) holds. (5.14) follows from (5.3) and (5.7).

(iii) (5.13) becomes

$$s_{m+n} = \frac{\zeta^2_{0,n+1,\lambda}}{\zeta_{1,n+1,\lambda} - s_{m+n+1}},\ n \geq 0, \quad c_{1,n,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}} - \frac{\frac{\zeta_{0,n+1,\lambda}}{\zeta_{0,n,\lambda}}}{c_{1,n+1,\lambda}},\ n \geq 1,\ c_{1,1,\lambda} = \frac{\zeta_{0,1,\lambda}}{\zeta_{1,0,\lambda}}.$$

(iv) From the theory of continued fractions, we can define the sequence $\{\varpi_{n,\lambda}\}_{n\geq 0}$ by $\varpi_{0,\lambda} = 1$ and $\varpi_{n+1,\lambda} = s_{m+n}\varpi_{n,\lambda}$ for $n \geq 0$, and, as a consequence, the first equation in (5.13) becomes

$$\varpi_{n+2,\lambda} = \zeta_{1,n+1,\lambda}\varpi_{n+1,\lambda} - \zeta^2_{0,n+1,\lambda}\varpi_{n,\lambda},\ n \geq 0, \quad \varpi_{1,\lambda} = \zeta_{1,0,\lambda}, \quad \varpi_{0,\lambda} = 1.$$

Thus, since $\zeta_{0,n+1,\lambda} \neq 0$ for $n \geq 0$, from Favard Theorem there exists a sequence of monic polynomials $\{\varpi_n(x;\lambda)\}_{n\geq 0}$ such that $\varpi_n(0;\lambda) = \varpi_{n,\lambda}$ for $n \geq 0$, that is orthogonal with respect to some positive definite linear functional because $\zeta_{1,n,\lambda}, \zeta_{0,n+1,\lambda} \in \mathbb{R}$ for $n \geq 0$. Furthermore, since $\varpi_{n,\lambda} \neq 0$ for $n \geq 0$, then (5.15) follows.

(v) It is a straightforward consequence of (5.4). □

Remark 5.9. Similarly to Theorem 5.8.*iv*., from second equation in (5.13) we can define recurrently the sequence $\theta_{n+1,\lambda} = \frac{\zeta_{0,n+1,\lambda}/\zeta_{0,n,\lambda}}{c_{1,n+1,\lambda}}\theta_{n,\lambda}$, $n \geq 1$, $\theta_{1,\lambda} = \frac{\zeta_{0,1,\lambda}}{c_{1,1,\lambda}}\theta_{0,\lambda}$ and $\theta_{0,\lambda} = 1$, and, as a consequence, it becomes $\theta_{0,\lambda} = 1$, $\theta_{1,\lambda} = \zeta_{1,0,\lambda}$,

$$\theta_{2,\lambda} = \frac{\zeta_{1,1,\lambda}}{\zeta_{0,1,\lambda}}\theta_{1,\lambda} - \zeta_{0,1,\lambda}\theta_{0,\lambda}, \quad \theta_{n+1,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}}\theta_{n,\lambda} - \frac{\zeta_{0,n,\lambda}}{\zeta_{0,n-1,\lambda}}\theta_{n-1,\lambda}, \quad n \geq 2.$$

Therefore, if $\zeta_{0,n,\lambda} \neq 0$ for $n \geq 1$, from Favard Theorem there exists a SMOP $\{\theta_n(x;\lambda)\}_{n\geq 0}$ which satisfies the TTRR

$$\theta_{n+1}(x;\lambda) = \left(x + \frac{\zeta_{1,n,\lambda}}{\zeta_{0,n,\lambda}}\right)\theta_n(x;\lambda) - \frac{\zeta_{0,n,\lambda}}{\zeta_{0,n-1,\lambda}}\theta_{n-1}(x;\lambda), \quad n \geq 2,$$

$$\theta_2(x;\lambda) = \left(x + \frac{\zeta_{1,1,\lambda}}{\zeta_{0,1,\lambda}}\right)\theta_1(x;\lambda) - \zeta_{0,1,\lambda}\theta_0(x;\lambda), \quad \theta_1(x;\lambda) = x + \zeta_{1,0,\lambda}, \quad \theta_0(x;\lambda) = 1,$$

with $\theta_n(0;\lambda) = \theta_{n,\lambda}$ for $n \geq 0$, and it is orthogonal with respect to some regular linear functional which is positive definite if $\zeta_{0,n,\lambda} > 0$ for $n \geq 1$. Besides, since $\theta_{n,\lambda} \neq 0$ for $n \geq 0$, then $c_{1,1,\lambda} = \zeta_{0,1,\lambda}\frac{\theta_0(0;\lambda)}{\theta_1(0;\lambda)}$ and

$$c_{1,n+1,\lambda} = \frac{\zeta_{0,n+1,\lambda}}{\zeta_{0,n,\lambda}}\frac{\theta_n(0;\lambda)}{\theta_{n+1}(0;\lambda)}, \quad s_{m+n} = \zeta_{0,n,\lambda}\frac{\theta_{n+1}(0;\lambda)}{\theta_n(0;\lambda)}, \quad n \geq 1.$$

Remark 5.10. When (μ_0, μ_1) is a $(1,0)$-coherent pair of order m, the previous Theorem holds taking $b_{1,n} = 0$, for $n \geq 0$. Besides, notice that for $n \geq 0$, $\zeta_{0,n,\lambda}$ and $\zeta_{1,n,\lambda}$ become a constant and a linear function of λ, respectively, and as a consequence, from (5.16) and by induction on n, $\varpi_n(0;\lambda)$ is a polynomial in λ of degree n with leading coefficient $\prod_{i=0}^{n-1}(i+1)_m^2\|Q_i\|_{\mu_1}^2$, for $n \geq 1$. Thus, (5.16) reads

$$\widetilde{\varpi}_{n+1}(\lambda) = (\lambda + \alpha_n)\widetilde{\varpi}_n(\lambda) - \beta_n\widetilde{\varpi}_{n-1}(\lambda), \;\; n \geq 0, \quad \widetilde{\varpi}_0(\lambda) = 1, \tag{5.17}$$

where $\widetilde{\varpi}_n(\lambda)$ is the monic polynomial $\varpi_n(0;\lambda)/(\prod_{i=0}^{n-1}(i+1)_m^2\|Q_i\|_{\mu_1}^2)$, $n \geq 1$, $\alpha_0 = \|P_m\|_{\mu_0}^2/[(1)_m^2\|Q_0\|_{\mu_1}^2]$, $\beta_0 = 0$, and

$$\alpha_n = \frac{\|P_{n+m}\|_{\mu_0}^2 + \frac{(n+m)^2}{n^2}a_{1,n}^2\|P_{n+m-1}\|_{\mu_0}^2}{(n+1)_m^2\|Q_n\|_{\mu_1}^2}, \quad \beta_n = \frac{a_{1,n}^2\|P_{m+n-1}\|_{\mu_0}^4}{(n)_m^4\|Q_n\|_{\mu_1}^2\|Q_{n-1}\|_{\mu_1}^2}, \quad n \geq 1.$$

Therefore, if $a_{1,n} \neq 0$ for $n \geq 1$, then the Sobolev norms $\{s_n\}_{n\geq 0}$ and the constants $\{c_{1,n,\lambda}\}_{n\geq 0}$ in (5.12) satisfy

$$s_{m+n} = \kappa_n\frac{\widetilde{\varpi}_{n+1}(\lambda)}{\widetilde{\varpi}_n(\lambda)}, \; n \geq 0, \quad c_{1,n,\lambda} = \widetilde{\kappa}_n\frac{\widetilde{\varpi}_{n-1}(\lambda)}{\widetilde{\varpi}_n(\lambda)}, \; n \geq 1,$$

with

$$\kappa_n = (n+1)_m^2\|Q_n\|_{\mu_1}^2, \quad \widetilde{\kappa}_n = a_{1,n}\frac{(n+1)_m}{(n)_m^3}\frac{\|P_{m+n-1}\|_{\mu_0}^2}{\|Q_{n-1}\|_{\mu_1}^2},$$

where $\{\widetilde{\varpi}_n(\lambda)\}_{n\geq 0}$ is a SMOP in λ with respect to some positive definite linear functional, such that the TTRR (5.17) holds.

5.2. A Numerical Example

Now, we deal with a numerical example in order to illustrate our Algorithm 5.5.

Example 5.11. Let us consider the Jacobi weight $d\mu^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta\chi_{(-1,1)}(x)dx$, $\alpha, \beta > -1$. Let $\{\widehat{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ be its corresponding SMOP. From [25, Example 5.1] and since $\frac{\left(\widehat{P}_{n+m}^{(\alpha,\beta)}(x)\right)^{(m)}}{(n+1)_m} = \widehat{P}_n^{(\alpha+m,\beta+m)}(x)$ for $n \geq 0$, it follows that

$$\frac{\left(\widehat{P}_{n+3}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n+1)_3} + a_{1,n}\frac{\left(\widehat{P}_{n+2}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n)_3} + a_{2,n}\frac{\left(\widehat{P}_{n+1}^{(\alpha-3,\beta-4)}(x)\right)'''}{(n-1)_3}$$

$$= \widehat{P}_n^{(\alpha-2,\beta)}(x) + b_{1,n}\widehat{P}_{n-1}^{(\alpha-2,\beta)}(x), \quad n \geq 0,$$

holds for $\alpha > 2$ and $\beta > 3$, where

$$b_{1,n} := \frac{2n(n+\alpha-2)}{(2n+\alpha+\beta-3)(2n+\alpha+\beta-2)}, \quad a_{1,n} := -\frac{4n(n+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-3)},$$
$$a_{2,n} := \frac{4n(n-1)(n+\beta-2)(n+\beta-1)}{(2n+\alpha+\beta-4)(2n+\alpha+\beta-3)^2(2n+\alpha+\beta-2)}.$$

Thus, the measures $d\mu_0 := d\mu^{\alpha-3,\beta-4}$ and $d\mu_1 := d\mu^{\alpha-2,\beta}$ form a $(2,1)$-coherent pair of order 3, with $P_n(x) := \widehat{P}_n^{(\alpha-3,\beta-4)}$ and $Q_n(x) := \widehat{P}_n^{(\alpha-2,\beta)}$, for $\alpha > 2$ and $\beta > 3$.

With the help of MAPLE, we applied Algorithm 5.5 and Remark 5.7 to the function $f : [-1,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) := e^{-3(x-\frac{1}{10})^2}\sin(10x),$$

in order to compute its Fourier-Sobolev coefficients with respect to the Sobolev SMOP associated with the inner product $\langle g(x), h(x)\rangle_\lambda = \int_{-\infty}^{\infty} g(x)h(x)d\mu_0 + \lambda\int_{-\infty}^{\infty} g'''(x)h'''(x)d\mu_1$, defined by the $(2,1)$-coherent pair of order 3, $(\mu_0,\mu_1) \equiv (\mu^{1,1},\mu^{2,5})$, i.e., $(\alpha,\beta) = (4,5)$. This is possible because $f \in L^2_{\mu_2}(-1,1)$ and $f''' \in L^2_{\mu_1}(-1,1)$.

For the choice $\lambda = 0.1$, Figures 5.1, 5.2, 5.3, and 5.4, simultaneously include plots of $f(x)$ and the partial sums of degree 30 of its Fourier-Jacobi and Fourier-Sobolev series, of $f'(x)$ and of the derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$, of $f''(x)$ and of the second derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$, and, of $f'''(x)$ and of the third derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$, respectively, in the intervals $[-1,1]$, $[0.9, 0.98]$, $[-1,-0.98]$ and $[0.98,1]$.

From them, there is a numerical evidence that the approximations for $f(x)$ and its derivatives $f'(x)$, $f''(x)$, $f'''(x)$, given by the partial sums of the Fourier-Sobolev series and its derivatives are better than the corresponding approximations given by the Fourier-Jacobi series and its derivatives. Indeed, Table 5.1 illustrates this statement, comparing the errors $\varepsilon^{(i)}_{J,L^2}$, $\epsilon^{(i)}_{J,\mu_0}$, and $E^{(i)}_{J,\lambda}$, with the errors $\varepsilon^{(i)}_{S,L^2}$, $\epsilon^{(i)}_{S,\mu_0}$, and $E^{(i)}_{S,\lambda}$, respectively, given by

$$\varepsilon^{(i)}_{\ell,L^2} := \|f^{(i)} - S^{(i)}_{30,\ell}(f)\|^2_{L^2} = \int_{-1}^{1} |f^{(i)}(x) - S^{(i)}_{30,\ell}(x;f)|^2 dx,$$
$$\epsilon^{(i)}_{\ell,\mu_0} := \|f^{(i)} - S^{(i)}_{30,\ell}(f)\|^2_{\mu_0} = \int_{-1}^{1} \left(f^{(i)}(x) - S^{(i)}_{30,\ell}(x;f)\right)^2 (1-x^2)dx,$$
$$E^{(i)}_{\ell,\lambda} := \|f^{(i)} - S^{(i)}_{30,\ell}(f)\|^2_{\lambda} = \int_{-1}^{1} \left(f^{(i)}(x) - S^{(i)}_{30,\ell}(x;f)\right)^2 (1-x^2)dx$$
$$+ (0.1)\int_{-1}^{1} \left(f^{(3+i)}(x) - S^{(3+i)}_{30,\ell}(x;f)\right)^2 (1-x)^2(1+x)^5 dx,$$

for $i = 0,1,2,3$, and $\ell = J,S$, when approaching the function $f(x)$ $(i=0)$ and its derivatives $f^{(i)}(x)$, $i = 1,2,3$, with the partial sums of degree 30 of the Fourier-Jacobi

Figure 5.1: $f(x)$ and the partial sums of degree 30 of its Fourier-Jacobi and Fourier-Sobolev series

(a) in $[-1, 1]$ (b) in $[0.9, 0.98]$ (c) in $[-1, -0.98]$ (d) in $[0.98, 1]$

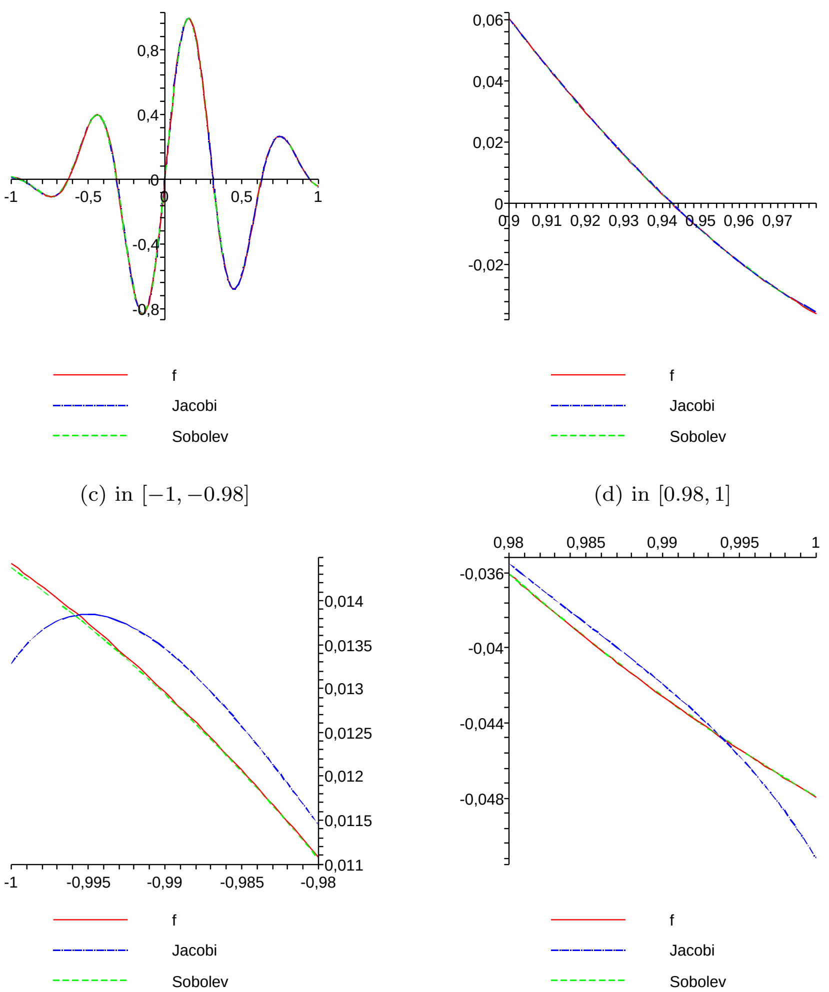


Figure 5.2: $f'(x)$ and the derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$

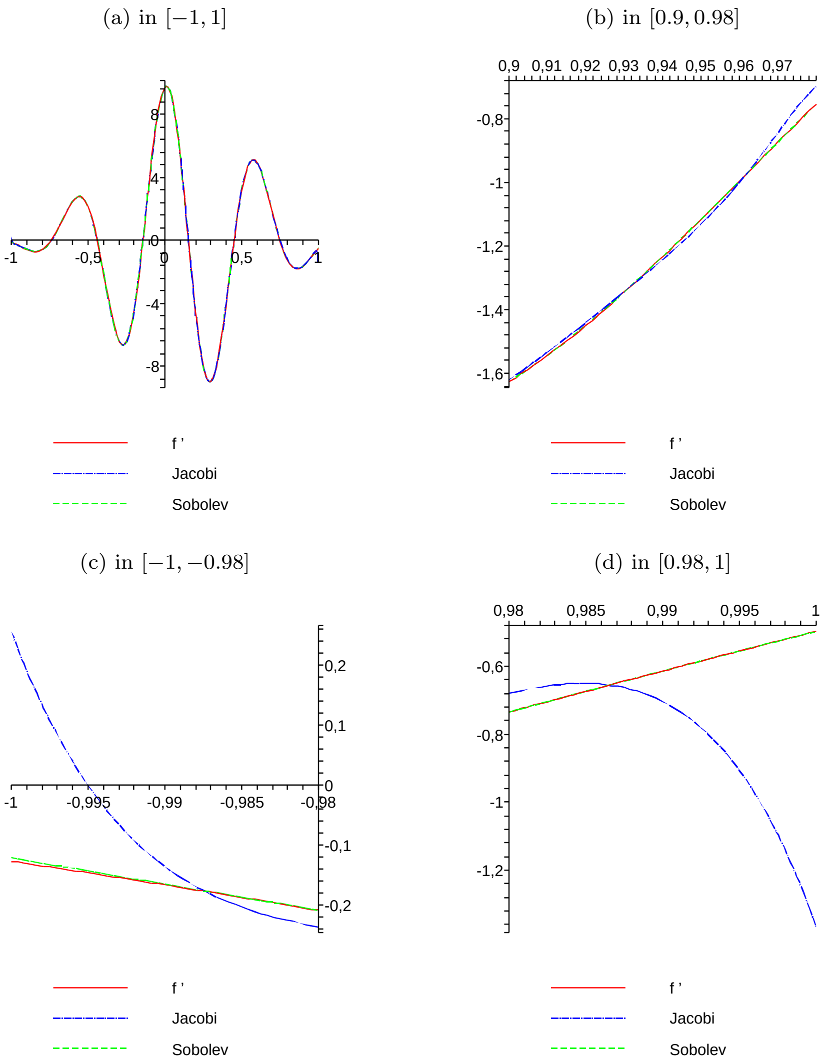


Figure 5.3: $f''(x)$ and the second derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$

(a) in $[-1, 1]$ (b) in $[0.9, 0.98]$

(c) in $[-1, -0.98]$ (d) in $[0.98, 1]$

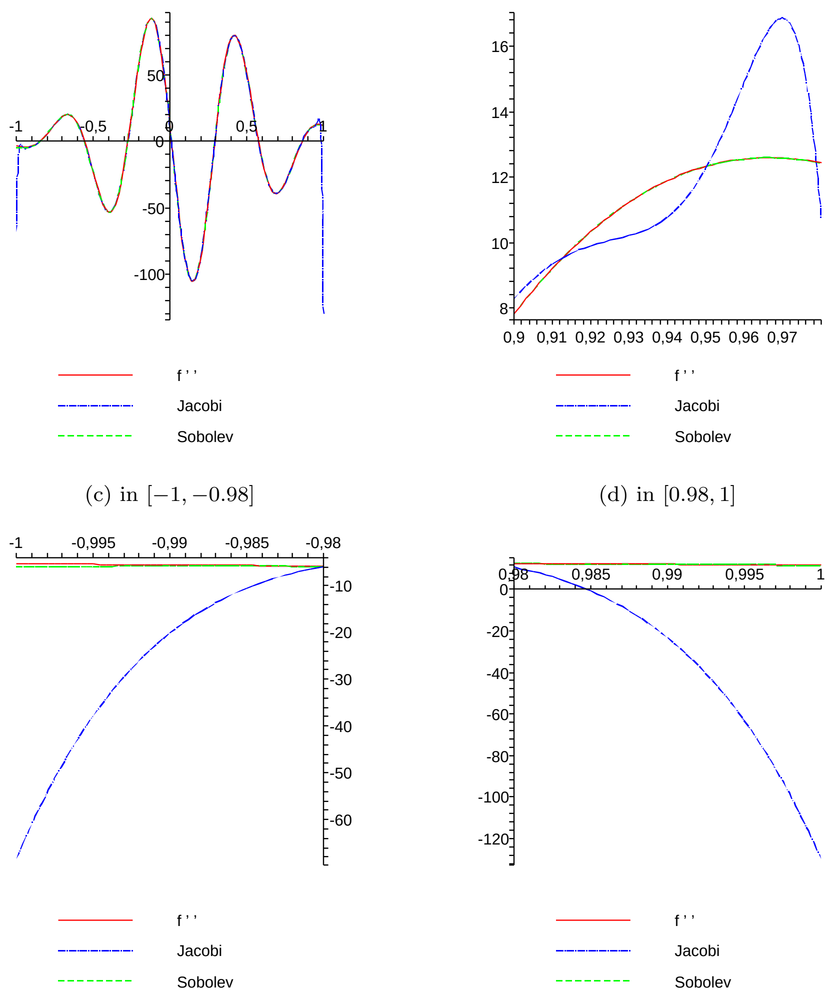
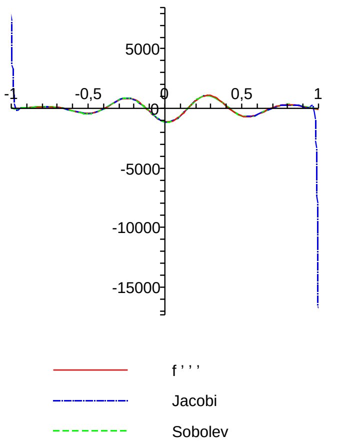
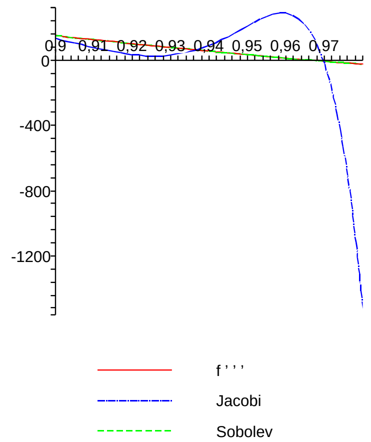


Figure 5.4: $f'''(x)$ and the third derivatives of both the partial sums of degree 30 of the Fourier-Jacobi and Fourier-Sobolev series of $f(x)$

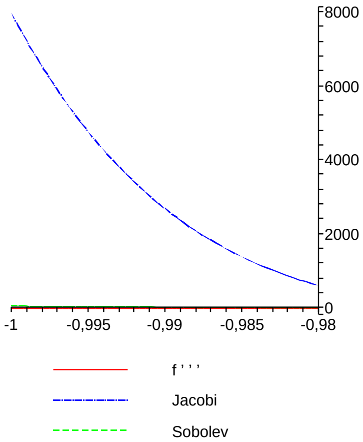
(a) in $[-1, 1]$



(b) in $[0.9, 0.98]$



(c) in $[-1, -0.98]$



(d) in $[0.98, 1]$

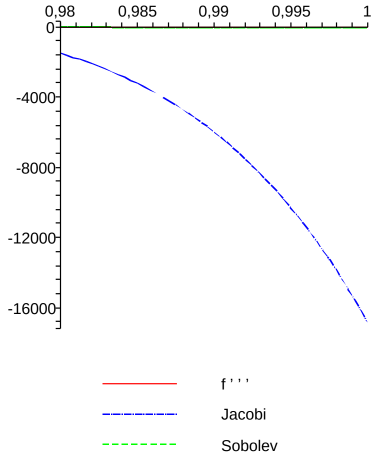


Table 5.1: Errors of the approximations of $f(x)$ ($i = 0$) and its derivatives ($i = 1, 2, 3$) with the partial sums of degree 30 of the Fourier-Jacobi (J) and Fourier-Sobolev (S) series of $f(x)$ and their derivatives

(a) for the norm $\|\cdot\|_{L^2}$

i	$\varepsilon^{(i)}_{J,L^2}$	$\varepsilon^{(i)}_{S,L^2}$
0	1.05×10^{-7}	1.33×10^{-10}
1	2.76×10^{-3}	2.61×10^{-7}
2	89.93	2.35×10^{-3}
3	1.65×10^{6}	29.3

(b) for the norm $\|\cdot\|_{\mu_0}$

i	$\epsilon^{(i)}_{J,\mu_0}$	$\epsilon^{(i)}_{S,\mu_0}$
0	1.92×10^{-9}	1.85×10^{-11}
1	1.87×10^{-5}	3.93×10^{-9}
2	0.645	2.15×10^{-5}
3	14682.775	0.305

(c) for the norm $\|\cdot\|_{\lambda=0.1}$

i	$E^{(i)}_{J,\lambda}$	$E^{(i)}_{S,\lambda}$
0	157.59	4.07×10^{-6}
1	2.54×10^{6}	1.39×10^{-3}
2	2.62×10^{10}	4.3
3	1.822×10^{14}	63500.24

and Fourier-Sobolev series of $f(x)$ and their derivatives, $S^{(i)}_{30,\ell}(x; f)$, $i = 1, 2, 3$, $\ell = J, S$, for norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{\mu_0}$, and $\|\cdot\|_{\lambda}$.

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