Orthogonal polynomials and perturbations on measures supported on the real line and on the unit circle. A matrix perspective

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Abstract

The connection between measures supported on the real line (resp. on the unit circle), Hankel (resp. Toeplitz) matrices, Jacobi (resp. Hessenberg and CMV) matrices, Stieltjes (resp. Carathéodory) functions constitutes a key element in the analysis of orthogonal polynomials on the real line (resp. on the unit circle). In the present contribution, we focus our attention on perturbations of the measures supported either on the real line or the unit circle and their consequences on the behavior of the corresponding sequences of orthogonal polynomials and the matrices associated with the multiplication operator in terms on those polynomial bases. The matrix perspective related to such perturbations from the point of view of factorizations (LU and QR) is emphasized. Finally, we show the role of spectral transformations in the analysis of some integrable systems.

Keywords: Orthogonal polynomials, Jacobi matrices, CMV matrices, Stieltjes functions, Carathéodory functions, Spectral transformations, Szegő transformation, Uvarov-Chihara integrable systems, Toda lattices.
2010 MSC: 42C05, 15A23

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1. Introduction

A positive measure μ is said to be nontrivial if its support has an infinite number of points. Given such a measure, a sequence of orthogonal polynomials $\{p_n\}_{n\geqslant 0}$ can be defined by applying the Gram-Schmidt orthogonalization process in the $L^2(\mu)$ space. The first examples of orthogonal polynomials with respect to positive Borel measures supported on an infinite subset of the real line (OPRL) appeared at the end of the eighteen century. Although the origin of the most widely known orthogonal families, the so-called classical polynomials (i.e. Jacobi, Laguerre and Hermite), lies in the context of problems of mathematical physics, their applications in areas such as numerical integration, integrable systems, spectral methods for the study of boundary value problems for ordinary and partial differential equations, and graph theory, among many others, has motivated researchers to continue to develop the theory of orthogonal polynomials to this date, from the wider perspective of orthogonality with respect to linear functionals on the space of polynomials.

Analogously, given a measure σ with support on the unit circle $\mathbb{T}$, orthogonal polynomials on the unit circle (OPUC) are defined by applying the Gram-Schmidt process in $L^2(\sigma)$. The study of orthogonal polynomials with respect to nontrivial measures on the unit circle is more recent. It was initiated by G. Szegő in the beginning of the twentieth century (see [1]) in the framework of the theory of analytic functions on the unit circle, and later on they were generalized to the more general case of orthogonality with respect to linear functionals in the space of Laurent polynomials by Ya L. Geronimus (see [2], [3]). Orthogonal polynomials on the unit circle are deeply connected with linear prediction of discrete stationary stochastic processes in statistics and with linear filtering methods in signal theory (see, for instance, [4]), among other applications.

In both cases, the study of the spectral properties of the matrix representation of the multiplication operator with respect to the basis of orthogonal polynomials has attracted the attention of many researchers. Indeed, for OPRL, this matrix turns out to be a symmetric tridiagonal matrix (Jacobi matrix) when one deals with orthonormal polynomials taking into account such an operator is symmetric. For OPUC, there exist two different representations of the multiplication operator. The first one (GGT named after Geronimus, Gragg, and Teplyaev) is a Hessenberg matrix, and it is associated with the orthogonal polynomial basis obtained by applying the Gram-Schmidt orthogonalization process to the canonical basis $\{z^n\}_{n\geqslant 0}$. The handicap of the GGT representation is that its unitary character depends on the properties of the orthogonality measure. On the other hand, a more recent representation was obtained by Cantero, Moral, and Velázquez by using a basis obtained from a particular orthogonalization method of the Laurent polynomial basis $\{z^n\}_{n\in\mathbb{Z}}$. This yields a five-diagonal matrix (the CMV matrix) which is always unitary independently of the orthogonality measure.

Given a sequence $\{p_n\}_{n\geqslant 0}$ of OPRL, there exists a deep connection between them and the following mathematical objects

(i) The nontrivial probability measure (orthogonality measure) μ supported on the real line, such that one has the corresponding OPRL by applying the Gram-Schmidt orthogonalization process to the monomial basis $\{x^n\}_{n\geqslant 0}$.

(ii) The sequence of moments $\{\mu\}_{n\geqslant 0}$ associated with such a measure, defined by

$$\mu_n = \int_E x^n d\mu(x), \quad n \geqslant 0,$$

which are assumed to be finite. They give precise information about the measure.

(iii) The Stieltjes function $S(x)$, defined by

$$S(x) = \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}} = \int_E \frac{d\mu(t)}{x-t},$$

i.e. the Z-transform of the moments or, equivalently, as the Cauchy transform of the measure, when the measure has a compact support, which is an analytic function in a neighborhood of infinity.

(iv) The coefficients of the three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \quad p_{-1}(x) = 0,$$

i.e. the sequences $\{a_n\}_{n\geqslant 1}$, $\{b_n\}_{n\geqslant 0}$, which can be read as the entries of a Jacobi matrix.

In the case of OPUC, the sequence $\{\Phi_n\}_{n\geqslant 0}$ yields some objects which are analogues of the previous ones, as follows

(*i*) The nontrivial probability measure (orthogonality measure) σ, supported on the unit circle, such that one has the corresponding OPUC.

(*ii*) The sequence of (trigonometric) moments $\{c_n\}_{n\in\mathbb{Z}}$ associated with such a measure, defined by

$$c_n = \int_{\mathbb{T}} z^n d\sigma(x), \quad n \in \mathbb{Z}.$$

(*iii*) The Carathéodory function $F(z)$, defined by

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k = \int_{\mathbb{T}} \frac{w+z}{w-z} d\sigma(w),$$

i.e. as a power series in terms of the moments or, equivalently, as the Riesz-Herglotz transform of the measure, which is an analytic function in a neighborhood of the origin.

(*iv*) The Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$, i.e the coefficients on the recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \geqslant 1,$$

which are strongly related with the entries of the GGT and CMV matrices.

In the theory of orthogonal polynomials (both for OPRL and OPUC), there are different types of perturbations that have been considered in the literature, as follows.

(*i*) Perturbations of the orthogonality measure. In particular, three canonical perturbations have been considered. They consist in a multiplication of the measure by a polynomial of degree 1, the addition of a Dirac delta, and the division by a polynomial of degree 1 with the addition of a Dirac delta at the zero of the polynomial. They are called, respectively, the Christoffel, Uvarov, and Geronimus transformation.

(*ii*) Perturbations of the sequence of moments either in a finite or infinite number of them.

(*iii*) Perturbations of the Stieltjes / Carathéodory functions, which can be classified in two types. A rational spectral transformation of $S(x)$ is another Stieltjes function defined by

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{C(x)S(x) + D(x)},$$

where A, B, C, and D are polynomials such that $AD - CB \neq 0$ and $\widetilde{S}(x)$ is again an analytic function. If $C \equiv 0$, the transformation is said to be linear. Similar perturbations are defined for the Carathéodory functions.

(*iv*) Perturbations of the coefficients of the recurrence relations. The most studied cases are the associated polynomials of order k (when the first k coefficients are removed) and the anti-associated polynomials of order k (when k new coefficients are introduced), as well as finite perturbations either on the entries of the Jacobi matrix or in the Verblunsky parameters.

Indeed, perturbations of type (*i*) can be related to certain factorizations of the matrix associated with the multiplication operator. Namely, for OPRL, the Christoffel, Uvarov, and Geronimus transformations of a measure can be expressed by means of the LU and UL factorizations of the corresponding Jacobi matrix. Such perturbations have been studied in the context of Darboux transformations, which arise from the so-called bispectral problem: to find all situations in which a pair of differential operators in two different variables have a common eigenfunction. For OPUC, on the other hand, the analog transformations can be obtained by means of the QR-type factorization of the corresponding Hessenberg matrix.

Additive perturbations on the moments (type (*ii*)) are interesting from a numerical point of view in the analysis of the spectral properties of Hankel and Toeplitz matrices, as well as in the behavior of zeros of the corresponding orthogonal polynomials.

The analysis of perturbations based on the Stieltjes and Carathéodory functions (type (*iii*)) has been presented, respectively, in [5] and [6]. An interesting question is related with the generators of the sets of linear and rational spectral transformations for each case. In the first one, it was solved in [5]. The second one remains an open problem.

Finally, finite perturbations on the entries of matrices associated with the multiplication operator (i.e. type (*iv*)) have been studied in [7], as well as in [8] for Jacobi matrices and in [9] for OPUC.

The structure of the manuscript is as follows. In Section 2, we present a summary of the theory of OPRL from the point of view of Jacobi matrices and Stieltjes functions. In such a way, linear and rational spectral transformations are analyzed in the framework of the factorization of Jacobi matrices. In Section 3, we show the connection of spectral transformations with the explicit solution of some integrable systems (Volterra, Toda, and Uvarov-Chihara). Section 4 is focused on the theory of OPUC from the point of view of GGT matrices and Carathéodory functions. Linear and rational spectral transformations are emphasized, as well as their connection with the QR factorization of the corresponding GGT matrices. The role of CMV matrices is also analyzed. Furthermore, we study some linear transformations of measures supported on the unit circle from the canonical linear transformations of measures supported on the real line by using the Szegő transformation. In Section 5, we deal with some interesting examples of integrable systems (Schur flows) associated with OPUC. Finally, an updated list of references concerning the discussed topics is presented.

2. OPRL: Jacobi matrices and Stieltjes functions

Let μ be a nontrivial probability measure supported on some infinite subset E of the real line. Then, a sequence of polynomials $\{p_n\}_{n\geqslant 0}$ satisfying

$$\int_E p_m(x)p_n(x)d\mu(x) = \delta_{m,n}, \quad n, m \geqslant 0, \tag{1}$$

where $\delta_{m,n}$ is the Kronecker delta, is said to be a sequence of orthonormal polynomials with respect to μ. In order to have uniqueness in such a sequence, we will write

$$p_n(x) = \gamma_n x^n + \zeta_n x^{n-1} + \text{lower degree terms}, \tag{2}$$

with $\gamma_n > 0$ for every $n \geqslant 0$. From the inner product

$$\langle f, g\rangle_\mu = \int_E f(x)g(x)d\mu(x)$$

in the linear space $\mathbb{P}$ of polynomials with real coefficients, $\{p_n\}_{n\geqslant 0}$ can be obtained by using the Gram-Schmidt orthonormalization process for the monomial basis $\{x^n\}_{n\geqslant 0}$. They can also be expressed as a quotient of determinants. If $\mu_k = \int_E x^k d\mu(x)$, $k \geqslant 0$, are the moments associated with μ, then the Gram matrix of the inner product in terms of the monomial basis is the Hankel matrix

$$H = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} & \cdots \\ \vdots & & \ddots & & \\ \mu_n & \mu_{n+1} & & \mu_{2n} & \\ \vdots & \vdots & & & \ddots \end{pmatrix}.$$

It is easy to prove that the existence of $\{p_n\}_{n\geqslant 0}$ is equivalent to the fact that $\det H_n > 0$, $n \geqslant 0$, where H_n is the $(n+1)\times(n+1)$ leading principal submatrix of H. This leads to the well known (see [10]) Heine's formula

$$p_n(x) = \frac{1}{\sqrt{\det H_n \det H_{n-1}}} \det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \cdots & \mu_{n+1} \\ \vdots & & \ddots & & \vdots \\ \mu_{n-1} & \mu_n & & \mu_{2n-2} & \mu_{2n-1} \\ 1 & x & \cdots & \cdots & x^n \end{pmatrix}, \quad n \geqslant 0,$$

and the convention $\det H_{-1} = 1$. However, there is a more easy way to compute them. Starting from the initial conditions $p_0(x) = 1$ (since μ is a nontrivial probability measure) and $p_{-1}(x) = 0$, one can obtain this family of polynomials by means of the so-called three term recurrence relation [10]

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0. \tag{3}$$

Here, the recurrence coefficients are

$$a_n = \int_E xp_{n-1}(x)p_n(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0, \quad n \geqslant 1, \tag{4}$$

$$b_n = \int_E xp_n^2(x)d\mu(x) = \frac{\zeta_n}{\gamma_n} - \frac{\zeta_{n+1}}{\gamma_{n+1}}, \quad n \geqslant 0. \tag{5}$$

In matrix form, we have

$$xp(x) = \hat{\mathbf{J}}p(x),$$

where $p(x) = [p_0(x), p_1(x), \ldots]^t$ and $\hat{\mathbf{J}}$ is the symmetric tridiagonal matrix

$$\hat{\mathbf{J}} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \ddots \\ 0 & 0 & a_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

which is known in the literature as Jacobi matrix [10]. It represents the multiplication operator with respect to the basis of orthonormal polynomials.

On the other hand, the monic orthogonal polynomials with respect to μ are given by $P_n(x) = p_n(x)/\gamma_n$, $n \geqslant 0$. In such a case, (3) becomes

$$P_{n+1}(x) = (x - b_n)P_n(x) - d_n P_{n-1}(x), \quad n \geqslant 0, \tag{6}$$

with $d_n = a_n^2$, and its corresponding matrix representation is

$$\mathbf{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ d_1 & b_1 & 1 & 0 & \cdots \\ 0 & d_2 & b_2 & 1 & \ddots \\ 0 & 0 & d_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{7}$$

which is known as monic Jacobi matrix. Notice that μ defines two sequences of real numbers $\{a_n\}_{n\geqslant 1}$, $\{b_n\}_{n\geqslant 0}$, given by (4) and (5), or, alternatively, the sequences $\{\gamma_n\}_{n\geqslant 1}$, $\{\zeta_n\}_{n\geqslant 0}$ defined by (2). The converse result is known as Favard's theorem (see [10]), and establishes that given arbitrary sequences of numbers $\{a_n\}_{n\geqslant 1}$, $\{b_n\}_{n\geqslant 0}$, with $a_n > 0$ and $b_n \in \mathbb{R}$ for every n, if a sequence of polynomials is computed via the recurrence relation, then there exists a nontrivial probability measure $d\mu$ such that (1) holds.

On the other hand, the Stieltjes function associated with μ is defined by

$$S(x) = \int_E \frac{d\mu(t)}{x - t}.$$

In terms of the moments, the Stieltjes function can be expressed as

$$S(x) = \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}}.$$

There exists a close relation between the Jacobi matrix and the Stieltjes function associated with a measure. Namely, if we consider the continued fraction

$$\frac{d_1|}{|x-b_1} - \frac{d_2|}{|x-b_2} - \frac{d_3|}{|x-b_3} - \dots,$$

i.e. the coefficients are the entries of the monic Jacobi matrix, then the n-th partial denominator is precisely the $n-$th degree monic orthogonal polynomial P_n. This means (see [10]) that

$$Q_n(x) - P_n(x)S(x) = O(1/x^{n+1}),$$

where $Q_n(x)$ is an $n-1$ degree monic polynomial, called the polynomial of the second kind, given by

$$Q_n(x) = \int_E \frac{P_n(y) - P_n(x)}{y-x} d\mu(y).$$

This means that

$$\lim_{n\to\infty} \frac{Q_n(x)}{P_n(x)} = S(x).$$

in a compact subset contained in a neighborhood of infinity. This relation plays a central role in rational approximation.

More generally, a linear functional $\mathcal{L}$ in the space of polynomials can be defined by $\langle \mathcal{L}, x^n \rangle = \mu_n$, $n \geqslant 0$, where $\{\mu_n\}_{n\geqslant 0}$ are arbitrary complex numbers. Thus, the orthogonality of $\{P_n\}_{n\geqslant 0}$ can be stated in terms of $\mathcal{L}$ by

$$\langle \mathcal{L}, P_n(x)P_m(x) \rangle = K_n \delta_{n,m}, \quad K_n \neq 0, \quad n, m \geqslant 0.$$

If $\det \mathbf{H}_n > 0$ for every $n \geqslant 0$, then $\mathcal{L}$ is said to be positive definite. In such a case, it has the integral representation

$$\langle \mathcal{L}, p(x) \rangle = \int_E p(x) d\mu(x),$$

where μ is a nontrivial positive measure such that the numbers $\{\mu_n\}_{n\geqslant 0}$ are its moments. On the other hand, if $\det \mathbf{H}_n \neq 0$ for every $n \geqslant 0$, then $\mathcal{L}$ is said to be quasi-definite. Notice that in this case we can still obtain an orthogonal sequence of polynomials by using the Heine's formula, although they will not be orthonormal.

2.1. Spectral transformations

Spectral transformations of nontrivial probability measures supported on the real line have been widely studied during the last years associated with integrable systems, bispectral problems, matrix theory, group theory, and Stieltjes functions (see [11], [12], [13], [14] among others). In this subsection, we consider the effect of some particular cases of such transformations on the corresponding Jacobi matrices (see [15], [16], [17], [18], and [19] among others) and Stieltjes functions (see [5]).

We will consider the following canonical transformations

(*i*) Christoffel transformation

$$d\tilde{\mu} = (x-\beta)d\mu, \quad \beta \notin supp(\mu).$$

(*ii*) Uvarov transformation

$$d\tilde{\mu} = d\mu + M_r\delta(x-\beta), \quad M_r \in \mathbb{R}.$$

(*iii*) Geronimus transformation

$$d\tilde{\mu} = \frac{d\mu}{x-\beta} + M_r\delta(x-\beta), \quad \beta \notin supp(\mu), M_r \in \mathbb{R}.$$

We will denote them by $\mathcal{R}_C(\beta)$, $\mathcal{R}_U(\beta, M_r)$, and $\mathcal{R}_G(\beta, M_r)$, respectively. They are related by

$$\begin{aligned} \mathcal{R}_C(\beta) \circ \mathcal{R}_G(\beta, M_r) &= \mathcal{I} \quad \text{(Identity transformation)}, \\ \mathcal{R}_G(\beta, M_r) \circ \mathcal{R}_C(\beta) &= \mathcal{R}_U(\beta, M_r). \end{aligned}$$

Furthermore, these transformations can be expressed in terms of the corresponding Stieltjes functions by means of the so-called linear spectral transformations which are defined by

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{D(x)}, \tag{8}$$

where $\widetilde{S}(x)$ is the Stieltjes function associated with $\tilde{\mu}$, and $A(x), B(x)$, $D(x)$ are polynomials in the variable x, which are explicitly given (see [5]) for each of the above transformations. Notice that A, B, D are polynomials such that the right hand side of (8) is analytic around infinity. Namely, we have

Proposition 1. *[5] Let $S(x)$ be the Stieltjes function associated with μ, which is assumed to be normalized by $\mu_0 = 1$. Denote by $S_C(x)$, $S_U(x)$, and $S_G(x)$ the Stieltjes functions associated with $\mathcal{R}_C(\beta)$, $\mathcal{R}_U(\beta, M_r)$, and $\mathcal{R}_G(\beta, M_r)$, respectively. Then, we have*

$$\begin{aligned} S_C(x) &= \frac{(x-\beta)S(x) - 1}{\mu_1 - \beta}, \\ S_U(x) &= \frac{S(x) + \xi(\mu_1 - \beta)(x-\beta)^{-1}}{1 + \xi(\mu_1 - \beta)}, \\ S_G(x) &= \frac{S(\beta) + \xi - S(x)}{(x-\beta)(S(\beta) + \xi)}, \end{aligned}$$

where ξ is a free parameter that can be chosen in such a way that the perturbed measure is positive.

Proof. Notice that for the Christoffel transformation, the moments associated with $d\tilde{\mu}$ are given by

$$\tilde{\mu}_k = \int_E x^k d\tilde{\mu}(x) = \int_E x^k (x-\beta) d\mu(x) = \mu_{k+1} - \beta\mu_k.$$

This means that

$$S_C(x) = \sum_{k=0}^{\infty} \frac{\tilde{\mu}_k}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{\mu_{k+1} - \beta\mu_k}{x^{k+1}} = (x-\beta)S(x) - \mu_0,$$

which, after normalization, yields the first equation above. The other two expressions follow in a similar way. ■

Furthermore, [5] shows that linear spectral transformations (8), for arbitrary polynomials A, B, and D, can be expressed in terms of transformations (i) and (iii).

On the other hand, transformations of the Stieltjes functions like

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{C(x)S(x) + D(x)}, \tag{9}$$

where A, B, C, and D are polynomials in x and $AD - BC \neq 0$, are called rational spectral transformations, again assuming that $\widetilde{S}$ is analytic around infinity. Examples of this kind of transformations appear, for instance, when the coefficients of the recurrence relations are perturbed in a particular way. Namely, if for some positive integer k we define the sequence $\{P_n^{(k)}\}_{n\geqslant 0}$ by the shifted recurrence relation

$$P_{n+1}^{(k)}(x) = (x - b_{n+k})P_n^{(k)}(x) - d_{n+k}P_{n-1}^{(k)}(x), \quad n \geqslant 0,$$

i.e. removing the first k rows and columns of $\mathbf{J}$, then we get the monic associated polynomials of order k. Conversely, if we "push" the first k rows and columns of $\mathbf{J}$, and introduce new coefficients $b_{-i} \quad (i = k, k-1, ..., 1)$ and $d_{-i} \quad (i = k-1, k-2, ..., 0)$, then the anti-associated polynomials of order k are defined by

$$P_{n+1}^{(-k)}(x) = (x - \tilde{b}_{n+k})P_n^{(-k)}(x) - \tilde{d}_{n+k}P_{n-1}^{(-k)}(x), \quad n \geqslant 0,$$

where $\{\tilde{b}_i\}_{i\geqslant 0} = \{b_{-i}\}_{i=k}^{1} \bigcup \{b_i\}_{i\geqslant 0}$ and $\{\tilde{d}_i\}_{i\geqslant 1} = \{d_{-i}\}_{i=k-1}^{0} \bigcup \{d_i\}_{i\geqslant 1}$. In both cases, the corresponding Stieltjes function has the form (9). In [5] it was proved that all transformations of the form (9) can be obtained as a combination of Christoffel, Geronimus, associated and anti-associated transformations.

On the other hand, linear spectral transformations can also be expressed in terms of Jacobi matrices. More precisely, the Jacobi matrices associated with the perturbed measures can be obtained from the original Jacobi matrices. This process involves the Darboux transformation, that is related to LU and UL factorizations of tridiagonal matrices.

If $\mathbf{J}$ is the monic Jacobi matrix associated with a nontrivial probability measure μ, and all of its principal leading submatrices are nonsingular, then $\mathbf{J}$ has a unique LU factorization where $\mathbf{L}$ and $\mathbf{U}$ are bidiagonal matrices

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ l_1 & 1 & 0 & 0 & \cdots \\ 0 & l_2 & 1 & 0 & \ddots \\ 0 & 0 & l_3 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_2 & 1 & 0 & \cdots \\ 0 & 0 & u_3 & 1 & \ddots \\ 0 & 0 & 0 & u_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \qquad (10)$$

We then define a new matrix $\mathbf{J}_D := \mathbf{UL}$ that is called the *Darboux transformation of* $\mathbf{J}$ *without parameter*, by inverting the order in the product of $\mathbf{L}$ and $\mathbf{U}$. Notice that $\mathbf{J}_D$ is again a tridiagonal matrix with ones as entries on the upper diagonal and thus it is a monic Jacobi matrix associated with some nontrivial measure $\tilde{\mu}$, by Favard's theorem.

Similarly, we consider the UL factorization $\mathbf{J} = \mathbf{UL}$, where $\mathbf{U}$ and $\mathbf{L}$ are as in (10). The UL factorization is not unique. Indeed, it depends of the choice of the parameter u_1 taking into account $u_1 + l_1 = b_0$ and $u_2 l_1 = d_1$. Then, each choice of u_1 determines the value of l_1 and thus all the elements are determined. Defining $\mathbf{J}_d := \mathbf{LU}$, we obtain again a monic Jacobi matrix. In this situation, $\mathbf{J}_d$ is said to be the *Darboux transformation* with parameter of $\mathbf{J}$.

Remark 2. *A necessary and sufficient condition for the existence of the LU factorization of* $\mathbf{J}$ *is* $P_n(0) \neq 0$, $n \geqslant 1$, *or, equivalently, that* $supp(\mu)$ *does not contain the origin. Furthermore, assuming the LU factorization exists, the entries of* $\mathbf{L}$ *and* $\mathbf{U}$ *are given by*

$$l_1 = \frac{d_1}{b_0}, \quad l_n = \frac{d_n}{b_{n-1} - l_{n-1}}, \quad n \geqslant 2, \tag{11}$$

$$u_1 = b_0, \quad u_n = b_{n-1} - l_{n-1}, \quad n \geqslant 2. \tag{12}$$

The following statements express transformations $(i) - (iii)$ in terms of the corresponding monic Jacobi matrices.

Proposition 3. *[16] Let* μ *be a nontrivial probability measure and denote by* $\{P_n\}_{n\geqslant 0}$ *and* $\mathbf{J}$ *its corresponding sequence of monic orthogonal polynomials and monic Jacobi matrix, respectively. Let* $\beta \in \mathbb{R}$ *such that* $P_n(\beta) \neq 0$, $n \geqslant 1$. *Then, if we apply the transformation*

$$\mathbf{J} - \beta\mathbf{I} = \mathbf{LU}, \quad \mathbf{J}_1 := \mathbf{UL} + \beta\mathbf{I},$$

then $\mathbf{J}_1$ *is the monic Jacobi matrix associated with* $d\tilde{\mu} = (x-\beta)d\mu$, *i.e. its Christoffel transformation.*

In order to have $P_n(\beta) \neq 0$ for every $n \geq 1$, it suffices that β does not belong to the interior of the convex hull of $supp(\mu)$.

There are similar results for transformations (ii) and (iii), as follows.

Proposition 4. *[16] Let* $\mathbf{J}$ *be the monic Jacobi matrix associated with the nontrivial probability measure* μ. *Consider the following transformations of* $\mathbf{J}$

$$\begin{aligned} \mathbf{J} - \beta\mathbf{I} &= \mathbf{L_1 U_1}, & \mathbf{J}_1 &:= \mathbf{U_1 L_1}, \\ \mathbf{J}_1 &= \mathbf{U_2 L_2}, & \mathbf{J}_2 &:= \mathbf{L_2 U_2} + \beta\mathbf{I}. \end{aligned}$$

Then $\mathbf{J}_2$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = d\mu + M_r \delta(x - \beta),$$

i.e. the Uvarov transformation of μ*, where*

$$M_r = \frac{\mu_0(b_0 - \beta - s)}{s},$$

with $\mu_0 = \int_E d\mu(x)$ *and* s *is the free parameter associated with the* UL *factorization of* $\mathbf{J}_1$.

Proposition 5. *[16] Let* $\mathbf{J}_1$ *be the monic Jacobi matrix associated with the measure* $\hat{\mu}$*. Suppose there exists a positive measure* μ *such that* $d\hat{\mu} = (x - \beta)d\mu$*. If we apply the following transformation to* $\mathbf{J}_1$

$$\mathbf{J}_1 - \beta \mathbf{I} \quad = \quad \mathbf{U_1 L_1}, \quad \mathbf{J}_2 := \mathbf{L_1 U_1} + \beta \mathbf{I},$$

then $\mathbf{J}_2$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = \frac{d\hat{\mu}}{x - \beta} + M_r \delta(x - \beta),$$

i.e. the Geronimus transformation of $\hat{\mu}$*, where* $M_r = \frac{\int_E d\hat{\mu}}{s}$ *and* s *is the free parameter associated with the* UL *factorization of* $\mathbf{J}_1$.

Some other kind of perturbations have been studied recently. Given a sequence of moments $\{\mu_n\}_{n\geqslant 0}$ the analysis of the orthogonal polynomials associated with the perturbed sequence $\{\tilde{\mu}_n\}_{n\geqslant 0}$, where $\tilde{\mu}_j = \mu_j + m_j$ with $m_j \in \mathbb{R}$ and $\tilde{\mu}_k = \mu_k$ for $k \neq j$, was studied in [20]. On the other hand, co-dilated and co-recursive perturbations of the coefficients of the recurrence relation have been considered in [8].

3. Linear spectral transformations on the real line and integrable systems

Linear spectral transformations associated with measures on the real line play a central role in the problem of determining the integrability of some lattice systems. For instance, let us consider the non linear lattice describing a system of a sequence of particles interacting with each other under the action of a potential φ with exponential interaction

$$\varphi(r) = e^{-r} + r - 1,$$

where r is the system's total amount of movement and $r_n = y_{n+1} - y_n$ represents the displacement of a single particle. This is known as the Toda lattice (see [21], [22]) and was studied by Flaschka [23] (see also [24, 25]), who proved its complete integrability by expressing it in terms of some Jacobi matrix. Later on, Van Moerbeke (see [26], [27]) working on Hill's equation [28], used Jacobi matrices to define the Toda hierarchy for the periodic Toda lattices.

Flaschka's used the change of variable

$$a_n = \frac{1}{2} e^{-(y_{n+1}-y_n)/2}, \quad b_n = \frac{1}{2} \dot{y}_n,$$

where the dot notation denotes the derivative with respect to time, in such a way that the new variables obey the evolution equations

$$\begin{aligned} \dot{a}_n &= a_n(b_{n+1} - b_n), && (13) \\ \dot{b}_n &= 2(a_n^2 - a_{n-1}^2), \quad a_{-1} = 0, \quad n \geqslant 0, && (14) \end{aligned}$$

with initial conditions $b_n^0 = b_n(0) = \overline{b_n(0)}$, $a_n^0 = a_n(0) > 0$, which are assumed to be bounded.

In matrix form, we have

$$\mathbf{J}_t = \begin{bmatrix} b_0(t) & a_0(t) & 0 & 0 & \cdots \\ a_0(t) & b_1(t) & a_1(t) & 0 & \cdots \\ 0 & a_1(t) & b_2(t) & a_2(t) & \ddots \\ 0 & 0 & a_2(t) & b_3(t) & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

where the coefficients are now time dependent, and we denote by $\mathbf{J}_\mu = \mathbf{J}_0$, i.e. with entries $a_n(0) = a_n^0$ and $b_n(0) = b_n^0$. The equations (13)-(14) can be expressed in terms of the Jacobi matrix $\mathbf{J}_t$ as the Lax pair

$$\dot{\mathbf{J}}_t = [\mathbf{A}, \mathbf{J}_t] = \mathbf{A}\mathbf{J}_t - \mathbf{J}_t\mathbf{A},$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & a_0(t) & 0 & 0 & \cdots \\ -a_0(t) & 0 & a_1(t) & 0 & \cdots \\ 0 & -a_1(t) & 0 & a_2(t) & \ddots \\ 0 & 0 & -a_2(t) & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{J}_t^+ - \mathbf{J}_t^-,$$

where $\mathbf{J}_t^+$ resp. $\mathbf{J}_t-$ is the upper-triangular (resp. lower-triangular) projection of the matrix $\mathbf{J}_t$, and $[\cdot,\cdot]$ denotes the commutator. The Favard's theorem guarantees the existence of an orthogonality measure associated with $\mathbf{J}_t$, which in this case is given by the spectral transformation

$$d\mu(x,t) = \kappa(t)e^{-tx}d\mu(x,0), \quad t > 0, \tag{15}$$

where $\kappa(t)$ is a normalization factor so that $\mu(x,t)$ is a nontrivial probability measure. Notice that the solution of the Toda lattice involves a combination of the inverse spectral problem from $\{a_n^0\}_{n\geqslant 0}$, $\{b_n^0\}_{n\geqslant 0}$ associated with the measure $d\mu = d\mu(\cdot, 0)$, the spectral transformation (15), and the direct spectral problem from $\{a_n(t)\}_{n\geqslant 0}$, $\{b_n(t)\}_{n\geqslant 0}$ associated with the measure $d\mu(\cdot,t)$. More recently, a more general perturbation (15) was studied in [29], where the authors also describe the time evolution of the zeros of the corresponding orthogonal polynomials.

3.1. Integrable systems associated with symmetric measures

Another example of the connection between spectral transformations and integrable systems was given in [30]. Let μ be a symmetric nontrivial probability measure μ, supported on an infinite subset of the real line that is symmetric with respect to the origin. Because of the symmetry, the corresponding sequence of monic orthogonal polynomials satisfies the recurrence relation (6)

$$xP_n(x) = P_{n+1}(x) + d_nP_{n-1}(x), \quad n \geqslant 0. \tag{16}$$

Notice that $P_{2n}(x) = A_n(x^2)$, $P_{2n+1}(x) = xB_n(x^2)$, where $\{A_n\}_{n\geqslant 0}$ is a sequence of monic orthogonal polynomials and $\{B_n\}_{n\geqslant 0}$ is its Christoffel transformation with parameter $\beta = 0$.

Let us introduce a time parameter on the coefficients of the recurrence relation and assume that $d_n(t) > 0$, $t > 0$. If we denote by $\{P_n(x,t)\}_{n\geqslant 0}$ the sequence of monic orthogonal polynomials constructed from (16), then according to Favard's theorem there exists a symmetric (in the variable x) measure $\mu(x,t)$ which is the orthogonality measure for these polynomials. Notice that the derivation with respect to the time variable of the monic polynomial $P_n(x,t)$ yields

$$\dot{P}_n(x,t) = C_{n,n-2}(t)P_{n-2}(x,t) + \ldots + C_{n,n-2k}(t)P_{n-2k}(x,t) + \ldots, \tag{17}$$

where the constants $C_{n,n-2k}(t)$ for $k \geqslant 1$ define the behavior of $d_n(t)$ and $P_n(x,t)$ with respect to the time.

Of course, different conditions on the time dependent constants $C_{n,n-2k}(t)$ lead to different integrable systems and thus different orthogonality measures. The case $C_{n,n-2k}(t) = 0$ for $k \geqslant 2$ was considered in [30]). In such a case, $C_{n,n-1} = -d_n d_{n-1}$, which leads to

$$\dot{d}_n = d_n(d_{n+1} - d_{n-1}),$$

which is known in the literature as the Volterra chain. The corresponding orthogonality measure is

$$d\mu(x,t) = \kappa(t)e^{x^2 t}d\mu(x,0), \quad t > 0,$$

where $\kappa(t)$ is a normalization factor so that $\mu(x,t)$ is a nontrivial probability measure. The results can be summarized as follows.

Proposition 6. *[30]. Let μ be a symmetric measure and introduce a time parameter on the coefficients of the recurrence relation. Furthermore, assume that the corresponding orthogonal polynomials behavior with respect to a time parameter is given by*

$$\dot{P}_n(x,t) = C_{n,n-2}(t)P_{n-2}(x,t).$$

Then, we have $\dot{d}_n = d_n(d_{n+1} - d_{n-1})$ and the time dependent orthogonality measure is given by $d\mu(x,t) = \kappa(t)e^{x^2 t}d\mu(x,0)$, where $\kappa(t)$ is a normalization factor.

If an additional term is considered in (17), then we have the following result.

Proposition 7. *Assume the conditions on the previous Proposition hold, up to the fact that the behavior of orthogonal polynomials with respect to the time is now given by*

$$\dot{P}_n(x,t) = C_{n,n-2}(t)P_{n-2}(x,t) + C_{n,n-4}(t)P_{n-4}(x,t), \quad n \geqslant 2. \tag{18}$$

If we denote by $\mu(x,t) = \omega(x,t)dx$ the time dependent orthogonality measure, then we have

$$\omega(x,t) = \omega(x,0)e^{q(x,t)}, \tag{19}$$

where q is a polynomial of degree 4 of the form $q(x,t) = \int [a_4(t)x^4 + a_2(t)x^2 + a_0(t)]dt$, with

$$\begin{aligned} a_4(t) &= \frac{d_1(\dot{d}_1 + \dot{d}_2) - \dot{d}_1 d_3}{d_1 d_2 d_3 d_4} \neq 0, \\ a_2(t) &= \frac{\dot{d}_1}{d_1 d_2} - (d_1 + d_2 + d_3)a_4(t), \\ a_0(t) &= -\frac{\dot{d}_1}{d_2} + d_1 d_3 a_4(t). \end{aligned}$$

Proof. From Favard's theorem, $\omega(x,t)$ is a positive weight for $t > 0$. Without loss of generality, we can assume $\omega(x,t)$ is normalized so that $\int_E d\omega(x,t)dx = 1, t > 0$. Since $\{P_n(x,t)\}_{n\geqslant 0}$ is orthogonal with respect to $\omega(x,t)$, we have

$$\int_E P_n(x,t)P_j(x,t)d\mu(x,t) = 0, \quad 0 \leqslant j \leqslant n-1. \tag{20}$$

Taking the derivative with respect to t, we get

$$\int_E \dot{P}_n(x,t)P_j(x,t)\omega(x,t)dx + \int_E P_n(x,t)P_j(x,t)\dot{\omega}(x,t)dx = 0, \quad 0 \leqslant j \leqslant n-1. \tag{21}$$

Now, if $j = 0$, we get

$$\int_E \dot{P}_n(x,t)\omega(x,t)dx + \int_E P_n(x,t)\dot{\omega}(x,t)dx = 0, \quad n \geqslant 1,$$

and therefore, by (18), we obtain

$$\int_E P_n(x,t)\dot{\omega}(x,t)dx = 0, \quad n > 4.$$

Since $P_n(x,t)$ is orthogonal with respect to $\omega(x,t)$, this means that the linear functional associated with $\dot{\omega}$ can be represented as

$$\dot{\omega} = \sum_{k=0}^{4} \lambda_k \frac{P_k(x,t)\omega(x,t)}{||P_k(x,t)||^2}, \tag{22}$$

where

$$\lambda_k = \int_E P_k(x,t)\dot{\omega}(x,t)dx, \quad 0 \leqslant k \leqslant 4. \tag{23}$$

Now, from

$$\frac{d}{dt}\int_E P_k(x,t)\omega(x,t)dx = \int_E P_k(x,t)\dot{\omega}(x,t)dx + \int_E \dot{P}_k(x,t)\omega(x,t)dx,$$

since the term on the left hand side vanishes for $0 \leqslant k \leqslant 4$, we obtain

$$\lambda_k = -\int_E \dot{P}_k(x,t)\omega(x,t)dx, \quad 0 \leqslant k \leqslant 4. \tag{24}$$

For the sake of simplicity, we will drop the time dependence on the notation of $C_{n,k}(t)$ and $d_k(t)$. Notice that, since $P_0(x,t) = 1$ and $P_1(x,t) = x$, we have $\lambda_0 = \lambda_1 = 0$. We also have $\lambda_3 = 0$ by symmetry. Furthermore, using the recurrence formula, we have

$$\begin{aligned}
P_2(x,t) &= xP_1(x,t) - d_1P_0(x,t) = x^2 - d_1,\\
P_3(x,t) &= xP_2(x,t) - d_2P_1(x,t) = x^3 - (d_1 + d_2)x,\\
P_4(x,t) &= xP_3(x,t) - d_3P_2(x,t) = x^4 - (d_1 + d_2 + d_3)x^2 + d_1d_3,
\end{aligned}$$

and it is easily verified that

$$\begin{aligned}
\lambda_2 &= \dot{d}_1,\\
\lambda_4 &= (\dot{d}_1 + \dot{d}_2)d_1 - \dot{d}_1d_3.
\end{aligned}$$

As a consequence, from (22), we get (19). The expressions for a_0, a_1, and a_2 follow from the recurrence relation and the fact that $||P_k(x,t)||^2 = d_kd_{k-1}\ldots d_1$. ■

Notice that, in general, the polynomial $q(x,t)$ will depend on the initial conditions for d_1, d_2, d_3, and d_4. Furthermore, the additional term $C_{n,n-4}$ introduces a quartic polynomial on the exponential. Since $d_1 = \mu_2$ is fixed by the orthogonality measure, using the above equations d_2, d_3, d_4 can be expressed as

$$\begin{aligned}
d_2 &= -d_1 - \frac{a_2(t)}{a_4(t)} - \frac{a_0(t)}{d_1a_4(t)},\\
d_3 &= \frac{\dot{d}_1}{d_1d_2a_4(t)} + \frac{a_0(t)}{d_1a_4(t)},\\
d_4 &= \frac{(\dot{d}_1 + \dot{d}_2)d_1 - \dot{d}_1d_3}{d_1d_2d_3a_4(t)},
\end{aligned}$$

if $a_4(t) \neq 0$.

On the other hand, also from (18), we have

$$C_{n,i} = \frac{\int_E \dot{P}_n(x,t)P_i(x,t)\omega(x,t)dx}{\|P_i(x,t)\|^2} = \frac{\int_E P_n(x,t)P_i(x,t)\dot{\omega}(x,t)dx}{\|P_i(x,t)\|^2}, \quad i = n-2, n-4,$$

and thus using (19) we obtain that

$$\begin{aligned} C_{n,n-2} &= d_n d_{n-1}(d_{n+1} + d_n + d_{n-1} + d_{n-2})A_4(t) - d_n d_{n-1} A_2(t), && (25) \\ C_{n,n-4} &= -(d_n d_{n-1} d_{n-2} d_{n-3})A_4(t), && (26) \end{aligned}$$

where $A_4(t) = \int a_4(t)dt$, $A_2(t) = \int a_2(t)dt$. If $A_0(t) = \int a_0(t)dt$ and $\omega(x,t)$ is normalized so that $\int_E \omega(x,t)dx = 1$, then we have

$$A_0(t) = \ln \frac{1}{\int_E e^{A_4(t)x^4 + A_2(t)x^2}\omega(x,0)dx}.$$

The coefficients $C_{n,n-2}, C_{n,n-4}$ can also be obtained comparing (18) and the recurrence relation (16). For instance, we get $C_{2,0} = -\dot{d}_1$, $C_{3,1} = -(\dot{d}_1 + \dot{d}_2)$ and, in general

$$C_{n,n-2} = -(\dot{d}_1 + \ldots + \dot{d}_{n-1}), \quad n \geqslant 2. \tag{27}$$

We also get, for $n \geqslant 4$,

$$C_{n,n-4} = \frac{d}{dt}[d_3 d_1 + d_4(d_1 + d_2) + \ldots + d_{n-1}(d_1 + \ldots + d_{n-3})] + (d_1 + \ldots + d_{n-3})C_{n,n-2}. \tag{28}$$

There exists a matrix interpretation of the dynamics of the Jacobi matrix. Let $\mathbf{J}(t)$ be the time dependent Jacobi matrix associated with $\{P_n\}_{n\geqslant 0}$ and denote $\mathbf{P}(t) = [P_0(x,t), P_1(x,t), \ldots]^t$. Then,

$$x\mathbf{P}(t) = \mathbf{J}(t)\mathbf{P}(t).$$

Differentiating with respect to t, we obtain $x\dot{\mathbf{P}}(t) = \dot{\mathbf{J}}(t)\mathbf{P}(t) + \mathbf{J}(t)\dot{\mathbf{P}}(t)$, and using (18) we obtain

$$x\mathbf{H}(t)\mathbf{P}(t) = \dot{\mathbf{J}}(t)\mathbf{P}(t) + \mathbf{J}(t)\mathbf{H}(t)\mathbf{P}(t),$$

where $\mathbf{H}(t)$ is the lower triangular matrix

$$\mathbf{H}(t) = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ C_{2,0} & 0 & \ddots & \\ 0 & C_{3,1} & \ddots & \\ C_{4,0} & 0 & \ddots & \\ 0 & C_{5,1} & \ddots & \\ \vdots & \ddots & \ddots & \end{pmatrix}.$$

As a consequence, we have $\mathbf{H}(t)\mathbf{J}(t)\mathbf{P}(t) = (\dot{\mathbf{J}}(t) + \mathbf{J}(t)\mathbf{H}(t))\mathbf{P}(t)$ and thus $\mathbf{J}(t)$ satisfies the Lax type equation

$$\dot{\mathbf{J}}(t) = \mathbf{H}(t)\mathbf{J}(t) - \mathbf{J}(t)\mathbf{H}(t) = [\mathbf{J}, \mathbf{H}].$$

$\dot{\mathbf{J}}$ is a matrix with entries $\dot{\mathbf{J}}_{k+1,k} = \dot{d}_k$, and the other entries are zero. On the other hand, by direct computation, we have

$$\dot{d}_n = C_{n-1,n-3} - C_{n,n-2}, \tag{29}$$

$$C_{n-1,n-5} - C_{n,n-4} = d_{n-1}C_{n-2,n-4} - d_{n-3}C_{n-1,n-3}. \tag{30}$$

Notice that the addition of an extra term in (17) causes the expressions describing the dynamics of the coefficients of the recurrence relation to be rather complicated. However, if all coefficients $C_{n,n-2k}(t)$ are taken into account, the resulting relations are considerably more simple.

Proposition 8. *[30] Assume that* (17) *holds, and that the following conditions are satisfied*

$$\dot{P}_{2n+1}(x,t) = 0; \quad \dot{P}_{2n}(x,t) = \frac{r_n(t)}{x} P_{2n-1}(x,t),$$

for some time dependent coefficients $r_n(t)$. Then, the corresponding orthogonality measure is

$$d\tilde{\mu}(x) = [1 - M(t)]d\mu(x) + M(t)\delta(x), \tag{31}$$

where $M : \mathbb{R}_+ \to [0,1]$ *is the positive* C^1 *function*

$$M(t) = 1 - \exp\left(-\int_{t_0}^{t} \frac{dt}{d_1(t)}\right).$$

Notice that an Uvarov type perturbation appears. When a time parameter is introduced, the behavior of the corresponding orthogonal polynomials and the recurrence relation coefficients was analyzed in [30]. The behavior of the zeros, including an electrostatic interpretation of such behavior, was deduced in [31], [32] for some particular classes of orthogonality measures.

The dynamics of the corresponding orthogonal polynomials, $\{P_n(x,t)\}_{n\geqslant 0}$, can be obtained as follows. Let $K_n(x,y)$ be the n-th reproducing kernel defined by

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{||P_k||^2} = \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1)}(y)}{||P_n||^2(x-y)}.$$

The right hand side is known as the Christoffel-Darboux formula (for $x \neq y$). $P_n(x,t)$ is then given by (see [15], [17])

$$P_n(x,t) = P_n(x) - \frac{M(t)P_n(0)}{1 - M(t) + M(t)K_{n-2}(0,0)} K_{n-2}(x,0). \tag{32}$$

Notice that since $M(0) = 0$, we have $P_n(x,0) = P_n(x)$, i.e., the non perturbed polynomials. Moreover, according to the symmetry of the measure μ, $P_{2n+1}(0) = 0$, i.e. the polynomials of odd degree do not change with respect to the time and thus we have $P_{2n+1}(x,t) = P_{2n+1}(x)$, $n \geqslant 0$, and

$$P_{2n}(x,t) = P_{2n}(x) - \frac{M(t)P_{2n}(0)}{1 - M(t) + M(t)K_{2n-2}(0,0)} K_{2n-2}(x,0), \quad n \geqslant 0, t > 0.$$

Obviously, we have $\dot{P}_{2n+1}(x,t) = 0$. For the polynomials of even degree, by using the Christoffel-Darboux formula, we get

$$\begin{aligned} \dot{P}_{2n}(x,t) &= -\frac{\dot{M}(t)P_{2n}(0)}{[1 - M(t) + M(t)K_{2n-2}(0,0)]^2} K_{2n-2}(x,0), \\ &= r_n \frac{P_{2n-1}(x)}{x}, \end{aligned} \tag{33}$$

with

$$r_n = -\frac{\dot{M}(t)P_{2n}(0)P_{2n-2}(0)}{||P_{2n-2}||^2[1 - M(t) + M(t)K_{2n-1}(0,0)]^2}. \tag{34}$$

Furthermore, since $K_{2n-1}(x,0) = \frac{P_{2n-1}(x)P_{2n-2}(0)}{\|P_{2n-2}\|^2 x}$, we have $K_{2n-1}(0,0) = \frac{P'_{2n-1}(0)P_{2n}(0)}{\|P_{2n-2}\|^2}$ and therefore

$$r_n = -\frac{\dot{M}(t)P_{2n}(0)P_{2n-2}(0)\|P_{2n-2}\|^2}{[(1-M(t))\|P_{2n-2}\|^2 + M(t)P'_{2n-1}(0)P_{2n}(0)]^2}. \tag{35}$$

The behavior of the coefficients of the recurrence relation with respect to the time is given by (see [30])

$$\dot{d}_{2n} = r_n(t), \quad \dot{d}_{2n+1} = -r_{n+1}(t),$$

i.e. a non-local integrable chain with discrete space variable and continuous time. The connections with the so-called Uvarov-Chihara problem in the theory of orthogonal polynomials have been analyzed in [33].

3.2. Extension to non symmetric measures

Although the previous cases correspond to symmetric measures, the method can be extended to non symmetric measures by using a symmetrization process as follows. First, notice that a (not necessarily symmetric) measure μ can be used to define the linear functional

$$\mathcal{L}[q(x)] = \int_E q(x)d\mu(x), \quad q \in \mathbb{P}.$$

If μ is a nontrivial probability measure, then $\mathcal{L}$ will be positive definite (i.e. $\mathcal{L}[q(x)] > 0$ for any non negative and non identically zero polynomial q). Orthogonality can now be expressed in terms of $\mathcal{L}$. Now, let us define

$$\mathcal{L}_s[x^{2n}] := \mathcal{L}[x^n], \quad \mathcal{L}_s[x^{2n+1}] := 0, \quad n \geqslant 0,$$

i.e., the linear functional $\mathcal{L}_s$ is symmetric. It is well known (see [10]) that, if $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ are the sequences of monic polynomials orthogonal with respect to $\mathcal{L}$ and $\mathcal{L}_s$, respectively, then both families are related by

$$Q_{2n}(x) = P_n(x^2), \quad Q_{2n+1}(x) = x\tilde{P}_n(x^2), \quad n \geqslant 0,$$

where $\{\tilde{P}_n\}_{n\geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to the linear functional $\tilde{\mathcal{L}} = x\mathcal{L}$, where $x\mathcal{L}[q] := \mathcal{L}[xq]$ for any $q \in \mathbb{P}$ (i.e. the Christoffel transform of $\mathcal{L}$). We also have

$$\tilde{P}_n(x) = \frac{1}{x}\left(P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)}P_n(x)\right), \quad n \geqslant 0.$$

Notice that a necessary condition for their existence is that $P_n(0) \neq 0$, $n \geqslant 0$ (this is equivalent to $0 \notin supp(\mu)$). It turns out that the condition is also sufficient.

Now, if $\mathcal{L}$ is any positive definite linear functional, then let $\{P_n(x,t)\}_{n\geqslant 0}$ be the sequence of monic polynomials orthogonal with respect to the linear functional $\mathcal{L}_t := (1 - M(t))\mathcal{L} + M(t)\delta(x)$. Thus, applying the symmetrization process to $\mathcal{L}_t$, we get

$$P_n(x^2, t) = Q_{2n}(x, t), \quad n \geqslant 0,$$

where $\{Q_n(x,t)\}_{n\geqslant 0}$ are symmetric polynomials orthogonal with respect to the linear functional $\mathcal{L}_s$. Therefore,

$$\dot{P}_n(x^2, t) = \dot{Q}_{2n}(x, t) = r_n(t)\frac{Q_{2n-1}(x,t)}{x},$$

where $r_n(t)$ is given as in the previous section and is computed using the polynomials Q_n. Thus,

$$\dot{P}_n(x, t) = r_n(t)\tilde{P}_{n-1}(x, t), \quad n \geqslant 1,$$

where $\tilde{P}_n(x,t)$ is orthogonal with respect to the linear functional $x\mathcal{L}_t$, provided $P_n(0,t) \neq 0$, $n \geqslant 1$ and $t > 0$. As a consequence, we get

Proposition 9. *[34] Let $\{P_n(x)\}_{n\geqslant 0}$ be the sequence of monic polynomials with respect a nontrivial probability measure $d\mu$. Let $d\tilde{\mu}$ be defined as in* (31) *and denote by $\{P_n(x,t)\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Then,*

$$\dot{P}_n(x,t) = \frac{r_n(t)}{x}\left(P_n(x,t) - \frac{P_n(0,t)}{P_{n-1}(0,t)}P_{n-1}(x,t)\right), \quad n \geqslant 1.$$

3.3. Zeros dynamics for semiclassical orthogonal polynomials

In this section, we study the behavior of the zeros of the orthogonal polynomials associated with the time dependent Uvarov transformation considered above.

Let us introduce the positive definite linear functional $\mathcal{L}$ which is symmetric and semiclassical, i.e.,

$$\mathcal{D}(\phi(x)\mathcal{L}) = \Psi(x)\mathcal{L},$$

where $\mathcal{D}$ is the derivative operator and ϕ and Ψ are polynomials, which are even and odd functions, respectively, with $\deg \Psi \geqslant 1$ (see [35]).

Let us consider the linear functional $\mathcal{L}_t := (1 - M(t))\mathcal{L} + M(t)\delta(x)$. Then, we have

$$x^2\phi(x)\mathcal{L}_t = (1 - M(t))x^2\phi(x)\mathcal{L}.$$

Since

$$\begin{aligned}
\mathcal{D}[x^2\phi(x)\mathcal{L}_t] &= (1 - M(t))\mathcal{D}[x^2\phi(x)\mathcal{L}] \\
&= (1 - M(t))[2x\phi(x)\mathcal{L} + x^2\mathcal{D}(\phi\mathcal{L})] \\
&= 2x\phi\mathcal{L}_t + (1 - M(t))x^2\Psi\mathcal{L} \\
&= (2x\phi + x^2\Psi)\mathcal{L}_t,
\end{aligned}$$

we deduce that $\mathcal{L}_t$ is a symmetric and semiclassical linear functional. As a consequence, its corresponding sequence of monic orthogonal polynomials satisfies the structure relation ([17], [35])

$$x^2\phi(x)\frac{\partial}{\partial x}P_n(x;t) = A_n(x;t)P_n(x;t) + B_n(x;t)P_{n-1}(x;t), \tag{36}$$

where the functions $A_n(x;t), B_n(x;t)$ can be calculated explicitly by using the measure associated with $\mathcal{L}$ (see [36], [37], [29], [35]).

According to [29], if $x_{n,k}(t)$ is the k-th zero of $P_n(x;t)$, taking the derivate with respect t in $P_n(x_{n,k}(t), t) = 0$, we obtain

$$\frac{\partial}{\partial x}P_n(x;t)\bigg|_{x=x_{n,k}}\dot{x}_{n,k} + \dot{P}_n(x_{n,k}, t) = 0.$$

Thus, evaluating (36) with $n = 2m$ at $x = x_{2m,k}(t)$ we get

$$x_{2m,k}^2(t)\phi(x_{2m,k}(t))\frac{\partial}{\partial x}P_{2m}(x_{2m,k}(t);t) = B_n(x_{2m,k}(t);t)P_{2m-1}(x_{2m,k}(t);t).$$

Therefore, using (33), we obtain the following expression for the dynamics of the zeros for the orthogonal polynomials $P_{2m}(x,t)$

$$\dot{x}_{2m,k}(t) = -r_m\frac{x_{2m,k}(t)\phi(x_{2m,k}(t))}{B_{2m}(x_{2m,k}(t))}. \tag{37}$$

3.3.1. Example

Consider the generalized Hermite polynomials $\{H_n^{(\lambda)}\}_{n\geqslant 0}$ (see [10]). They are orthogonal with respect to the symmetric weight function

$$|x|^{2\lambda}e^{-x^2}, \quad -\infty < x < \infty, \tag{38}$$

where $\lambda > -1/2$. When $\lambda = 0$, of course, we get the Hermite polynomials. They satisfy the recurrence relation

$$H_{n+1}^{(\lambda)}(x) = 2xH_n^{(\lambda)}(x) - 2(n+\theta_n)H_{n-1}^{(\lambda)}(x), \quad n \geqslant 1, \tag{39}$$

where $\theta_{2k} = 0$ and $\theta_{2k+1} = 2\lambda$, and the structure relation

$$\frac{d}{dx}H_n^{(\lambda)}(x) = 2nH_{n-1}^{(\lambda)}(x) + 2(n-1)\theta_n H_{n-2}^{(\lambda)}(x). \tag{40}$$

If we perturb (38) adding a time-dependent mass at $x = 0$ as in (31), we obtain

$$H_{2n}^{(\lambda)}(x,t) = H_{2n}^{(\lambda)}(x) - \frac{M(t)H_{2n}^{(\lambda)}(0)}{1 - M(t) + M(t)K_{2n-2}(0,0)}K_{2n-2}(x,0), \quad n \geqslant 0, t > 0.$$

Using the Christoffel-Darboux formula and taking derivative with respect to x, we get From (32), we have

$$\frac{d}{dx}H_{2n}^{(\lambda)}(x,t) = H_{2n}'^{(\lambda)}(x) - N(t,n)\frac{xH_{2n-1}'^{(\lambda)}(x) - H_{2n-1}^{(\lambda)}(x)}{x^2}, \tag{41}$$

where

$$N(t,n) = \frac{M(t)H_{2n}^{(\lambda)}(0)H_{2n-2}^{(\lambda)}(0)}{[1 - M(t) + M(t)K_{n-2}(0,0)]||H_{2n-2}^{(\lambda)}||^2}.$$

We will obtain the relation (36) for these polynomials. In this case, we have $\phi(x) = x$, so that

$$x^3H_{2n}'^{(\lambda)}(x,t) = x^3H_{2n}'^{(\lambda)}(x) - N(t,n)x[xH_{2n-1}'^{(\lambda)}(x) - H_{2n-1}^{(\lambda)}(x)]. \tag{42}$$

where

$$N(t,n) = \frac{M(t)H_{2n}(0)H_{2n-2}^{(\lambda)}(0)}{[1 - M(t) + M(t)K_{n-2}(0,0)]||H_{2n-2}^{(\lambda)}||^2}.$$

Thus, from (39), (40) and (42), we obtain

$$x^3H_{2n}'^{(\lambda)}(x,t) = A_n(x,t)H_{2n}^{(\lambda)}(x,t) + B_n(x,t)H_{2n-1}^{(\lambda)}(x,t),$$

with

$$\begin{aligned} A_n(x,t) &= \left(1 - \frac{1}{2n-2}x^2\right)(2n-1)N(t,n), \\ B_n(x,t) &= \frac{N(t,n)A(n,t)}{x} + \left(4n - \frac{2(2n-1)N(t,n)}{2n-2}\right)x^3 \\ &+ \left(\frac{(2n-1)(4n-3+2\lambda)}{2n-2} - 1\right)N(t,n)x. \end{aligned}$$

The behavior of the zeros of $H_{2m}^{(\lambda)}(x,t)$ can therefore be described as

$$\dot{x}_{2m,k}(t) = -r_m(t)\frac{x_{2m,k}(t)}{B_{2m}(x_{2m,k}(t))}.$$

4. OPUC: Hessenberg matrices and Carathéodory functions

Let σ be a nontrivial probability Borel measure supported on the unit circle $\mathbb{T} = \{z : |z| = 1\}$, and let us consider the inner product

$$\langle p(z), q(z)\rangle = \int_{\mathbb{T}} p(z)\overline{q(z)}d\sigma(z), \quad p, q \in \mathbb{P}, \tag{43}$$

where $\mathbb{P}$ is the linear space of polynomials with complex coefficients. Applying the Gram-Schmidt orthonormalization process to the basis $\{z^n\}_{n\geqslant 0}$, a (unique) sequence of polynomials $\{\varphi_n\}_{n\geqslant 0}$, with $deg\,\varphi_n = n$ and positive leading coefficient can be obtained, such that

$$\langle \varphi_n, \varphi_m\rangle_\sigma = \int_{\mathbb{T}} \varphi_n(z)\overline{\varphi_m(z)}d\sigma(z) = \delta_{m,n}. \tag{44}$$

$\{\varphi_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to σ. The moments associated with σ are defined by

$$c_n = \int_{\mathbb{T}} z^n d\sigma(x), \quad n \in \mathbb{Z},$$

so that the corresponding Gram matrix is

$$\mathbf{T} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}, \tag{45}$$

known in the literature as a Toeplitz matrix [38]. Notice that $\mathbf{T}$ is an Hermitian matrix. The polynomials can also be computed via the determinant expression

$$\varphi_n(z) = \frac{1}{\sqrt{\det \mathbf{T}_n \det \mathbf{T}_{n-1}}} \det \begin{pmatrix} c_0 & c_1 & \ldots & \ldots & c_n \\ c_{-1} & c_0 & \ldots & \ldots & c_{n-1} \\ \vdots & & \ddots & & \vdots \\ c_{-n+1} & c_{-n+2} & & c_0 & c_1 \\ 1 & z & \ldots & \ldots & z^n \end{pmatrix}, \quad n \geqslant 0,$$

where $\mathbf{T_n}$ denotes the $(n+1) \times (n+1)$ leading principal submatrix of $\mathbf{T}$ and with the convention $\det \mathbf{T_{-1}} = 1$. Denoting by κ_n the leading coefficient of $\varphi_n(z)$, $\Phi_n(z) = \varphi_n(z)/\kappa_n$ is the corresponding sequence of monic orthogonal polynomials. Notice that we have

$$\kappa_n^2 = \frac{\det \mathbf{T}_{n-1}}{\det \mathbf{T}_n}. \tag{46}$$

These polynomials satisfy the following forward and backward recurrence relations (see [2], [38], [39], [1])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \geqslant 0, \tag{47}$$

$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \quad n \geqslant 0, \tag{48}$$

where $\Phi_n^*(z) = z^n \bar{\Phi}_n(z^{-1})$ is the so-called reversed polynomial and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are known as Verblunsky, Szegő, Schur, or reflection parameters. It is important to notice that in the positive definite case we get $|\Phi_n(0)| < 1$ for every $n \geqslant 1$. Conversely, given a sequence of complex numbers $\{\alpha_n\}_{n\geqslant 1}$ with $|\alpha_n| < 1$, for every $n \geqslant 1$, there exists a unique probability measure supported on the unit circle such that the corresponding sequence of monic orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$ satisfies $\Phi_n(0) = \alpha_n$, for every $n \geqslant 1$. This is known as Verblunsky theorem (see [39]).

As in the real line case, we can define the orthogonality in terms of a linear functional, as follows. Let $\mathcal{L}$ be a linear functional in the linear space of Laurent polynomials ($\Lambda = span\{z^k\}_{k\in\mathbb{Z}}$) such that $\mathcal{L}$ is Hermitian, i.e.

$$c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \overline{c}_{-n}, \quad n \in \mathbb{Z}.$$

The complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ are said to be the moments associated with $\mathcal{L}$. Under this conditions, a bilinear functional can be defined in the linear space $\mathbb{P} = span\{z^k\}_{k\in\mathbb{N}}$ of polynomials with complex coefficients by

$$\langle p(z), q(z) \rangle_{\mathcal{L}} = \langle \mathcal{L}, p(z)\bar{q}(z^{-1}) \rangle, \quad p, q \in \mathbb{P}.$$

Thus, $\{\Phi_n\}_{n\geqslant 0}$ will be an orthogonal sequence with respect to $\mathcal{L}$ if $\langle \Phi_n(z), \Phi_m(z) \rangle_{\mathcal{L}} = \mathbf{k}_n \delta_{n,m}$, for $n, m \geqslant 0$, where $\mathbf{k}_n = \langle \Phi_n(z), \Phi_n(z) \rangle_{\mathcal{L}} \neq 0$. As before, $\mathcal{L}$ is said to be positive definite if all principal leading submatrices of the Toeplitz matrix $\mathbf{T}$ have positive determinant (in this case, there will be an integral representation in terms of a positive measure supported on the unit circle), and $\mathcal{L}$ is said to be quasi definite if the Toeplitz determinants are nonzero.

The multiplication operator with respect to $\{\varphi_n\}_{n\geqslant 0}$ is represented in a matrix form by

$$z\varphi(z) = \mathbf{H}_\varphi \varphi(z), \tag{49}$$

where $\varphi(z) = [\varphi_0(z), \varphi_1(z), \ldots, \varphi_n(z), \ldots]^t$ and $\mathbf{H}_\varphi$ is a lower Hessenberg matrix whose entries are

$$h_{n,j} = \begin{cases} \frac{\kappa_n}{\kappa_{n+1}} & \text{if} \quad j = n+1, \\ -\frac{\kappa_j}{\kappa_n} \Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if} \quad j \leqslant n, \\ 0 & \text{if} \quad j > n+1. \end{cases} \tag{50}$$

This is called the GGT (after Geronimus, Gragg, and Teplyaev) representation for the multiplication operator (see [39]).

Proposition 10. $\mathbf{H}_\varphi$ *satisfies*

(i) $\mathbf{H}_\varphi \mathbf{H}_\varphi^* = \mathbf{I}$,

(ii) $\mathbf{H}_\varphi^* \mathbf{H}_\varphi = \mathbf{I} - \lambda_\infty(0)\varphi(0)\varphi(0)^*$,

where $\mathbf{I}$ *is the semi-infinite identity matrix and* $\lambda_\infty(0) = \left(\sum_{n=1}^{\infty} |\Phi_n(0)|^2\right)^{-1}$.

Remark 11. *Proposition 10 states that the infinite matrix* $\mathbf{H}_\varphi$ *is unitary if and only if* $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 = +\infty$*. In terms of the measure* σ *this fact is equivalent to* $\log \sigma' \notin L^1\left(\frac{d\theta}{2\pi}\right)$*. In other words,* σ *does not belong to the Szegő class (see [39]).*

Remark 12. *From* (49)*, is not difficult to show that* $\mathbf{H}_\Phi$*, the lower Hessenberg matrix associated with* $\{\Phi_n\}_{n\geqslant 0}$*, has as entries*

$$h_{n,j} = \begin{cases} 1 & \textit{if} \quad j = n+1, \\ -\frac{\mathbf{k}_n}{\mathbf{k}_j} \Phi_{n+1}(0)\overline{\Phi_j(0)} & \textit{if} \quad j \leqslant n, \\ 0 & \textit{if} \quad j > n+1, \end{cases} \tag{51}$$

where

$$\mathbf{k}_n = \frac{1}{\kappa_n^2}.$$

The conditions stated in Remark 11 for the unitary character of the GGT matrix yield a substantial constraint. There exists a simpler matrix representation for the multiplication operator, which is based on the use of an orthonormal basis for the Laurent polynomials. It is called the CMV representation (after Cantero, Moral, and Velázquez) and was introduced independently in [40] and [41]. Namely, let $\Lambda_{(k,l)}$ be the subspace of Laurent polynomials spanned by $\{z^j\}_{j=k}^{l}$ (with $k \leq l$), and $\Pi_{(k,l)}$ the orthogonal projection onto $\Lambda_{(k,l)}$ in $L^2(\mathbb{T}, d\sigma)$. Define

$$\Lambda^{(n)} = \begin{cases} \Lambda_{(-k,k)}, & n = 2k, \\ \Lambda_{(-k,k+1)}, & n = 2k+1, \end{cases} \tag{52}$$

and let $\Pi^{(n)}$ be the orthogonal projection onto $\Lambda^{(n)}$. Now, define $\chi_n^{(0)}$ by

$$\chi_n^{(0)} = \begin{cases} z^{-k}, & n = 2k, \\ z^{k+1}, & n = 2k+1. \end{cases} \tag{53}$$

Applying the Gram-Schmidt process to the basis $\left\{\chi_n^{(0)}\right\}_{n\geqslant 0}$, we obtain the CMV basis

$$\chi_n = \frac{(1-\Pi^{(n-1)})\chi_n^{(0)}}{\|(1-\Pi^{(n-1)})\chi_n^{(0)}\|}. \tag{54}$$

Therefore, the CMV matrix representation of the multiplication operator, $C(d\sigma)$, is defined by

$$C_{i,j}(d\sigma) = \left\langle z\chi_j, \chi_i \right\rangle. \tag{55}$$

This basis turns out to satisfy a five term recurrence relation and, as a consequence, we get

$$C = \begin{pmatrix} -\Phi_1(0) & -\Phi_2(0)\rho_1 & \rho_2\rho_1 & 0 & 0 & \cdots \\ \rho_1 & -\Phi_2(0)\overline{\Phi_1(0)} & \overline{\Phi_1(0)}\rho_2 & 0 & 0 & \cdots \\ 0 & -\Phi_3(0)\rho_2 & -\Phi_3(0)\overline{\Phi_2(0)} & -\Phi_4(0)\rho_3 & \rho_4\rho_3 & \cdots \\ 0 & \rho_3\rho_2 & \overline{\Phi_2(0)}\rho_3 & -\Phi_4(0)\overline{\Phi_3(0)} & \overline{\Phi_3(0)}\rho_4 & \cdots \\ 0 & 0 & 0 & -\Phi_5(0)\rho_4 & -\Phi_5(0)\overline{\Phi_4(0)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{56}$$

where $\rho_l = (1 - |\Phi_l(0)|^2)^{1/2}$, $l \geq 1$. Notice that we have a five diagonal matrix, and all entries can be expressed in terms of the Verblunsky coeficients. The following is a useful factorization of C involving block diagonal matrices.

Theorem 13. *[39] Let*

$$\Theta_j = \begin{pmatrix} -\Phi_{j+1}(0) & \rho_{j+1} \\ \rho_{j+1} & \overline{\Phi_{j+1}(0)} \end{pmatrix}. \tag{57}$$

Then, we have $C = \mathbb{L}\mathbb{M}$, with

$$\mathbb{M} = \begin{pmatrix} 1 & & & \\ & \Theta_1 & & \\ & & \Theta_3 & \\ & & & \ddots \end{pmatrix}, \qquad \mathbb{L} = \begin{pmatrix} \Theta_0 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix}. \tag{58}$$

We point out that C is always a unitary matrix, and this represents an advantage with respect to the GGT matrix, since it is unitary independently of the measure.

On the other hand, in terms of the moments, we can introduce an analytic function in a neighborhood of the origin, associated with the linear functional as follows

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k. \tag{59}$$

If $\mathcal{L}$ is a positive definite functional, then (59) is analytic in the open unit disc and its real part is positive therein. In such a case, (59) is called Carathéodory function and it can be represented by the Riesz-Herglotz transform

$$F(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\sigma(w),$$

where σ is the positive measure associated with $\mathcal{L}$. On the other hand, the sequence of polynomials of the second kind $\{\Omega_n\}_{n\geq 0}$ is defined by

$$\Omega_n(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} [\Phi_n(w) - \Phi_n(z)] d\sigma(w).$$

Let us notice that Ω_n is a monic polynomial of degree n, assuming σ is a nontrivial probability measure, and $\Omega_n(0) = -\Phi_n(0)$. Polynomials of the second kind play a central role in rational approximation. Indeed, we have (see [39])

$$\begin{aligned} \Phi_n(z)F(z) + \Omega_n(z) &= 2\|\Phi_n\|^2 z^n + O(z^{n+1}) \\ \Phi_n^*(z)F(z) - \Omega_n^*(z) &= O(z^{n+1}), \end{aligned}$$

where the last expression leads to

$$\lim_{n\to\infty}\frac{\Omega_n^*(z)}{\Phi_n^*(z)} = F(z), \quad z \in \mathbb{D}.$$

By extension, for a quasi-definite linear functional, we will call (59) its corresponding Carathéodory function.

4.1. Spectral transformations

Transformations $(i)-(iii)$ defined in the previous Section for measures supported on the real line can also be defined for measures supported on the unit circle ([42, 43, 44]), as follows

(i) Christoffel transformation ([45, 46, 47])

$$d\sigma_C = |z-\alpha|^2 d\sigma, \quad \alpha \in \mathbb{C}, |z| = 1.$$

(ii) Uvarov transformation with one mass point ([43])

$$d\sigma_U = d\sigma + M_c\delta(z-\alpha), \quad |\alpha| = 1, M_c \in \mathbb{R}_+.$$

(iii) Uvarov transformation with two mass points which are symmetric with respect to the unit circle ([44])

$$d\sigma_U = d\sigma + M_c\delta(z-\alpha) + \bar{M}_c\delta(z-\bar{\alpha}^{-1}), \quad |\alpha| \in \mathbb{R}_+ \setminus \{0,1\}, M_c \in \mathbb{C}.$$

(iv) Geronimus transformation ([48, 49])

$$d\sigma_G = \frac{1}{|z-\alpha|^2}d\sigma + M_c\delta(z-\alpha) + \bar{M}_c\delta(z-\bar{\alpha}^{-1}), \quad |\alpha| > 1, M_c \in \mathbb{C}.$$

We will denote these transformations by $\mathcal{F}_C(\alpha)$, $\mathcal{F}_U(\alpha, M_c)$, and $\mathcal{F}_G(\alpha, M_c)$, respectively. Some other perturbations for measures have been considered in [50] and [51]. As in the case of the real line, these transformations are related by

$$\begin{aligned} \mathcal{F}_C(\alpha)\circ\mathcal{F}_G(\alpha, M_c) &= \mathcal{I} \quad \text{(Identity transformation)}, \\ \mathcal{F}_G(\alpha, M_c)\circ\mathcal{F}_C(\alpha) &= \mathcal{F}_U(\alpha, M_c). \end{aligned}$$

Furthermore, these transformations can be expressed in terms of the Carathéodory functions as

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)}, \tag{60}$$

where $\widetilde{F}(z)$ is the Carathéodory function associated with the perturbed measure and $A(z), B(z), D(z)$ are polynomials that are explicitly given (see [44]) for the above transformations. Indeed,

Proposition 14. *[44] Let $F(z)$ be the Carathéodory function associated with σ, which is assumed to be normalized by $c_0 = 1$. Denote by $F_C(x)$, $F_U(x)$, and $F_G(x)$ the Carathéodory functions associated with $\mathcal{F}_C(\alpha)$, $\mathcal{F}_U(\alpha, M_c)$, and $\mathcal{F}_G(\alpha, M_c)$, respectively. Then, we have*

$$\begin{aligned} F_C(x) &= \frac{(-\bar{\alpha}z^2 + (1+|\alpha|^2)z - \alpha)F(z) - \bar{\alpha}z^2 + (\alpha c_{-1} - \bar{\alpha}c_1)z + \alpha}{z}, && (i) \\ F_U(x) &= F(z) + \frac{(\alpha - \bar{\alpha}z^2)(M_c + \bar{M}_c) + (1-|\alpha|^2)(M_c - \bar{M}_c)z}{(z-\alpha)(\bar{\alpha}z - 1)}, && (ii) \\ F_G(x) &= \frac{zF(z) + \alpha\tilde{c}_0 z^2 - 2i\mathfrak{Im}(q_0)z - \alpha\tilde{c}_0}{-\bar{\alpha}z^2 + (1+|\alpha|^2)z - \alpha}, && (iii) \end{aligned}$$

where q_0 is a free parameter that can be chosen in such a way the perturbed measure is positive.

Proof. Notice that the Christoffel transformation can be expressed, in terms of the inner product, as

$$\langle p, q \rangle_{\sigma_C} = \langle (z-\alpha)p, (z-\alpha)q \rangle_{\sigma},$$

and thus the perturbed moments are

$$\tilde{c}_{-k} = \left\langle 1, z^k \right\rangle_{\sigma_C} = \left\langle (z-\alpha), (z-\alpha)z^k \right\rangle_{\sigma} = (1+|\alpha|^2)c_{-k} - \alpha c_{-(k+1)} - \bar{\alpha}c_{-(k-1)}.$$

Thus,

$$\begin{aligned} F_C(z) &= \tilde{c}_0 + 2\sum_{k=1}^{\infty} \tilde{c}_{-k} z^k \\ &= (1+|\alpha|^2)F(z) - \bar{\alpha}\left(c_1 + 2\sum_{k=0}^{\infty} c_{-k} z^{k+1}.\right) - \alpha\left(c_{-1} + 2\sum_{k=2}^{\infty} c_{-k} z^{k-1}\right), \end{aligned}$$

which can be arranged as

$$F_C(z) = \frac{(-\bar{\alpha}z^2 + (1+|\alpha|^2)z - \alpha)F(z) + -\bar{\alpha}z^2 + (\alpha c_{-1} - \bar{\alpha}c_1)z + \alpha}{z},$$

which is the first equation. The other statements follow in a similar way. ■

If the perturbed Carathéodory function has the form

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{C(z)F(z) + D(z)}, \tag{61}$$

where $AD - BC \neq 0$, then the spectral transformation is said to be rational. The most notable examples are

- Aleksandrov transformation: Define $\{\Phi_n^{\lambda}(0)\}_{n\geqslant 1}$, where $\Phi_n^{\lambda}(0) = \lambda\Phi_n(0)$, with $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then,

  $$\Phi_{n+1}^{\lambda}(z) = z\Phi_n^{\lambda}(z) + \Phi_{n+1}^{\lambda}(0)(\Phi_n^{\lambda})^*(z),$$

  are called Aleksandrov polynomials. In such a case, we have (see [39])

  $$\widetilde{F}(z) = \frac{(1+\lambda)F(z) + (1-\lambda)}{(1-\lambda)F(z) + (1+\lambda)}.$$

  If $\lambda = -1$, we obtain the polynomials of the second kind defined above, and we get $\widetilde{F}(z) = 1/F(z)$.

- Associated polynomials of order N: Denote by $\{\Phi_n^{(N)}\}_{n\geqslant 0}$ the associated polynomials of order N, defined by

  $$\Phi_{n+1}^{(N)}(z) = z\Phi_n^{(N)}(z) + \Phi_{n+N+1}(0)(\Phi_n^{(N)})^*(z), \quad n \geqslant 0,$$

  i.e. the first N Verblunsky coefficients are removed. Then, we have (see [6])

  $$\widetilde{F}(z) = \frac{(\Phi_N(z) + \Phi_N^*(z))F(z) + \Omega_N(z) - \Omega_N^*(z)}{(\Phi_N(z) - \Phi_N^*(z))F(z) + \Omega_N(z) + \Omega_N^*(z)},$$

  where $\Omega_N(z)$ is the $N-th$ degree second kind polynomial.

- Anti-associated polynomials of order N: Let $\nu_1, \nu_2, \ldots, \nu_N \in \mathbb{C}$ with $|\nu_j| < 1$, $1 \leqslant j \leqslant N$. Define $\{\hat{\Phi}_n(0)\}_{n\geqslant 1} = \{\nu_j\}_{j=1}^N \bigcup \{\Phi_j(0)\}_{j=1}^{\infty}$. Then, the polynomials

  $$\Phi_{n+1}^{(-N)}(z) = z\Phi_n^{(-N)}(z) + \hat{\Phi}_{n+1}(0)(\Phi_n^{(-N)})^*(z), \quad n \geqslant 0,$$

are called anti-associated polynomials of order N, and [6]

$$\widetilde{F}(z) = \frac{(\Omega_N^{(-N)}(z) + (\Omega_N^{(-N)})^*(z))F(z) + (\Omega_N^{(-N)})^*(z) - \Omega_N^{(-N)}(z)}{((\Phi_N^{(-N)})^*(z) - \Phi_N^{(-N)}(z))F(z) + \Phi_N^{(-N)}(z) + (\Phi_N^{(-N)})^*(z)}.$$

The above linear spectral transformations can also be expressed in terms of the corresponding Hessenberg matrices, by means of a QR factorization. Next, we describe some results in this direction.

We begin with the Christoffel transformation. Let us denote by $\{\psi_n\}_{n\geqslant 0}$ the sequence of orthonormal polynomials with respect to σ_C. The connection formula between both families of polynomials is (see [45])

$$(z-\alpha)\psi_n(z) = \sqrt{\frac{K_n(\alpha,\alpha)}{K_{n+1}(\alpha,\alpha)}}\varphi_{n+1}(z) - \sum_{j=0}^{n} \frac{\varphi_{n+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{n+1}(\alpha,\alpha)K_n(\alpha,\alpha)}}\varphi_j(z), \tag{62}$$

where $K_n(z,y) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(y)}$. In the sequel,

$$K_n^{(i,j)}(z,y) = \sum_{k=0}^{n} \varphi_k^{(i)}(z)\overline{\varphi_k^{(j)}(y)},$$

i.e., the $i-th$ ($resp.\ j-th$) partial derivative of $K_n(z,y)$ with respect to the variable z ($resp.\ y$) is taken.

On the other hand, If

$$\varphi(z) = [\varphi_0(z),\ \ldots,\varphi_n(z),\ldots]^T \quad \text{and} \quad \psi(z) = [\psi_0(z),\ \ldots,\psi_n(z),\ldots]^T,$$

then the matrix representation of (62) is

$$(z-\alpha)\psi(z) = \mathbf{M}_C\varphi(z), \tag{63}$$

where $\mathbf{M}_C$ is a lower Hessenberg matrix with entries

$$m_{i,j} = \begin{cases} -\frac{\varphi_{i+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{i+1}(\alpha,\alpha)K_i(\alpha,\alpha)}}, & \text{if} \quad j \leqslant i, \\ \sqrt{\frac{K_i(\alpha,\alpha)}{K_{i+1}(\alpha,\alpha)}}, & \text{if} \quad j = i+1, \\ 0, & \text{if} \quad j > i+1. \end{cases} \tag{64}$$

Notice that $(\mathbf{M}_C)_n$, the $n\times n$ principal leading submatrix of $\mathbf{M}_C$, is a quasi-unitary matrix, i.e. its first $n-1$ rows constitute an orthonormal set, and the last row is orthogonal with respect to this set, but it is not normalized in the sense that its norm is not 1.

Furthermore, if we denote by $\mathbf{L}_{\varphi\psi}$ the lower triangular matrix such that $\varphi(z) = \mathbf{L}_{\varphi\psi}\psi(z)$, then $\mathbf{L}_{\varphi\psi}$ can be computed from $\mathbf{H}_{\varphi}$ and $\mathbf{M}_{C}$ as follows

Proposition 15. *[42],[47]*

$$\mathbf{L}_{\varphi\psi} = (\mathbf{H}_{\varphi} - \alpha\mathbf{I})\mathbf{M}_{C}^{*}. \tag{65}$$

Now it is possible to determine the relation between $\mathbf{H}_{\psi}$, the Hessenberg matrix associated with σ_C, and $\mathbf{H}_{\varphi}$.

Proposition 16. *[42],[47]*

$$\mathbf{H}_{\varphi} - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_{C}, \tag{66}$$

$$\mathbf{H}_{\psi} - \alpha\mathbf{I} = \mathbf{M}_{C}\mathbf{L}_{\varphi\psi}. \tag{67}$$

Notice that an "almost" *QR* factorization appears, since $\mathbf{M}_C$ is not a unitary matrix (but it is very close). The iteration of the canonical Christoffel transformation has been analyzed in [45], [46], and [43].

The results for the Christoffel transformation can be used to obtain a similar result for the Uvarov transformation. If we assume that there exists a sequence of polynomials $\{\upsilon_n\}_{n\geqslant 0}$ orthonormal with respect to σ_U as defined in (*ii*), then the relation between the Hessenberg matrices $\mathbf{H}_{\varphi}$ and $\mathbf{H}_{\upsilon}$ associated with σ and σ_U, respectively, is (see [42], [44])

Proposition 17.

$$\mathbf{H}_{\varphi} - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_{C}, \tag{68}$$

$$\mathbf{H}_{\upsilon} - \alpha\mathbf{I} = \mathbf{L}_{U}\mathbf{M}_{U}, \tag{69}$$

where $\mathbf{L}_U = \mathbf{L}_{\upsilon\varphi}\mathbf{L}_{\varphi\psi}$, $\mathbf{M}_U = \mathbf{M}_C\mathbf{L}_{\upsilon\varphi}^{-1}$, *and* $\mathbf{L}$ *are the matrices of change of bases for the orthonormal polynomial families denoted by their subindices.*

The iteration of the canonical Uvarov transformation, with $|\alpha| = 1$, has been studied in [2] and [43].

Finally, let us consider the Geronimus transformation defined in (*iii*). Necessary and sufficient conditions for the existence of a sequence of monic polynomials orthogonal with respect to σ_G were studied in [52], as well as the relation between the corresponding families of monic orthogonal polynomials and their associated Hessenberg matrices. If we denote by $\{G_n\}_{n\geqslant 0}$ the sequence of monic polynomials orthogonal with respect to σ_G and by $\mathbf{M}_G$ the corresponding Hessenberg matrix, from the relation between $\{G_n\}_{n\geqslant 0}$ and $\{\Phi_n\}_{n\geqslant 0}$, then we get

Proposition 18. *[52] Let $\mathbf{L}_G$ be the lower triangular matrix with 1 on the diagonal entries such that $G(z) = \mathbf{L}_G\Phi(z)$ and denote by $\mathbf{H}_G$ the Hessenberg matrix associated with $\{G_n\}_{n\geqslant 0}$. Then,*

$$\mathbf{H}_\Phi - \alpha\mathbf{I} = \mathbf{M}_G\mathbf{L}_G \tag{70}$$

and

$$\mathbf{H}_G - \alpha\mathbf{I} = \mathbf{L}_G\mathbf{M}_G. \tag{71}$$

Remark 19. *Notice that the GGT matrix is associated with the representation of the multiplication operator in terms of the orthogonal polynomials on the unit circle. Thus, we have discussed above GGT matrices corresponding to linear spectral transformations. The corresponding analysis for such a kind of perturbations for CMV matrices (i.e when the multiplication operator is associated with the Laurent polynomials basis) has been recently addressed in [53].*

A perturbation of the sequence of moments associated with an Hermitian linear functional on the space of Laurent polynomials was analyzed in [54], where the authors obtain connection formulas between the corresponding families of orthogonal polynomials and also deduce existence conditions.

Finally, let us consider an example of a perturbation to the Verblunsky parameters, which was studied in [9]. Given the sequence $\{\Phi_n(0)\}_{n\geqslant 1}$ and an arbitrary positive integer k, a new sequence of Verblunsky parameters $\{\beta_n\}_{n\geqslant 1}$ is defined by $\beta_n = \Phi_n(0)$, $n \neq k$, and β_k, with $|\beta_k| < 1$, an arbitrary complex number. From the Verblunsky theorem, we can construct a family of monic orthogonal polynomials $\{\tilde{\Phi}_n\}_{n\geqslant 0}$ using the recurrence relation (47) with $\tilde{\Phi}_n(0) := \beta_n$, for every $n \geqslant 1$. The connection formula between both families of orthogonal polynomials is shown in the following result.

Proposition 20. *[9] For $n > k$, we have*

$$2z^{k+2}\|\Phi_k\|^2 \begin{pmatrix} -\tilde{\Omega}_{n+1}(z) \\ \tilde{\Phi}_{n+1}(z) \end{pmatrix} = \mathbf{C}(z) \begin{pmatrix} -\Omega_{n+1}(z) \\ \Phi_{n+1}(z) \end{pmatrix},$$

where $\tilde{\Omega}_{n+1}$ is the polynomial of the second kind associated with Φ_{n+1}, and

$$\mathbf{C}(z) = \begin{pmatrix} r(z,k)\Omega_k^*(z) + zr^*(z,k)\Omega_k(z) & s(z,k)\Omega_k^*(z) - zs^*(z,k)\Omega_k(z) \\ r(z,k)\Phi_k^*(z) - zr^*(z,k)\Phi_k(z) & zs(z,k)\Phi_k^*(z) + zs^*(z,k)\Phi_k(z) \end{pmatrix},$$

with

$$\begin{aligned} r(z,k) &= (1 - \bar{\beta}_k\Phi_k(0))z\Phi_k(z) - \overline{(\Phi_k(0) - \beta_k)}\Phi_k^*(z), \\ s(z,k) &= (1 - \bar{\beta}_k\Omega_k(0))z\Phi_k(z) + \overline{(\Phi_k(0) - \beta_k)}\Omega_k^*(z). \end{aligned}$$

The result can be easily extended for a finite composition of such perturbations. Furthermore, the author shows that the corresponding Carathéodory functions are related through a linear spectral transformation.

4.2. Linear spectral transformations and the Szegő transformation

If μ is a nontrivial probability measure supported on $[-1, 1]$, then it is very well known ([1]) that it induces a nontrivial positive measure σ supported on the unit circle in such a way that if $d\mu(x) = \omega(x)dx$, then

$$d\sigma(\theta) = \frac{1}{2}\omega(\cos\theta)|\sin\theta|d\theta. \tag{72}$$

This process is called the Szegő transformation. On the other hand, if σ is induced through the Szegő transformation, then their corresponding orthogonal polynomials Φ_n have real coefficients, and the Verblunsky coefficients are also real. Furthermore, if $x = \frac{(z+z^{-1})}{2}$, then

$$p_n(x) = \frac{\kappa_{2n}(\sigma)}{2(\sqrt{1 + \Phi_{2n}(0)})}[z^{-n}\Phi_{2n}(z) + z^n\Phi_{2n}(z^{-1})],$$

and

$$\begin{aligned} 2a_n &= \sqrt{[1 - \Phi_{2n}(0)][1 - \Phi_{2n-1}^2(0)][1 + \Phi_{2n-2}(0)]}, \quad n \geqslant 1, &(73) \\ 2b_n &= \Phi_{2n-1}(0)[1 - \Phi_{2n}(0)] - \Phi_{2n+1}(0)[1 + \Phi_{2n}(0)], \quad n \geqslant 0. &(74) \end{aligned}$$

Suppose that a canonical linear spectral transformation is applied to a measure μ, and then the Szegő transformation is applied to the perturbed measure. The problem of determining what kind of transformation is obtained for the corresponding measure on the unit circle was considered in [48], where the authors use the well known relation between the corresponding Stieltjes and Carathéodory functions (see [6])

$$F(z) = \frac{1-z^2}{2z} S(x), \quad z = x + \sqrt{x^2-1}.$$

It turns out that a linear spectral transformation of the same kind appears. More precisely,

Theorem 21. *[48] Let μ be a nontrivial probability measure supported on $[-1,1]$, and let σ be the measure obtained in (72) via the Szegő transformation.*

1. *If a Christoffel transformation with parameter β is applied to μ, we get a Christoffel transformation of σ with parameters $\alpha_\pm = \beta \pm \sqrt{\beta^2-1}$, $|\beta| > 1$.*
2. *If an Uvarov transformation with parameters β and M_r is applied to μ, then an Uvarov transformation of σ with parameters $\alpha_\pm = \beta \pm \sqrt{\beta^2-1}$, $|\beta| > 1$ and $M_c = M_r/2$ is obtained.*
3. *If a Geronimus transformation with parameters β and M_r is applied to μ, then a Geronimus transformation of σ with parameters $\alpha_\pm = \beta \pm \sqrt{\beta^2-1}$, $|\beta| > 1$ and $M_c = \frac{S(\beta)+M_r}{\alpha}$ is obtained.*

On the other hand, in [48], the authors show an algorithm to obtain the Verblunsky coefficients associated with σ by using the LU factorization of the Jacobi matrix associated with μ. If we define the sequence of real numbers $\{u_k\}_{k\geqslant 1}$, with

$$u_k = \frac{1}{2}[1-\Phi_k(0)][1+\Phi_{k-1}(0)], \tag{75}$$

then, taking into account (73) and (74), we have

$$a_k^2 = u_{2k}u_{2k-1}, \quad k \geqslant 1, \tag{76}$$

$$b_k + 1 = u_{2k} + u_{2k+1}, \quad k \geqslant 0, \tag{77}$$

with $u_0 := 0$. This means that we have the (unique) LU factorization $\mathbf{J} + \mathbf{I} = LU$, with

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ u_2 & 1 & 0 & 0 & \cdots \\ 0 & u_4 & 1 & 0 & \ddots \\ 0 & 0 & u_6 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_3 & 1 & 0 & \cdots \\ 0 & 0 & u_5 & 1 & \ddots \\ 0 & 0 & 0 & u_7 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{78}$$

Notice that, from (75), the Verblunsky coefficients can be computed recursively by

$$\Phi_k(0) = 1 + \frac{2u_k}{1 + \Phi_{k-1}(0)}, \quad n \geqslant 1. \tag{79}$$

As a consequence, as established in Theorem 21, a Christoffel transformation of a measure μ on $[-1, 1]$ becomes a Christoffel transformation of the measure σ on the unit circle, under the Szegő transformation. We can use Proposition 3 to obtain an algorithm to compute the Verblunsky coefficients of the perturbed measure on the unit circle. We argue as follows. If $\mathbf{J}$ is the Jacobi matrix associated with μ, apply the LU factorization to the matrix $\mathbf{J}-\beta\mathbf{I}$, where β is the parameter of the Christoffel transformation, to obtain matrices $\hat{\mathbf{L}}$ and $\hat{\mathbf{U}}$ as in (78) with elements $\{\hat{u}_k\}_{n\geqslant 1}$ given by

$$\begin{aligned} a_k^2 &= \hat{u}_{2k}\hat{u}_{2k-1}, \quad k \geqslant 1, \\ b_k - \beta &= \hat{u}_{2k} + \hat{u}_{2k+1}, \quad k \geqslant 0, \end{aligned}$$

with $\hat{u}_o := 0$. That is, the sequence $\{\hat{u}_k\}_{n\geqslant 1}$ can be recursively computed from the *known* sequences $\{a_k^2\}_{n\geqslant 1}$ and $\{b_k\}_{n\geqslant 0}$. Next, we define the matrix $\tilde{\mathbf{J}} := \hat{\mathbf{U}}\hat{\mathbf{L}} + \beta\mathbf{I}$. This is,

$$\tilde{\mathbf{J}} = \begin{pmatrix} \hat{u}_1 + \hat{u}_2 + \beta & 1 & 0 & 0 & \cdots \\ \hat{u}_2\hat{u}_3 & \hat{u}_3 + \hat{u}_4 + \beta & 1 & 0 & \cdots \\ 0 & \hat{u}_4\hat{u}_5 & \hat{u}_5 + \hat{u}_6 + \beta & 1 & \ddots \\ 0 & 0 & \hat{u}_6\hat{u}_7 & \hat{u}_7 + \hat{u}_8 + \beta & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since, by Proposition 3, $\tilde{\mathbf{J}}$ is the Jacobi matrix associated with the Christoffel transformation of μ, if we denote by $\{\tilde{a}_k^2\}_{n\geqslant 1}$ and $\{\tilde{b}_k\}_{n\geqslant 0}$ the sequences of coefficients of the recurrence relation of such a perturbed measure, we have

$$\tilde{a}_k^2 = \hat{u}_{2k}\hat{u}_{2k+1}, \quad k \geqslant 1,$$
$$\tilde{b}_k = \hat{u}_{2k+1} + \hat{u}_{2k+2} + \beta, \quad k \geqslant 0,$$

and, using again (76) and (77), we can define a sequence $\{\tilde{u}_k\}_{n\geqslant 1}$ by

$$\tilde{a}_k^2 = \tilde{u}_{2k}\tilde{u}_{2k-1}, \quad k \geqslant 1,$$
$$\tilde{b}_k + 1 = \tilde{u}_{2k} + \tilde{u}_{2k+1}, \quad k \geqslant 0,$$

with $\tilde{u}_0 := 0$. Finally, this sequence can be used to compute the Verblunsky coefficients of the perturbed measure on the unit circle using (79). This can be summarized as follows.

Proposition 22. *Let μ be a positive Borel measure supported in $[-1, 1]$ and let $\{a_k^2\}_{n\geqslant 1}$ and $\{b_k\}_{n\geqslant 1}$ be the corresponding sequence of coefficients of the three term recurrence relation. Denote by σ the measure in the unit circle, defined by the Szegő transformation, and let $\{\Phi_k(0)\}_{k\geqslant 1}$ be the sequence of Verblunsky coefficients. Let $d\tilde{\mu} = (x-\beta)d\mu$ be the Christoffel transformation of μ. Then, the sequence $\{\tilde{\Phi}_k(0)\}_{k\geqslant 1}$ of Verblunsky coefficients associated with the Christoffel transformation of σ can be computed as follows.*

Data: *$\{a_k^2\}_{n\geqslant 1}$, $\{b_k\}_{n\geqslant 1}$, β.*

1. *Compute the sequence $\{\hat{u}_k\}_{n\geqslant 1}$ by*

$$\hat{u}_{2k+1} = b_k - \beta - \hat{u}_{2k}, \quad k \geqslant 0, \quad \hat{u}_{2k} = \frac{a_k^2}{\hat{u}_{2k-1}}, \quad k \geqslant 1,$$

   with $\hat{u}_0 := 0$

2. *Compute the sequences $\{\tilde{a}_k^2\}_{n\geqslant 1}$ and $\{\tilde{b}_k\}_{n\geqslant 0}$ as*

$$\tilde{a}_k^2 = \hat{u}_{2k}\hat{u}_{2k+1}, \quad k \geqslant 1,$$
$$\tilde{b}_k = \hat{u}_{2k+1} + \hat{u}_{2k+2} + \beta, \quad k \geqslant 0.$$

   They are the coefficients of the recurrence relation associated with $\tilde{\mu}$.

3. *Compute the sequence* $\{\tilde{u}_k\}_{n\geqslant 1}$, *by*

$$\tilde{u}_{2k+1} = \tilde{b}_k + 1 - \tilde{u}_{2k}, \quad k \geqslant 0, \quad \tilde{u}_{2k} = \frac{\tilde{a}_k^2}{\tilde{u}_{2k-1}}, \quad k \geqslant 1,$$

with $\tilde{u}_0 := 0$.

4. *Compute the Verblunsky coefficients* $\{\tilde{\Phi}_k(0)\}_{k\geqslant 1}$ *by*

$$\tilde{\Phi}_k(0) = 1 + \frac{2\tilde{u}_k}{1 + \tilde{\Phi}_{k-1}(0)}, \quad n \geqslant 1.$$

The connection via the Szegő transformation of the perturbations on the moments considered above was analyzed in [55]. Therein, the authors show that when the j-th moment is additively perturbed in the Hankel matrix, the entries of the corresponding Toeplitz matrix are additively perturbed as follows.

Proposition 23. *[55] Let* μ *be a nontrivial positive Borel measure supported in* $[-1,1]$ *and let* σ *be its associated measure supported in the unit circle, defined through the Szegő transformation. Let* $\{\mu_n\}_{n\geq 0}$ *and* $\{c_n\}_{n\in\mathbb{Z}}$ *be their corresponding sequences of moments. If we consider a new measure* $\tilde{\mu}$ *such that its moments* $\{\tilde{\mu}_n\}_{n\geq 0}$ *are defined by* $\tilde{\mu}_j = \mu_j + m_j$, *and* $\tilde{\mu}_k = \mu_k$ *for* $k \neq j$, *then the corresponding perturbed measure in the unit circle* $\tilde{\sigma}$ *is a linear spectral transformation of* σ, *and its moments are given by*

$$\tilde{c}_{-n} = c_{-n} + i^{n-j} 2^{j-1} m_j \left(\binom{(n+j)/2}{j} + \binom{(n+j-2)/2}{j} \right), \quad n = j + 2k, \quad k \geq 0,$$

with the convention $\binom{j-1}{j} := 0$. *The remaining moments remain unchanged. Furthermore, the perturbation affects only the singular part of* σ.

Conversely, if an additive perturbation in the entries of a Toeplitz matrix associated with σ is considered, then the entries of the Hankel matrix associated with the perturbation of μ by using the inverse Szegő transformation are as follows.

Proposition 24. *[55] Let* σ *be a positive nontrivial Borel measure supported in the unit circle with real moments, and let* μ *be its corresponding measure in* $[-1,1]$, *obtained through the inverse Szegő transformation. Let* $\{c_n\}_{n\in\mathbb{Z}}$ *and* $\{\mu_n\}_{n\geq 0}$ *be their corresponding sequences of moments. Assume that the measure* $\tilde{\sigma}$, *defined by* (**??**) *with*

$m \in \mathbb{R}$, is positive. Then, the measure $\tilde{\mu}$, obtained by applying the inverse Szegő transformation to $\tilde{\sigma}$, is given by

$$d\tilde{\mu} = d\mu + 2m\frac{T_j(x)}{\pi}\frac{dx}{\sqrt{1-x^2}},$$

and its corresponding sequence of moments is given by

$$\tilde{\mu}_n = \begin{cases} \mu_n + mB(n,j), & \text{if} \quad n \geqslant j \text{ and } n+j \text{ is even}, \\ \\ \mu_n, & \text{otherwise}, \end{cases} \tag{80}$$

where

$$B(n,j) = j\sum_{k=0}^{[j/2]}\left(\frac{(-1)^k(j-k-1)!(2)^{j-2k}}{k!(j-2k)!}\prod_{i=1}^{(j+n-2k)/2}\frac{j+n-2k-(2i-1)}{j+n-2k-2(i-1)}\right).$$

Here $T_j(x)$ denotes the j-th Chebyshev polynomial of the first kind and degree j.

The analysis through the Szegő transformation of the co-polynomials described at the end of Section 2 is included in [8]. Namely, the authors describe the perturbation obtained on the Verblunsky coefficients associated with the corresponding orthogonal polynomials on the unit circle. As expected, an additive perturbation is introduced on the Verblunsky coefficients $\Phi_n(0)$, for $n > k$. Such a perturbation can be expressed in terms of the evaluations at $x = \pm 1$ of the orthogonal polynomials on the real line (see [8]).

5. Integrable systems on the unit circle

Although much of the research devoted to the connection of orthogonal polynomials and integrable systems has been restricted to measures supported on the real line, there are some contributions in which integrable systems associated with orthogonal polynomials on the unit circle are considered. In particular, the following case is analyzed in [56]. Consider the orthogonality measure

$$d\sigma(z,t) = \frac{1}{2\pi I_0(t)}e^{\frac{1}{2}t[z+z^{-1}]}\frac{dz}{iz}, \quad t > 0,$$

where I_ν is the modified Bessel function of νth kind defined, as usual, by $I_\nu(t) = i^{-\nu}J_\nu(it)$, where $J_\nu(t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}(\frac{t}{2})^{2k+\nu}$ is the Bessel function of νth kind, with ν

a real number. The corresponding orthonormal polynomials are called modified Bessel orthogonal polynomials, Then, using (46), the leading coefficient of $\varphi_n(z,t)$, which now is time dependent, can be expressed as

$$\kappa_n^2(t) = I_0(t)\frac{\det(I_{j-k}(t))_{0\leqslant j,k\leqslant n-1}}{\det(I_{j-k}(t))_{0\leqslant j,k\leqslant n}}.$$

On the other hand, from the previous expression and the recurrence relations, the relation between the κ_n and $\varphi_n(0,t)$ can be obtained. In particular, if we define the "reflection" coefficients

$$r_n(t) = \frac{\varphi_n(0,t)}{\kappa_n(t)}, \quad n \geqslant 1,$$

then it is shown in [56] that they satisfy the Painlevé type equation

$$-2\frac{n}{t}\frac{r_n(t)}{1-r_n^2(t)} = r_{n+1}(t) - r_{n-1}(t),$$

with initial conditions $r_0(t) = 1$ and $r_1(t) = -I_1(t)/I_0(t)$. Furthermore, by differentiating the recurrence relation with respect to the time, one deduces the following differential equation for the modified Bessel orthogonal polynomials.

Proposition 25. *[56] The modified Bessel orthogonal polynomials satisfy the differential equation*

$$2\frac{d}{dt}\varphi_n(z,t) = \left(\frac{I_1(t)}{I_0(t)} + \frac{r_{n+1}(t)}{r_n(t)}\right)\varphi_n(z,t) - \frac{\kappa_{n-1}(t)}{\kappa_n(t)}\left(1 + \frac{r_{n+1}(t)}{r_n(t)}z\right)\varphi_{n-1}(z,t), \quad n \geqslant 1,$$

and $\frac{d}{dt}\varphi_0(z,t) = 0$*. Furthermore, we also have the differential equations*

$$\frac{2}{\kappa_n(t)}\frac{d}{dt}\kappa_n(t) = \frac{I_1(t)}{I_0(t)} + \frac{r_{n+1}(t)}{r_n(t)}, \quad n \geqslant 1,$$

$$\frac{2}{\varphi_n(0,t)}\frac{d}{dt}\varphi_n(0,t) = \frac{I_1(t)}{I_0(t)} + \frac{r_{n+1}(t)}{r_n(t)} - \frac{\varphi_{n-1}(0,t)\kappa_{n-1}(t)}{\varphi_n(0,t)\kappa_n(t)}, \quad n \geqslant 1.$$

Finally, the reflection coefficients $r_n(t)$ satisfy a second order nonlinear differential equation. Their connection with a certain type of Painlevé equations is stated in the following result.

Proposition 26. *[56] The sequence* $\{r_n(t)\}_{n\geqslant 1}$ *satisfies*

$$\frac{d^2}{dt^2}r_n(t) = \frac{1}{2}\left(\frac{1}{r_n(t)+1} + \frac{1}{r_n(t)-1}\right)\left(\frac{d}{dt}r_n(t)\right)^2 - \frac{1}{t}\frac{d}{dt}r_n(t) - r_n(t)(1-r_n^2(t)) + \frac{n^2}{t^2}\frac{r_n(t)}{1-r_n^2(t)},$$

with boundary conditions

$$r_n(t) \sim \frac{(-t/2)^n}{n!}\left[1+\left(\frac{n}{n+1}-\delta_{n,1}\right)\frac{1}{4}t^2+O(t^4)\right],\quad n\geqslant 1,\quad t\to 0.$$

Furthermore, if $z_n(t)$ satisfies the Painlevé transcendent $P-V$ equation with parameters $\alpha=\beta=n^2/8$, $\gamma=0$, $\delta=-2$, then we have

$$r_n(t)=\frac{z_n(t)+1}{z_n(t)-1}.$$

Some other examples related with Lax pairs of the CMV matrix associated with the orthogonality measure have been studied in [57] and [58].

5.1. An Uvarov-Chihara problem in the unit circle

The results shown in Section 3.1 can be used to describe the dynamics of the Verblunsky coefficients associated with a probability measure supported on the unit circle, obtained via the Szegő transformation defined in the previous section.

Proposition 27. *[34] Let us assume σ is defined from a measure in the interval* $[-1,1]$ *via the Szegő transformation* (72) *and consider the perturbation*

$$d\tilde{\sigma}(z,t)=(1-M(t))d\sigma(z)+M(t)\delta(z-1),$$

i.e., a time dependent mass is added at the point $z=1$, where $M:\mathbb{R}_+\to[0,1]$ is a positive C^1 function. Then, the dynamics of the Verblunsky coefficients associated with σ are described by

$$\dot{\Phi}_n(0;t)=-\frac{\dot{M}(t)\Phi_n^2(1)(1-\Phi_n(0))}{\|\Phi_n\|^2[1-M(t)+M(t)K_{n-1}(1,1)]^2}. \tag{81}$$

Proof. If $\{\Phi_n(z;t)\}_{n\geqslant 0}$ is the MOPS with respect to $\tilde{\sigma}$, then

$$\Phi_n(z;t)=\Phi_n(z)-\frac{M(t)\Phi_n(1)}{1-M(t)+M(t)K_{n-1}(1,1)}K_{n-1}(z,1), \tag{82}$$

where $K_n(z,y)$, the n-th reproducing polynomial kernel for σ, now defined as (see [39], [1])

$$K_n(z,y)=\sum_{k=0}^{n}\frac{\Phi_k(z)\overline{\Phi_k(y)}}{\|\Phi_k\|^2}=\frac{\Phi_{n+1}^*(z)\overline{\Phi_{n+1}^*(y)}-\Phi_{n+1}(z)\overline{\Phi_{n+1)}(y)}}{\|\Phi_{n+1}\|^2(1-z\bar{y})},$$

provided $z\bar{y} \neq 1$. Notice that (82) is the unit circle analog of (32). Setting $z = 0$ in (82), we obtain

$$\Phi_n(0;t) = \Phi_n(0) - \frac{M(t)\Phi_n(1)}{1 - M(t) + M(t)K_{n-1}(1,1)} K_{n-1}(0,1), \tag{83}$$

and since we have real coefficients and $\Phi_n^*(0) = 1$,

$$\Phi_n(0;t) = \Phi_n(0) - \frac{M(t)\Phi_n^2(1)(1 - \Phi_n(0))}{||\Phi_n||^2[1 - M(t) + M(t)K_{n-1}(1,1)]}. \tag{84}$$

(81) is then obtained by derivating the above expression with respect to the time. ■

Remark 28. *Notice that $\Phi_n(1)$ and $K_{n-1}(1,1)$ can be expressed in terms of Verblunsky coefficients. From the recurrence relation, we have*

$$\Phi_n(1) = \Phi_{n-1}(1) + \Phi_n(0)\Phi_{n-1}^*(1),$$

but since $\Phi_{n-1}(z)$ has real coefficients, we get

$$\Phi_n(1) = [1 + \Phi_n(0)]\Phi_{n-1}(1),$$

so we have

$$\Phi_n(1) = \prod_{k=1}^{n}(1 + \Phi_k(0)).$$

On the other hand, we have

$$\begin{aligned} K_{n-1}(z,1) &= \frac{\Phi_n^*(z)\Phi_n^*(1) - \Phi_n(z)\Phi_n(1)}{||\Phi_n||^2(1-z)} \\ &= \frac{\overline{\Phi_n(1)}}{||\Phi_n||^2} \frac{\Phi_n^*(z) - \Phi_n(z)}{(1-z)}, \end{aligned}$$

and thus we get

$$K_{n-1}(1,1) = \frac{\overline{\Phi_n(1)}}{||\Phi_n||^2}[\Phi_n'(1) - (\Phi_n^*)'(1)].$$

Furthermore, in terms of the Verblunksy parameters, the previous expression reads as

$$K_{n-1}(1,1) = \sum_{k=0}^{n-1} \frac{\Phi_k^2(1)}{||\Phi_k||^2} = \sum_{k=0}^{n-1} \frac{\prod_{j=1}^{k}(1 + \Phi_j(0))^2}{\prod_{j=1}^{k}(1 - \Phi_j^2(0))} = \sum_{k=0}^{n-1} \frac{\prod_{j=1}^{k}(1 + \Phi_j(0))}{\prod_{j=1}^{k}(1 - \Phi_j(0))}.$$

As a conclusion, in order to describe the dynamics of $\Phi_n(0;t)$, the values of $\{\Phi_k(0)\}_{k=1}^n$ are required.

Example 29. *Consider the perturbation of the Lebesgue measure on the real line defined by*

$$d\tilde{\mu}(x,t) = dx + \frac{1}{M(t)}\delta(x+1) + \frac{1}{M(t)}\delta(x-1),$$

which is symmetric. Applying the Szegő transformation to $d\tilde{\mu}(x)$ will induce a measure $d\sigma(z,t)$ on the unit circle which is also symmetric. It was shown in [59] that in such a case, the Verblunsky coefficients associated with $d\sigma$ are

$$\begin{aligned} \Phi_{2n}(0,t) &= \frac{-1}{2n+1}\frac{3n^2(n+1)^2 + 2n(n+1)M(t) - M^2(t)}{n^2(n+1)^2 + 2n(n+1)M(t) + M^2(t)}, \\ \Phi_{2n+1}(0,t) &= 0, \end{aligned}$$

and thus the dynamics of $\Phi_n(0)$ can be obtained in terms of $M(t)$ only.

Acknowledgements

The work of the first author has been supported by Conacyt, México, grant 156668, and the Department of Mathematics of Universidad Nacional de Colombia, Bogotá. The work of the second author was supported by Dirección General de Investigación Científica y Técnica, Ministerio de Economía y Competitividad of Spain, grant MTM2012-36732-C03-01. Both authors thank the anonymous referee for his/her valuable comments and suggestions. They greatly contributed to improve the manuscript.

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