Asymptotic behavior of varying discrete Jacobi–Sobolev orthogonal polynomials

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Abstract

In this contribution we deal with a varying discrete Sobolev inner product involving the Jacobi weight. Our aim is to study the asymptotic properties of the corresponding orthogonal polynomials and the behavior of their zeros. We are interested in Mehler–Heine type formulae because they describe the essential differences from the point of view of the asymptotic behavior between these Sobolev orthogonal polynomials and the Jacobi ones. Moreover, this asymptotic behavior provides an approximation of the zeros of the Sobolev polynomials in terms of the zeros of other well–known special functions. We generalize some results appeared very recently in the literature.

Keywords: Sobolev orthogonal polynomials, Jacobi polynomials, Mehler–Heine formulae, Asymptotics, Zeros.
2010 MSC: 33C47, 42C05

1. Introduction

One of the aims of this paper is the study of the asymptotic behavior of sequences of polynomials $\{Q_n^{(\alpha,\beta,M_n)}\}_{n\geq 0}$ orthogonal with respect to the inner product

$$(f,g)_S = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx + M_n f^{(j)}(1)g^{(j)}(1), \tag{1}$$

where $\alpha > -1$, $\beta > -1$, and $j \geq 0$.

We assume that $\{M_n\}_{n\geq 0}$ is a sequence of nonnegative real numbers satisfying

$$\lim_{n\to\infty} M_n n^{\gamma} = M > 0, \tag{2}$$

where γ is a fixed real number.

☆The author FM is partially supported by Dirección General de Investigación, Ministerio de Economía y Competitividad Innovación of Spain, Grant MTM2012-36732-C03-01. The author JJMB is partially supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain and European Regional Development Found, grant MTM2011–28952–C02–01, and Junta de Andalucía, Research Group FQM–0229 (belonging to Campus of International Excellence CEIMAR) and project P11–FQM–7276.

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The main motivation to study this type of inner product arises from the papers [3] and [4]. In [3] the authors work with a measure supported on $[-1,1]$. However, in [4] the authors deal with measures supported on an unbounded interval. In both cases the authors consider measures with nonzero absolutely continuous part, i.e., they work with the so–called continuous Sobolev orthogonal polynomials. The main topic in those papers is how to balance the Sobolev inner product to equilibrate the influence of the two measures in the asymptotic behavior of the corresponding orthogonal polynomials. This inspires us to consider the discrete Sobolev inner product

$$(f,g)_S = \int fg d\mu_0 + M \int f^{(j)} g^{(j)} d\mu_1 = \int fg d\mu_0 + M f^{(j)}(c) f^{(j)}(c),$$

which is a perturbation of a standad inner product. Now, making M depend on n we can study the influence of the perturbation on the asymptotic behavior of the orthogonal polynomials.

From here, very recently, in [7] the authors found the asymptotic behavior of a family of orthogonal polynomials with respect to a varying Sobolev inner product similar to (1), involving the Laguerre weight $w(x) = x^\alpha e^{-x}, \alpha > -1$. We remark that the techniques used in [7] are not useful in this case, and now we need to use more powerful techniques based on those considered in [9].

Previously, in [8] J. J. Moreno–Balcázar obtained some results in this direction but only for the case $j = 0$. Again, the method used in such a paper does not allow to tackle our problem.

We want to emphasize that our objective is to establish that the size of the sequence $\{M_n\}_{n\geq 0}$ has an essential influence on the asymptotic behavior of the orthogonal polynomials with respect to (1), but this influence is only local, that is, around the point where we have introduced the perturbation. In our case, this point is located at $x = 1$. Furthermore, we prove that this influence depends on the size of the sequence $\{M_n\}_{n\geq 0}$ and its relation with the parameter α in the Jacobi weight and the order of the derivative in (1). It is important to remark that for a sequence $\{M_n\}_{n\geq 0}$, we have a sequence of orthogonal polynomials for each n, so we have a square tableau $\{Q_k^{(\alpha,\beta,M_n)}\}_{k\geq 0}$. Here, we deal with the diagonal of this tableau, i.e. $\{Q_n^{(\alpha,\beta,M_n)}\}_{n\geq 0} = \{Q_0^{(\alpha,\beta,M_0)}(x), Q_1^{(\alpha,\beta,M_1)}(x), \ldots, Q_i^{(\alpha,\beta,M_i)}(x), \ldots\}$. In this moment, in order to simplify the notation, we will denote $Q_n^{(\alpha,\beta,M_n)}(x) = Q_n(x)$.

A second aim of this paper is to establish a simple asymptotic relation between the zeros of the Sobolev polynomials which are orthogonal with respect to (1) and the zeros of combinations of Bessel functions of the first kind. This relation is deduced as an immediate consequence of Mehler–Heine formulae (Theorem 2) and they have a numerical interest since we provide an estimate of the zeros of these polynomials.

Since Jacobi classical orthogonal polynomials are involved in the varying inner product (1), we recall some of their basic properties. Jacobi polynomials are orthogonal with respect to the standard inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)(1-x)^\alpha (1+x)^\beta dx, \quad \alpha, \beta > -1.$$

In the sequel, we will work with the sequence $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$, $\alpha > -1$ and $\beta > -1$, normalized by (see [10, f. (4.1.1)])

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}. \tag{3}$$

The derivatives of Jacobi polynomials satisfy (see, [10, f. (4.21.7)])

$$(P_n^{(\alpha,\beta)}(x))^{(k)} = \frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}P_{n-k}^{(\alpha+k,\beta+k)}(x), \quad k \geq 0. \tag{4}$$

Using (3) and (4), we deduce

$$(P_n^{(\alpha,\beta)}(1))^{(k)} = \frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)}, \tag{5}$$

where $(P_n^{(\alpha,\beta)}(1))^{(k)}$ denotes the kth derivative of $P_n^{(\alpha,\beta)}$ evaluated at $x = 1$.

We also note that the squared norm of a Jacobi polynomial is (see, [10, f(4.3.3)]):

$$||P_n^{(\alpha,\beta)}||^2 = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{6}$$

Finally, we will use the Mehler–Heine formula for classical Jacobi polynomials

Theorem 1 *([10, Th. 8.1.1]) Let $\alpha, \beta > -1$. Then,*

$$\lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}\left(\cos\left(\frac{x}{n}\right)\right) = \lim_{n\to\infty}\frac{1}{n^\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{x^2}{2n^2}\right) = (x/2)^{-\alpha}J_\alpha(x),$$

uniformly on compact subsets of $\mathbb{C}$. Here $J_\alpha(x)$ denotes the Bessel function of the first kind, i.e.,

$$J_\alpha(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k+\alpha}.$$

We will also use the following limit related to Stirling formula (see, for example, [5, f.(5.11.13)])

$$\lim_{n\to\infty}\frac{n^{b-a}\Gamma(n+a)}{\Gamma(n+b)} = 1. \tag{7}$$

We introduce the following notation: If a_n and b_n are two sequences of real numbers, then $a_n \approx b_n$ means that the sequence $\frac{a_n}{b_n}$ converges to 1.

2. Varying Jacobi–Sobolev Orthogonal Polynomials

It is well known that the classical Jacobi orthogonal polynomials, $\{P_i^{(\alpha,\beta)}\}_{i=0}^n$, constitute a basis of the linear space $\mathbb{P}_n[x]$ of polynomials with real coefficients and degree at most n. Therefore, the Jacobi-Sobolev orthogonal polynomial of degree n, $Q_n(x)$, can be expressed as

$$Q_n(x) = P_n^{(\alpha,\beta)}(x) + \sum_{i=0}^{n-1} a_{n,i}P_i^{(\alpha,\beta)}(x),$$

where $a_{n,i}$ can be obtained in the usual way. For $0 \leq i \leq n-1$,

$$
\begin{aligned}
0 &= (Q_n(x), P_i^{(\alpha,\beta)}(x))_S \\
&= \sum_{i=0}^{n} a_{n,i} \left(P_k^{(\alpha,\beta)}(x), P_i(x)^{(\alpha,\beta)} \right) + M_n Q_n^{(j)}(1) \left(P_n^{(\alpha,\beta)}(1) \right)^{(j)}, \\
a_{n,i} &= \frac{-M_n Q_n^{(j)}(1) \left(P_i^{(\alpha,\beta)}(1) \right)^{(j)}}{||P_i^{(\alpha,\beta)}(x)||^2}. \qquad (8)
\end{aligned}
$$

So, using (8), the polynomial $Q_n(x)$ can be written as

$$
\begin{aligned}
Q_n(x) &= P_n^{(\alpha,\beta)}(x) - M_n Q_n^{(j)}(1) \sum_{i=0}^{n-1} \frac{\left(P_i^{(\alpha,\beta)}(1) \right)^{(j)} P_i^{(\alpha,\beta)}(x)}{||P_i^{(\alpha,\beta)}(x)||^2} \\
&= P_n^{(\alpha,\beta)}(x) - M_n Q_n^{(j)}(1) K_{n-1}^{(j,0)}(1,x), \qquad (9)
\end{aligned}
$$

with

$$
K_n^{(j,k)}(x,y) = \sum_{i=0}^{n} \frac{\left(P_i^{(\alpha,\beta)}(x) \right)^{(j)} \left(P_i^{(\alpha,\beta)}(y) \right)^{(k)}}{||P_i^{(\alpha,\beta)}(x)||^2}.
$$

Taking the jth derivative in (9)

$$
Q_n^{(j)}(x) = \left(P_n^{(\alpha,\beta)}(x) \right)^{(j)} - M_n Q_n^{(j)}(1) K_{n-1}^{(j,j)}(1,x). \qquad (10)
$$

Thus, evaluating (10) in $x = 1$ we obtain

$$
Q_n^{(j)}(1) = \frac{\left(P_n^{(\alpha,\beta)}(1) \right)^{(j)}}{1 + M_n K_{n-1}^{(j,j)}(1,1)}. \qquad (11)
$$

So, combining (10) and (11) we get

$$
Q_n(x) = P_n^{(\alpha,\beta)}(x) - \frac{M_n \left(P_n^{(\alpha,\beta)}(1) \right)^{(j)}}{1 + M_n K_{n-1}^{(j,j)}(1,1)} K_{n-1}^{(j,0)}(1,x). \qquad (12)
$$

Lemma 1 *Let $\{Q_n\}_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to (1) and $0 \leq k \leq n$, then*

a)

$$
\lim_{n\to\infty} \frac{(Q_n)^{(k)}(1)}{\left(P_n^{(\alpha,\beta)}(1) \right)^{(k)}} = \begin{cases} \frac{k-j}{\alpha+j+k+1}, & \textit{if} \quad \gamma < 2(\alpha+2j+1), \\ \theta_{\alpha,\beta,j,k}, & \textit{if} \quad \gamma = 2(\alpha+2j+1), \\ 1, & \textit{if} \quad \gamma > 2(\alpha+2j+1), \end{cases}
$$

where

$$
\theta_{\alpha,\beta,j,k} = \frac{M(k-j) + \Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)(\alpha+j+k+1)}{(\alpha+j+k+1)\left(M + \Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)\right)}. \qquad (13)
$$

b) $(Q_n, Q_n)_S \approx ||P_n^{(\alpha,\beta)}||^2$.

Proof. First, by using the Stolz's criterion we will prove that the following limit exists,

$$\lim_{n\to\infty} \frac{K_{n-1}^{(j,k)}(1,1)}{n^{2\alpha+2j+2k+2}} \in \mathbb{R}.$$

We also use that

$$n^{2\alpha+2j+2k+2} - (n-1)^{2\alpha+2j+2k+2} \approx (2\alpha+2j+2k+2)n^{2\alpha+2j+2k+1}.$$

We will compute

$$\begin{aligned}\lim_{n\to\infty} \frac{K_{n-1}^{(j,k)}(1,1)}{n^{2\alpha+2j+2k+2}} &= \lim_{n\to\infty} \frac{K_{n-1}^{(j,k)}(1,1) - K_{n-2}^{(j,k)}(1,1)}{n^{2\alpha+2j+2k+2} - (n-1)^{2\alpha+2j+2k+2}} \\ &= \lim_{n\to\infty} \frac{\left(P_{n-1}^{(\alpha,\beta)}(1)\right)^{(k)} \left(P_{n-1}^{(\alpha,\beta)}(1)\right)^{(j)}}{||P_{n-1}^{(\alpha,\beta)}||^2(2\alpha+2j+2k+2)n^{2\alpha+2j+2k+1}}.\end{aligned}$$

By using $2n+\alpha+\beta+1 \approx 2n$ and (5), (6), and (7), we get

$$\begin{aligned}&\lim_{n\to\infty} \frac{\left(P_{n-1}^{(\alpha,\beta)}(1)\right)^{(k)} \left(P_{n-1}^{(\alpha,\beta)}(1)\right)^{(j)}}{||P_{n-1}^{(\alpha,\beta)}||^2(2\alpha+2j+2k+2)n^{2\alpha+2j+2k+1}} = \\ &\lim_{n\to\infty} \frac{\frac{1}{2^j}\frac{\Gamma(n+\alpha+\beta+j)}{\Gamma(n+\alpha+\beta)}\frac{\Gamma(n+\alpha)}{\Gamma(n-j)\Gamma(\alpha+j+1)}\frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k)}{\Gamma(n+\alpha+\beta)}\frac{\Gamma(n+\alpha)}{\Gamma(n-k)\Gamma(\alpha+k+1)}}{\frac{2^{\alpha+\beta+1}}{2(n+\alpha/2+\beta/2-1/2)}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n)\Gamma(n+\alpha+\beta)}2(\alpha+j+k+1)n^{2\alpha+2j+2k+1}} = \\ &\lim_{n\to\infty} \frac{\Gamma(n+\alpha+\beta+j)\Gamma(n+\alpha+\beta+k)\Gamma(n+\alpha)\Gamma(n)\cdot C_{j,k}}{\Gamma(n-j)\Gamma(n+\alpha+\beta)\Gamma(n-k)\Gamma(n+\beta)n^{2\alpha+2j+2k}} = \\ &C_{j,k} \in \mathbb{R},\end{aligned}$$

where

$$C_{j,k} = \frac{1}{\Gamma(\alpha+j+1)\Gamma(\alpha+k+1)2^{\alpha+\beta+j+k+1}(\alpha+j+k+1)}.$$

So we have proved that

$$\lim_{n\to\infty} \frac{K_{n-1}^{(j,k)}(1,1)}{n^{2\alpha+2j+2k+2}} = C_{j,k}. \tag{14}$$

We will now prove part a) of the lemma, by (12)

$$\lim_{n\to\infty} \frac{Q_n^{(k)}(1)}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(k)}} =$$

$$\lim_{n\to\infty} \left(1 - \frac{M_n K_{n-1}^{(j,k)}(1,1)}{1+M_n K_{n-1}^{(j,j)}(1,1)} \frac{\left(P_n^{(\alpha,\beta)}(1)\right)^{(j)}}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(k)}} \right) =$$

$$\lim_{n\to\infty} \left(1 - \frac{M_n n^\gamma \frac{1}{2^j} \frac{\Gamma(n+\alpha+\beta+j+1)}{\Gamma(n-j+1)\Gamma(\alpha+j+1)} \frac{K_{n-1}^{(j,k)}(1,1)}{n^{2\alpha+2j+2k+2}} n^{2\alpha+2j+2k+2-\gamma}}{\frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)} \left(1+M_n n^\gamma \frac{K_{n-1}^{(j,j)}}{n^{2\alpha+4j+2}} n^{2\alpha+4j+2-\gamma}\right)} \right).$$

To simplify the computations we introduce the following notation

$$a_n := M_n n^\gamma, \quad \text{by (2) we have} \quad \lim_{n\to\infty} a_n = M,$$

and

$$b_{n,j,k} := \frac{K_{n-1}^{(j,k)}(1,1)}{n^{2\alpha+2j+2k+2}}, \quad \text{by (14) we have} \quad \lim_{n\to\infty} b_{n,j,k} = C_{j,k}.$$

Then, the above limit becomes

$$\lim_{n\to\infty} \left(1 - \frac{a_n b_{n,j,k} \frac{1}{2^j} \frac{\Gamma(n+\alpha+\beta+j+1)}{\Gamma(n-j+1)\Gamma(\alpha+j+1)} n^{2\alpha+2j+2k+2-\gamma}}{\frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)} (1+a_n b_{n,j,j} n^{2\alpha+4j+2-\gamma})} \right) =$$
$$1 - \frac{2^{k-j}\Gamma(\alpha+k+1)}{\Gamma(\alpha+j+1)} \times \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_{n,j,k} \times$$
$$\lim_{n\to\infty} \frac{n^{2\alpha+2j+2k+2-\gamma}}{n^{2k-2j}(1+a_n b_{n,j,j} n^{2\alpha+4j+2-\gamma})} =$$
$$1 - \frac{2^{k-j}\Gamma(\alpha+k+1)C_{j,k}M}{\Gamma(\alpha+j+1)} \times \lim_{n\to\infty} \frac{n^{2\alpha+4j+2-\gamma}}{(1+a_n b_{n,j,j} n^{2\alpha+4j+2-\gamma})} =$$
$$1 - \frac{2^{k-j}\Gamma(\alpha+k+1)C_{j,k}M}{\Gamma(\alpha+j+1)} \times \lim_{n\to\infty} \frac{1}{\frac{1}{n^{2\alpha+4j+2-\gamma}} + MC_{j,j}}.$$

Then, it is necessary to distinguish three cases according to the value of the parameter γ. The value of this limit is:

Case $\gamma > 2(\alpha+2j+1)$.

$$1 - \frac{2^{k-j}\Gamma(\alpha+k+1)C_{j,k}M}{\Gamma(\alpha+j+1)} \times \lim_{n\to\infty} \frac{1}{\frac{1}{n^{2\alpha+4j+2-\gamma}} + MC_{j,j}} = 1.$$

Case $\gamma < 2(\alpha+2j+1)$.

$$
\begin{aligned}
&1-\frac{2^{k-j}\Gamma(\alpha+k+1)C_{j,k}M}{\Gamma(\alpha+j+1)}\times\lim_{n\to\infty}\frac{1}{\frac{1}{n^{2\alpha+4j+2-\gamma}}+MC_{j,j}}\\
&=1-\frac{2^{k-j}\Gamma(\alpha+k+1)C_{j,k}M}{\Gamma(\alpha+j+1)}\times\frac{1}{MC_{j,j}}\\
=\quad&1-\frac{\alpha+2j+1}{\alpha+j+k+1}\\
&=\frac{k-j}{\alpha+j+k+1}.
\end{aligned}
$$

Case $\gamma=2(\alpha+2j+1)$.

$$
\begin{aligned}
&1-\frac{2^{k-j}\Gamma(\alpha+k+1)C_{j,k}M}{\Gamma(\alpha+j+1)}\times\frac{1}{1+MC_{j,j}}\\
&=1-\frac{M(\alpha+2j+1)}{(\alpha+j+k+1)\left(M+\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)\right)}\\
&=\frac{M(k-j)+\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)(\alpha+j+k+1)}{(\alpha+j+k+1)\left(M+\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)\right)}\\
&=\theta_{\alpha,\beta,j,k}.
\end{aligned}
$$

Thus, we have proved a). Now, we are going to prove b). Let us recall that

$$
Q_n(x)=P_n^{(\alpha,\beta)}(x)+\sum_{i=0}^{n-1}a_{n,i}P_i^{(\alpha,\beta)}(x).
$$

From (11) we deduce

$$
\begin{aligned}
(Q_n,Q_n)_S &= (Q_n,P_n^{(\alpha,\beta)}+\sum_{i=0}^{n-1}a_{n,i}P_i^{(\alpha,\beta)})_S\\
&= (Q_n,P_n^{(\alpha,\beta)})_S\\
&= (P_n^{(\alpha,\beta)}+\sum_{i=0}^{n-1}a_{n,i}P_i^{(\alpha,\beta)},P_n^{(\alpha,\beta)})+M_nQ_n^{(j)}(1)\left(P_n^{(\alpha,\beta)}\right)^{(j)}(1)\\
&= (P_n^{(\alpha,\beta)},P_n^{(\alpha,\beta)})+\frac{M_n\left(\left(P_n^{(\alpha,\beta)}(1)\right)^{(j)}\right)^2}{1+M_nK_{n-1}^{(j,j)}(1,1)}\\
&= ||P_n^{(\alpha,\beta)}||^2+\frac{M_n\left(\left(P_n^{(\alpha,\beta)}(1)\right)^{(j)}\right)^2}{1+M_nK_{n-1}^{(j,j)}(1,1)}.
\end{aligned}
$$

Then,

$$
\lim_{n\to\infty}\frac{(Q_n,Q_n)_S}{||P_n^{(\alpha,\beta)}||^2} = \lim_{n\to\infty}\left(1+\frac{\left(\left(P_n^{(\alpha,\beta)}(1)\right)^{(j)}\right)^2}{||P_n^{(\alpha,\beta)}||^2}\frac{M_n}{1+M_nK_{n-1}^{(j,j)}(1,1)}\right).
$$

To deduce b) it is enough to prove that

$$\lim_{n\to\infty}\left(\frac{M_n\left(\left(P_n^{(\alpha,\beta)}(1)\right)^{(j)}\right)^2}{||P_n^{(\alpha,\beta)}||^2\left(1+M_nK_{n-1}^{(j,j)}(1,1)\right)}\right)=0.$$

Indeed, from (5) this limit can be expressed as

$$\lim_{n\to\infty}\left(\frac{M_n\left(\left(P_n^{(\alpha,\beta)}(1)\right)^{(j)}\right)^2}{||P_n^{(\alpha,\beta)}||^2\left(1+M_nK_{n-1}^{(j,j)}(1,1)\right)}\right)=$$

$$\lim_{n\to\infty}\left(\frac{M_n\frac{1}{2^{2j}}\frac{\Gamma(n+\alpha+\beta+j+1)}{\Gamma(n-j+1)\Gamma(\alpha+j+1)}\frac{\Gamma(n+\alpha+\beta+j+1)}{\Gamma(n+\alpha+\beta+1)}\frac{n^{-4j-2\alpha-\beta+\gamma}\Gamma(n+\alpha+1)}{\Gamma(n-j+1)\Gamma(\alpha+j+1)}\frac{1}{n^{-4j-2\alpha-\beta+\gamma}}}{\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{n^{-\beta}\Gamma(n+\beta+1)}{\Gamma(n+1)}\frac{1}{n^{-\beta}}\left(1+M_nK_{n-1}^{(j,j)}(1,1)\frac{n^{2\alpha+4j+2-\gamma}}{n^{2\alpha+4j+2-\gamma}}\right)}\right).$$

Again, to simplify the computations we take into account

$$\begin{aligned}
a_n &:= M_nn^{\gamma} \quad\to\quad \lim_{n\to\infty}a_n=M,\\
b_n &:= \frac{\Gamma^2(n+\alpha+\beta+j+1)\Gamma(n+\alpha+1)n^{-4j-2\alpha-\beta}}{\Gamma(n-j+1)\Gamma^2(n+\alpha+\beta+1)} \quad\to\quad \lim_{n\to\infty}b_n=1,\\
c_n &:= \frac{\Gamma(n+\beta+1)n^{-\beta}}{\Gamma(n+1)} \quad\to\quad \lim_{n\to\infty}c_n=1,\\
d_n &:= M_nn^{\gamma}\frac{K_{n-1}^{(j,j)}(1,1)}{n^{2\alpha+4j+2}} \quad\to\quad \lim_{n\to\infty}d_n=MC_{j,j},
\end{aligned}$$

and we define

$$E_{\alpha,j}:=\frac{1}{2^{2j}}\frac{1}{\Gamma^2(\alpha+j+1)}.$$

In this way, for every γ, the above limit is

$$\begin{aligned}
&\lim_{n\to\infty}\frac{E_{\alpha,j}a_nb_nn^{4j+2\alpha+\beta-\gamma}}{c_n\frac{2^{\alpha+\beta+1}n^{\beta}}{2n+\alpha+\beta+1}\left(1+d_nn^{4j+2\alpha+2-\gamma}\right)}=\\
&\lim_{n\to\infty}\frac{E_{\alpha,j}a_nb_n(2n+\alpha+\beta+1)n^{4j+2\alpha+\beta-\gamma}}{2^{\alpha+\beta+1}c_n\left(1+d_nn^{4j+2\alpha+2-\gamma}\right)}=\\
&\lim_{n\to\infty}\frac{E_{\alpha,j}a_nb_n(2n+\alpha+\beta+1)n^{4j+2\alpha+\beta-\gamma}}{2^{\alpha+\beta+1}n^{4j+2\alpha+2-\gamma}\left(\frac{c_n}{n^{4j+2\alpha+2-\gamma}}+c_nd_n\right)}=\\
&\lim_{n\to\infty}\frac{E_{\alpha,j}a_nb_n(2n+\alpha+\beta+1)}{2^{\alpha+\beta+1}n^{2}\left(\frac{c_n}{n^{4j+2\alpha+2-\gamma}}+c_nd_n\right)}=0
\end{aligned}$$

and we have just proved b). $\square$

Corollary 1 *Let $\{q_n\}_{n\geq0}$ be the sequence of orthonormal polynomials with respect to (1) and $\{p_n^{(\alpha,\beta)}\}_{n\geq0}$ be the sequence of orthonormal Jacobi polynomials. Then*

$$\lim_{n\to\infty}\frac{(q_n)^{(k)}(1)}{\left(p_n^{(\alpha,\beta)}(1)\right)^{(k)}}=\begin{cases}\frac{k-j}{\alpha+j+k+1}, & \textit{if} \quad \gamma<2(\alpha+2j+1),\\ \theta_{\alpha,\beta,j,k}, & \textit{if} \quad \gamma=2(\alpha+2j+1),\\ 1, & \textit{if} \quad \gamma>2(\alpha+2j+1),\end{cases}$$

where the constants $\theta_{\alpha,\beta,j,k}$ are given in (13).

Proof. According to Lemma 1,

$$\lim_{n\to\infty}\frac{(q_n)^{(k)}(1)}{\left(p_n^{(\alpha,\beta)}(1)\right)^{(k)}}=\lim_{n\to\infty}\frac{\frac{Q_n^{(k)}(1)}{\sqrt{(Q_n,Q_n)_S}}}{\frac{\left(P_n^{(\alpha,\beta)}(1)\right)^{(k)}}{||P_n^{(\alpha,\beta)}||}}=\lim_{n\to\infty}\frac{Q_n^{(k)}(1)}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(k)}}. \quad \square$$

Proposition 1 *There exists a family of real numbers $\{b_i(n)\}_{i=0}^{j+1}$, not identically zero, such that the following connection formula holds*

$$Q_n(x)=\sum_{i=0}^{j+1}b_i(n)(1-x)^iP_{n-i}^{(\alpha+2i,\beta)}(x),\quad n\geq j+1. \tag{15}$$

Proof. We will show that there exist real numbers $\{d_i(n)\}_{i=0}^{j+1}$, such that the polynomial $R_n(x)$ defined by $R_n(x):=\sum_{i=0}^{j+1}d_i(n)(1-x)^iP_{n-i}^{(\alpha+2i,\beta)}(x)$, satisfies

$$(R_n(x),(1-x)^k)_S=0,\quad \text{with} \quad 0\leq k\leq n-1.$$

Actually, for $j+1\leq k\leq n-1$, the product $(R_n(x),(1-x)^k)_S$ is always zero, independently of the value of $d_i(n)$.

If we want $(R_n(x),(1-x)^k)_S=0$ for $k=0,\dots,j$, then we obtain the following homogeneous linear system $Ab=0$, where

$$A=\begin{pmatrix} 0 & I_{1,0} & I_{2,0} & I_{3,0} & \cdots & I_{j,0} & I_{j+1,0}\\ 0 & 0 & I_{2,1} & I_{3,1} & \cdots & I_{j,1} & I_{j+1,1}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & I_{j,j-1} & I_{j+1,j-1}\\ c_{0,j} & c_{1,j} & c_{2,j} & c_{3,j} & \cdots & c_{j,j} & I_{j+1,j}+c_{j+1,j}\end{pmatrix},$$

$$b=\begin{pmatrix} d_0(n)\\ d_1(n)\\ \vdots\\ d_j(n)\\ d_{j+1}(n)\end{pmatrix},\quad 0=\begin{pmatrix}0\\0\\ \vdots\\0\\0\end{pmatrix},$$

and (see, [6] formula ET II 284 (2)),

$$\begin{aligned} I_{i,k} &= \int_{-1}^{1}P_{n-i}^{(\alpha+2i,\beta)}(x)(1-x)^{\alpha+i+k}(1+x)^{\beta}dx\\ &= \frac{2^{\beta+\alpha+i+k+1}\Gamma(\alpha+i+k+1)\Gamma(\beta+n-i+1)\Gamma(n-k)}{(n-i)!\Gamma(i-k)\Gamma(\beta+\alpha+k+n+2)},\\ c_{i,j} &= (-1)^j j! M_n\left((1-x)^iP_{n-i}^{(\alpha+2i,\beta)}(x)\right)^{(j)}|_{x=1}. \end{aligned}$$

Thus, we have a homogeneous linear system of $j+1$ equations and $j+2$ unknowns. Then, this system has a nontrivial solution for $\{d_i(n)\}_{i=0}^{j+1}$. Therefore, $R_n(x) = c(n)Q_n(x)$, and we finally deduce that

$$Q_n(x) = \sum_{i=0}^{j+1} b_i(n)(1-x)^i P_{n-i}^{(\alpha+2i,\beta)}(x), \quad b_i(n) = \frac{d_i(n)}{c(n)}. \quad \square$$

Lemma 2 *Let $\{b_i(n)\}_{i=0}^{j+1}$ be the coefficients in (15). Then*

$$\lim_{n\to\infty} b_i(n) = b_i \in \mathbb{R}, \qquad i \in \{0, 1, \ldots, j+1\}.$$

Proof. We take the kth derivative in (15) and we evaluate the corresponding expression at $x = 1$,

$$\begin{aligned} Q_n^{(k)}(x) &= \sum_{i=0}^{j+1} b_i(n) \sum_{s=0}^{k} \binom{k}{s} \left((1-x)^i\right)^{(s)} \left(P_{n-i}^{(\alpha+2i,\beta)}(x)\right)^{(k-s)} \\ &= \sum_{i=0}^{j+1} b_i(n) \sum_{s=0}^{\min\{i,k\}} \binom{k}{s} (-1)^s \frac{i!}{(i-s)!} (1-x)^{i-s} \left(P_{n-i}^{(\alpha+2i,\beta)}(x)\right)^{(k-s)}. \end{aligned}$$

Then,

$$Q_n^{(k)}(1) = \sum_{i=0}^{k} b_i(n) \binom{k}{i} (-1)^i i! \left(P_{n-i}^{(\alpha+2i,\beta)}(1)\right)^{(k-i)}.$$

From Lemma 1 $\lim_{n\to\infty} \frac{Q_n^{(k)}(1)}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(k)}}$ exists, so

$$\frac{Q_n^{(k)}(1)}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(k)}} = \sum_{i=0}^{k} b_i(n) \binom{k}{i} (-1)^i i! A_i(k,n) \tag{16}$$

with $A_i(k,n) = \frac{\left(P_{n-i}^{(\alpha+2i,\beta)}(1)\right)^{(k-i)}}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(k)}}$.

It only remains to prove that there exists $\lim_{n\to\infty} A_i(k,n) \in \mathbb{R}$ and, therefore, we get that the coefficients $\{b_i(n)\}_{i=0}^{j+1}$ are convergent. Indeed

$$\begin{aligned} \lim_{n\to\infty} A_i(k,n) &= \lim_{n\to\infty} \frac{\frac{1}{2^{k-i}} \frac{\Gamma(n-i+\alpha+2i+\beta+k-i+1)}{\Gamma(n-i+\alpha+2i+\beta+1)} \frac{\Gamma(n-i+\alpha+2i+1)}{\Gamma(n-i-k+i+1)\Gamma(\alpha+2i+k-i+1)}}{\frac{1}{2^k} \frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)}} \\ &= \frac{2^i \Gamma(\alpha+k+1)}{\Gamma(\alpha+i+k+1)} := A_i(k,\alpha). \quad \square \end{aligned}$$

Remark 1 *Let $b_i := \lim_{n\to\infty} b_i(n)$ with $i \in \{0, 1, \ldots, j+1\}$. (16) is a recursive algorithm to compute b_i.*

- *Step 1. For $k = 0$ we obtain b_0 in a straightforward way.*
- *Step 2. For $k = 1$ we deduce the value of b_1 from (16) using the step 1. Similarly, for $k \geq 2$ we apply (16) in a recursive way.*

3. Asymptotics and zeros of varying Jacobi–Sobolev

We focus our attention on the analysis of Mehler-Heine formulas for these discrete Jacobi–Sobolev orthogonal polynomials because we want to know how the discrete part in the inner product (1) influences the asymptotic behavior of the corresponding orthogonal polynomials. Furthermore, we will prove that this influence is related to the size of the sequence $\{M_n\}_{n\geq 0}$.

Theorem 2 *For the sequence $\{Q_n\}_{n\geq 0}$ the following Mehler–Heine formula holds*

$$\lim_{n\to\infty}\frac{Q_n(\cos(x/n))}{n^\alpha}=\lim_{n\to\infty}\frac{Q_n\left(1-\frac{x^2}{2n^2}\right)}{n^\alpha}=\begin{cases}\phi_\alpha(x), & \textit{if} \quad \gamma>2(\alpha+2j+1),\\ \psi_{\alpha,j}(x), & \textit{if} \quad \gamma=2(\alpha+2j+1),\\ \varphi_{\alpha,j}(x), & \textit{if} \quad \gamma<2(\alpha+2j+1),\end{cases} \tag{17}$$

uniformly on compact subsets of $\mathbb{C}$, where

$$\begin{aligned}\phi_\alpha(x) &:= \left(\frac{x}{2}\right)^{-\alpha}J_\alpha(x),\\ \psi_{\alpha,j}(x) &:= \sum_{i=0}^{j+1} b_i 2^i\left(\frac{x}{2}\right)^{-\alpha}J_{\alpha+2i}(x),\\ \varphi_{\alpha,j}(x) &:= \sum_{i=0}^{j+1} c_i 2^i\left(\frac{x}{2}\right)^{-\alpha}J_{\alpha+2i}(x),\end{aligned}$$

and the coefficients b_i and c_i are given by

$$\begin{aligned}c_k &= \frac{\frac{k-j}{\alpha+j+k+1}-\sum_{i=0}^{k-1} b_i\binom{k}{i}(-1)^i i!\frac{2^i\Gamma(\alpha+k+1)}{\Gamma(\alpha+k+i+1)}}{(-1)^k k!\frac{2^k\Gamma(\alpha+k+1)}{\Gamma(\alpha+2k+1)}},\\ c_0 &= \frac{-j}{\alpha+j+1},\\ b_k &= (-1)^k\frac{\frac{M(k-j)-\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)(\alpha+j+k+1)}{(\alpha+j+k+1)(M+\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1))}-\sum_{i=0}^{k-1} b_i\binom{k}{i}(-1)^i i!\frac{2^i\Gamma(\alpha+k+1)}{\Gamma(\alpha+k+i+1)}}{k!\frac{2^k\Gamma(\alpha+k+1)}{\Gamma(\alpha+2k+1)}},\\ b_0 &= \frac{-jM-\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)(\alpha+j+1)}{(\alpha+j+1)(M+\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1))},\end{aligned}$$

with $1\leq k\leq j+1$.

Proof. Scaling and taking limits in (15)

$$\begin{aligned}\lim_{n\to\infty}\frac{Q_n\left(1-\frac{x^2}{2n^2}\right)}{n^\alpha} &= \lim_{n\to\infty}\frac{\sum_{i=0}^{j+1} b_i(n)\left(1-\left(1-\frac{x^2}{2n^2}\right)\right)^i P_{n-i}^{(\alpha+2i,\beta)}\left(1-\frac{x^2}{2n^2}\right)}{n^\alpha}\\ &= \sum_{i=0}^{j+1}\lim_{n\to\infty} b_i(n)\times\lim_{n\to\infty}\frac{\left(1-\left(1-\frac{x^2}{2n^2}\right)\right)^i P_{n-i}^{(\alpha+2i,\beta)}\left(1-\frac{x^2}{2n^2}\right)}{n^\alpha}\\ &= \sum_{i=0}^{j+1} b_i 2^i\left(\frac{x}{2}\right)^{-\alpha}J_{\alpha+2i}(x),\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$. Notice that in the last inequality we have used Theorem 1 written in the following way

$$\lim_{n\to\infty} \frac{\left(\frac{x^2}{2n^2}\right)^i P_{n-i}^{(\alpha+2i,\beta)}\left(1-\frac{x^2}{2n^2}\right)}{n^\alpha} = 2^i \left(\frac{x}{2}\right)^{-\alpha} J_{\alpha+2i}(x),$$

uniformly on compact subsets of $\mathbb{C}$, where i is a fixed nonnegative integer number.

Now, we distinguish three cases according to the value of the parameter γ.

- If $\gamma > 2(\alpha+2j+1)$, we are going to prove that $b_0 = 1$ and $b_i = 0$ if $i \in \{1, 2, \ldots, j+1\}$.

We can compute b_i from (16). If $k = 0$, then

$$\frac{Q_n(1)}{P_n^{(\alpha,\beta)}(1)} = b_0(n)A_0(0,n),$$

Using Lemma 1 and taking limits, we obtain $b_0 = 1$. If $k = 1$, then according to Lemma 1 we have

$$\frac{Q_n^{(1)}(1)}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(1)}} = b_0(n)A_0(1,n) - b_1(n)A_1(1,n).$$

Taking limits,

$$1 = 1 - b_1 A_1(1,\alpha) \Rightarrow b_1 = 0.$$

Applying a recursive procedure we get $b_i = 0$ for $i \in \{1, 2, \ldots, j+1\}$. To illustrate this procedure we consider the case $k = j+1$. Thus, we have $b_i = 0$ for $i \in \{1, 2, \ldots, j\}$. Then,

$$\begin{aligned}\frac{Q_n^{(j+1)}(1)}{\left(P_n^{(\alpha,\beta)}(1)\right)^{(j+1)}} &= b_0(n)A_0(1,n) + \sum_{i=1}^{j} b_i(n)\binom{j+1}{i}(-1)^i i! A_i(j+1,n) \\ &+ b_{j+1}(n)(-1)^{j+1}(j+1)!A_{j+1}(j+1,n).\end{aligned}$$

Taking limits,

$$1 = 1 + b_{j+1}(-1)^{j+1}(j+1)!A_{j+1}(j+1,\alpha) \Rightarrow b_{j+1} = 0.$$

- Case $\gamma = 2(\alpha+2j+1)$. From (16) and $k = 0$, we have

$$\frac{Q_n(1)}{P_n^{(\alpha,\beta)}(1)} = b_0(n)A_0(0,n).$$

Taking limits when n tends to infinity in the above expression, we get

$$b_0 = \frac{-jM - \Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)(\alpha+j+1)}{(\alpha+j+1)(M + \Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1))}.$$

For $k \geq 1$, again we use Lemma 1 and take limits. Thus, we deduce the coefficients b_i in a recursive way from (16), obtaining

$$b_k = \frac{\frac{M(k-j)-\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1)(\alpha+j+k+1)}{(\alpha+j+k+1)(M+\Gamma^2(\alpha+j+1)2^{\alpha+\beta+2j+1}(\alpha+2j+1))} - \sum_{i=0}^{k-1} b_i \binom{k}{i}(-1)^i i! \frac{2^i\Gamma(\alpha+k+1)}{\Gamma(\alpha+k+i+1)}}{(-1)^k k! \frac{2^k\Gamma(\alpha+k+1)}{\Gamma(\alpha+2k+1)}}.$$

- Case $\gamma < 2(\alpha + 2j + 1)$. We can tackle this case in the same way as the case $\gamma = 2(\alpha + 2j + 1)$. $\square$

Next, we are going to study the zeros of the polynomials $\{Q_n\}_{n\geq 0}$ orthogonal with respect to (1).

Proposition 2 *The polynomial $Q_n(x)$, $n \geq 1$, has n real and simple zeros and at most one of them is located outside the interval $[-1, 1]$.*

Proof. For $n < j+1$ the result is obvious because the polynomial $Q_n(x) = P_n^{(\alpha,\beta)}(x)$, and therefore all their zeros lie on the interval [-1,1]. Thus, we consider $n \geq j + 1$. Then, by Lemma 2.1 of [2] we can claim that there exists $\varphi(x)$, a polynomial with $\deg(\varphi) = 1$, so that for every polynomial q with real and simple zeros on the interval $[-1, 1]$, we get

$$(q\varphi)^{(j)}(1) = 0.$$

In addition, $\varphi(x)$ does not change sign on the interval $[-1, 1]$.

Let $x_1, x_2, \ldots, x_k$ be the simple zeros of Q_n in the interval $[-1, 1]$ with $k < n-1$. Then, we define $q(x) := (x - x_1)(x - x_2)\ldots(x - x_k)$. Thus, $\deg(q) < n - 1$ and therefore there exists $\varphi(x)$ so that $(q\varphi)^{(j)}(1) = 0$. Since $\deg(q\varphi) \leq n - 1$, taking into account that $Q_n(x)$ is orthogonal with respect to (1), we get

$$\begin{aligned} 0 = (q\varphi, Q_n)_S &= \int_{-1}^{1} (q(x)\varphi(x))Q_n(x)(1-x)^\alpha(1+x)^\beta dx + M_n(q\varphi)^{(j)}(1)Q_n(1) \\ &= \int_{-1}^{1} (q(x)\varphi(x))Q_n(x)(1-x)^\alpha(1+x)^\beta dx, \end{aligned}$$

but the integrand in the above integral does not change sign on the interval [-1,1] and, therefore it cannot be zero. This implies that $k = n - 1$ or $k = n$ which proves the result. $\square$

We can give more information about the location of the zeros. The case $j = 0$ was considered in [8]. We notice that in that case all the zeros are in the interval $(-1, 1)$. Thus, next we will assume $j > 0$ and we will denote by $y_{n,1} > y_{n,2} > \cdots > y_{n,n-1} > y_{n,n}$ the zeros of $Q_n(x)$.

Proposition 3 *Let $\{Q_n\}_{n\geq 0}$ be a sequence of orthogonal polynomials with respect to (1). For n large enough and $j > 0$, we have*

- *If $\gamma > 2(\alpha + 2j + 1)$, then all zeros of $Q_n(x)$ are located in $(-1, 1)$.*
- *If $\gamma < 2(\alpha + 2j + 1)$, then $y_{n,1} > 1$.*

- *If $\gamma = 2(\alpha + 2j + 1)$, then $y_{n,1} > 1$ if and only if*

$$M > \frac{2^{\alpha+\beta+2j+1}(\alpha + j + 1)(\alpha + 2j + 1)\Gamma^2(\alpha + j + 1)}{j}$$

Proof. We distinguish three cases, but essentially we use Lemma 1 a) with $k = 0$, and the fact that the leading coefficient of Q_n is positive. Then

- If $\gamma > 2(\alpha+2j+1)$, then by Lemma 1 $Q_n(1) > 0$ for n large enough. Therefore, taking into account Proposition 2 we get that all the zeros are located in $(-1, 1)$.
- If $\gamma < 2(\alpha + 2j + 1)$, then $Q_n(1) < 0$ for n large enough, which implies that there is a zero of Q_n greater than 1 and by Proposition 2 it is the only one.
- If $\gamma = 2(\alpha + 2j + 1)$, then $y_{n,1} > 1$ if and only if $Q_n(1) < 0$ for n large enough, and this only happens if and only if $M > \frac{2^{\alpha+\beta+2j+1}(\alpha+j+1)(\alpha+2j+1)\Gamma^2(\alpha+j+1)}{j}$. □

Now we deduce the asymptotic behavior of the zeros of $Q_n(x)$.

Proposition 4 *Let $y_{n,1} > y_{n,2} > \cdots > y_{n,n-1} > y_{n,n}$ be the zeros of $Q_n(x)$ and $\phi_\alpha(x)$, $\varphi_{\alpha,j}(x)$, and $\psi_{\alpha,j}(x)$ the functions defined in Theorem 2. We assume $j > 0$.*

1. *If $\gamma > 2(\alpha + 2j + 1)$, then*
$$\lim_{n\to\infty} n\sqrt{2(1 - y_{n,i})} = j_{\alpha,i}, \quad i \geq 1,$$
where $j_{\alpha,i}$ denotes the ith positive zero of the Bessel function of the first kind.
2. *If $\gamma < 2(\alpha + 2j + 1)$, then*
$$\lim_{n\to\infty} y_{n,1} = 1, \quad \lim_{n\to\infty} n\sqrt{2(1 - y_{n,i})} = s_{\alpha,i-1}, \quad i \geq 2,$$
where $s_{\alpha,i}$ denotes the ith positive zero of the function $\varphi_{\alpha,j}(x)$.
3. *If $\gamma = 2(\alpha + 2j + 1)$, we have two cases:*
   (a) *If $M \leq \frac{2^{\alpha+\beta+2j+1}(\alpha+j+1)(\alpha+2j+1)\Gamma^2(\alpha+j+1)}{j}$, then $y_{n,1} \leq 1$, for n large enough, and*
   $$\lim_{n\to\infty} n\sqrt{2(1 - y_{n,i})} = t_{\alpha,i}, \quad i \geq 1,$$
   where $t_{\alpha,i}$ denotes the ith positive zero of the function $\psi_{\alpha,j}(x)$.
   (b) *If $M > \frac{2^{\alpha+\beta+2j+1}(\alpha+j+1)(\alpha+2j+1)\Gamma^2(\alpha+j+1)}{j}$, then*
   $$\lim_{n\to\infty} y_{n,1} = 1, \quad \lim_{n\to\infty} n\sqrt{2(1 - y_{n,i})} = t_{\alpha,i-1}, \quad i \geq 2,$$
   where $t_{\alpha,i}$ denotes the ith positive zero of the function $\psi_{\alpha,j}(x)$.

Proof. It follows from Theorem 2, Proposition 3, and Hurwitz's Theorem (see [10, Th. 1.91.3]). $\square$

To illustrate Theorem 2 we are going to recover the case $j = 0$ obtained in [8]. In that paper the author uses monic polynomials, and here we are considering a different normalization, i.e. the leading coefficient of Q_n is

$$\frac{\Gamma(2n+\alpha+\beta+1)}{2^n\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

Therefore, it is necessary to do some easy computations. We use the relations (see, [5, f.10.6.1], [1, 6.1.18])

$$J_\alpha(x) - \frac{2(\alpha+1)}{x} J_{\alpha+1}(x) = -J_{\alpha+2}(x), \tag{18}$$

as well as

$$\Gamma(2x) = \frac{\Gamma(x)\Gamma(x+\frac{1}{2})}{2^{1-2x}\sqrt{\pi}}. \tag{19}$$

First, using (7) and (19) we get

$$\begin{aligned}\frac{\Gamma(2n+\alpha+\beta+1)}{2^n\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} &\approx \frac{2^{n+\alpha+\beta}}{\sqrt{\pi}} \frac{\Gamma\left(n+\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right)}{\Gamma(n+1)} \frac{\Gamma\left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)}{\Gamma(n+\alpha+\beta+1)} \\ &\approx \frac{2^{n+\alpha+\beta}}{n^{\frac{1}{2}}\sqrt{\pi}}.\end{aligned}$$

In [8] it was obtained

$$\lim_{n\to\infty} \frac{2^n \hat{P}_n^{(\alpha,\beta,M_n)}(\cos(x/n))}{n^{\alpha+1/2}} = \begin{cases} -\frac{\sqrt{\pi}}{2^\beta} x^2 z_{\alpha+2}(x), & \text{if } \gamma < 2\alpha+2, \\ -\frac{\sqrt{\pi}}{2^\beta}(z_\alpha(x) + a_{\alpha,\beta,M} z_{\alpha+1}(x)), & \text{if } \gamma = 2\alpha+2, \\ \frac{\sqrt{\pi}}{2^\beta} z_\alpha(x), & \text{if } \gamma > 2\alpha+2, \end{cases}$$

where

$$\begin{aligned} z_\alpha(x) &= x^{-\alpha} J_\alpha(x), \\ a_{\alpha,\beta,M} &= \frac{-2M(\alpha+1)}{M + 2^{\alpha+\beta+1}\Gamma(\alpha+2)\Gamma(\alpha+1)}, \end{aligned}$$

and $\{\hat{P}_n^{(\alpha,\beta,M_n)}\}_{n\geq 0}$ denotes the sequence of monic polynomials which are orthogonal with respect to (1) with $j = 0$. This result can be written as follows

$$\lim_{n\to\infty} \frac{2^{n+\alpha+\beta} \hat{P}_n^{(\alpha,\beta,M_n)}(\cos(x/n))}{n^{\alpha+1/2}\sqrt{\pi}} = \begin{cases} -\frac{1}{2^{-\alpha}} x^2 z_{\alpha+2}(x), & \text{if } \gamma < 2\alpha+2, \\ -\frac{1}{2^{-\alpha}}(z_\alpha(x) + a_{\alpha,\beta,M} z_{\alpha+1}(x)), & \text{if } \gamma = 2\alpha+2, \\ \frac{1}{2^{-\alpha}} z_\alpha(x), & \text{if } \gamma > 2\alpha+2. \end{cases} \tag{20}$$

We can observe that

$$\frac{2^{n+\alpha+\beta}\hat{P}_n^{(\alpha,\beta,M_n)}(\cos(x/n))}{n^{\alpha+1/2}\sqrt{\pi}} \approx \frac{Q_n(\cos(x/n))}{n^\alpha}.$$

Therefore, it only remains to compare the limit functions in (20) and (17). The case $\gamma > 2\alpha+2$ is trivial. We pay attention to the other two cases.

- $\gamma < 2\alpha+2$.

  In this case $c_0 = 0$ and $c_1 = -1/2$. Thus we have

$$\varphi_{\alpha,0}(x) = -\left(\frac{x}{2}\right)^{-\alpha} J_{\alpha+2}(x) = -\frac{1}{2^{-\alpha}}x^2x^{-\alpha-2}J_{\alpha+2}(x) = \frac{-x^2}{2^{-\alpha}}z_{\alpha+2}.$$

- $\gamma = 2\alpha+2$.

  In this case,

$$\begin{aligned} b_0 &= -\frac{\Gamma(\alpha+1)^2 2^{\alpha+\beta+1}(\alpha+1)}{M+\Gamma(\alpha+1)^2 2^{\alpha+\beta+1}(\alpha+1)}, \\ b_1 &= \frac{M}{2(M+\Gamma(\alpha+1)^2 2^{\alpha+\beta+1}(\alpha+1))}. \end{aligned}$$

  By using (18) we deduce

$$\begin{aligned} \psi_{\alpha,0}(x) &= b_0\left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x) + 2b_1\left(\frac{x}{2}\right)^{-\alpha} J_{\alpha+2}(x) \\ &= \frac{-\Gamma(\alpha+1)^2 2^{\alpha+\beta+1}}{M+\Gamma(\alpha+1)^2 2^{\alpha+\beta+1}(\alpha+1)}\left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x) \\ &+ \frac{M}{M+\Gamma(\alpha+1)^2 2^{\alpha+\beta+1}(\alpha+1)}\left(\frac{x}{2}\right)^{-\alpha} J_{\alpha+2}(x) \\ &= -\left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x) + \frac{M(\alpha+1)}{M+2^{\alpha+\beta+1}\Gamma(\alpha+1)^2(\alpha+1)}\left(\frac{x}{2}\right)^{-\alpha-1} J_{\alpha+1}(x) \\ &= -\frac{1}{2^{-\alpha}}(z_\alpha(x) + a_{\alpha,\beta,M}z_{\alpha+1}(x)). \end{aligned}$$

4. Numerical Experiments

In this section we illustrate the previous results on the zeros of the polynomials Q_n with some numerical experiments where we have taken $j = 3$ for all of them. Thus, we are dealing with the varying Sobolev inner product

$$(f,g)_S = \int_{-1}^{1} f(x)g(x)(1-x)^\alpha(1+x)^\beta dx + M_n f^{(3)}(1)g^{(3)}(1).$$

We have used the mathematical software *Mathematica®8.0* for our computations. In all the numerical experiments we have computed the four largest zeros of the polynomials $Q_n(x)$ and the corresponding scaled zeros for several values of n. In the tables about the scaled zeros we show their asymptotic behavior such as it is described in Proposition 4.

- Case $\gamma > 2(\alpha + 2j + 1)$.

We choose the following values:

$$\alpha = 3, \quad \beta = 1, \quad \gamma = 25, \quad \text{and} \quad M_n = \frac{3e^n}{(6e^n + 4)n^\gamma}.$$

It was proven in Theorem 2 that in this case the Mehler–Heine formula for the polynomials Q_n is the same one as for the classical Jacobi polynomials. This behavior is due to the negligible influence of the sequence of masses $\{M_n\}_{n\geq 0}$ on the asymptotics. Obviously, as it was stated in Proposition 4, this determines the asymptotic behavior of the zeros which is illustrated in Table 1 and Table 2.

	$y_{n,4}$	$y_{n,3}$	$y_{n,2}$	$y_{n,1}$
$n = 150$	0.994346	0.99636	0.997952	0.999125
$n = 250$	0.997937	0.998672	0.999254	0.999681
$n = 500$	0.999479	0.999665	0.999811	0.999919

Table 1: **Case** $\gamma = 25 > 2(\alpha + 2j + 1)$,
$j = 3,\ \alpha = 3,\ \beta = 1,\ \gamma = 25,\ M_n = \frac{3e^n}{(6e^n+4)n^\gamma}$.

	$n\sqrt{2(1-y_{n,1})}$	$n\sqrt{2(1-y_{n,2})}$	$n\sqrt{2(1-y_{n,3})}$	$n\sqrt{2(1-y_{n,4})}$
$n = 150$	6.27524	9.59956	12.7982	15.9503
$n = 250$	6.31687	9.66386	12.885	16.0602
$n = 500$	6.34839	9.71233	12.9501	16.1421
Limit	$\mathbf{j_{3,1}} = 6.38016$	$\mathbf{j_{3,2}} = 9.76102$	$\mathbf{j_{3,3}} = 13.0152$	$\mathbf{j_{3,4}} = 16.2235$

Table 2: **Case** $\gamma = 25 > 2(\alpha + 2j + 1)$,
$j = 3,\ \alpha = 3,\ \beta = 1,\ \gamma = 25,\ M_n = \frac{3e^n}{(6e^n+4)n^\gamma}$.

- Case $\gamma < 2(\alpha + 2j + 1)$.

According to Theorem 2 the limit function in the Mehler–Heine formula is given by $\varphi_{\alpha,3}(x) := \sum_{i=0}^{4} c_i 2^i \left(\frac{x}{2}\right)^{-\alpha} J_{\alpha+2i}(x)$, where

$$\begin{aligned}
c_0 &= \frac{-3}{\alpha+4}, \\
c_1 &= -\frac{1}{2} + \frac{3}{\alpha+4} - \frac{3}{\alpha+5}, \\
c_2 &= \frac{3(\alpha+7)}{4(\alpha+5)(\alpha+6)}, \\
c_3 &= \frac{-3}{4(\alpha+4)(\alpha+5)}, \\
c_4 &= \frac{3}{8(\alpha+4)(\alpha+5)(\alpha+6)}.
\end{aligned}$$

We choose the following values:

$$\alpha = 3, \quad \beta = 1, \quad \gamma = 4, \quad \text{and} \quad M_n = \frac{7\ln(n+1)+5}{(3+\ln(n^2))n^\gamma}.$$

In Table 3 we can see that the largest zero is greater than 1 for n large enough according to Proposition 3. Table 4 shows the asymptotic behavior of the scaled zeros given in Proposition 4.

	$y_{n,4}$	$y_{n,3}$	$y_{n,2}$	$y_{n,1}$
$n = 150$	0.994574	0.996593	0.998169	0.999286
$n = 250$	0.998176	0.998915	0.999497	1.0016
$n = 500$	0.999554	0.999739	0.999883	1.0014

Table 3: **Case** $\gamma = 4 < 2(\alpha + 2j + 1)$, $j = 3,\ \alpha = 3,\ \beta = -1/2,\ \gamma = 4,\ M_n = \frac{7\ln(n+1)+5}{(3+\ln(n^2))n^\gamma}$.

	$n\sqrt{2(1-y_{n,2})}$	$n\sqrt{2(1-y_{n,3})}$	$n\sqrt{2(1-y_{n,4})}$
$n = 150$	9.07735	12.382	15.6257
$n = 250$	7.92964	11.6463	15.1011
$n = 500$	7.6415	11.4238	14.9355
Limit	$\mathbf{s_{3,1}} = 7.64622$	$\mathbf{s_{3,2}} = 11.4432$	$\mathbf{s_{3,3}} = 14.9699$

Table 4: **Case** $\gamma = 4 < 2(\alpha + 2j + 1)$, $j = 3,\ \alpha = 3,\ \beta = -1/2,\ \gamma = 4,\ M_n = \frac{7\ln(n+1)+5}{(3+\ln(n^2))n^\gamma}$.

- Case $\gamma = 2(\alpha + 2j + 1)$.

  According to Theorem 2 the limit function in the Mehler–Heine formula is given by $\psi_{\alpha,3}(x) := \sum_{i=0}^{4} b_i 2^i \left(\frac{x}{2}\right)^{-\alpha} J_{\alpha+2i}(x)$, where

$$\begin{aligned}
b_0 &= -1 + \frac{M(\alpha+1)}{(\alpha+4)(M + 2^{\alpha+\beta+7}(\alpha+7)\Gamma^2(\alpha+4))}, \\
b_1 &= -\frac{-M(\alpha+2)(\alpha+7)}{2(\alpha+4)(\alpha+5)(M + 2^{\alpha+\beta+7}(\alpha+7)\Gamma^2(\alpha+4))}, \\
b_2 &= \frac{3M(\alpha+7)\Gamma(\alpha+5)}{4M\Gamma(\alpha+7) + 2^{\alpha+\beta+9}\Gamma^2(\alpha+4)\Gamma(\alpha+8)}, \\
b_3 &= \frac{-3M(\alpha+6)\Gamma(\alpha+4)}{4M\Gamma(\alpha+7) + 2^{\alpha+\beta+9}\Gamma^2(\alpha+4)\Gamma(\alpha+8)}, \\
b_4 &= \frac{3M}{8(\alpha+4)(\alpha+5)(\alpha+6)(M + 2^{\alpha+\beta+7}(\alpha+7)\Gamma^2(\alpha+4))}.
\end{aligned}$$

We choose the following values:

$$\alpha = \beta = -9/10, \quad \gamma = 61/5 = 12.2,$$

and we denote by V the quantity which appears in Proposition 4, i.e.

$$V := \frac{2^{\alpha+\beta+2j+1}(\alpha+j+1)(\alpha+2j+1)\Gamma^2(\alpha+j+1)}{j}.$$

Thus, with this data

$$V = 2^{1/5}\frac{15128}{75}\Gamma^2\left(\frac{31}{10}\right) \simeq 1119.0037947.$$

Now we take

$$M_n = \frac{Mn^2(n-1/2)(n+2)}{n^{\gamma+4}} = \frac{Mn^2(n-1/2)(n+2)}{n^{81/5}}.$$

According to Proposition 4 we have two possible choices of M which determine two different asymptotic behaviors of the zeros. In Table 5 and Table 6 we show the case $M \leq V$ where $M = 5$. We can see that the largest zero of Q_n is always less than 1. However, when $M > V$ then $y_{n,1} > 1$ for n large enough and this is illustrated in Table 7 for $M = 10^6$. In Table 8 the asymptotic behavior of the scaled zeros is shown.

	$y_{n,4}$	$y_{n,3}$	$y_{n,2}$	$y_{n,1}$
$n = 150$	0.99778	0.99854	0.999585	0.999991
$n = 250$	0.999142	0.999585	0.999871	0.999997
$n = 500$	0.999786	0.99985	0.999968	0.999999

Table 5: **Case** $\gamma = 61/5 = 2(\alpha+2j+1),\ M = 5 \leq V$
$j = 3,\ \alpha = -9/10,\ \beta = -9/10,\ \gamma = 61/5,\ M_n = \frac{5n^2(n-1/2)(n+2)}{n^{\gamma+4}}$.

	$n\sqrt{2(1-y_{n,1})}$	$n\sqrt{2(1-y_{n,2})}$	$n\sqrt{2(1-y_{n,3})}$	$n\sqrt{2(1-y_{n,4})}$
$n = 150$	0.649565	4.02672	7.20558	10.3659
$n = 250$	0.64887	4.02249	7.19831	10.3561
$n = 500$	0.64853	4.01929	7.19273	10.3484
Limit	$\mathbf{t_{0,1}} = 0.648561$	$\mathbf{t_{0,2}} = 4.01985$	$\mathbf{t_{0,3}} = 7.19169$	$\mathbf{t_{0,4}} = 10.3446$

Table 6: **Case** $\gamma = 61/5 = 2(\alpha+2j+1),\quad M = 5 \leq V$
$j = 3,\ \alpha = -9/10,\ \beta = -9/10,\ \gamma = 61/5,\ M_n = \frac{5n^2(n-1/2)(n+2)}{n^{\gamma+4}}$.

	$y_{n,4}$	$y_{n,3}$	$y_{n,2}$	$y_{n,1}$
$n = 150$	0.996412	0.999306	0.999978	1.00042
$n = 250$	0.99931	0.999739	0.999991	1.00009
$n = 500$	0.999818	0.999928	0.999999	1.000001

Table 7: **Case** $\gamma = 61/5 = 2(\alpha+2j+1),\ M = 10^6 > V$
$j = 3,\ \alpha = -9/10,\ \beta = -9/10,\ \gamma = 61/5,\ M_n = \frac{10^6n^2(n-1/2)(n+2)}{n^{\gamma+4}}$.

	$n\sqrt{2(1-y_{n,2})}$	$n\sqrt{2(1-y_{n,3})}$	$n\sqrt{2(1-y_{n,4})}$
$n=150$	1.77464	6.0132	9.53661
$n=250$	1.10344	5.71202	9.35539
$n=500$	1.00403	5.58651	9.27349
Limit	$\mathbf{t_{0,1}} = 0.903528$	$\mathbf{t_{0,2}} = 5.34057$	$\mathbf{t_{0,3}} = 9.07889$

Table 8: **Case** $\gamma = 61/5 = 2(\alpha+2j+1)$, $M = 10^6 > V$

$$j = 3,\ \alpha = -9/10,\ \beta = -9/10,\ \gamma = 61/5,\ M_n = \frac{10^6 n^2 (n-1/2)(n+2)}{n^{\gamma+4}}.$$

Finally, we illustrate Theorem 2 plotting the curves corresponding to the limit functions and to the scaled polynomials $Q_n\left(1-\frac{x^2}{2n^2}\right)$ with $n=150$ and $n=500$. In all the figures we have used the same values for the parameters as those ones taken previously in the numerical experiments about the zeros.

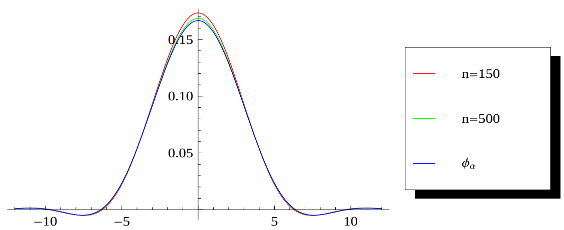

Figure 1: Case $\gamma > 2(\alpha+2j+1)$. Limit function and scaled polynomials $Q_n(1-x^2/(2n^2))$.

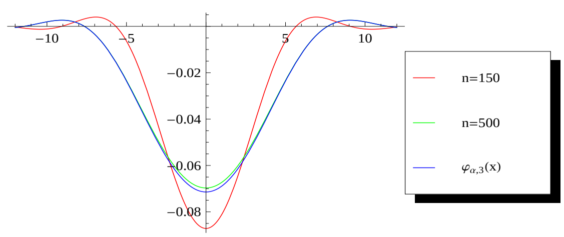

Figure 2: Case $\gamma < 2(\alpha+2j+1)$. Limit function and scaled polynomials $Q_n(1-x^2/(2n^2))$.

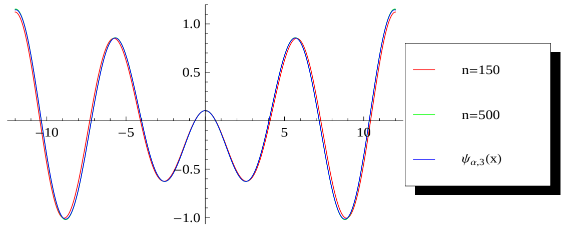

Figure 3: Case $\gamma = 2(\alpha + 2j + 1)$. Limit function and scaled polynomials $Q_n(1 - x^2/(2n^2))$ with $M < V$.

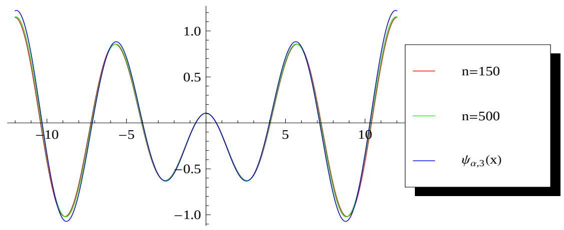

Figure 4: Case $\gamma = 2(\alpha + 2j + 1)$ Limit function and scaled polynomials $Q_n(1 - x^2/(2n^2))$ with $M > V$.

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