A matrix approach for the semiclassical and coherent orthogonal polynomials

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Abstract

We obtain a matrix characterization of semiclassical orthogonal polynomials in terms of the Jacobi matrix associated with the multiplication operator in the basis of orthogonal polynomials, and the lower triangular matrix that represents the orthogonal polynomials in terms of the monomial basis of polynomials. We also provide a matrix characterization for coherent pairs of linear functionals.

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1. Introduction

Let us consider a linear functional $\mathcal{U} : \mathbb{P} \to \mathbb{C}$ defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients. A sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ such that

$$\deg(P_n(x)) = n \quad \text{and} \quad \langle \mathcal{U}, P_n(x)P_m(x)\rangle = k_n\delta_{n,m} \quad \text{with} \quad k_n \neq 0, \quad n, m \geq 0,$$

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is said to be the sequence of monic orthogonal polynomials (SMOP) associated with $\mathcal{U}$. The existence of a SMOP can be characterized in terms of the infinite Hankel matrix $H = [u_{i+j}]_{i,i\geqslant 0}$, where $u_n = \langle \mathcal{U}, x^n \rangle$, $n \geq 0$, are called the moments associated with $\mathcal{U}$. Indeed, $\{P_n(x)\}_{n\geq 0}$ exists if and only if the leading principal submatrices

$$H_n = [u_{i+j}]_{i,j=0}^{n}, \quad n \geqslant 0,$$

of H are nonsingular. In this situation, $\mathcal{U}$ is said to be a quasi-definite or regular ([3]). On the other hand, if for every $n \geq 0$, $\det H_n > 0$, $\mathcal{U}$ is said to be positive definite and it has the integral representation

$$\langle \mathcal{U}, q(x) \rangle = \int_E q(x) d\mu(x),$$

where μ is a nontrivial positive Borel measure supported on some infinite subset $E \subset \mathbb{R}$. Assuming $u_0 = 1$, the most familiar sequences of orthogonal polynomials are the so-called *classical* families: Jacobi, Laguerre and Hermite polynomials. They correspond to the cases when E has bounded support ($E = [-1, 1]$), E is the positive real axis, and $E = \mathbb{R}$, respectively, and the corresponding probability measures are the Beta, Gamma and normal distributions. There are several ways to characterize the classical orthogonal polynomials: as polynomial solutions of a hypergeometric differential equation, as polynomials expressed by a Rodrigues formula, and as the only sequences of orthogonal polynomials whose derivatives also constitute an orthogonal family.

One of the most important properties of orthogonal polynomials is that they satisfy the three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n P_{n-1}(x), \quad n \geqslant 0, \tag{1}$$

where $P_{-1}(x) := 0$, $b_n \in \mathbb{R}$, $n \geqslant 0$, and $a_n \neq 0$, $n \geqslant 1$. If $\mathcal{U}$ is positive definite, we have $a_n > 0$, $n \geqslant 1$. In a matrix form,

$$xP(x) = JP(x),$$

where $P(x) = [P_0(x), P_1(x), \ldots]^T$ and J is the tridiagonal infinite matrix

$$J = \begin{pmatrix} b_0 & 1 & 0 & \ldots \\ a_1 & b_1 & 1 & \ddots \\ 0 & a_2 & b_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

called the monic Jacobi matrix associated with $\{P_n(x)\}_{n\geq 0}$. It is straightforward to see that the zeros of P_n are the eigenvalues of J_n, the $n \times n$ principal leading submatrix of J. On the other hand, given arbitrary sequences $\{b_n\}_{n\geqslant 0}$ and $\{a_n\}_{n\geqslant 1}$, with $b_n \in \mathbb{R}$ and $a_n \neq 0$, you can define J as in (2) and construct $\{P_n(x)\}_{n\geq 0}$ by using (1). Then, $\{P_n(x)\}_{n\geq 0}$ is orthogonal with respect to some linear funcional $\mathcal{U}$. This relevant fact is known in the literature as Favard's theorem (see [3]).

Recently, in [20], a matrix characterization for classical orthogonal polynomials was introduced. Let write

$$P_n(x) = \sum_{j=0}^{n} a_{n,j} x^j, \quad n \geq 0,$$

and let define the infinite matrix A with entries $a_{n,j}$, for $0 \leqslant j \leqslant n, n \geqslant 0$, and zero otherwise. Notice that A is a lower triangular matrix whose n-th row contains the coefficients of the n-th degree orthogonal polynomial with respect to the canonical basis $\{x^n\}_{n\geq 0}$. Furthermore, since P_n is monic, the diagonal entries are $a_{n,n} = 1$ and, therefore, A is nonsingular. We say that A is the matrix associated with the sequence $\{P_n(x)\}_{n\geq 0}$. If the polynomials are classical, we will say that A is classical.

Following the notation used in [20], we say that a matrix B is a lower semi-matrix if there exists an integer m such that $b_{i,j} = 0$ whenever $i - j < m$. The entry $b_{i,j}$ is in the $m-$th diagonal $i - j = m$. If B is non zero, we say that B has index m, $\operatorname{ind}(B) = m$, if m is the minimum integer such that B has at least one nonzero entry in the $m-$th diagonal, also if all the entries in its diagonal of index m are equal to 1, B is called monic. Finally, B is said to be $(n, m)-$banded if there exists a pair of integers (n, m) with $n \leq m$ and all the nonzero entries of B lie between the diagonals of indices n and m. It is easy to see that the set of banded matrices is closed under addition and multiplication, despite the fact that the inverse of a banded matrix might not be banded.

Let define the matrices

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 1/3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

then we get the following matrix characterization for the orthogonality of a sequence of polynomials.

Theorem 1. *Let $\{P_n(x)\}_{n\geq 0}$ be a monic polynomial sequence and let A be its associated matrix. Then, the sequence $\{P_n(x)\}_{n\geq 0}$ is orthogonal with respect to some linear functional if and only if $J = AXA^{-1}$ is a $(-1, 1)$-banded matrix whose entries in the diagonals of indices 1 and -1 are all nonzero.*

The proof can be found in [20]. Notice that this is a matrix version of the Favard's theorem, and the entries of J, i.e., the coefficients of the recurrence relation for the orthogonal polynomials, can be obtained from the matrix A. On the other hand, AD has index 1 and its k-th row is the vector $[a_{k,1}, 2a_{k,2}, 3a_{k,3}, \ldots, ka_{k,k}, 0, \ldots]$, which corresponds to the derivative of $P_k(x)$. Therefore the matrix $\tilde{A} = \hat{D}AD$ is a monic matrix of index zero and it is associated with the sequence $\{P_n^{[1]}(x)\}_{n\geqslant 0}$, where $P_n^{[1]}(x) = P'_{n+1}(x)/(n+1)$. Using the fact that a sequence of orthogonal polynomials is classical if and only if the sequence of their derivatives is also orthogonal, the following matrix characterization for classical polynomials is also given in [20].

Theorem 2. *Let A be the matrix associated with $\{P_n(x)\}_{n\geq 0}$. Then A is classical if and only if $A\tilde{A}^{-1}$ is a (0,2)-banded monic matrix.*

2. A matrix characterization for semiclassical polynomials

Let $\phi(x) = a_t x^t + \ldots + a_0, \psi(x) = b_l x^l + \cdots + b_0$ be non zero polynomials such that $a_t b_l \neq 0$, $t \geq 0, l \geq 1$. (ϕ, ψ) is called an admissible pair if either $t - 1 \neq l$ or, $t - 1 = l$ and $na_{l+1} + b_l \neq 0, n \geq 0$. A quasi-definite linear functional $\mathcal{U}$ is said to be semiclassical if there exists an admissible pair (ϕ, ψ) such that $\mathcal{U}$ satisfies

$$\mathcal{D}(\phi\mathcal{U}) = \psi\mathcal{U}, \tag{3}$$

where $\mathcal{D}$ denotes the distributional derivative. The corresponding sequence of orthogonal polynomials is called semiclassical.
The class of a semiclassical linear functional is the non negative integer

$$s = \min\{\, \max\{\deg(\phi) - 2, \deg(\psi) - 1\} : (\phi, \psi) \text{ is an admissible pair}\,\}.$$

The class of a semiclassical SMOP has been characterized as follows.

Proposition 3 ([18]). *Let $\mathcal{U}$ a semiclassical linear functional given by* (3). *The class of $\mathcal{U}$ is s if and only if one of the following statements holds*

i. *The polynomials $\phi(x)$ and $\psi(x) - \phi'(x)$ are coprime.*

ii. *If c is a common zero of $\phi(x)$ and $\psi(x) - \phi'(x)$, then*

$$\langle \mathcal{U}, \widetilde{\psi}_c(x) + \phi'_c(x) \rangle \neq 0,$$

where

$$\phi(x) = (x - c)\phi_c(x) \quad \textit{and} \quad \psi(x) - \phi'(x) = (x - c)\widetilde{\psi}_c(x).$$

The previous conditions can be written as

$$\prod_{\{c \in \mathbb{C} : \phi(c) = 0\}} \left(|\psi(c) - \phi'(c)| + \left| \left\langle \mathcal{U}, \widetilde{\psi}_c(x) + \phi'_c(x) \right\rangle \right| \right) > 0,$$

or, equivalently,

$$\prod_{\{c \in \mathbb{C} : \phi(c) = 0\}} \left(|\psi(c) - \phi'(c)| + \left| \left\langle \mathcal{U}, \theta_c \psi(x) - \theta_c^2 \phi(x) \right\rangle \right| \right) > 0,$$

where $\theta_c p(x) = \frac{p(x) - p(c)}{x - c}$, for $p \in \mathbb{P}$.

There are several characterizations of semiclassical orthogonal polynomials in terms of the so called structure relations. Some of them are listed in the following theorem.

Theorem 4. *Let $\mathcal{U}$ be a quasi-definite linear functional and let $\{P_n(x)\}_{n \geq 0}$ be its corresponding SMOP. Then, the following statements are equivalent*

- *There exist non zero polynomials ϕ, ψ of degrees $t \geq 0, l \geq 1$, respectively, such that* (3) *holds.*

- *[18] (First structure relation) There exist a polynomial ϕ of degree t and sequences $\{a_{n,k}\}$ such that $\{P_n(x)\}$ satisfies*

$$\phi(x) P_n^{[1]}(x) = \sum_{k=n-s}^{n+t} a_{n,k} P_k(x), \quad n \geq s, \quad a_{n,n-s} \neq 0, n \geq s + 1, \tag{4}$$

where s is a positive integer such that $t \leq s + 2$.

- *[17] (Second structure relation) There exist non-negative integers t, s, and sequences $\{\tilde{a}_{n,k}\}$, $\{\tilde{b}_{n,k}\}$, such that*

$$\sum_{k=n-s}^{n+s} \tilde{a}_{n,k} P_k(x) = \sum_{k=n-t}^{n+s} \tilde{b}_{n,k} P_k^{[1]}(x), \quad n \geq \max\{s, t\},$$

  holds, where $\tilde{a}_{n,n+s} = \tilde{b}_{n,n+s} = 1, n \geq \max\{s, t+1\}$,

- *[2] There exist a non-negative integer s and sequences $\{b_{n,j}\}$ and $\{c_{n,j}\}$ such that $\{P_n(x)\}$ satisfies the structure relation*

$$\sum_{j=0}^{s} b_{n,n-j} P_{n-j}(x) = \sum_{j=0}^{s+2} c_{n,n-j} P_{n-j}^{[1]}(x), \quad b_{n,n} = c_{n,n} = 1, \quad n \geq s+1. \tag{5}$$

Notice that (5) can be expressed in matrix form as

$$BA = C\tilde{A}, \tag{6}$$

where B is a $(0, s)-$banded monic matrix, and C is a $(0, s+2)-$banded monic matrix. Indeed, the entries on the $j-th$ diagonal of B (resp. C) are the coefficients $b_{n,n-j}$ (resp. $c_{n,n-j}$), for $0 \leqslant j \leqslant s$ (resp. $s+2$). Thus, the previous theorem means that $\{P_n(x)\}_{n\geqslant 0}$ is semiclassical (of class at most s) if and only if there exist those matrices B and C such that (6) holds.

On the other hand, if $\{P_n(x)\}_{n\geqslant 0}$ is semiclassical of class at most s, it follows from (6) that $BA\tilde{A}^{-1}$ is a $(0, s+2)-$banded monic matrix. Conversely, if there exists a $(0, s)-$banded monic matrix B such that $BA\tilde{A}^{-1}$ is a $(0, s+2)-$banded monic matrix, then (6) holds, and therefore, $\{P_n(x)\}_{n\geqslant 0}$ is semiclassical. As a consequence, we have the following straightforward generalization of Theorem 2 for semiclassical polynomials.

Theorem 5. *Let $\{P_n(x)\}_{n\geqslant 0}$ be a SMOP with respect to some linear functional $\mathcal{U}$. Then, $\mathcal{U}$ is semiclassical of class at most s if and only if there exists a semi-infinite $(0, s)-$banded monic matrix B such that $BA\tilde{A}^{-1}$ is a $(0, s+2)-$banded monic matrix.*

Remark 6. *In the classical case, i.e. $s = 0$, we get that B is a $(0, 0)-$banded monic matrix, i.e. B is the identity matrix. Hence, $\{P_n(x)\}_{n\geqslant 0}$ is classical if and* *only if $BA\tilde{A}^{-1} = A\tilde{A}^{-1}$ is a $(0,2)-$banded monic matrix (result obtained in [20]) or, equivalently, $\{P_n(x)\}_{n\geqslant 0}$ satisfies the following structure relation (proved in [11])*

$$P_n(x) = P_n^{[1]}(x) + c_{n,n-1}P_{n-1}^{[1]}(x) + c_{n,n-2}P_{n-2}^{[1]}(x), \quad n \geq 1.$$

Now, let assume that the linear functional $\mathcal{U}$ in (3) is positive definite, and it has an associated absolutely continuous positive measure μ supported in $[a,b] \subset \mathbb{R}$, which can be expressed as $d\mu(x) = \omega(x)dx$, with the weight ω satisfying $\lim_{x\to a_+} \phi(x)\omega(a) = \lim_{x\to b_-} \phi(x)\omega(b) = 0$. The Pearson equation (3) can be expressed in terms of the weight as

$$(\phi\omega)' = \psi\omega.$$

In such a case, there exists a sequence of orthonormal polynomials $\{p_n(x)\}_{n\geqslant 0}$, and the corresponding Jacobi matrix is the symmetric matrix

$$\tilde{J} = \begin{pmatrix} b_0 & a_1 & 0 & \dots \\ a_1 & b_1 & a_2 & \ddots \\ 0 & a_2 & b_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{7}$$

satisfying $xp(x) = \tilde{J}p(x)$, with $p(x) = \begin{pmatrix} p_0(x) & p_1(x) & \cdots \end{pmatrix}^T$.

In this context, the first structure relation for semiclassical polynomials given in (4) also holds for the corresponding sequence of orthonormal polynomials $\{p_n(x)\}_{n\geqslant 0}$ associated with the semiclassical functional $\mathcal{U}$ and can be expressed in a matrix form as

$$\phi(x)p'(x) = X^T\tilde{H}p(x), \tag{8}$$

where $\tilde{H}$ is a $(-t,s)-$banded matrix whose elements, starting from the row s, are the coefficients appearing in (4) given in terms of $\{p_n(x)\}_{n\geqslant 0}$, and $p'(x) = [p_0'(x), p_1'(x), \dots]^T$. The following result establishes a relation between $\tilde{H}$ and $\tilde{J}$.

Theorem 7. *Let $\{p_n(x)\}_{n\geqslant 0}$ be a semiclassical sequence of orthonormal polynomials and let $\tilde{H}$ be the $(-t,s)-$banded matrix associated with the first structure relation* (8)*. Then, we have*

(*i*) $[\tilde{J}, X^T\tilde{H}] = \phi(\tilde{J})$,

(*ii*) $\tilde{H} + \tilde{H}^T = -\psi(\tilde{J})$, *where* $[\tilde{J}, X^T\tilde{H}] = \tilde{J}X^T\tilde{H} - X^T\tilde{H}\tilde{J}$ *and* ϕ, ψ *are the polynomials appearing in the Pearson equation.*

Proof. Notice that taking the derivative with respect to the variable x in (7), we get

$$xp'(x) + p(x) = \tilde{J}p'(x).$$

Multiplying by $\phi(x)$ and using (8), we obtain

$$xX^T\tilde{H}p(x) + \phi(x)p(x) = \tilde{J}X^T\tilde{H}p(x).$$

Taking into account (7), we get

$$X^T\tilde{H}\tilde{J}p(x) + \phi(\tilde{J})p(x) = \tilde{J}X^T\tilde{H}p(x).$$

Therefore, (i) follows. In order to prove (ii), from the Pearson equation we have

$$\int_a^b p_n(x)p_m(x)(\phi\omega)'(x)dx = \int_a^b p_n(x)p_m(x)\psi(x)\omega(x)dx,$$

and after integration by parts we get

$$\begin{aligned}\int_a^b p_n(x)p_m(x)\psi(x)\omega(x)dx &= p_n(x)p_m(x)(\phi\omega)(x)\Big|_{x=a}^{x=b} - \int_a^b \phi(x)p_n'(x)p_m(x)\omega(x)dx \\ &- \int_a^b \phi(x)p_m'(x)p_n(x)\omega(x)dx.\end{aligned}$$

Notice that the first term in the right hand side vanishes, and the second and third terms are the entries n, m and m, n of $\tilde{H}$, respectively. Furthermore, the integral in the left hand side is the m, n entry of $\psi(\tilde{J})$. As a consequence, (ii) follows. ■

Remark 8. *Let us remember the symmetric and skew-symmetric components of a matrix* M, *i.e.,* $M_1 = (M + M^T)/2$ *and* $M_2 = (M - M^T)/2$, *respectively, so* $M = M_1 + M_2$. *Then,*

- (ii) *becomes*

$$\tilde{H}_1 = -\frac{1}{2}\psi(\tilde{J}).$$

- *If $\widehat{H} = X^T \tilde{H}$, then from* (i) *we get*

$$\tilde{J}\widehat{H}_1 - \widehat{H}_1\tilde{J} = \frac{1}{2}\left[\phi(\tilde{J}) - \phi(\tilde{J})^T\right] = 0,$$

and $\tilde{J}\widehat{H}_1$ is a symmetric matrix. On the other hand,

$$\tilde{J}\widehat{H}_2 - \widehat{H}_2\tilde{J} = \frac{1}{2}\left[\phi(\tilde{J}) + \phi(\tilde{J})^T\right] = \phi(\tilde{J}).$$

Finally, taking into account that $P(x) = AY$ with $Y = (1, x, x^2, \dots)^T$, we get $P'(x) = ADY$. Then, multiplying by $\phi(x)$ and using (4) in a matrix form as (8), this is $\phi(x)P'(x) = X^T HP(x)$ (or $\phi(x)XP'(x) = HP(x)$), we obtain

$$X^T HAY = AD\phi(x)Y \qquad (\text{or } HAY = XAD\phi(x)Y).$$

If $\phi(x) = \sum_{k=0}^{\deg(\phi)} \gamma_k x^k$, then

$$\phi(x)Y = \left(\sum_{k=0}^{\deg(\phi)} \gamma_k x^k\right) Y = \sum_{k=0}^{\deg(\phi)} \gamma_k (x^k, x^{k+1}, \dots)^T = \sum_{k=0}^{\deg(\phi)} \gamma_k X^k Y = \phi(X)Y.$$

As a consequence, we have the following result.

Proposition 9. *Let $\{P_n(x)\}_{n\geqslant 0}$ be a semiclassical SMOP with associated matrix A. Then, if H is the matrix associated with the first structure relation* (4)*, we have*

$$X^T H = AD\phi(X)A^{-1} \qquad (\textit{or } H = XAD\phi(X)A^{-1}).$$

3. A matrix characterization for the coherence of orthogonal polynomials

We say that two non-trivial probability measures, $d\mu_0$ and $d\mu_1$, constitute a $(k, 0)$-coherent pair of order m, with $k, m \in \mathbb{N}_0$ fixed constants, if for each $n \in \mathbb{N}$, the monic orthogonal polynomial $P_n(\cdot; d\mu_1)$ can be expressed as a linear combination of the set $P_{n+m}^{(m)}(\cdot; d\mu_0), \dots, P_{n+m-k}^{(m)}(\cdot, d\mu_0)$. The coherence is classified in terms of k and m. The concept of coherence was introduced by A. Iserles, P.E. Koch, S. P. Norsett and J. M. Sanz-Serna in [9] and deeply analyzed in [10]. They established that a pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ in the linear space of polynomials with complex coefficients is said to be a $(1, 0)$-*coherent pair of order 1*, or simply $(1, 0)$-coherent pair, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy the structure relation

$$P_n^{[1]}(x) + c_n P_{n-1}^{[1]}(x) = Q_n(x), \quad n \geq 0, \tag{9}$$

where

$$P_n^{[1]}(x) = \frac{P'_{n+1}(x)}{n+1},$$

$\{c_n\}_{n\geq 0}$ is a sequence of complex numbers such that $c_n \neq 0$ for $n \geq 1$, and c_0 is a free parameter. In this context, they also introduced the concept of symmetrically coherent pair, when the two measures are symmetric and the subscripts in (9) are changed appropriately.

The main reason why they studied these relations, was that (9) gives a sufficient condition for the existence of a relation

$$P_{n+1}(x) + \frac{n+1}{n} c_n P_n(x) = S_{n+1}(x;\lambda) + c_{n,\lambda} S_n(x;\lambda), \quad n \geq 1, \tag{10}$$

where $\{c_{n,\lambda}\}_{n\geq 1}$ are rational functions in $\lambda > 0$ and $\{S_n(x;\lambda)\}_{n\geq 0}$ is the SMOP associated with the Sobolev inner product

$$\langle p(x), r(x)\rangle_\lambda = \int_{-\infty}^{\infty} p(x)r(x)d\mu_0(x) + \lambda \int_{-\infty}^{\infty} p'(x)r'(x)d\mu_1(x), \quad \lambda > 0,$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients. They studied the case when the first measure $d\mu_0$ is either the Gamma or the Beta distribution, whose corresponding sequences of orthogonal polynomials are the Laguerre and Jacobi SMOP, respectively.

They implemented an algorithm to compute the Fourier-Sobolev coefficients $\{f_n(\lambda)/s_n(\lambda)\}_{n\geq 0}$ with

$$f_n(\lambda) = \langle f(x), S_n(x;\lambda)\rangle_\lambda, \quad \text{and} \quad s_n(\lambda) = \langle S_n(x;\lambda), S_n(x;\lambda)\rangle_\lambda, \quad n \geq 0,$$

for the Fourier expansion

$$f(x) \equiv \sum_{n=0}^{\infty} \frac{f_n(\lambda)}{s_n(\lambda)} S_n(x;\lambda),$$

for a smooth function $f(x)$ in the Sobolev space

$$W^{1,2}[I,\mu_0,\mu_1] = \{f : I \to \mathbb{R} | f \in L^2_{\mu_0}(I), f' \in L^2_{\mu_1}(I)\},$$

where I is an open interval of $\mathbb{R}$. It is important to mention that this algorithm does not need the explicit expressions of the Sobolev orthogonal polynomials $S_n(x;\lambda)$, $n \geq 0$. The authors in [9] have tested the algorithm for a comparison between the Legendre-Fourier expansion and Legendre -Sobolev-Fourier expansion and their behavior at the ends of the interval for a smooth function. It reveals that Gibbs phenomenon does not appear in the second one and thus you have a better understanding how the Fourier expansion reflects the behavior of the function and its derivatives. From the point of view of applications, the potential interest of such Sobolev orthogonal polynomials appears when you consider spectral (Galerkin and collocation) methods for boundary value problems associated with Schrödinger equations whose potentials are related with such coherent pairs.

In 1997, in [19], H. G. Meijer determined all (1,0)-coherent pairs $(\mathcal{U}, \mathcal{V})$ of regular linear functionals. He proved that at least one of the linear functionals ($\mathcal{U}$ or $\mathcal{V}$) must be classical (Laguerre or Jacobi). Moreover, he showed that there exist non zero polynomials $\sigma(x)$ and $\varrho(x)$, with $\deg(\sigma(x)) \leq 2$ and $\deg(\varrho(x)) = 1$, such that the linear functionals $\mathcal{U}$ and $\mathcal{V}$ are related by

$$\sigma(x)\mathcal{U} = \varrho(x)\mathcal{V}.$$

Later on, in 2005, A. Delgado and F. Marcellán [8] extended the notion of coherent pair to *generalized coherent pairs* (we call them (1,1)-coherent pairs) studying the relation

$$P_n^{[1]}(x) + c_n P_{n-1}^{[1]}(x) = Q_n(x) + b_n Q_{n-1}(x), \quad c_n \neq 0, \; n \geq 1.$$

They proved that this is a necessary and sufficient condition for the relation (10). They also determined all the (1,1)-coherent pairs of linear functionals (b_n can be zero). They showed that at least one of the regular linear functionals must be semiclassical of class at most 1, generalizing the results by H. G. Meijer for (1,0)-coherent pairs. In addition, they showed that the linear functionals $\mathcal{U}$ and $\mathcal{V}$ satisfy the relation $\tilde{\sigma}(x)\mathcal{U} = \tilde{\varrho}(x)\mathcal{V}$, where $\tilde{\sigma}(x)$ and $\tilde{\varrho}(x)$ non zero polynomials such that $\deg(\tilde{\sigma}(x)) \leq 3$ and $\deg(\tilde{\varrho}(x)) = 1$.

Finally, in a recent work by M. N. de Jesús, F. Marcellán, J. Petronilho and N. C. Pinzón-Cortés ([5]), the more general case for coherence was characterized. They studied the structure relation

$$\sum_{i=0}^{M} c_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

where, M, N, m, k are non negative integers and the constants $\{c_{i,n}\}, \{b_{i,n}\}$ satisfy some natural conditions. That relation was called (M, N)-coherence of order (m, k). They concluded that the corresponding functionals $\mathcal{U}$ and $\mathcal{V}$ are semiclassical, whenever $m \neq k$, and they are related by an expression of rational type. When $k = 0$, they also generalized the obtained results in the framework of Sobolev orthogonal polynomials and their connections with coherent pairs, considering the Sobolev inner product

$$\langle p(x), r(x)\rangle_{\lambda,m} = \int_{-\infty}^{\infty} p(x)r(x)d\mu_0(x) + \lambda \int_{-\infty}^{\infty} p^{(m)}(x)r^{(m)}(x)d\mu_1(x), \quad \lambda > 0. \quad (11)$$

On the other hand, in [15], F. Marcellán and N. C. Pinzón-Cortés considered a matrix interpretation of (M, N)-coherence of order m. They established a relation between the Jacobi matrices associated with (M, N)-coherent pairs of linear functionals of order m and the Hessenberg matrix associated with the multiplication operator in terms of the basis of monic polynomials orthogonal with respect to the Sobolev inner product (11).

The aim of our contribution is to provide a matrix characterization of coherent pairs of measures and to show with some illustrative examples how you can implement, from a numerical point of view, the matrices involved therein.

3.1. (1,0)-Coherence

In [19], a complete classification of $(1, 0)$-coherent pairs of regular linear functionals was given. However, the $(1, 0)$-coherent pairs have been also studied in [1, 12, 13, 14].

We can establish a relation between the matrices corresponding to sequences of orthogonal polynomials associated with a coherent pair of linear functionals, i.e., that satisfy (9).

Lemma 10. *If $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are SMOP with associated matrices A and Q, respectively, then*

$$(\tilde{A}X\tilde{A}^{-1})^2\tilde{A}Q^{-1} = \tilde{A}Q^{-1}N^2 \quad (12)$$

holds, where $\tilde{A} = \hat{D}AD$ and N is the Jacobi matrix associated with Q.

Proof. From Theorem 1 we can see that $QX = NQ$ holds, hence $X = Q^{-1}NQ$ and, as a consequence, $X^2 = Q^{-1}NQQ^{-1}NQ = Q^{-1}N^2Q$. Thus $\tilde{A}X^2Q^{-1} = \tilde{A}Q^{-1}N^2$, or equivalently

$$(\tilde{A}X\tilde{A}^{-1})^2\tilde{A}Q^{-1} = \tilde{A}Q^{-1}N^2.$$

■

Theorem 11. *Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be SMOP with associated matrices A and Q, respectively. Then $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ constitute a $(1,0)-$Coherent pair if and only if $Q\tilde{A}^{-1}$ is lower bidiagonal with ones in the main diagonal and nonzero entries in the subdiagonal.*

Proof. Assume $(\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0})$ is a $(1,0)-$coherent pair, i.e., the relation

$$P_n^{[1]}(x) + c_n P_{n-1}^{[1]}(x) = Q_n(x), \quad c_n \neq 0, \quad n \geq 0, \tag{13}$$

holds. Since $\tilde{A}$ is the matrix associated with $\{P_n^{[1]}(x)\}_{n\geq 0}$, then (13) can be written in matrix form as

$$\tilde{A} + CX^T\tilde{A} = Q,$$

where $C = \mathrm{diag}(c_0, c_1, \dots)$. Another way to write the above equation is

$$\left(I + CX^T\right)\tilde{A} = Q,$$

and, since $\tilde{A}$ is nonsingular,

$$I + CX^T = Q\tilde{A}^{-1}.$$

So $Q\tilde{A}^{-1}$ is clearly lower bidiagonal with ones in the diagonal and non zero entries in the subdiagonal, since $c_n \neq 0$, $n \geq 0$.

For the converse, if $Q\tilde{A}^{-1} = T$ is bidiagonal with ones in the main diagonal and non zero elements in the subdiagonal, then

$$T\tilde{A} = Q,$$

so $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ constitute a $(1,0)-$Coherent pair of SMOP. ■

Remark 12. *Notice that, in the particular case when $\{P_n(x)\}_{n\geq 0}$ is an orthogonal classical family, then $\{P_n^{[1]}(x)\}_{n\geq 0}$ is also orthogonal (and classical), and it has an associated Jacobi matrix $M = \tilde{A}X\tilde{A}^{-1}$ (after monic normalization). In this situation* (12) *becomes*

$$M^2T - TN^2 = 0,$$

which is a particular case of a Sylvester equation. As a consequence, the coherence coefficients c_n can be obtained using the Bartels-Stewart algorithm.

Remark 13. *Notice that $T = Q\tilde{A}^{-1}$ is a $(0,1)$-banded matrix.*

Example 14. *(1,0)-Coherence example*

As an illustrative example, we consider the coherent measures $d\mu_1 = |x-1|(1-x)^{-.5}(1+x)^{-.5}dx$ and $d\mu_2 = (1-x)^{.5}(1+x)^{.5}dx$ and the corresponding SMOP, a perturbation of Jacobi polynomials with parameters $\alpha = \beta = -.5$ for $d\mu_1$ and the Jacobi polynomials with parameters $\alpha = \beta = .5$ for $d\mu_2$. We obtained the explicit expresions for the perturbed polynomials by using the formula in [3].

We construct the A and Q matrices with the coefficients of $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$,

$$A = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.2500 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.6786 & 2.1429 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3413 & 2.5385 & 3.1346 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.1707 & 2.3402 & 5.4201 & 4.1340 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0.0854 & 1.8760 & 6.9762 & 9.3039 & 5.1340 & 1.0000 & 0 & 0 & 0 & 0 \\
0.0427 & 1.3763 & 7.4971 & 15.2463 & 14.1877 & 6.1340 & 1.0000 & 0 & 0 & 0 \\
0.0213 & 0.9500 & 7.1295 & 20.4177 & 28.1510 & 20.0718 & 7.1340 & 1.0000 & 0 & 0 \\
0.0107 & 0.6273 & 6.2052 & 23.7353 & 45.0212 & 46.6892 & 26.9559 & 8.1340 & 1.0000 & 0 \\
0.0053 & 0.4005 & 5.0501 & 24.8362 & 61.7192 & 86.6923 & 71.8613 & 34.8398 & 9.1340 & 1.0000
\end{bmatrix},$$

$$Q = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.7500 & 2.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5000 & 2.5000 & 3.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3125 & 2.5000 & 5.2500 & 4.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0.1875 & 2.1875 & 7.0000 & 9.0000 & 5.0000 & 1.0000 & 0 & 0 & 0 & 0 \\
0.1094 & 1.7500 & 7.8750 & 15.0000 & 13.7500 & 6.0000 & 1.0000 & 0 & 0 & 0 \\
0.0625 & 1.3125 & 7.8750 & 20.6250 & 27.5000 & 19.5000 & 7.0000 & 1.0000 & 0 & 0 \\
0.0352 & 0.9375 & 7.2188 & 24.7500 & 44.6876 & 45.5000 & 26.2500 & 8.0000 & 1.0000 & 0 \\
0.0195 & 0.6445 & 6.1875 & 26.8125 & 62.5625 & 85.3125 & 69.9999 & 34.0000 & 9.0000 & 1.0000
\end{bmatrix},$$

with this, we compute the matrix T

$$T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0714 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.0897 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.1005 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.1072 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.1116 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.1148 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.1172 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1191 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1206 & 1
\end{bmatrix},$$

and we can see that it is an (0,1)-banded matrix with the coherence coefficients in the diagonals.

3.2. (1,0)-Coherence of order m

The notion of coherence can be generalized for higher order of derivatives as follows. A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ in the linear space of polynomials $\mathbb{P}$ with complex coefficients is said to be a $(1,0)$-*coherent pair of order* m, if their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy the structure relation

$$P_n^{[m]}(x) + c_n P_{n-1}^{[m]}(x) = Q_n(x), \quad n \geq 0, \tag{14}$$

where $\{c_n\}_{n\geq 0}$ is a sequence of complex numbers such that $c_n \neq 0$ for $n \geq 1$, c_0 is a free parameter, $P_{-1}(x) = 0$, and $P_n^{[m]}(x)$ denotes the monic polynomial of degree n

$$P_n^{[m]}(x) = \frac{P_{n+m}^{(m)}(x)}{(n+1)_m}, \quad n \geq 0,$$

where $(n+1)_m$ is the Pochhammer symbol defined by $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, $n \geq 1$, and $(\alpha)_0 = 1$.

A particular case of higher order derivatives was studied by A. Branquinho and M. N. Rebocho in [2], where they consider $m = 2$. The general case was studied by F. Marcellán and N. C. Pinzón-Cortés in [16], where they characterized the (1,0)-coherence of order m, and deduced the connection with Sobolev orthogonal polynomials (which depends on m), the relations between these functionals and their corresponding formal Stieltjes series.

To see the structure relation (14) from a matrix point of view, we define $D_m := D^m$, and $\hat{D}_m := \hat{D}^m$, they are the diagonal matrices of index m and index $-m$,

$$D_m = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & 0 & \cdots \\ (n+1)_m & \vdots & 0 & 0 & \cdots \\ 0 & (n)_m & \vdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}, \qquad \hat{D}_m = \begin{bmatrix} 0 & \cdots & 1/(n+1)_m & 0 & \cdots \\ 0 & 0 & \cdots & 1/(n)_m & \ddots \\ 0 & 0 & 0 & \cdots & \ddots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

respectively, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, $n \geq 1$, $(\alpha)_0 = 1$ is the Pochhammer symbol and then, using the same argument as before, we obtain the following results.

Lemma 15. *If $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are SMOP with associated matrices A and Q, respectively, then*

$$(A^{[m]}XA^{[m]^{-1}})^2 A^{[m]}Q^{-1} = A^{[m]}Q^{-1}N^2$$

holds, where $A^{[m]} = \hat{D}_m A D_m$ is the matrix associated with the sequence $\{P_n^{[m]}(x)\}_{n\geq 0}$ and N is the Jacobi matrix associated with Q.

Theorem 16. *Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be SMOP with associated matrices A and Q, respectively. Then $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ constitute a $(1,0)-$coherent pair of order m if and only if $QA^{[m]^{-1}}$ is lower bidiagonal with ones on the main diagonal and nonzero entries in the subdiagonal.*

Example 17. *(1,0)-Coherence of order m,* $(m = 4)$

In this example we consider the monic Laguerre polynomials that satisfy the following structure relation

$$\frac{1}{(n+4)(n+3)(n+2)(n+1)} \frac{d^4}{dx^4} \left(L_{n+4}^{\alpha+1}(x) + nL_{n+3}^{\alpha+1}(x) \right) = L_n^{\alpha+4}(x), \quad n \geq 0. \quad (15)$$

Notice that we are in presence of a $(1,0)$-coherent pair of order 4 with coherence coefficients equal to n. We can represent (15) *in a matrix form as*

$$C\tilde{A}^{[4]} = Q.$$

We calculate the explicit monic Laguerre polynomials in (15) *by using the computational software Mathematica and, thus, construct their associated matrices $\tilde{A}^{[4]}$ and Q*

$$\tilde{A}^{[4]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 72 & -18 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -720 & 270 & -30 & 1 & 0 & 0 & 0 & 0 & 0 \\ 7920 & -3960 & 660 & -44 & 1 & 0 & 0 & 0 & 0 \\ -95040 & 59400 & -13200 & 1320 & -60 & 1 & 0 & 0 & 0 \\ 1235520 & -926640 & 257400 & -34320 & 2340 & -78 & 1 & 0 & 0 \\ -17297280 & 15135120 & -5045040 & 840840 & -76440 & 3822 & -98 & 1 & 0 \\ 259459200 & -259459200 & 100900800 & -20180160 & 2293200 & -152880 & 5880 & -120 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 56 & -16 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -504 & 216 & -27 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5040 & -2880 & 540 & -40 & 1 & 0 & 0 & 0 & 0 \\ -55440 & 39600 & -99000 & 1100 & -55 & 1 & 0 & 0 & 0 \\ 665280 & -570240 & 178200 & -26400 & 1980 & -72 & 1 & 0 & 0 \\ -8648640 & 8648640 & -3243240 & 600600 & -60060 & 3276 & -91 & 1 & 0 \\ 121080960 & -138378240 & 60540480 & -13453440 & 1681680 & -122304 & 5096 & -112 & 1 \end{bmatrix}.$$

Finally we compute the matrix $C = Q\tilde{A}^{-[4]}$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 1 \end{bmatrix}$$

which is monic lower bidiagonal and contains the coherence coefficients in the sub diagonal that turn out to be exactly n as in the structure relation formula (15).

3.3. (M,0)-Coherence

Another generalization of the notion of coherence can be obtained by adding a finite number of terms on the left hand side of (13). In this case, the pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a $(M, 0)$-*coherent pair*. Indeed, their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy the structure relation

$$\sum_{i=0}^{M} c_{i,n} P_{n-i}^{[1]}(x) = Q_n(x), \quad n \geq 0, \tag{16}$$

where $\{c_{i,n}\}_{n\geq 0}$, $0 \leq i \leq M$, is a sequence of complex numbers such that $c_{M,n} \neq 0$ if $n \geq M$ and $c_{i,n} = 0$ if $i > n$. This case has been studied in [2, 4, 5, 6, 7]. We characterize this relation in a matrix form as is shown in the following result.

Theorem 18. *Let* $\{P_n(x)\}_{n\geq 0}$ *and* $\{Q_n(x)\}_{n\geq 0}$ *be the SMOP with associated matrices A and Q, respectively. Then* $\{P_n(x)\}_{n\geq 0}$ *and* $\{Q_n(x)\}_{n\geq 0}$ *constitute a* $(M, 0)-$*Coherent pair if and only if* $Q\tilde{A}^{-1}$ *is* $(0, M)-$*banded with ones on the main diagonal.*

Proof. Let assume $(\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0})$ is a $(M, 0)-$coherent pair, i.e., the relation (16) holds. Since $\tilde{A}$ is the matrix associated with $\{P_n^{[1]}(x)\}_{n\geq 0}$, then (16) can be written in a matrix form as

$$\tilde{A} + Y\tilde{A} = Q,$$

where $Y = \left[C_1 X^T + C_2 {X^T}^2 + \cdots + C_M {X^T}^M\right]$ and C_i is a diagonal matrix of index 0 with entries $c_{n+i,n}$, $n \geq 0$, $1 \leq i \leq M$. Another way to write the above equation is

$$(I + Y)\tilde{A} = Q,$$

and, since $\tilde{A}$ is nonsingular,

$$(I + Y) = Q\tilde{A}^{-1},$$

so, $Q\tilde{A}^{-1}$ is clearly $(0, M)$-banded.
For the converse, since $Q\tilde{A}^{-1} = T$ is $(0, M)-$banded with ones in the main diagonal, we have

$$T\tilde{A} = Q,$$

so $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are $(1, 0)-$Coherent. ■

Example 19. *(M,0)-Coherence example (M=2)*

For this example we consider $M = 2$ and, following Remark 6, we use the monic Jacobi polynomials with parameters $\alpha = \beta = 0.5$ and its corresponding derivatives. The matrix equation then is $Q = C\widetilde{Q}$, with

$$Q = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.7500 & 2.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5000 & 2.5000 & 3.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0.3125 & 2.5000 & 5.2500 & 4.0000 & 1.0000 & 0 & 0 & 0 & 0 \\
0.1875 & 2.1875 & 7.0000 & 9.0000 & 5.0000 & 1.0000 & 0 & 0 & 0 \\
0.1094 & 1.7500 & 7.8750 & 15.0000 & 13.7500 & 6.0000 & 1.0000 & 0 & 0 \\
0.0625 & 1.3125 & 7.8750 & 20.6250 & 27.5000 & 19.5000 & 7.0000 & 1.0000 & 0 \\
0.0352 & 0.9375 & 7.2188 & 24.7500 & 44.6876 & 45.5000 & 26.2500 & 8.0000 & 1.0000
\end{bmatrix}$$

and

$$\widetilde{Q} = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.8333 & 2.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.6250 & 2.6250 & 3.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0.4375 & 2.8000 & 5.4000 & 4.0000 & 1.0000 & 0 & 0 & 0 & 0 \\
0.2916 & 2.6250 & 7.5000 & 9.1666 & 5.0000 & 1.0000 & 0 & 0 & 0 \\
0.1875 & 2.2500 & 8.8392 & 15.7143 & 13.9286 & 6.0000 & 1.0000 & 0 & 0 \\
0.1171 & 1.8047 & 9.2812 & 22.3438 & 28.4375 & 19.6875 & 7.0000 & 1.0000 & 0 \\
0.0716 & 1.3750 & 8.9375 & 27.8056 & 47.3958 & 46.6666 & 26.4444 & 8.0000 & 1.0000
\end{bmatrix},$$

from where we can easily obtain $C = Q\widetilde{Q}^{-1}$ by using Mathematica to compute it

$$C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0833 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.6250 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.1500 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.1666 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.1786 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.1875 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -.1944 & 0 & 1
\end{bmatrix}.$$

We can see that this is a $(0,2)$*-banded matrix as it should and we can explicitly obtain the coherence coefficients from the entries of the matrix.*

3.4. (M,0)-Coherence of order m

The case when we take m derivatives for $(M,0)$-coherence, i. e.

$$\sum_{i=0}^{M} c_{i,n} P_{n-i}^{[m]}(x) = Q_n(x), \quad n \geq 0,$$

with $P_n^{[m]}$ defined as in section 3.2. This case of coherence is considered by M.N. de Jesús, F. Marcellán, J.Petronilho and N. C. Pinzón-Cortés in [5, 6, 7]. They show that the linear functionals associated with the corresponding SMOP are semiclassical and they are related by a rational factor.

Following the notation used in section 3.2, we establish the following result for $(M,0)$-coherence of order m in a matrix form.

Theorem 20. *Let* $\{P_n(x)\}_{n\geq 0}$ *and* $\{Q_n(x)\}_{n\geq 0}$ *be the SMOP with associated matrices A and Q, respectively. Then* $\{P_n(x)\}_{n\geq 0}$ *and* $\{Q_n(x)\}_{n\geq 0}$ *constitute a* $(M,0)-$*Coherent pair of order m if, and only if* $Q\tilde{A}^{[m]^{-1}}$ *is* $(0,M)-$*banded with ones on the main diagonal.*

The proof uses the same arguments as in Theorem 16.

Example 21. *(M,0)-Coherence of order m,* $(M = 2, m = 2)$

In this case we consider the monic Laguerre polynomials that satisfy the following structure relation

$$\frac{1}{(n+2)(n+1)} \frac{d^2}{dx^2} \left(L_{n+2}^{\alpha+2}(x) + c_{i,n} L_{n+1}^{\alpha+2}(x) + c_{i,n-1} L_n^{\alpha+2}(x) \right) = L_n^{\alpha+2}(x), \quad n \geq 0, \tag{17}$$

where $\{c_{i,n}\}_{n\geq 0}$ *are the coherence coefficients for this* $(2,0)$*-coherent pair of order* 2*. We can express* (17) *in a matrix form as*

$$C\tilde{A}^{[2]} = A,$$

where A is the lower triangular matrix associated with the monic Laguerre polynomials $\{L_n^{\alpha+2}(x)\}_{n\geq 0}$ *and* $\tilde{A}^{[2]}$ *is defined as in Section 3.2.*

Using Mathematica we compute explicitly the monic Laguerre polynomials and construct the matrices A and $\tilde{A}^{[2]}$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & 90 & -18 & 1 & 0 & 0 & 0 & 0 & 0 \\ 840 & -840 & 252 & -28 & 1 & 0 & 0 & 0 & 0 \\ -6720 & 8400 & -3360 & 560 & -40 & 1 & 0 & 0 & 0 \\ 60480 & -90720 & 45360 & -10080 & 1080 & -54 & 1 & 0 & 0 \\ -604800 & 1058400 & -635040 & 176400 & -25200 & 1890 & -70 & 1 & 0 \\ 6652800 & -13305600 & 9313920 & -3104640 & 554400 & -55440 & 3080 & -88 & 1 \end{bmatrix},$$

$$\tilde{A}^{[2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 42 & -14 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -336 & 168 & -24 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3024 & -2016 & 432 & -36 & 1 & 0 & 0 & 0 & 0 \\ -30240 & 25200 & -7200 & 900 & -50 & 1 & 0 & 0 & 0 \\ 332640 & -332640 & 118800 & -19800 & 1650 & -66 & 1 & 0 & 0 \\ -3991680 & 4656960 & -1995840 & 415800 & -46200 & 2772 & -84 & 1 & 0 \\ 51891840 & -69189120 & 34594560 & -8648640 & 1201200 & -96096 & 4368 & -104 & 1 \end{bmatrix}$$

and, in a straightforward way, we obtain $C = A\tilde{A}^{-[2]}$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 10 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 30 & 12 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 42 & 14 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 56 & 16 & 1 \end{bmatrix},$$

which is a $(0,2)$-banded matrix with ones in the main diagonal as expected and contains the coherence coefficients in the sub diagonals.

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References

[1] M. Alfaro, F. Marcellán, and M. L. Rezola , *Orthogonal polynomials on Sobolev spaces: Old and new directions*. J. Comput. Appl. Math. **48** (1993), 113-131.

[2] A. Branquinho and M. N. Rebocho, *On the Semiclassical Character of Orthogonal Polynomials Satisfying Structure Relations*. J. Difference Equ. Appl. **18** (2012), 111-138.

[3] T. S. Chihara, *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York-London-Paris, 1978.

[4] M. G. de Bruin and H. G. Meijer, *Zeros of orthogonal polynomials in a non-discrete Sobolev space*. Annals Numer. Math. **2** (1995), 233-246.

[5] M. N. de Jesús, F. Marcellán, J. Petronilho, and N. C. Pinzón-Cortés, *(M,N)-coherent pairs of order (m,k) and Sobolev orthogonal polynomials*. J. Comput. Appl. Math. **256** (2014), 16-35.

[6] M. N. de Jesús and J. Petronilho, *On linearly related sequences of derivatives of orthogonal polynomials*. J. Math. Anal. Appl. **347** (2008), 482-492.

[7] M. N. de Jesús and J. Petronilho, *Sobolev orthogonal polynomials and (M,N)-coherent pairs of measures*. J. Comput. Appl. Math. **237** (2013), 83-101.

[8] A. M. Delgado, *Ortogonalidad No Estándar: Problemas Directos e Inversos.* Doctoral Dissertation, Universidad Carlos III de Madrid, 2006. In Spanish.

[9] A. Iserles, P. E. Koch, S. P. Norsett, and J. M. Sanz-Serna, *Orthogonality and Approximation in a Sobolev Space*. In Algorithms for Approximations, J. C. Mason and M. G. Cox Editors. Chapman and Hall, London 1990. 117–124.

[10] A. Iserles, P. E. Koch, S. P. Norsett, and J. M. Sanz-Serna, *On Polynomials Orthogonal with Respect to Certain Sobolev Inner Products.* J. Approx. Theory **65** (1991), 151-175.

[11] F. Marcellán, A. Branquinho, and J. Petronilho, *Classical Orthogonal Polynomials: A Functional Approach*. Acta Appl. Math. **34** (1994), 283-303.

[12] F. Marcellán, T. E. Pérez, and M. A. Piñar, *Orthogonal Polynomials on Weighted Sobolev Spaces: The Semiclassical Case.* Annals Numer. Math. **2** (1995), 93-122.

[13] F. Marcellán, T. E. Pérez, J. C. Petronilho and M. A. Piñar, *What is beyond Coherent Pairs of Orthogonal Polynomials?* J. Comput. Appl. Math. **65** (1995), 267-277.

[14] F. Marcellán and J. C. Petronilho, *Orthogonal Polynomials and Coherent Pairs: The Classical Case.* Indag. Math. N. S. **6** (1995), 287-307.

[15] F. Marcellán and N. C. Pinzón-Cortés. (M,N)-*Coherent Pairs of Linear Functionals and Jacobi Matrices*. Appl. Math. Comput. **232** (2014), 76-83.

[16] F. Marcellán and N. Pinzón, *Higher order coherent pairs*. Acta Appl. Math. **121** (2012), 105-135.

[17] F. Marcellán and R. Sfaxi, *Second Structure Relation for Semiclassical Orthogonal Polynomials*. J. Comput. Appl. Math. **200** (2007), 537-554.

[18] P. Maroni, *Une Théorie Algébrique des Polynômes Orthogonaux. Application aux Polynômes Orthogonaux Semi-Classiques*. In Orthogonal Polynomials and their Applications, C. Brezinski, L. Gori, and A. Ronveaux (Eds.). IMACS Annals Comput. Appl. Math. **9** (1991), 95-130.

[19] H. G. Meijer, *Determination of All Coherent Pairs.* J. Approx. Theory **89** (1997), 321-343.

[20] L. Verde-Star, *Characterization and construction of classical orthogonal polynomials using a matrix approach.* Linear Algebra Appl. **438** (2013), 3635-3648.