Discrete semiclassical orthogonal polynomials of class one

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Abstract

We study the discrete semiclassical orthogonal polynomials of class $s = 1$. By considering particular solutions of the Pearson equation, we obtain five canonical families. We also consider limit relations between these and other families of orthogonal polynomials.

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1 Introduction

Discrete orthogonal polynomials with respect to uniform lattices have attracted the interest of researchers from many points of view [26]. A first approach comes from the discretization of hypergeometric second order linear differential equations and thus the classical discrete orthogonal polynomials Charlier, Krawtchouk, Meixner and Hahn appear in a natural way. As a consequence of the symmetrization problem for the above second order difference equations, you can deduce the (discrete) measure with respect to such polynomials are orthogonal. This yields the so-called Pearson equation that the measure satisfies.

In the last twenty years, new families of discrete orthogonal polynomials have been considered in the literature taking into account the so-called canonical spectral transformations of the orthogonality measure. In particular, when you add mass points to the discrete measure (Uvarov transformation) the sequences of orthogonal polynomials with respect to the new measure have been studied extensively (see [12], [3], [2], among others). When you multiply the discrete measure by a polynomial (Christoffel transformation), some results are known [30].

From a structural point of view, some effort has been done in order to translate to the discrete case the well stated theory of semiclassical orthogonal polynomials (see [24]). In particular, characterizations of such polynomials in terms of structure relations of the first and second kind, as well as discrete holonomic equations (second order linear difference equations with polynomial coefficients of fixed degree and where the degree of the polynomial appears as a parameter) have been done [22]. Linear spectral perturbations of semiclassical linear functionals have been studied in the Uvarov case [18].

On the other hand, we must point out that the linear canonical spectral transformations (Christoffel, Uvarov, Geronimus) of classical discrete orthogonal polynomials yield discrete semiclassical orthogonal polynomials. But, as a first step, it remains open the problem of classification of discrete semiclassical linear functional of class one. The symmetric discrete semiclassical linear functionals of class one have been described in [25]. Notice that the classification of D- semiclassical linear functional of class one was done in [7] and for class two in [23].

The aim of this work is to provide a constructive method of D_w-semiclassical orthogonal polynomials based on the Pearson equation that the corresponding linear functional satisfies. We will focus our attention on the classification of D_1-semiclassical linear functionals of class $s = 1$. In such a way, new families of linear functionals appear. Notice that an alternative method is based on the Laguerre-Freud equations satisfied by the coefficients of the three term recurrence relations associated with these orthogonal polynomials. Their complexity increases with the class of the linear functional and the solution is cumbersome. Basic references concerning this approach are [16] as well as [25].

The structure of the manuscript is as follows. Section 2 deals with the basic definitions and the theoretical background we will need in the sequel. In Section 3 we describe the D_1-classical linear functionals, as D_1-semiclassical of class $s = 0$. It will be very useful in the sequel taking into account that most of the semiclassical linear functionals of class $s = 1$ are related to them. Indeed, in Section 4, the classification of such semiclassical linear functionals is given. Some of them are not known in the literature, as far as we know. Finally, in Section 5, limit relations in terms of their parameters for semiclassical orthogonal polynomials are studied.

2 Preliminaries and basic background

Definition 1 *Let $\{\mu_n\}_{n\geq 0}$ be a sequence of complex numbers and let $\mathcal{L}$ be a linear complex valued function defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients by*

$$\langle \mathcal{L}, x^n \rangle = \mu_n.$$

Then, $\mathcal{L}$ is called the moment functional determined by the moment sequence $\{\mu_n\}_{n\geq 0}$ and μ_n is called the moment of order n.

Given a moment functional $\mathcal{L}$, the formal Stieltjes function of $\mathcal{L}$ is defined by

$$S_{\mathcal{L}}(z) = -\sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}.$$

For any moment functional $\mathcal{L}$ and any polynomial $q(x)$, we define the moment functional $q\mathcal{L}$ by

$$\langle q\mathcal{L}, P \rangle = \langle \mathcal{L}, qP \rangle, \quad P \in \mathbb{P}.$$

In the sequel, we will denote the set of nonnegative integers by $\mathbb{N}_0$.

Definition 2 *Let $\mathcal{L}$ be the linear functional associated with the moment sequence $\{\mu_n\}_{n\geq 0}$ and*

$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}.$$

Then, $\mathcal{L}$ is said to be regular/quasidefinite (resp. positive definite) if $\Delta_n \neq 0$ (resp. > 0) for all $n \in \mathbb{N}_0$.

Definition 3 *A sequence of polynomials $\{P_n(x)\}_{n\geq 0}$, $\deg(P_n) = n$, is said to be an orthogonal polynomial sequence with respect to a regular linear functional $\mathcal{L}$, if there exists a sequence of nonzero real numbers $\{\zeta_n\}_{n\geq 0}$ such that*

$$\langle \mathcal{L}, P_k P_n \rangle = \zeta_n \delta_{k,n}, \quad k, n \in \mathbb{N}_0.$$

If $\zeta_n = 1$, then $\{P_n(x)\}_{n\geq 0}$ is said to be an orthonormal polynomial sequence. Notice that if the linear functional is positive definite there exists a unique sequence of orthonormal polynomials assuming the leading coefficient is a positive real number.

If the leading coefficient of $P_n(x)$ is 1 for every $n \in \mathbb{N}_0$, then the sequence is said to be a monic orthogonal sequence (MOPS, in short).

Theorem 4 *Let $\{b_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ with $\gamma_n \neq 0$ for every $n \in \mathbb{N}_0$ be arbitrary sequences of complex numbers and let $\{P_n(x)\}$ a sequence of monic polynomials defined by the three term recurrence relation (TTRR)*

$$P_{n+1}(x) = (x - b_n) P_n(x) - \gamma_n P_{n-1}(x), \tag{1}$$

with $P_{-1} = 0$ and $P_0 = 1$. Then, there is a unique linear functional $\mathcal{L}$ such that $\mathcal{L}(1) = \gamma_0$ and

$$\langle \mathcal{L}, P_k(x) P_n(x) \rangle = \gamma_0 \gamma_1 \cdots \gamma_n \delta_{k,n}.$$

Proof. See [11, Theorem 4.4] ■

If the linear functional is positive definite and $\{p_n(x)\}_{n\geq 0}$ is the corresponding orthonormal polynomial sequence, then the above TTRR formula becomes

$$a_{n+1} p_{n+1}(x) = (x - b_n) p_n(x) - a_n p_{n-1}(x),$$

where a_n is a real number and $a_n^2 = \gamma_n$.

Definition 5 *Let $\mathcal{L}$ be a linear functional and $U^* : \mathbb{P} \to \mathbb{P}$ be a linear operator. The linear functional $U\mathcal{L}$ is defined by*

$$\langle U\mathcal{L}, P\rangle = -\langle \mathcal{L}, U^* P\rangle, \quad P \in \mathbb{P}.$$

Example 6 *If U is the standard derivative operator D, we have $U^* = U = D$.*

Definition 7 *A regular linear functional $\mathcal{L}$ is called U-semiclassical if it satisfies the Pearson equation $U(\phi\mathcal{L}) + \psi\mathcal{L} = 0$ or, equivalently,*

$$\langle U(\phi\mathcal{L}) + \psi\mathcal{L}, P\rangle = 0, \quad P \in \mathbb{P},$$

where ϕ, ψ are two polynomials, and ϕ is monic. The corresponding orthogonal sequence $\{P_n(x)\}_{n\geq 0}$ is called U-semiclassical.

In the literature, semiclassical linear functionals with respect to different choices of operators have been studied. In particular, if $U = D$ (the standard derivative operator), the theory of D-semiclassical linear functionals has been exhaustively studied by P. Maroni and co-workers (see [24] for an excellent survey on this topic).

If $U = D_\omega$, where

$$D_\omega f(x) = \frac{f(x+\omega) - f(x)}{\omega}, \quad \omega \neq 0,$$

a regular linear functional $\mathcal{L}$ is said to be D_ω-semiclassical if there exist polynomials ϕ, ψ, where ϕ is monic and $\deg\psi \geq 1$ such that $D_\omega(\phi\mathcal{L}) + \psi\mathcal{L} = 0$.

Notice that

$$D_1 f(x) = f(x+1) - f(x) = \Delta f(x),$$
$$D_{-1} f(x) = f(x) - f(x-1) = \nabla f(x),$$

are the forward and backward difference operators, respectively, and

$$\lim_{\omega\to 0} D_\omega f(x) = Df(x) = f'(x).$$

If $U = D_\omega$, we define $U^* = D_{-\omega}$. With this definition, we have $\Delta^* = \nabla$ and when $\omega \to 0$ we recover the identity $U^* = D = U$.

The concept of class of a D_ω-semiclassical linear functional plays a central role in order to give a constructive theory of such linear functionals.

Definition 8 *If $\mathcal{L}$ is a D_ω- semiclassical linear functional, then the class s of $\mathcal{L}$ is defined by*

$$s = \min_{\phi,\psi} \max \{\deg \phi - 2, \deg \psi - 1\}.$$

among all polynomials ϕ, ψ such that the Pearson equation holds. Notice that the class s is always nonnegative.

For any complex number c, we introduce the linear application $\theta_c : \mathbb{P} \to \mathbb{P}$ defined by

$$\theta_c(p)(x) = \frac{p(x) - p(c)}{x - c}.$$

We have the following result [24].

Theorem 9 *The regular linear functional $\mathcal{L}$ satisfying the Pearson equation*

$$D_\omega (\phi \mathcal{L}) + \psi \mathcal{L} = 0$$

is of class s if and only if

$$\prod_{c \in Z(\phi)} \left(|\psi(c-\omega) + (\theta_c \phi)(c-\omega)| + |\langle \mathcal{L}, \theta_{c-\omega}(\psi + \theta_c \phi) \rangle| \right) > 0,$$

where $Z(\phi)$ denotes the set of zeros of the polynomial $\phi(x)$.

When there exists $c \in Z(\phi)$ such that

$$\psi(c-\omega) + (\theta_c \phi)(c-\omega) = \langle \mathcal{L}, \theta_{c-\omega}(\psi + \theta_c \phi) \rangle = 0,$$

the Pearson equation becomes

$$D_\omega [(\theta_c \phi) \mathcal{L}] + [\theta_{c-\omega}(\psi + \theta_c \phi)] \mathcal{L} = 0.$$

Remark 10 *When $s = 0$, the D_ω-classical orthogonal polynomials appear (see [1]). When $\omega = 1$ several characterizations of classical orthogonal polynomials have been done in [17]. Indeed, we explain with more detail in the next section the main characteristics of these polynomials and their corresponding linear functionals.*

The D_1-semiclassical linear functionals have been studied by F. Marcellán and L. Salto and they are characterized following the same ideas as in the D case. P. Maroni and M. Mejri deduced the Laguerre-Freud equations for the coefficients of the TTRR of D_w- semiclassical orthogonal polynomials of class $s = 1$. In the symmetric case, i.e., the moments of odd order vanish, they deduce the explicit values of such coefficients and the integral representations of the corresponding linear functionals are given.

On the other hand, the Pearson equation yields a difference equation for the moments of the linear functional and, as consequence, we get a linear difference equation with polynomial coefficients satisfied by the Stieltjes function associated with the linear functional. Indeed,

Theorem 11 *If $\mathcal{L}$ is a D_ω-semiclassical moment functional, then the formal Stieltjes function of $\mathcal{L}$ satisfies a non homogeneous first order linear difference equation*

$$\phi(z)\, D_\omega S_{\mathcal{L}}(z) = a(z) S_{\mathcal{L}}(z) + b(z),$$

where $a(z)$ and $b(z)$ are polynomials depending on ϕ and ψ, with $\deg(a) \leq s+1$ and $\deg(b) \leq s$.

3 Discrete semiclassical orthogonal polynomials

In the sequel, we will consider linear functionals

$$\langle \mathcal{L}, P \rangle = \sum_{x=0}^{\infty} P(x) \rho(x),$$

for some positive weight function $\rho(x)$ supported on a countable subset of the real line. With this choice, the Pearson equation

$$\langle \Delta(\phi \mathcal{L}) + \psi \mathcal{L}, P \rangle = 0, \quad P \in \mathbb{P},$$

yields

$$\Delta(\phi\rho) + \psi\rho = 0. \tag{2}$$

We rewrite this equation as

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\phi(x) - \psi(x)}{\phi(x+1)} = \frac{\lambda(x)}{\phi(x+1)}, \tag{3}$$

with

$$\phi(x) = x(x+\beta_1)(x+\beta_2)\cdots(x+\beta_r),$$

and

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2)\cdots(x+\alpha_l).$$

Since the Pochhammer symbol $(\alpha)_x$ defined by $(\alpha)_0 = 1$ and

$$(a)_x = a\,(a+1)\cdots(a+x-1)\,, \quad x \in \mathbb{N}, \tag{4}$$

satisfies the identity

$$\frac{(\alpha)_{x+1}}{(\alpha)_x} = x + \alpha, \quad x \in \mathbb{N}_0,$$

we obtain

$$\rho\,(x) = \frac{(\alpha_1)_x \cdots (\alpha_l)_x}{(\beta_1+1)_x \cdots (\beta_r+1)_x}\frac{c^x}{x!}. \tag{5}$$

We will denote the orthogonal polynomials associated with $\rho\,(x)$ by

$$P_n^{(l,r)}\,(x;\alpha_1,\ldots,\alpha_l,\beta_1,\ldots,\beta_r;c)\,.$$

The moments of the weight function (5) are given by

$$\mu_n = \sum_{x=0}^{\infty} x^n \frac{(\alpha_1)_x \cdots (\alpha_l)_x}{(\beta_1+1)_x \cdots (\beta_r+1)_x}\frac{c^x}{x!}, \quad n = 0, 1, \ldots.$$

They exist if [27, 16.2]:

1.
$$l \le r, \quad \text{and} \quad c \in \mathbb{C}. \tag{6}$$

2.
$$l \ge r+1, \tag{7}$$

   one or more of the top parameters α_i is a nonpositive integer, and $c \in \mathbb{C}$.

3.
$$l = r+1, \quad \text{and} \quad |c| < 1. \tag{8}$$

4.
$$l = r+1, \quad |c| = 1, \quad \text{and} \quad \operatorname{Re}\,(\beta_1 + \cdots + \beta_r - \alpha_1 - \cdots - \alpha_l) > 0. \tag{9}$$

3.1 Discrete classical polynomials

When $s = 0$, we solve the Pearson equation (3) with $\deg(\psi) = 1$ and

$$1 \leq \deg(\phi) \leq 2.$$

Three canonical cases appear [26]:

$\deg(\lambda)$	$\deg(\phi)$	$\deg(\psi) = \deg(\phi - \lambda)$
0	1	1
1	1	1
2	2	1

1. If $\deg(\lambda) = 0$ and $\deg(\phi) = 1$,we can take

$$\lambda(x) = c, \quad \phi(x) = x, \quad \psi(x) = \phi(x) - \lambda(x) = x - c, \tag{10}$$

and from (5) we obtain

$$\rho(x) = \frac{c^x}{x!}, \quad c > 0, \quad x \in \mathbb{N}_0. \tag{11}$$

The family of orthogonal polynomials associated with the weight function (11) are known as the Charlier polynomials, and we denote them by $P_n^{(0,0)}(x; c)$ .They have the hypergeometric representation [20, 9.14.1]

$$P_n^{(0,0)}(x; c) = {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array} ; -\frac{1}{c} \right), \tag{12}$$

where the hypergeometric function ${}_pF_q(z)$ is defined by

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{13}$$

The monic Charlier polynomials $\widehat{P}_n^{(0,0)}(x; c)$ are given by

$$\widehat{P}_n^{(0,0)}(x; c) = (-c)^n P_n^{(0,0)}(x; c). \tag{14}$$

It is usual to denote these polynomials by

$$C_n(x; a) = P_n^{(0,0)}(x; a). \tag{15}$$

2. If $\deg(\lambda) = 1$ and $\deg(\phi) = 1$,we can take

$$\lambda(x) = c(x+\alpha), \quad \phi(x) = x, \quad \psi(x) = (1-c)x - c\alpha, \tag{16}$$

and from (5) we have

$$\rho(x) = (\alpha)_x \frac{c^x}{x!}, \quad \alpha > 0, \quad 0 < c < 1, \quad x \in \mathbb{N}_0. \tag{17}$$

From (8), the condition $0 < c < 1$ is needed for the moments to exist. The first moment μ_0 is given by

$$\mu_0 = \sum_{x=0}^{\infty} (\alpha)_x \frac{c^x}{x!} = (1-c)^{-\alpha}. \tag{18}$$

The family of orthogonal polynomials associated with the weight function (17) are known as the Meixner polynomials, and we denote them by $P_n^{(1,0)}(x;\alpha;c)$. They have the hypergeometric representation [20, 9.10.1]

$$P_n^{(1,0)}(x;\alpha;c) = {}_2F_1\left(\begin{array}{c} -n,-x \\ \alpha \end{array}; 1 - \frac{1}{c}\right) \tag{19}$$

and the monic Meixner polynomials $\widehat{P}_n^{(1,0)}(x;\alpha;c)$ are given by

$$\widehat{P}_n^{(1,0)}(x;\alpha;c) = (\alpha)_n \left(\frac{c}{c-1}\right)^n P_n^{(1,0)}(x;\alpha;c). \tag{20}$$

It is usual to denote these polynomials by

$$M_n(x;\beta,c) = P_n^{(1,0)}(x;\beta;c).$$

If we want c to be unbounded, we can use (7) and set $\alpha = -N$, with $N \in \mathbb{N}$. For the weight function to be positive we need $c < 0$, and we obtain the Krawtchouk polynomials $P_n^{(1,0)}(x;-N;c)$ with

$$\rho(x) = (-N)_x \frac{c^x}{x!}, \quad c < 0, \quad N \in \mathbb{N}, \quad x \in [0,N], \tag{21}$$

and

$$\phi(x) = x, \quad \psi(x) = (1-c)x + cN. \tag{22}$$

It is usual to denote these polynomials by

$$K_n(x;p,N) = P_n^{(1,0)}\left(x;-N;\frac{p}{p-1}\right).$$

3. If $\deg(\lambda) = 2$ and $\deg(\phi) = 2$,we can take

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2), \quad \phi(x) = x(x+\beta).$$

Thus,

$$\psi(x) = \phi(x) - \lambda(x) = (1-c)x^2 + x(\beta - c\alpha_1 - c\alpha_2) - c\alpha_1\alpha_2,$$

and since $\deg(\psi) = 1$, we must have $c = 1$. Hence,

$$\phi(x) = x(x+\beta), \quad \psi(x) = x(\beta-\alpha_1-\alpha_2) - \alpha_1\alpha_2, \tag{23}$$

and

$$\rho(x) = \frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{1}{x!}, \quad x \in \mathbb{N}_0. \tag{24}$$

From (9), we need $\operatorname{Re}(\beta+1-\alpha_1-\alpha_2) > 0$ for the moments to exist. The first moment μ_0 is given by [27, 15.4.20]

$$\mu_0 = \sum_{x=0}^{\infty}\frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{1}{x!} = \frac{\Gamma(\beta+1)\Gamma(\beta+1-\alpha_1-\alpha_2)}{\Gamma(\beta+1-\alpha_1)\Gamma(\beta+1-\alpha_2)}.$$

Thus, we need $\alpha_1, \alpha_2 > 0$ and $\beta+1 > \alpha_1+\alpha_2$. The family of orthogonal polynomials associated with the weight function (24) are known as the Hahn polynomials, and we denote them by $P_n^{(2,1)}(x;\alpha_1,\alpha_2,\beta;1)$. They have the hypergeometric representation [13, 10.23.12]

$$P_n^{(2,1)}(x;\alpha_1,\alpha_2,\beta;1) = {}_3F_2\left(\begin{matrix} -n,-x,n+\alpha_1+\alpha_2-\beta-1 \\ \alpha_1,\alpha_2 \end{matrix};1\right). \tag{25}$$

In the literature (see (9.5.1) in [20]), the so called Hahn polynomials $Q_n(x;\alpha,\gamma,N)$ correspond to the choice $\alpha_1 = \alpha+1$, and $\alpha_2 = -N$, $\gamma = -N-\beta-1$,with $N \in \mathbb{N}$.

On the other hand, another family of Hahn polynomials is (see page 34 in [26])

$$h_n(x;\alpha,\beta,N) = P_n^{(2,1)}(x;\beta+1,1-N,-N-\alpha;1). \tag{26}$$

Notice that in [26] the authors consider two different families of Hahn's polynomials. The polynomials involved in the corresponding Pearson equations are related by using negative signs for the variable x. Indeed, for the second family you also have a relation

$$\tilde{h}_n(x;\mu,\nu,N) = P_n^{(2,1)}(x;1-N-\nu,1-N,\mu;1), \tag{27}$$

as well as

$$h_n(x;\alpha,\beta,N) = \tilde{h}_n(x;-N-\alpha,-N-\beta,N). \tag{28}$$

4 Discrete semiclassical polynomials of class 1

When $s = 1$, we solve the Pearson equation (2) with $\deg(\psi) = 2$ and

$$1 \leq \deg(\phi) \leq 3,$$

and obtain the following five canonical cases:

$\deg(\lambda)$	$\deg(\phi)$	$\deg(\psi) = \deg(\phi - \lambda)$
0	2	2
1	2	2
2	1	2
2	2	2
3	3	2

1. If $\deg(\lambda) = 0$ and $\deg(\phi) = 2$,we can take

$$\lambda(x) = c, \quad \phi(x) = x(x+\beta), \quad \psi(x) = x^2 + \beta x - c,$$

and from (5) we have

$$\rho(x) = \frac{1}{(\beta+1)_x}\frac{c^x}{x!}, \quad x \in \mathbb{N}_0, \tag{29}$$

where $\beta > -1$ and $c > 0$.The family of orthogonal polynomials associated with the weight function (29) are known as the Generalized Charlier polynomials, and we denote them by $P_n^{(0,1)}(x;\beta;c)$.

2. If $\deg(\lambda) = 1$ and $\deg(\phi) = 2$,we can take

$$\lambda(x) = c(x+\alpha), \quad \phi(x) = x(x+\beta),$$

and

$$\psi(x) = x^2 + (\beta - c)x - c\alpha.$$

From (5), we have

$$\rho(x) = \frac{(\alpha)_x}{(\beta+1)_x}\frac{c^x}{x!}, \quad x \in \mathbb{N}_0, \tag{30}$$

where $\alpha(\beta+1) > 0$ and $c > 0$.The family of orthogonal polynomials associated with the weight function (30) are known as the Generalized Meixner polynomials, and we denote them by $P_n^{(1,1)}(x;\alpha,\beta;c)$.

3. If $\deg(\lambda) = 2$ and $\deg(\phi) = 1$,we can take

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2), \quad \phi(x) = x.$$

From (5), we have

$$\rho(x) = (\alpha_1)_x(\alpha_2)_x\frac{c^x}{x!},$$

and from (7) we need $\alpha_2 = -N$, with $N \in \mathbb{N}$ for the moments to exist. Setting $\alpha_1 = \alpha$, we get

$$\lambda(x) = c(x+\alpha)(x-N), \quad \phi(x) = x,$$

and

$$\psi(x) = -cx^2 + x(Nc - c\alpha + 1) + Nc\alpha.$$

We call the family of orthogonal polynomials associated with the weight function

$$\rho(x) = (\alpha)_x(-N)_x\frac{c^x}{x!}, \quad x \in [0,N], \tag{31}$$

with $c < 0$ and $\alpha > 0$, "Generalized Krawtchouk polynomials", and we denote them by $P_n^{(2,0)}(x;\alpha,-N;c)$.

4. If $\deg(\lambda) = 2$ and $\deg(\phi) = 2$,we can take

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2), \quad \phi(x) = x(x+\beta),$$

and

$$\psi(x) = (1-c)x^2 + (\beta - c\alpha_1 - c\alpha_2)x - c\alpha_1\alpha_2.$$

From (5), we have

$$\rho(x) = \frac{(\alpha_1)_x (\alpha_2)_x}{(\beta+1)_x} \frac{c^x}{x!}, \quad x \in \mathbb{N}_0, \tag{32}$$

and from (8) we need $0 < c < 1$, with $\alpha_1\alpha_2(\beta+1) > 0$. We call the family of orthogonal polynomials associated with the weight function (32) "Generalized Hahn polynomials of type I", and we denote them by $P_n^{(2,1)}(x;\alpha_1,\alpha_2;\beta;c)$.

5. If $\deg(\lambda) = 3$ and $\deg(\phi) = 3$,we can take

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2)(x+\alpha_3), \quad \phi(x) = x(x+\beta_1)(x+\beta_2)$$

and

$$\psi(x) = x(x+\beta_1)(x+\beta_2) - c(x+\alpha_1)(x+\alpha_2)(x+\alpha_3).$$

For $\psi(x)$ to be of second degree we need $c = 1$. Thus,

$$\lambda(x) = (x+\alpha_1)(x+\alpha_2)(x+\alpha_3), \quad \phi(x) = x(x+\beta_1)(x+\beta_2),$$

$$\psi(x) = -x^2(\alpha_1+\alpha_2-\beta_1+\alpha_3-\beta_2) - x(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3-\beta_1\beta_2) - \alpha_1\alpha_2\alpha_3,$$

and from (5) we obtain

$$\rho(x) = \frac{(\alpha_1)_x (\alpha_2)_x (\alpha_3)_x}{(\beta_1+1)_x (\beta_2+1)_x} \frac{1}{x!}, \quad x \in \mathbb{N}_0. \tag{33}$$

For the moments to exist, (9) gives $\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3 > 0$, while positivity demands that $\alpha_1\alpha_2\alpha_3(\beta_1+1)(\beta_2+1) > 0$. We call the family of orthogonal polynomials associated with the weight function (33) "Generalized Hahn polynomials of type II", and we denote them by $P_n^{(3,2)}(x;\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,;1)$.

In the next sections, we study these polynomials in detail.

4.1 Generalized Charlier polynomials

Recall that for these polynomials the weight function is given by

$$\rho(x) = \frac{1}{(\beta+1)_x} \frac{c^x}{x!}, \quad x \in \mathbb{N}_0,$$

with $\beta > -1$ and $c > 0$. The first moments are

$$\mu_0 = c^{-\frac{\beta}{2}} I_\beta\left(2\sqrt{c}\right) \Gamma(\beta+1), \quad \mu_1 = c^{\frac{1-\beta}{2}} I_{\beta+1}\left(2\sqrt{c}\right) \Gamma(\beta+1),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind [27, 10.25.2].

In [19] Hounkonnou, Hounga and Ronveaux studied the semiclassical polynomials associated with the weight function

$$\rho_r(x) = \frac{c^x}{(x!)^r}, \quad r = 0, 1, \ldots. \tag{34}$$

When $r = 2$, they derive the Laguerre-Freud equations for the recurrence coefficients and a second-order difference equation. Note that from (34) we have

$$\frac{\rho_r(x+1)}{\rho_r(x)} = \frac{c}{(x+1)^r},$$

and from (3) we conclude that

$$\lambda_r(x) = c, \quad \phi_r(x) = x^r, \quad \psi_r(x) = x^r - c,$$

and therefore the orthogonal polynomials associated with $\rho_r(x)$ are of class $r-1$. The case $r = 2$ is a particular example of (29) with $\beta = 0$.

In [33] Van Assche and Foupouagnigni also consider (34) with $r = 2$. They simplify the Laguerre-Freud equations obtained in [19] and get

$$\begin{aligned} u_{n+1} + u_{n-1} &= \frac{1}{\sqrt{c}} \frac{n u_n}{1 - u_n^2}, \\ v_n &= \sqrt{c}\, u_{n+1} u_n \end{aligned}$$

with $\gamma_n = c\left(1 - u_n^2\right)$ and $\beta_n = v_n + n$. They show that these equations are related to the discrete Painlevé II equation $\mathrm{dP_{II}}$ [32]

$$x_{n+1} + x_{n-1} = \frac{(an+b)\, x_n + c}{1 - x_n^2}.$$

They also obtain the asymptotic behavior

$$\lim_{n\to\infty}\gamma_n = c, \quad \lim_{n\to\infty} v_n = 0,$$

and conclude that the asymptotic zero distribution is given by the uniform distribution on $[0,1]$, as is the case for the usual Charlier polynomials [21].

In [31], Smet and Van Assche studied the orthogonal polynomials associated with the weight function (29). They obtained the Laguerre-Freud equations

$$\begin{aligned}\left(a_{n+1}^2 - c\right)\left(a_n^2 - c\right) &= c\left(b_n - n\right)\left(b_n - n + \beta\right), \\ b_n + b_{n-1} &= n - 1 - \beta + \frac{cn}{a_n^2},\end{aligned} \tag{35}$$

for the orthonormal polynomials. They showed that these equations are a limiting case of the discrete Painlevé IV equation $\mathrm{dP_{IV}}$ [32]

$$\begin{aligned}x_{n+1}x_n &= \frac{\left(y_n - \delta n - E\right)^2 - A}{y_n^2 - B}, \\ y_n + y_{n-1} &= \frac{\delta n + E - \delta/2 - C}{1 + Dx_n} + \frac{\delta n + E - \delta/2 + C}{1 + x_n/D}.\end{aligned}$$

Finally, in [15] Filipuk and Van Assche related the system 35 to the (continuous) fifth Painlevé equation $\mathrm{P_V}$.

4.2 Generalized Meixner polynomials

For these polynomials the weight function is given by

$$\rho(x) = \frac{(\alpha)_x}{(\beta+1)_x}\frac{c^x}{x!}, \quad x \in \mathbb{N}_0,$$

with $\alpha(\beta+1) > 0$ and $c > 0$. The first moments are

$$\mu_0 = M(\alpha, \beta+1; c), \quad \mu_1 = \frac{\alpha c}{\beta+1} M(\alpha+1, \beta+2; c),$$

where $M(a, b; z)$ is the confluent hypergeometric function [27, 13.2.2].

In [28] Ronveaux considered the semiclassical polynomials associated with the weight function

$$\rho_r(x) = \prod_{j=1}^{r} (\alpha_j)_x \; \frac{c^x}{(x!)^r}, \quad r = 1, 2, \ldots,$$

and in [29] he made some conjectures on the asymptotic behavior of the recurrence coefficients.

In [31], Smet and Van Assche studied the orthogonal polynomials associated with the weight function (30). They obtained the Laguerre-Freud equations

$$\begin{aligned}\left(u_n+v_n\right)\left(u_{n+1}+v_n\right) &= \frac{\alpha-1}{c^2}v_n\left(v_n-c\right)\left(v_n-c\frac{\alpha-1-\beta}{\alpha-1}\right), \\ \left(u_n+v_n\right)\left(u_{n+1}+v_{n-1}\right) &= \frac{u_n}{u_n-\frac{cn}{\alpha-1}}\left(u_n+c\right)\left(u_n+c\frac{\alpha-1-\beta}{\alpha-1}\right),\end{aligned} \tag{36}$$

for the orthonormal polynomials, with

$$\begin{aligned}a_n^2 &= cn-\left(\alpha-1\right)u_n, \\ b_n &= n+\alpha+c-\beta-1-\frac{\alpha-1}{c}v_n.\end{aligned}$$

They also proved that the system (36) is a limiting case of the asymmetric discrete Painlevé IV equation $\alpha-\mathrm{dP_{IV}}$ [32].

In [14] Filipuk and Van Assche showed that the system (36) can be obtained from the Bäcklund transformation of the fifth Painlevé equation $\mathrm{P_V}$. The particular case of (30) when $\beta=0$ was considered by Boelen, Filipuk, and Van Assche in [9].

If we set $\alpha=-N$, $N\in\mathbb{N}$ in (30), then we obtain

$$\rho\left(x\right)=\frac{\left(-N\right)_x}{\left(\beta+1\right)_x}\frac{c^x}{x!},$$

where we now have $\beta>-1$ and $c<0$. This case was analyzed by Boelen, Filipuk, Smet, Van Assche, and Zhang in [8].

4.2.1 Singular limits

If we let $\alpha\to0$ and $\beta\to-1$ in (30), we have $\rho\left(x\right)\to\widetilde{\rho}\left(x\right)$ where $\widetilde{\rho}\left(x\right)$ is a new weight function satisfying the Pearson equation

$$\Delta\left[\left(x-1\right)x\widetilde{\rho}\right]+\left[x-\left(c+1\right)\right]x\widetilde{\rho}=0. \tag{37}$$

Assuming that $\widetilde{\rho}\left(x\right)$ satisfies $x\widetilde{\rho}(x)=xu(x)$, for some weight function $u(x)$, we get

$$\Delta\left[\left(x-1\right)xu\right]+\left[x-\left(c+1\right)\right]xu=0. \tag{38}$$

Using the product rule

$$\Delta(fg) = f\Delta g + g\Delta f + \Delta f \Delta g \tag{39}$$

in (38), we have

$$xu + (x-1)\,\Delta(xu) + \Delta(xu) + [x-(c+1)]\,xu = 0,$$

or

$$x\Delta(xu) + [x-(c+1)+1]\,xu = 0.$$

Dividing by x, we obtain

$$\Delta(xu) + (x-c)\,u = 0.$$

Comparing with (10), we see that $u(x)$ is the weight function corresponding to the Charlier polynomials (11), and therefore (37) implies that

$$\widetilde{\rho}(x) = \frac{c^x}{x!} + M\delta(x), \tag{40}$$

where $\delta(x)$ is the Dirac delta function.

The orthogonal polynomials $P_n^{(1,1)}(x;0,-1;c)$ associated with the weight function (40) were first studied by Chihara in [12]. He showed that they satisfy a three term recurrence relation (1) with

$$b_n = c\frac{n}{n+1}\frac{D_n}{D_{n+1}} + (n+1)\frac{D_{n+1}}{D_n},$$

and

$$\gamma_n = c\frac{n^2}{n+1}\frac{D_n^2}{D_{n-1}D_{n+1}},$$

where

$$D_n = \frac{c^n}{n!}\frac{M}{e^c + MK_{n-1}},$$

and

$$K_n = \sum_{j=0}^{n}\frac{c^j}{j!}, \quad K_{-1} = 0.$$

Note that in order that D_n is well defined for all n, we need $M > -1$, since $K_n \nearrow e^c$.

In [5], Bavinck and Koekoek obtained a difference equation satisfied by these polynomials and in [2] Álvarez-Nodarse, García, and Marcellán found the hypergeometric representation

$$P_n^{(1,1)}(x;0;-1;c) = (-c)^n \ {}_3F_1\left(\begin{array}{c} -n, -x, 1+\frac{x}{D_n} \\ \frac{x}{D_n} \end{array} ; -\frac{1}{c} \right).$$

Since

$$\lim_{z\to\infty} \frac{(1+z)_x}{(z)_x} = 1,$$

we see that

$$\lim_{M\to 0} P_n^{(1,1)}(x;0,-1;c) = \widehat{C}_n(x;c),$$

where $\widehat{C}_n(x;c)$ is the monic Charlier polynomial (14).

4.3 Generalized Krawtchouk polynomials

For these polynomials the weight function is given by

$$\rho(x) = (\alpha)_x (-N)_x \frac{c^x}{x!}, \quad x \in \ [0,N],$$

with $c < 0$, $N \in \mathbb{N}$ and $\alpha > 0$. The first moments are

$$\mu_0 = C_N\left(-\alpha; -\frac{1}{c}\right), \quad \mu_1 = -c\alpha N C_{N-1}\left(-\alpha-1; -\frac{1}{c}\right),$$

where $C_n(x;a)$ is the Charlier polynomial (15).

To our knowledge, these polynomials have not appeared before in the literature.

4.4 Generalized Hahn polynomials of type I

For these polynomials the weight function is given by

$$\rho(x) = \frac{(\alpha_1)_x (\alpha_2)_x}{(\beta+1)_x} \frac{c^x}{x!}, \quad x \in \mathbb{N}_0,$$

with $0 < c < 1$, with $\alpha_1 \alpha_2 (\beta+1) > 0$.

The first moments are

$$\mu_0 = {}_2F_1\left(\begin{matrix}\alpha_1, \alpha_2 \\ \beta+1\end{matrix}; c\right),$$
$$\mu_1 = c\frac{\alpha_1\alpha_2}{\beta+1}\, {}_2F_1\left(\begin{matrix}\alpha_1+1, \alpha_2+1 \\ \beta+2\end{matrix}; c\right),$$

where ${}_2F_1\left(\begin{matrix}a, b \\ c\end{matrix}; z\right)$ is the hypergeometric function.

4.4.1 Singular limits

1) If we let $\alpha_2 \to 0, \beta \to -1$ and $\alpha_1 = \alpha$ in (32), we have $\rho(x) \to \widetilde{\rho}(x)$ where $\widetilde{\rho}(x)$ is a new weight function satisfying the Pearson equation

$$\Delta\left[(x-1)\, x\widetilde{\rho}\right] + \left[(1-c)\, x - (1+c\alpha)\right] x\widetilde{\rho} = 0. \tag{41}$$

Assuming that $\widetilde{\rho}(x)$ satisfies $x\widetilde{\rho}(x) = xu(x)$, for some weight function $u(x)$, we get

$$\Delta\left[(x-1)\, xu\right] + \left[(1-c)\, x - (1+c\alpha)\right] xu = 0. \tag{42}$$

Using the product rule (39) in (42), we have

$$xu + (x-1)\,\Delta(xu) + \Delta(xu) + \left[(1-c)\, x - (1+c\alpha)\right] xu = 0,$$

or

$$x\Delta(xu) + \left[(1-c)\, x - (1+c\alpha) + 1\right] xu = 0.$$

Dividing by x, we obtain

$$\Delta(xu) + \left[(1-c)\, x - c\alpha\right] u = 0.$$

Comparing with (16), we see that $u(x)$ is the weight function corresponding to the Meixner polynomials (17), and therefore (37) implies that

$$\widetilde{\rho}(x) = (\alpha)_x \frac{c^x}{x!} + M\delta(x). \tag{43}$$

The orthogonal polynomials associated with the weight function (43) were first studied by Chihara in [12]. He showed that they satisfy a three term recurrence relation (1) with

$$b_n = \frac{c(\alpha+n)}{c-1}\frac{n}{n+1}\frac{B_n}{B_{n+1}} + \frac{n+1}{c-1}\frac{B_{n+1}}{B_n},$$

and

$$\gamma_n = \frac{c}{(c-1)^2} \frac{n^2 (\alpha+n)}{n+1} \frac{B_n^2}{B_{n-1}B_{n+1}},$$

where

$$B_n = \frac{c^n \ (\alpha)_n}{(1-c)\, n!} \frac{M}{(1-c)^{-\alpha} + M K_{n-1}},$$

and

$$K_n = \sum_{j=0}^{n} (\alpha)_j \ \frac{c^j}{j!}, \quad K_{-1} = 0.$$

Note that in order that B_n is well defined for all n, we need $M > -1$, since $K_n \nearrow (1-c)^{-\alpha}$.

In [10], Richard Askey proposed the problem of finding a second-order difference equation satisfied by these polynomials. The problem was solved by Bavinck and van Haeringen in [6], and in [2] Álvarez-Nodarse, García and Marcellán found the hypergeometric representation

$$P_n^{(2,1)}(x;\alpha,0,-1;c) = (\alpha)_n \left(\frac{c}{c-1}\right)^n {}_3F_2 \left(\begin{matrix} -n, -x, 1+\frac{x}{B_n} \\ \alpha, \frac{x}{B_n} \end{matrix} ; 1 - \frac{1}{c} \right).$$

In this case,

$$\lim_{M\to 0} P_n^{(2,1)}(x;\alpha,0,-1;c) = \widehat{M}_n(x;\alpha,c),$$

where $\widehat{M}_n(x;\alpha,c)$ is the monic Meixner polynomial (20).

2) If $\alpha_1 = -N$, $N \in \mathbb{N}$, we can remove the restriction that $0 < c < 1$ and take any $c < 0$, with $\alpha_2 \notin [-N, 0]$, $\beta \notin [-N-1, -1]$, and $\alpha_2 (\beta+1) > 0$. If we let $\alpha_2 \to -(N-1)$ and $\beta \to -N$, we have $\rho(x) \to \widetilde{\rho}(x)$ where $\widetilde{\rho}(x)$ is a new weight function satisfying the Pearson equation

$$\psi(x) = (1-c)\, x^2 + (\beta - c\alpha_1 - c\alpha_2)\, x - c\alpha_1\alpha_2,$$

$$\Delta [x(x-N)\, \widetilde{\rho}] + [(1-c)\, x + c(N-1)]\,(x-N)\,\widetilde{\rho} = 0. \tag{44}$$

Assuming that $\widetilde{\rho}(x)$ satisfies $(x-N)\,\widetilde{\rho}(x) = (x-N)\, u(x)$, for some weight function $u(x)$, we get

$$\Delta [x(x-N)\, u] + [(1-c)\, x + c(N-1)]\,(x-N)\, u = 0. \tag{45}$$

Using the product rule (39) in (45), we have

$$xu + (x-N+1)\,\Delta(xu) + [(1-c)\, x + c(N-1)]\,(x-N)\, u = 0,$$

or

$$(x-N+1)\,\Delta\,(xu)+(x-N+1)\,(x+Nc-cx)\,u=0.$$

Dividing by $x-N+1$, we obtain

$$\Delta\,(xu)+[(1-c)\,x+cN]\,u=0.$$

Comparing with (22), we see that $u(x)$ is the weight function corresponding to the Krawtchouk polynomials (21), and therefore (44) implies that

$$\widetilde{\rho}\,(x)=(-N)_x\,\frac{c^x}{x!}+M\delta\,(x-N)\,.$$

4.5 Generalized Hahn polynomials of type II

For these polynomials the weight function is given by

$$\rho\,(x)=\frac{(\alpha_1)_x\,(\alpha_2)_x\,(\alpha_3)_x}{(\beta_1+1)_x\,(\beta_2+1)_x}\frac{1}{x!},\quad x\in\mathbb{N}_0,$$

with $\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3>0$, and $\alpha_1\alpha_2\alpha_3\,(\beta_1+1)\,(\beta_2+1)>0$. The first moments are

$$\begin{aligned}\mu_0&={}_3F_2\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_3\\ \beta_1+1,\beta_2+1\end{array};1\right),\\ \mu_1&=\frac{\alpha_1\alpha_2\alpha_3}{(\beta_1+1)\,(\beta_2+1)}\,{}_3F_2\left(\begin{array}{c}\alpha_1+1,\alpha_2+1,\alpha_3+1\\ \beta_1+2,\beta_2+2\end{array};1\right),\end{aligned}$$

where ${}_3F_2\left(\begin{array}{c}a_1,a_2,a_3\\ b_1,b_2\end{array};z\right)$ is the hypergeometric function.

To our knowledge, these polynomials have not appeared before in the literature.

4.5.1 Singular limits

1) If we let $\alpha_3\to 0,\beta_2\to -1$, $\beta_1=\beta$ in (33), we have $\rho\,(x)\to\widetilde{\rho}\,(x)$ where $\widetilde{\rho}\,(x)$ is a new weight function satisfying the Pearson equation

$$\Delta\,[(x-1)\,(x+\beta)\,x\widetilde{\rho}]+[(\beta-1-\alpha_1-\alpha_2)\,x-\alpha_1\alpha_2-\beta]\,x\widetilde{\rho}=0. \qquad (46)$$

Assuming that $\widetilde{\rho}\,(x)$ satisfies $x\widetilde{\rho}(x)=xu(x)$, for some weight function $u(x)$, we get

$$\Delta\,[(x-1)\,(x+\beta)\,xu]+[(\beta-1-\alpha_1-\alpha_2)\,x-\alpha_1\alpha_2-\beta]\,xu=0. \qquad (47)$$

Using the product rule (39) in (47), we have

$$(x+\beta)\,xu + x\Delta\left[(x+\beta)\,xu\right] + \left[(\beta-1-\alpha_1-\alpha_2)\,x - \alpha_1\alpha_2 - \beta\right] xu = 0,$$

or

$$x\Delta\left[(x+\beta)\,xu\right] + \left[(\beta-\alpha_1-\alpha_2)\,x - \alpha_1\alpha_2\right] xu = 0.$$

Dividing by x, we obtain

$$\Delta\left[(x+\beta)\,xu\right] + \left[(\beta-\alpha_1-\alpha_2)\,x - \alpha_1\alpha_2\right] u = 0.$$

Comparing with (23), we see that $u(x)$ is the weight function corresponding to the Hahn polynomials (24) and, therefore, (46) implies that

$$\widetilde{\rho}\,(x) = \frac{(\alpha_1)_x\,(\alpha_2)_x}{(\beta+1)_x}\frac{1}{x!} + M\delta\,(x)\,. \tag{48}$$

2) Similarly, if we let $\alpha_3 = -N, \alpha_2 \to -(N-1)\,, \beta_2 \to -N,\ \alpha_1 = \alpha,$ $\beta_1 = \beta,\ \alpha\,(\beta+1) < 0$ in (33), we have $\rho\,(x) \to \widetilde{\rho}\,(x)$ where $\widetilde{\rho}\,(x)$ is a new weight function satisfying the Pearson equation

$$\Delta\left[x\,(x+\beta)\,(x-N)\,\widetilde{\rho}\right] + \left[(\beta-\alpha+N-1)\,x + \alpha\,(N-1)\right](x-N)\,\widetilde{\rho} = 0. \tag{49}$$

Assuming that $\widetilde{\rho}\,(x)$ satisfies $(x-N)\,\widetilde{\rho}(x) = (x-N)\,u(x)$, for some weight function $u(x)$, we get

$$\Delta\left[x\,(x+\beta)\,(x-N)\,u\right] + \left[(\beta-\alpha+N-1)\,x + \alpha\,(N-1)\right](x-N)\,u = 0. \tag{50}$$

Using the product rule (39) in (50), we have

$$\begin{gathered}(x+\beta)\,xu + (x-N+1)\,\Delta\left[(x+\beta)\,xu\right]\\ + \left[(\beta-\alpha+N-1)\,x + \alpha\,(N-1)\right](x-N)\,u = 0,\end{gathered}$$

or

$$\begin{gathered}(x-N+1)\,\Delta\left[(x+\beta)\,xu\right]\\ + (x-N+1)\left[(\beta-\alpha+N)\,x + \alpha N\right] u = 0.\end{gathered}$$

Dividing by $x-N+1$, we obtain

$$\Delta\left[(x+\beta)\,xu\right] + \left[(\beta-\alpha+N)\,x + \alpha N\right] u = 0.$$

Comparing with (23), we see that $u(x)$ is the weight function corresponding to the truncated Hahn polynomials (26), and therefore (49) implies that

$$\widetilde{\rho}(x) = \frac{(\alpha)_x (-N)_x}{(\beta+1)_x} \frac{1}{x!} + M\delta(x-N). \tag{51}$$

The orthogonal polynomials associated with the weight functions (48) and (51) were first studied by Álvarez-Nodarse and Marcellán in [4].

5 Limit relations between polynomials

From the identities [20]

$$\lim_{\lambda\to\infty} \frac{(\lambda\alpha)_x}{\lambda^x} = \alpha^x,$$

and

$$\lim_{\lambda\to\infty} \frac{(\lambda\alpha)_x}{(\lambda\beta)_x} = \left(\frac{\alpha}{\beta}\right)^x,$$

the following limit relations follow:

1. Generalized Hahn polynomials of type II to generalized Hahn polynomials of type I

$$\lim_{\alpha\to\infty} P_n^{(3,2)}\left(x;\alpha_1,\alpha_2,\alpha,\beta,\frac{\alpha}{c};1\right) = P_n^{(2,1)}(x;\alpha_1,\alpha_2,\beta;c).$$

2. Generalized Hahn polynomials of type I to generalized Krawtchouk polynomials

$$\lim_{\beta\to\infty} P_n^{(2,1)}(x;\alpha,-N,\beta;c\beta) = P_n^{(2,0)}(x;\alpha,-N;c).$$

3. Generalized Hahn polynomials of type I to generalized Meixner polynomials

$$\lim_{\alpha_2\to\infty} P_n^{(2,1)}\left(x;\alpha,\alpha_2,\beta;\frac{c}{\alpha_2}\right) = P_n^{(1,1)}(x;\alpha,\beta;c).$$

4. Generalized Meixner polynomials to generalized Charlier polynomials

$$\lim_{\alpha\to\infty} P_n^{(1,1)}\left(x;\alpha,\beta;\frac{c}{\alpha}\right) = P_n^{(0,1)}(x;\beta;c).$$

5. Generalized Meixner polynomials to Meixner polynomials

$$\lim_{\beta\to\infty} P_n^{(1,1)}(x;\alpha,\beta;c\beta) = M_n(x;\alpha;c).$$

6. Generalized Charlier polynomials to Charlier polynomials

$$\lim_{\beta\to\infty} P_n^{(0,1)}(x;\beta;c\beta) = C_n(x;c).$$

We also have the singular limits:

1. Generalized Meixner polynomials to Charlier-Dirac polynomials

$$\lim_{\substack{\alpha\to 0\\ \beta\to -1}} P_n^{(1,1)}(x;\alpha,\beta;c) = C_n(x;c) \oplus \delta(x),$$

2. Generalized Hahn polynomials of type I to truncated Hahn polynomials

$$\lim_{\substack{\alpha_2\to -N\\ c\to 1}} P_n^{(2,1)}(x;\alpha,\alpha_2,\beta;c) = Q_n(x;\alpha,\beta,N),$$

3. Generalized Hahn polynomials of type I to Meixner-Dirac polynomials

$$\lim_{\substack{\alpha_2\to 0\\ \beta\to -1}} P_n^{(2,1)}(x;\alpha,\alpha_2,\beta;c) = M_n(x;\alpha;c) \oplus \delta(x),$$

4. Generalized Hahn polynomials of type I to Krawtchouk-Dirac polynomials

$$\lim_{\substack{\alpha_2\to -N+1\\ \beta\to -N}} P_n^{(2,1)}(x;-N,\alpha_2,\beta;c) = K_n(x;-N;c) \oplus \delta(x-N),$$

5. Generalized Hahn polynomials of type II to Hahn-Dirac polynomials

$$\lim_{\substack{\alpha_2\to 0\\ \beta_2\to -1}} P_n^{(3,2)}(x;\alpha,\alpha_2,-N,\beta,\beta_2;1) = Q_n(x;\alpha,\beta,N) \oplus \delta(x),$$

6. Generalized Hahn polynomials of type II to Hahn-Dirac polynomials

$$\lim_{\substack{\alpha_2 \to -N+1 \\ \beta_2 \to -N}} P_n^{(3,2)}\left(x;\alpha,\alpha_2,-N,\beta,\beta_2;1\right) = Q_n\left(x;\alpha,\beta,N\right) \oplus \delta(x-N),$$

where we use the notation "$\oplus\ \delta\left(x-x_0\right)$" to denote the addition of a delta function to the measure of orthogonality at the point x_0.

We can summarize these results in the following scheme:

$$\begin{array}{cccccc}
P_n^{(3,2)} & \to & \text{Hahn} \oplus \delta(x) & \text{Hahn} \oplus \delta(x-N) & & \\
\downarrow & & \downarrow & \downarrow & & \\
P_n^{(2,1)} & \to & \text{Meixner} \oplus \delta(x) & \text{Krawtchouk} \oplus \delta(x-N) & & \text{Hahn} \\
\downarrow & \searrow & \searrow & \downarrow & & \downarrow \\
P_n^{(2,0)} & P_n^{(1,1)} & \to & \text{Charlier} \oplus \delta(x) & & \\
 & \downarrow & \searrow & & & \\
 & P_n^{(0,1)} & & & & \text{Meixner - Krawtchouk} \\
 & & \searrow & & & \downarrow \\
 & & & & & \text{Charlier}
\end{array}$$

6 Concluding remarks

We have described the discrete semiclassical orthogonal polynomials of class $s=1$ using the different choices for the polynomials in the canonical Pearson equation that the corresponding linear functional satisfies. We have focused our attention when the linear functional has a representation in terms of a discrete positive measure supported on a countable subset of the real line. Some new families of orthogonal polynomials appear as well as some families of orthogonal polynomials (generalized Charlier, generalized Krawtchouk, and generalized Meixner) which have attracted the interest of researchers in the last years taking into account the connection of the coefficients of the three term recurrence relations with discrete and continuous Painlevé equations. We have also studied limit relations between such families of orthogonal polynomials having in mind an analogue of the Askey tableau for classical orthogonal polynomials. It would be very interesting to find the equations satisfied by the coefficients of the three term recurrence relations for the above new sequences of semiclassical orthogonal polynomials. Furthermore, an analysis of the class $s = 2$ will also be welcome in order to get a complete classification of such a class as well as to check if new families of orthogonal polynomials appear as in the case of the D-semiclassical orthogonal polynomials pointed out in [23].

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