A classification of symmetric $(1,1)-$coherent pairs of linear functionals

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Abstract

In this paper we study a classification of symmetric $(1,1)-$coherent pairs by using a symmetrization process. In particular, the positive-definite case is carefully described .

1 Introduction

The concept of coherent pair of measures on the real line was introduced by A. Iserles et al. [20] in the framework of the theory of polynomials orthogonal with respect to a Sobolev inner product associated with a pair of nontrivial positive measures (ν_0,ν_1) supported on the real line. This Sobolev inner product is defined by

$$\langle p,q\rangle_S=\int_{\mathbb{R}}p(x)q(x)d\nu_0(x)+\lambda\int_{\mathbb{R}}p'(x)q'(x)d\nu_1(x), \tag{1}$$

where p and q are polynomials with real coefficients and λ is a nonnegative real number.

The pair of measures (ν_0,ν_1) is said to be coherent if the corresponding sequences of monic orthogonal polynomials $\{P_n(\nu_0;x)\}_{n\geq 0}$ and $\{P_n(\nu_1;x)\}_{n\geq 0}$ satisfy

$$nP_{n-1}(\nu_1;x)=P'_n(\nu_0;x)+a_nP'_{n-1}(\nu_0;x),\quad n\geq 1, \tag{2}$$

with $a_n\neq 0$ for $n\geq 2$. Assuming (2), if $\{S_n(\nu_0,\nu_1;\lambda;x)\}_{n\geq 0}$ denotes the sequence of monic orthogonal polynomials associated with the Sobolev inner product, then there exists a nice algebraic relation with the sequence of monic orthogonal polynomials $\{P_n(\nu_0;x)\}_{n\geq 0}$ with respect to the measure ν_0. Indeed,

$$S_n(\nu_0,\nu_1;\lambda;x)+b_n(\lambda)S_{n-1}(\nu_0,\nu_1;\lambda;x)=P_n(\nu_0;x)+a_nP_{n-1}(\nu_0;x),\quad n\geq 1. \tag{3}$$

H. G. Meijer in [33] proved that if (ν_0,ν_1) is a coherent pair of positive measures supported on the real line, i.e. (2) holds, then one of the measures is classical (Laguerre or Jacobi) and its companion is rational perturbation of it.

What was proved by Meijer [33] is slightly more general than what is stated above. He deals with orthogonal polynomials with respect to a pair of quasi-definite linear functionals on the set of polynomials with real coefficients and he proves that one of such linear functionals must be classical, i.e. either

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Laguerre or Jacobi or Bessel linear functional. Notice that positive definite linear functionals are associated with nontrivial probability measures supported on the real line (see [9]). Thus, Meijer [33] also determines all the possible coherent pairs of positive measures supported in the real line.

The relation (3) is very useful when we study analytic properties of the corresponding Sobolev orthogonal polynomials. In particular, outer relative asymptotics have been deeply analyzed in the literature (see [31], [32] as well as the recent survey [25], where an updated list of references concerning this topic is presented).

In [7], the authors show that there are Sobolev inner products of the type (1) where the pair of measures (ν_0, ν_1) is not coherent but the relation (3) still holds ([7, Thm. 4.1]) or, in other words, a combination of Sobolev orthogonal polynomials as

$$S_n(\nu_0, \nu_1; \lambda; x) + b_n(\lambda) S_{n-1}(\nu_0, \nu_1; \lambda; x), \tag{4}$$

can be written as a linear combination of orthogonal polynomials $P_n(\nu; x)$ and $P_{n-1}(\nu; x)$, where the measure ν is closely related to the measures ν_0 and ν_1 ([7, Thm. 3.1]).

The results obtained in [7] can be covered by extending the concept of coherence (see [22]). It is important to observe that given the Sobolev inner product as (1) if the sequences $\{S_n(\nu_0, \nu_1; \lambda; x)\}_{n\geq 0}$ and $\{P_n(\nu_0; x)\}_{n\geq 0}$ satisfy (3), then

$$nP_{n-1}(\nu_1; x) + c_n P_{n-2}(\nu_1; x) = P_n'(\nu_0; x) + a_n P_{n-1}'(\nu_0; x), \quad n \geq 1, \tag{5}$$

with $a_n \neq 0$, for $n \geq 2$. When (5) holds (see [13]), the pair (ν_0, ν_1) is referred as a $(1,1)$-coherent pair. In this case, one of the measures must be semiclassical of class at most 1 and the other one is a rational perturbation of it. Semiclassical orthogonal polynomials have been introduced in [19] and [29]. A nice survey about this topic is [30]. In particular, the concept of class of the corresponding linear functional plays a central role in the study of the algebraic properties of semiclassical orthogonal polynomials. The class $s = 0$ is constituted by the classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel). The classification of semiclassical orthogonal polynomials of class 1 appears in [5].

If the measures involved in (1) are symmetric, i.e. their odd moments are zero, in [20] the concept of symmetrically coherent pair of measures is introduced. Indeed, a pair of symmetric measures (ν_0, ν_1) is said to be symmetrically coherent if their corresponding sequences of monic orthogonal polynomials $\{P_n(\nu_0; x)\}_{n\geq 0}$ and $\{P_n(\nu_1; x)\}_{n\geq 0}$ satisfy

$$nP_{n-1}(\nu_1; x) = P_n'(\nu_0; x) + c_n P_{n-2}'(\nu_0; x), \quad n \geq 2, \tag{6}$$

with $c_n \neq 0$ for $n \geq 2$.

In [33], H. G. Meijer proved that if (ν_0, ν_1) is a symmetrically coherent pair of positive measures supported on the real line, i.e. (6) holds, then one of such measures is symmetric and classical (Hermite or Gegenbauer) and the other one is a rational perturbation of it. Analytic properties of the corresponding sequences of Sobolev orthogonal polynomials have been studied in the literature (see [25] and the references therein). Indeed, the main tool is the existence of an algebraic relation

$$S_n(\nu_0, \nu_1; \lambda; x) + v_n(\lambda) S_{n-2}(\nu_0, \nu_1; \lambda; x) = P_n(\nu_0; x) + c_n P_{n-2}(\nu_0; x), \quad n \geq 2, \tag{7}$$

where $c_n \neq 0$, for $n \geq 2$ and $\{S_n(\nu_0, \nu_1; \lambda; x)\}_{n\geq 0}$ denotes the sequence of monic orthogonal polynomials associated with the Sobolev inner product (1), where ν_0 and ν_1 are symmetric measures. (7) is an important tool to study such Sobolev orthogonal polynomials. Indeed, in [14] was proved that if (7) holds, then

$$(n+1)P_n(\nu_1; x) + r_n P_{n-2}(\nu_1; x) = P_{n+1}'(\nu_0; x) + c_n P_{n-1}'(\nu_0; x), \quad n \geq 2, \tag{8}$$

with $c_n \neq 0$, for $n \geq 2$. Some examples of symmetric measures whose sequences of orthogonal polynomials satisfy (8) have been studied in [14]. Asymptotic properties of the corresponding sequences of orthogonal polynomials and the location of their zeros are analyzed in [2], [3] for the Gegenbauer case and in [10] and [16] for the Hermite case. The aim of the present contribution is to find all the symmetric pairs of measures such that (8) holds.

Semiclassical symmetric linear functionals of order at most two are the natural framework of our study. They have been analyzed by many authors (see [1], [18], [35], [15], among others). On the other hand, the so called symmetrization process for linear functionals (see [9]) will play a central role in this contribution. In [4], the class of the symmetrized linear functional associated with a semiclassical linear functional has been studied. Notice that this process can also be considered in the framework of Sobolev inner products (see [8]).

Both coherent and symmetrically coherent pairs can be included in a more general problem analyzed in [21] and [26], where the concept of $(M, N)-$ coherence between two sequences of orthogonal polynomials $\{P_n(\nu_0; x)\}_{n\geq 0}$ and $\{P_n(\nu_1; x)\}_{n\geq 0}$ with respect to measures ν_0, ν_1 is defined as follows

$$(n+1)P_n(\nu_1; x) + \sum_{k=1}^{N} r_{n,k}P_{n-k}(\nu_1; x) = P'_{n+1}(\nu_0; x) + \sum_{j=0}^{M} u_{n,j}P'_{n-j}(\nu_0; x), \quad n \geq max\{M, N\}, \tag{9}$$

with $r_{n,N}u_{n,M} \neq 0$. In this context, in [21] and [26], the analysis of the sequences of Sobolev orthogonal polynomials with respect to the inner product (1) has been done.

The structure of this manuscript is the following. In Section 2, the basic background about linear functionals and orthogonal polynomials is presented. A special emphasis on semiclassical linear functionals is given. The symmetrization process for linear functionals is also analyzed. Moreover, the main results about $(1, 1)-$ coherent pairs of measures are summarized. By using a symmetrization process, in Section 3 we study pairs $(\widetilde{u}, \widetilde{v})$ whose respective symmetrized pairs (u, v) are symmetric $(1, 1)--$coherent pairs. Finally, in Section 4 we deduce all positive-definite symmetric $(1, 1)-$coherent pairs (u, v) when either u or v are of class $s \leq 2$.

2 Preliminaries

Let $\mathbf{P}$ be the linear space of polynomials with complex coefficients. Its topological dual space will be denoted by $\mathbf{P}'$. $\mathbf{P}_n$ will represent the linear subspace of polynomials of degree at most n. If $U \in \mathbf{P}'$, then $\langle U, p\rangle$ will denote the action of the linear functional U on the polynomial $p \in \mathbf{P}$. $\{u_n\}_{n\geq 0}$, with $u_n = \langle U, x^n\rangle$ is said to be the sequence of moments associated with U.

Definition 1. *For any polynomial q and $a \in \mathbb{C}$, we define the operator $\theta_a : \mathbf{P} \to \mathbf{P}$ as follows*

$$(\theta_a q)(x) = \frac{q(x) - q(a)}{x - a}. \tag{10}$$

If $U \in \mathbf{P}'$ and $a, b \in \mathbb{C}$, $b \neq 0$, a displacement of U, denoted by $(\tau_a \circ h_b) U$, is defined as follows

$$\langle (\tau_a \circ h_b) U, p(x)\rangle = \langle U, (h_b \circ \tau_{-a}) p(x)\rangle = \langle U, p(bx + a)\rangle, \tag{11}$$

for every $p \in \mathbf{P}$. If $q \in \mathbf{P}$, then the linear functional qU is defined by

$$\langle qU, p\rangle := \langle U, qp\rangle, \quad p \in \mathbf{P}. \tag{12}$$

The linear functional $\delta(x - c)$ given by $\langle \delta(x - c), p\rangle := p(c)$, $p \in \mathbf{P}$, $c \in \mathbb{C}$, is said to be the *Dirac delta* linear functional at c. Let $U \in \mathbf{P}'$ and let $\sigma \in \mathbf{P}$ be a polynomial of degree n with zeros $x_i \in \mathbb{C}$, $1 \leq i \leq r$, of multiplicities n_k, respectively, i.e. $\sum_{k=1}^{r} n_k = n$. Then for every $p \in \mathbf{P}$, we define $\sigma^{-1}(x)U \in \mathbf{P}'$ as follows,

$$\left\langle \sigma^{-1}(x)U, p(x)\right\rangle := \left\langle U, \frac{p(x) - L_\sigma(x; p)}{\sigma(x)}\right\rangle, \tag{13}$$

where $L_\sigma(x; p)$ is the interpolatory polynomial

$$L_\sigma(x; p) = \sum_{i=1}^{r} \sum_{j=0}^{n_i-1} p^{(j)}(x_i)L_{i,j}(x). \tag{14}$$

There $L_{i,j}(x)$ is the polynomial of degree at most $n - 1$ such that $L_{i,j}^{(k)}(x_l) = \delta_{i,l}\delta_{k,j}$, $i, l = 1, \cdots, r$, and $0 \leq k, j \leq n_i - 1$. As an illustrative example when $\sigma(x) = x^2 - \zeta$, with $\zeta > 0$, the zeros of σ are $\pm\sqrt{\zeta}$. Then if $q \in \mathbf{P}$ we get

$$L_{x^2-\zeta}(x;q) = \sum_{i=1}^{2} q(x_i)\frac{x^2-\zeta}{(x-x_i)2x_i} = \frac{x^2-\zeta}{2\sqrt{\zeta}}\left(\frac{q(\sqrt{\zeta})}{(x-\sqrt{\zeta})} - \frac{q(-\sqrt{\zeta})}{(x+\sqrt{\zeta})}\right).$$

Furthermore, if $q(x) = p(x^2)$ we deduce

$$L_{x^2-\zeta}(x;q) = p(\zeta)\frac{x^2-\zeta}{-(x+\sqrt{\zeta})2\sqrt{\zeta}} + p(\zeta)\frac{x^2-\zeta}{(x-\sqrt{\zeta})2\sqrt{\zeta}} = p(\zeta). \tag{15}$$

Besides, if $\sigma(x) = x-\zeta$, then $L_{x-\zeta}(x;p(x)) = p(\zeta)$ and we conclude that

$$L_{x^2-\zeta}(x;p(x^2)) = \ L_{x-\zeta}(x;p(x)) = p(\zeta). \tag{16}$$

On the other hand, if $\sigma(x) = (x-\xi)^n$, i.e. σ has a zero of multiplicity n, then for any linear functional U

$$\left\langle (x-\xi)^{-n}U, p(x)\right\rangle = \left\langle U, \frac{p(x)-T^{\xi}_{n-1}(p)(x)}{(x-\xi)^n}\right\rangle, \tag{17}$$

where $T^{\xi}_{n-1}(p)$ denotes the Taylor polynomial of degree $n-1$ of the polynomial p around $x=\xi$. When $\xi = 0$, we will write $T_{n-1}(p)$.

Definition 2. *Given $a \in \mathbb{C}$ the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)(a+2)\dots(a+n-1)$, $n \geq 1$, and $(a)_0 = 1$.*

Lemma 3. *Let $p \in \mathbf{P}$ and $q(x) = p\left((x-\xi)^2\right)$. Then for $n \geq 0$ we get*

$$T_n(p)((x-\xi)^2) = T^{\xi}_{2n}(q)(x). \tag{18}$$

Remark 4. *Since $T^{\xi}_{2n+1}(q)(x) = T^{\xi}_{2n}(q)(x)$, then also $T_n(p)((x-\xi)^2) = T^{\xi}_{2n+1}(q)(x)$.*

If $U \in \mathbf{P}'$, then the (distributional) derivative of U, denoted by DU, is the linear functional such that

$$\langle DU, p\rangle = \langle U, -p'\rangle, \quad p \in \mathbf{P}.$$

Given $U \in \mathbf{P}'$, U is said to be *quasi-definite* or *regular* (see [9], [30]) if the leading principal submatrices of the Hankel matrix $(u_{i+j})^{\infty}_{i,j=0}$ are non-singular. If all of them have a positive determinant, then U is said to be a *positive definite linear functional.* In this case, there exists a positive Borel measure μ supported on an infinite set $E \subseteq \mathbb{R}$ such that

$$\langle U, p\rangle = \int_E p(x)d\mu(x), \quad p \in \mathbf{P}.$$

Proposition 5. *([9]). Let $U \in \mathbf{P}'$. U is quasi-definite if and only if there exists a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, with $\deg P_n = n$, such that $\langle U, P_nP_m\rangle = 0$, for $n \neq m$, and $\left\langle U, P_n^2\right\rangle \neq 0$, for every $n \in \mathbb{N}$. Such a sequence is said to be a monic orthogonal polynomial sequence, (MOPS in short), with respect to the functional U.*

Proposition 6. *([9]). Let $U \in \mathbf{P}'$ be a quasi-definite linear functional and let $\{P_n\}_{n\geq 0}$ be the corresponding MOPS. If $P_n(0) \neq 0$, for every $n \geq 1$, then $xU \in \mathbf{P}'$ is quasi-definite. Furthermore, if $\{\widetilde{P}_n\}_{n\geq 0}$ is the corresponding MOPS, then*

$$\widetilde{P}_n(x) = x^{-1}\left(P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)}P_n(x)\right).$$

Moreover, if U is positive-definite in $[a,b]$, then xU is also positive-definite on $[a,b]$ if and only if $a \geq 0$. The polynomial $\widetilde{P}_n$ is called the n-th monic Kernel polynomial corresponding to U whit $\kappa-$parameter 0.

The above proposition defines a mapping in the linear space of quasi-definite linear functionals. A natural question can be posed. ¿Is this mapping one-to-one ?. The answer is no. It is well known that there exist infinitely many MOPS generating the same sequence of Kernel polynomials of $\kappa-$parameter. The next result gives the answer to this question.

Theorem 7. *([23]). Let $u \in \mathbf{P}'$ be a quasi-definitelinear functional and $P_n\}_{n\geq 0}$ its corresponding MOPS. Let $v \in \mathbf{P}'$ be the linear functional $v := u + M\delta(x-a)$, with $M \in \mathbb{C}$, $a \in \mathbb{R}$. Then v is quasi-definite if and only if $d_n := 1+MK_n(a,a) \neq 0$, where $K_n(x,y)$ is the $n-th$ Kernel polynomial associated with u. Besides $\{R_n\}_{n\geq 0}$, the MOPS associated with v, satisfies*

$$R_n(x) = P_n(x) - M\frac{P_n(a)}{d_{n-1}}K_{n-1}(x,a), \quad n \geq 0,$$

with $d_1 = 1$ and $K_{-1}(x,y) = 0$.

To conclude this section we state a lemma which will be needed later on.

Lemma 8. *If $u, v \in \mathbf{P}'$ are related by $xv = u + M\delta(x-a) + N\delta(x)$, $M, a \neq 0$, then $v = x^{-1}u + \frac{M}{a}\delta(x-a) + (\langle v, 1\rangle - \frac{M}{a})\delta(x) - N\delta'(x)$.*

Proof. For any polynomial p, it is enough to consider the action of the linear functional xv, defined as above, on $q(x) := \frac{p(x)-p(0)}{x}$. □

2.1 Semiclassical and classical linear functionals

Let ϕ and σ be two nonzero polynomials such that $\deg(\phi) = m \geq 0$, and $\deg(\phi) = n \geq 1$ with leading coefficients a_m and b_n respectively. (ϕ, σ) is said to be an *admissible pair* if either $m-1 \neq n$ or, if $m-1 = n$ then $ka_{n+1} - b_n \neq 0$ for every $k \in \mathbb{N}$. $U \in \mathbf{P}'$ is said to be a *semiclassical* linear functional if there exists an admissible pair $(\phi\ , \psi)$, where ϕ is monic, such that the following differential relation holds,

$$D\left(\phi U\right) + \psi U = 0, \qquad \text{(Pearson equation)} \tag{19}$$

If $U \in \mathbf{P}'$ is a semiclassical linear functional, then the nonnegative integer number

$$s = \min_{\Phi} \max\{\deg\phi - 2, \deg\psi - 1\},$$

is said to be the class of U. Here Φ denotes the set of all admissible pairs of nonzero polynomials (ϕ, σ) such that (19) holds. With respect to the class of a semiclassical linear functional we describe the next irreducibility condition.

Proposition 9. *([30]). Suppose that $U \in \mathbf{P}'$ is semiclassical and $D\left(\phi U\right) + \psi(x)U = 0$. The class of U is a non-negative real number $s = \max\{\deg\phi - 2, \deg\psi - 1\}$ if and only*

$$|\phi'(c) + \psi(c)| + \left|\left\langle U, \theta_c\psi + \theta_c^2\phi\right\rangle\right| > 0, \tag{20}$$

for every zero c of ϕ.

Next, we summarize some characterizations of semiclassical linear functionals.

Theorem 10. *(see [30]). Let u be a quasi-definite linear functional and $\{P_n\}_{n\geq 0}$ the corresponding MOPS. u is semiclassical of class $\underset{\sim}{s}$ if and only if one of the next equivalent conditions holds.*

A). There exists a polynomial $\widetilde{\phi}$, with $\deg(\widetilde{\phi}) = t \leq s+2$, such that the MOPS $\{P_n\}_{n\geq 0}$ satisfies

$$\widetilde{\phi}(x)\frac{P'_{n+1}(x)}{n+1} = \sum_{k=n-s}^{n+t} a_{n,k}P_k(x), \quad n \geq s, \tag{21}$$

with $a_{n,n-s} \neq 0$ and $n \geq s+1$.

B). There exists a monic polynomial $\widetilde{\phi}$ such that the sequence $\left\{\frac{P'_{n+1}(x)}{n+1}\right\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to $\widetilde{\phi}u$, i.e.

$$\left\langle \widetilde{\phi}u, x^k\frac{P'_{n+1}(x)}{n+1}\right\rangle = 0, \quad k \leq n-s-1,$$

and

$$\left\langle \widetilde{\phi}u, x^{n-s}\frac{P'_{n+1}(x)}{n+1}\right\rangle \neq 0.$$

Remark 11. *Notice that the classification of semiclassical quasi-definite linear functionals of class $s = 1$ is given in [5]. The semiclassical linear functionals of class 2 are described in [27].*

$U \in \mathbf{P}'$ is said to be *classical* if its class is $s = 0$, i.e. there exist non zero polynomials ϕ and ψ, with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$, such that (19) holds. In this case, the MOPS associated with U is called a *classical* MOPS. Up to an affine transformation on the variable, Hermite, Laguerre, Bessel and Jacobi polynomials are the classical MOPS, (see Table 1). Besides, except the Bessel polynomials, if U is classical then, under certain restrictions on the parameters, it is positive-definite and it has a integral representation with respect to a weight function ω on an interval (a, b), as described in Table 2.

Linear Functional	$\phi; \psi$	Restriction on the parameters
$\mathcal{H}$, (*Hermite*)	$1; 2x$	$--$
$\mathcal{L}^{(\alpha)}$, (*Laguerre*)	$x; -x-\alpha-1,$	$-\alpha \notin \mathbb{N},$
$\mathcal{B}^{(\alpha)}$, (*Bessel*)	$x^2; -2(\alpha x+1),$	$-\alpha \notin \mathbb{N}$
$\mathcal{J}^{(\alpha,\beta)}$, (*Jacobi*)	$x^2-1;$ $-(\alpha+\beta+2)x+\alpha-\beta$	$-\alpha, -\beta \notin \mathbb{N},$ $-\alpha-\beta \notin \mathbb{N}\backslash\{1\}$

Table 1: Quasi-definite Classical Orthogonal Polynomials

Linear Functional	(a, b)	$\omega(x)$	Restriction on the parameters
$\mathcal{H}$	$(-\infty, \infty)$	e^{-x^2}	$--$
$\mathcal{L}^{(\alpha)}$	$[0, \infty)$	$x^\alpha e^{-x},$	$\alpha > -1$
$\mathcal{J}^{(\alpha,\beta)}$	$[-1, 1]$	$(1-x)^\alpha(1+x)^\beta,$	$\alpha, \beta > -1$

Table 2: Positive- Definite Classical Orthogonal Polynomials

The shifted Jacobi functional on a finite interval $[a, b]$ will be denoted by $\mathcal{J}_{[a,b]}^{(\alpha,\beta)}$, and $\mathcal{J}_{[-1,1]}^{(\alpha,\beta)} := \mathcal{J}^{(\alpha,\beta)}$. Also, the shifted Laguerre functional on $[a, \infty)$ will be denoted by $\mathcal{L}_{[a,\infty)}^{(\alpha)}$, and $\mathcal{L}_{[0,\infty)}^{(\alpha)} := \mathcal{L}^{(\alpha)}$. In this way, the Jacobi functional $\mathcal{J}_{[a,b]}^{(\alpha,\beta)}$ satisfies

$$D\left[(x-a)(x-b))\mathcal{J}_{[a,b]}^{(\alpha,\beta)}\right] = ((\alpha+\beta+2)x + [a(\alpha+1)+b(\beta+1)])\,\mathcal{J}_{[a,b]}^{(\alpha,\beta)},$$

and

$$\left\langle \mathcal{J}_{[a,b]}^{(\alpha,\beta)}, p(x)\right\rangle = \int_a^b p(x)\,(b-x)^\alpha\,(x-a)^\beta\,dx, \qquad p \in \mathbf{P}.$$

The Laguerre functional $\mathcal{L}_{[a,\infty)}^{(\alpha,\beta)}$ satisfies

$$D\left((x-a)\mathcal{L}_{[a,\infty)}^{\alpha}\right) = (-x+\alpha+a+1)\,\mathcal{L}_{[a,\infty)}^{\alpha},$$

and

$$\left\langle \mathcal{L}_{[a,\infty)}^{\alpha}, p(x)\right\rangle = \int_a^\infty p(x)e^{-x}(x-a)^\alpha dx, \qquad p \in \mathbf{P}.$$

2.2 Symmetric Linear Functionals

A linear functional $U \in \mathbf{P}'$ is called symmetric if $u_{2n+1} = \left\langle U, x^{2n+1}\right\rangle = 0$, for every $n \in \mathbb{N}$. (See [9] for other characterizations of symmetric regular linear functionals). If $U \in \mathbf{P}'$ is symmetric and quasi-definite and $\{P_n\}_{n\geq 0}$ is its corresponding MOPS, then we can define $\widetilde{u} \in \mathbf{P}'$ by

$$\langle \widetilde{u}, x^n \rangle = \left\langle U, x^{2n}\right\rangle, \quad n \in \mathbb{N}, \tag{22}$$

and the sequences of monic polynomials $\{A_n\}_{n\geq 0}$ and $\{\widetilde{A}_n\}_{n\geq 0}$, by

$$P_{2n}(x) = A_n(x^2) \quad \text{and} \quad P_{2n+1}(x) = x\widetilde{A}_n(x^2). \tag{23}$$

Theorem 12. *([9]). If $U \in \mathbf{P}'$ is a symmetric and quasi-definite linear functional and $\{P_n\}_{n\geq 0}$ is its corresponding MOPS, then $\widetilde{u}$, defined by (22), is quasi-definite. Besides, $\{A_n\}_{n\geq 0}$ and $\{\widetilde{A}_n\}_{n\geq 0}$ defined by (23) are the MOPS with respect to $\widetilde{u}$ and $x\widetilde{u}$, respectively.*

Conversely, if $\widetilde{u} \in \mathbf{P}'$ is quasi-definite, we can define the symmetric linear functional $U \in \mathbf{P}'$ given by

$$\left\langle U, x^{2n}\right\rangle = \left\langle \widetilde{u}, x^n\right\rangle \text{ and } \left\langle U, x^{2n+1}\right\rangle = 0, \quad n \geq 0. \tag{24}$$

Theorem 13. *([9]). If $\widetilde{u}$ and $x\widetilde{u}$ are a quasi-definite linear functionals on $\mathbf{P}'$ and $\{A_n\}_{n\geq 0}$ and $\{\widetilde{A}_n\}_{n\geq 0}$ are their corresponding MOPS, then the symmetric linear functional $U \in \mathbf{P}'$ defined by (24) is quasi-definite and its MOPS $\{P_n\}_{n\geq 0}$ is given by (23).*

Remark 14. *Notice that $\{\widetilde{A}_n\}_{n\geq 0}$ are the Kernel polynomials with $\kappa-$parameter 0, associated with $\widetilde{u}$. Besides U is called the symmetrized linear functional of $\widetilde{u}$.*

Theorem 15. *([9]). U is positive definite on $\left[-\sqrt{b}, \sqrt{b}\right]$ if and only if $\widetilde{u}$ and $x\widetilde{u}$ are positive-definite on $[a, b]$ with $a \geq 0$.*

Now, we deduce some interesting consequences of (13).

Lemma 16. *Let U be the symmetrization of $\widetilde{u} \in \mathbf{P}'$. Let σ be a polynomial with nonzero simple zeros. Then for every polynomial q we get*

$$\left\langle \sigma^{-1}(x^2)U, q(x^2)\right\rangle = \left\langle \sigma^{-1}(x)\widetilde{u}, q(x)\right\rangle. \tag{25}$$

Proof. If $\sigma(x) = \prod_{i=1}^{k}(x - x_i)$, let $\overline{\sigma}(x) = \sigma(x^2) = \prod_{i=1}^{2k}(x - y_i)$, where $y_{2j} = \sqrt{x_j}$ and $y_{2j-1} = -\sqrt{x_j}$ for $j = 1, ..., k$. Then from (13)

$$\left\langle \sigma^{-1}(x^2)U, q(x^2)\right\rangle = \left\langle \overline{\sigma}^{-1}(x)U, q(x^2)\right\rangle = \left\langle U, \frac{q(x^2) - L_{\overline{\sigma}}(x; q(x^2))}{\overline{\sigma}(x)}\right\rangle,$$

and from (14)

$$\begin{aligned} L_{\overline{\sigma}}(x; q(x^2)) &= \sum_{i=1}^{2k} q(y_i^2)\frac{\overline{\sigma}(x)}{(x - y_i)\overline{\sigma}'(y_i)} \\ &= \sum_{i=1}^{k} \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)2y_i}\left[\frac{1}{(x - \sqrt{x_i})} - \frac{1}{(x + \sqrt{x_i})}\right] \\ &= \sum_{i=1}^{k} \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)(x^2 - x_i)}, \end{aligned}$$

then

$$\left\langle \sigma^{-1}(x^2)U, q(x^2)\right\rangle = \left\langle U, \frac{q(x^2) - \sum_{i=1}^{k} \frac{q(x_i)\sigma(x^2)}{\sigma'(x_i)(x^2 - x_i)}}{\sigma(x^2)}\right\rangle.$$

Besides, since U is the symmetrization of $\widetilde{u}$, then for any polynomial p, $\left\langle U, p(x^2)\right\rangle = \left\langle \widetilde{u}, p(x)\right\rangle$, and, as a consequence,

$$\begin{aligned} \left\langle \sigma^{-1}(x^2)U, q(x^2)\right\rangle &= \left\langle \widetilde{u}, \frac{q(x) - \sum_{i=1}^{k} \frac{q(x_i)\sigma(x)}{\sigma'(x_i)(x - x_i)}}{\sigma(x)}\right\rangle \\ &= \left\langle \widetilde{u}, \frac{q(x) - L_{\sigma}(x; q(x))}{\sigma(x)}\right\rangle \\ &= \left\langle \sigma^{-1}(x)\widetilde{u}, q(x)\right\rangle. \end{aligned}$$

□

Given a semiclassical quasi-definite linear functional $\widetilde{u}$, the semiclassical character of the symmetrized linear functional of $\widetilde{u}$, its class and the respective Pearson equation are described in the next theorem.

Theorem 17. *([4]). Let $\widetilde{u} \in \mathbf{P}'$ be semiclassical of class $\widetilde{s}$ satisfying the Pearson equation*

$$D\left[\widetilde{\phi}(x)\widetilde{u}\right] + \widetilde{\psi}(x)\widetilde{u} = 0, \tag{26}$$

and $x\widetilde{u}$ is a quasi-definite linear functional, and $\widehat{\Psi}(x) := \widetilde{\phi}'(x) + 2\widetilde{\psi}(x)$. Then U, the symmetrization of $\widetilde{u}$, is semiclassical of class s satisfying the Pearson equation

$$D\left[\phi(x)U\right] + \psi(x)U = 0, \tag{27}$$

where the number s and the polynomials ϕ and ψ are defined according with next cases:
***i)**. If*

$$\widetilde{\phi}(0) = 0 \quad \textit{and} \quad \widehat{\Psi}(0) = 0, \tag{28}$$

then

$$\phi(x) = (\theta_0\widetilde{\phi})(x^2), \quad \psi(x) = x\left[2(\theta_0\widetilde{\psi})(x^2) + (\theta_0^2\widetilde{\phi})(x^2)\right] \tag{29}$$

and $s = 2\widetilde{s}$.
***ii)**. If*

$$\widetilde{\phi}(0) = 0 \quad \textit{and} \quad \widehat{\Psi}(0) \neq 0, \tag{30}$$

then

$$\phi(x) = x(\theta_0\widetilde{\phi})(x^2), \;\; \psi(x) = 2\widetilde{\psi}(x^2)$$

and $s = 2\widetilde{s} + 1$.
***iii)**. If $\widetilde{\phi}(0) \neq 0$, then*

$$\phi(x) = x\widetilde{\phi}(x^2), \quad \psi(x) = 2\left[x^2\widetilde{\psi}(x^2) - \widetilde{\phi}(x^2)\right]$$

and $s = 2\widetilde{s} + 3$.

Corollary 18. *If s is odd, the polynomials ϕ and ψ in (27) are, respectively, odd and even functions. If s is even, the polynomials ϕ and ψ in (27) are, respectively, even and odd functions.*

In the cases $\mathcal{J}_{[0,1]}^{(\alpha,\beta)}$ and $\mathcal{L}^{(\alpha)}$, where the weight functions are $\omega(x) = (1-x)^\alpha x^\beta$ on $[0,1]$ and $\omega(x) = x^\alpha e^{-x}$ on $[0,\infty)$, respectively, the new weight functions associated with the symmetrized linear functionals $\overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}$ and $\overline{\mathcal{L}}^{(\alpha)}$ are $\omega(x) = (1-x^2)^\alpha |x|^{2\beta+1}$ on $[-1,1]$ (the Generalized Gegenbauer weight) and $\omega(x) = |x|^{2\alpha+1} e^{-x^2}$ on $\mathbb{R}$ (the Generalized Hermite weight), respectively. Notice that $\left\langle \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}, p(x^2)\right\rangle = \left\langle \mathcal{J}_{[0,1]}^{(\alpha,\beta)}, p(x)\right\rangle$ and $\left\langle \overline{\mathcal{L}}^{(\alpha)}, p(x^2)\right\rangle = \left\langle \mathcal{L}^{(\alpha)}, p(x)\right\rangle$ for any polynomial p.

If u is a positive-definite linear functional, with weight function ω on the interval I, and if p and q are polynomials, then the linear functional with weight $\frac{|p(x)|}{|q(x)|}\omega(x)$, (provided this is a weight function), will be represented by $\frac{|p(x)|}{|q(x)|}u$.

Remark 19. *In the symmetric framework, the quasi-definite semiclassical linear functionals of class 1 are described in [4] as well as in [15] the symmetric quasi-definite linear functionals of class 2 are given. Finally, examples of symmetric semiclassical linear functionals of class 3 are studied in [28].*

2.3 $(1,1)-$coherent pairs

In [13] the $(1,1)-$coherence relation

$$\frac{T_{n+1}'(x)}{n+1} + \widetilde{a}_{n-1}\frac{T_n'(x)}{n} = Q_n(x) + \widetilde{b}_{n-1}Q_{n-1}(x), \quad \widetilde{a}_{n-1} \neq 0, \quad n \geq 1. \tag{31}$$

is studied, where $\{T_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are orthogonal with respect to quasi-definite linear functionals u and v, respectively. In such a paper, the following result is proved.

Theorem 20. *([13]). If (u,v) is a $(1,1)-$coherent pair given by (31), such that $\widetilde{a}_0 \neq \widetilde{b}_0$, (or, equivalently, $Q_n \neq \frac{T'_{n+1}}{n+1}$, for $n \geq 1$), then*

i). *Either u is a semiclassical linear functional of class at most 1, i. e. there exist polynomials $\widetilde{\beta}$ and φ with $\deg(\widetilde{\beta}) \leq 3$ and $\deg(\varphi) \leq 2$ such that*

$$D[\widetilde{\beta}u] = -\varphi u.$$

ii). *Or v is a semiclassical linear functional of class at most 1, i. e. there exist polynomials $\widetilde{\beta}$ and φ with $\deg(\widetilde{\beta}) \leq 3$ and $\deg(\varphi) \leq 2$ such that*

$$D[\widetilde{\beta}v] = -\varphi v.$$

Furthermore there exists a constant ζ such that the pair (u,v) satisfy

$$(x-\zeta)v = \widetilde{\beta}u. \tag{32}$$

Moreover, in [13] all $(1,1)-$ coherent pairs of linear functionals were determined. Besides, in each case **i)** and **ii)** of the above theorem the pair (u,v) is called either *type I* or *type II*, respectively.

2.4 Symmetric $(1,1)-$coherent pairs

From now on, U and V will denote two symmetric quasi-definite linear functionals and $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ will be their corresponding MOPS, respectively. For the above linear functionals the normalization $\langle U,1\rangle = \langle V,1\rangle = 1$ is assumed as well as the existence of sequences of non-zero real numbers $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, with $a_n \neq 0$, such that

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x), \quad n \geq 0, \tag{33}$$

holds. In this case, (U,V) is said to be a *symmetric* $(1,1)-$*coherent pair.*

In [14] the relation (33) is studied. Indeed, when U is a classical linear functional the authors obtain the coefficients of the three term recurrence relation that the MOPS $\{R_n\}_{n\geq 0}$ satisfies. Besides, its companion linear functional is a rational modification of the U.

Lemma 21. *([13]) Let (U,V) be a symmetric $(1,1)-$coherent pair given by (33). $a_0 \neq b_0$ and $a_1 \neq b_1$ if and only if $R_n \neq \frac{P'_{n+1}}{n+1}$ for $n \geq 2$.*

Since U and V are symmetric quasi-definite linear functionals, from (23) we can define

$$P_{2m}(x) = A_m(x^2) \quad \text{and} \quad P_{2m+1}(x) = x\widetilde{A}_m(x^2),$$
$$R_{2m}(x) = B_m(x^2) \quad \text{and} \quad R_{2m+1}(x) = x\widetilde{B}_m(x^2),$$

where $\{A_n\}_{n\geq 0}$, $\{\widetilde{A}_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, and $\{\widetilde{B}_n\}_{n\geq 0}$ are the MOPS with respect to $\widetilde{u}$, $x\widetilde{u}$, $\widetilde{v}$ and $x\widetilde{v}$, respectively.

Next, we will deduce some relevant results to be used in the sequel. For $n \geq 0$ and from (33) we obtain

$$\frac{P'_{2n+3}(x)}{2n+3} = R_{2n+2}(x) + \sum_{j=0}^{n}(-1)^j \left(\prod_{k=0}^{j} \widetilde{u}_{2n-2(k-1)}\right)(s_{2n-2j} - u_{2n-2j})\, R_{2n-2j}(x),$$

where

$$\widetilde{u}_{2n-2(k-1)} = \begin{cases} u_{2n-2(k-1)}, & 1 \leq k \leq j, \\ 1, & k = 0, \end{cases}$$

and for $n \geq 1$

$$\frac{P'_{2n+2}(x)}{2n+2} = R_{2n+1}(x) + \sum_{j=0}^{n-1}(-1)^j \left(\prod_{k=0}^{j} \widetilde{u}_{2n-2k+1}\right)(s_{2n-1-2j} - u_{2n-1-2j})\, R_{2n-1-2j}(x),$$

where

$$\widetilde{u}_{2n-2k+1} = \begin{cases} u_{2n-2k+1}, & 1 \le k \le j, \\ 1, & k = 0. \end{cases}$$

Let define $r_{2n+1}(x) := R_{2n+1}(x) + A_{2n+1}x$, $n \ge 1$. Then

$$\begin{aligned} 0 = & \left\langle \left(R_{2n+1} + A_{2n+1}x\right) v, \frac{P'_{2n+4}}{2n+4} \right\rangle \\ = & \left(s_{2n+1} - u_{2n+1}\right) \left\langle v, R_{2n+1}^2 \right\rangle + A_{2n+1}(-1)^n \left(\prod_{k=0}^{n-1} u_{2n+1-2k} \right) (s_1 - u_1) \left\langle v, R_1^2 \right\rangle \end{aligned}$$

if and only if

$$A_{2n+1} = \frac{\left(s_{2n+1} - u_{2n+1}\right) \left\langle v, R_{2n+1}^2 \right\rangle}{(-1)^{n+1} \left(\prod_{k=0}^{n-1} u_{2n+1-2k} \right) (s_1 - u_1) \left\langle v, R_1^2 \right\rangle}, \quad n \ge 1,$$

and, inductively, we can prove that $\left\langle \left(R_{2n+1} + A_{2n+1}x\right) v, P'_{2n+2k} \right\rangle = 0$ for $k \ge 2$. On the other hand, for $n \ge 1$ we define $r_{2n}(x) := R_{2n}(x) + A_{2n}$. Then

$$\begin{aligned} & \left\langle r_{2n} v, \frac{P'_{2n+3}}{2n+3} \right\rangle \\ = & \left(s_{2n} - u_{2n}\right) \left\langle v, R_{2n}^2 \right\rangle + A_{2n} \left\langle v, (-1)^n \left(\prod_{k=0}^{n} \widetilde{u}_{2n-2(k-1)} \right) (s_0 - u_0) \right\rangle = 0 \end{aligned}$$

if and only if

$$A_{2n} = \frac{\left(s_{2n} - u_{2n}\right) \left\langle v, R_{2n}^2 \right\rangle}{(-1)^{n+1} \left(\prod_{k=1}^{n} u_{2n-2(k-1)} \right) (s_0 - u_0)}.$$

Also, $\left\langle r_{2n} v, \frac{P'_k}{k} \right\rangle = 0$, $k \ge 2n+2$. On the other hand, let consider the linear functional $r_{2n+1}v$ and its expansion in terms of the dual basis $\{\widehat{U}_n\}_{n\ge 0}$ associated with $\{\frac{P'_{n+1}}{n+1}\}_{n\ge 0}$. Namely,

$$r_{2n+1} v = \sum_{k=0}^{\infty} \widetilde{\lambda}_{nk} \widehat{U}_k, \qquad \widetilde{\lambda}_{nk} = \left\langle r_{2n+1} v, \frac{P'_{k+1}}{k+1} \right\rangle = 0 \quad , \quad k \ge 2n+2,$$

where $\widetilde{\lambda}_{nk} = \left\langle r_{2n+1} v, \frac{P'_{k+1}}{k+1} \right\rangle = 0$, if $k+1$ is odd. If $r_{2n+1} v = \sum_{k=0}^{n} \widetilde{\lambda}_{n,2k+1} \widehat{U}_{2k+1}$, then we can apply the distributional derivative in both hand sides and we obtain

$$D\left[r_{2n+1} v\right] = -\sum_{k=0}^{n} \widetilde{\lambda}_{n,2k+1} (2k+2) U_{2k+2},$$

where $\{U_n\}_{n\ge 0}$ is the dual basis associated with $\{P_n\}_{n\ge 0}$. Since $U_m = \frac{P_m}{\langle u, P_m^2 \rangle} u$, then

$$D\left[r_{2n+1} v\right] = -\left(\sum_{k=0}^{n} \widetilde{\lambda}_{n,2k+1} (2k+2) \frac{P_{2k+2}}{\left\langle u, P_{2k+1}^2 \right\rangle} \right) u.$$

In an analog way, for $n \ge 1$, we consider $r_{2n}v$ and, as above,

$$r_{2n} v = \sum_{k=0}^{\infty} \lambda_{nk} \widehat{U}_k, \qquad \lambda_{nk} = \left\langle r_{2n} v, \frac{P'_{k+1}}{k+1} \right\rangle = 0, \text{ if } k \ge 2n+3,$$

and $\lambda_{nk} = \left\langle r_{2n} v, \frac{P'_{k+1}}{k+1} \right\rangle = 0$, if $k+1$ is even. Then $D\left[r_{2n} v\right] = -\sum_{k=0}^{n} \lambda_{n,2k} (2k+1) U_{2k+1}$, and

$$D\left[r_{2n} v\right] = -\left(\sum_{k=0}^{n} \lambda_{n,2k} (2k+1) \frac{P_{2k+1}(x)}{\left\langle u, P_{2k+1}^2 \right\rangle} \right) u.$$

Next we summarize the above results.

Proposition 22. *For $m \geq 2$ there exist polynomials r_m and ϕ_{m+1}, with $\deg r_m = m$, $\deg \phi_{m+1} \leq m+1$, and such that*

$$D[r_m v] = -\phi_{m+1} u, \qquad m \geq 2, \tag{34}$$

with $r_m(x) := R_m(x) + A_m x^{\left(\frac{1-(-1)^m}{2}\right)}$,

$$A_m = \frac{(s_m - u_m)\langle v, R_m^2 \rangle}{(-1)^{[m/2]+1} \left(\prod_{k=1}^{n} u_{m+2-2k} \right) \left(s_{\frac{1-(-1)^m}{2}} - u_{\frac{1-(-1)^m}{2}} \right) \left\langle v, R^2_{\frac{1-(-1)^m}{2}} \right\rangle}.$$

Moreover

$$\phi_{2n+2}(x) = \sum_{k=0}^{n} \frac{\widetilde{\lambda}_{n,2k+1}(2k+2)}{\langle u, P^2_{2k+2} \rangle} P_{2k+2}(x), \qquad n \geq 1, \tag{35}$$

and

$$\phi_{2n+1}(x) = \sum_{k=0}^{n} \frac{\lambda_{n,2k}(2k+1)}{\langle u, P^2_{2k+1} \rangle} P_{2k+1}(x), \qquad n \geq 1. \tag{36}$$

3 Symmetric $(1,1)-$coherent pairs and symmetrization

The concept of symmetric $(1,1)-$coherent pair was introduced in [13] where, among others, the relation between connection coefficients in the coherence relation and recurrence coefficients for the MOPS, and the particular case when u is classical, are deeply studied. The associated inverse problem is solved in [12], namely,

Theorem 23. *Let (u, v) be a symmetric (1,1)-coherent pair. There exist polynomials A, B and C with* $\deg(A) = 4$, $\deg(B) \leq 5$ *and* $\deg(C) \leq 6$, *such that*

$$A(x)Dv = B(x)u, \tag{37}$$

$$B(x)v = C(x)Dv, \tag{38}$$

$$xC(x)u = xA(x)v, \tag{39}$$

where

$$A(x) = \frac{r_4'(x)r_2(x) - r_4(x)r_2'(x)}{x}, \tag{40}$$

$$B(x) = \frac{r_2'(x)\phi_5(x) - r_4'(x)\phi_3(x)}{x}, \tag{41}$$

$$C(x) = \frac{r_4(x)\phi_3(x) - r_2(x)\phi_5(x)}{x}. \tag{42}$$

Depending on the nature of the zeros of A, it is possible to refine the rational relation (39). Besides, according to (40), A is an even function. In this way, either $A(x) = 2(x^2 - \xi_1^2)(x^2 - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$, or $A(x) = 2\left(x^2 - \xi^2\right)^2$. In the sequel, we will assume that $\xi_1^2, \xi_2^2 \in \mathbb{R}$. Next, we study each case. Before that some technical lemma that describe how the symmetrization process allows us to take the current problem to simpler case.

Definition 24. *Given an even polynomial p of degree n, the polynomial p^E, with of deg$(p^E) = n/2$, is defined as $p^E(x^2) := p(x)$*

Lemma 25. ***i).*** *Let u and v be the symmetrized of $\widetilde{u}$ and $\widetilde{v}$, respectively. If ϕ and ψ are even polynomials such that $\phi u = \psi v$, holds, then $\phi^E \widetilde{u} = \psi^E \widetilde{v}$. Besides the converse also holds.*

ii). *If u and v satisfy $D(x\phi u) = \psi v$, where ϕ and ψ are even polynomials, then $D\left(x\phi^E \widetilde{u}\right) = \frac{1}{2}\psi^E \widetilde{v}$.*

3.1 Case $A(x)=2\left(x^2-\xi^2\right)^2$

If $A(x)=2\left(x^2-\xi^2\right)^2$, then (39) can be written as $xC(x)u=2x\left(x^2-\xi^2\right)^2 v$ as well as $r_2(x)=x^2-a^2$ and $r_4(x)=x^4+fx^2+g$. As a consequence, $A(x)=2x^4-4x^2\xi^2+2\xi^4$, where $\xi^2=a^2$ and $fa^2+g=\xi^4$. Thus, $r_2(x)=x^2-\xi^2$. Since $(xA(x))'=r_4''(x)r_2(x)-r_4(x)r_2''(x)$, we deduce $r_4''(\xi)r_2(\xi)-2r_4(\xi)=0$. From this expression, taking into account $r_2(\xi)=0$, we get $r_4(\xi)=0$. As a consequence, $r_4(x)=r_2(x)\rho_2(x)$, where $\rho_2(x):=x^2-\delta$. From (42),

$$C(x)=r_4(x)\frac{\phi_3(x)}{x}-r_2(x)\frac{\phi_5(x)}{x}=r_2(x)\left(\rho_2(x)\frac{\phi_3(x)}{x}-\frac{\phi_5(x)}{x}\right)=r_2(x)\sigma_4(x).$$

According to (41), $B(x)=2\phi_5(x)-2\left(\rho(x)+r_2(x)\right)\phi_3(x)$, then from (37) we get

$$r_2^2(x)Dv=\left(\phi_5(x)-\left(x^2+r_2(x)\right)\phi_3(x)\right)u.$$

For $m=2$, multiplying (34) by $r_2(x)$ we deduce, $r_2^2(x)Dv=-r_2(x)\phi_3(x)u-2xr_2(x)v$. On the other hand, from the above expressions we get

$$\left(\phi_5(x)-\left(\rho(x)+r_2(x)\right)\phi_3(x)\right)u+r_2(x)\phi_3(x)u+2xr_2(x)v=0,$$

i.e.

$$\left(\phi_5(x)-\rho(x)\phi_3(x)\right)u+2xr_2(x)v=0.$$

Thus, we get

$$x\sigma_4(x)u=2xr_2(x)v, \tag{43}$$

where

$$\sigma_4(x)=\rho(x)\frac{\phi_3(x)}{x}-\frac{\phi_5(x)}{x}. \tag{44}$$

From (43), if a symmetric $(1,1)-$coherent pair (u,v) satisfies (43), then

$$x^2\sigma_4(x)u=2x^2r_2(x)v. \tag{45}$$

Through the symmetrization process, we can find pairs (u,v) of symmetric linear functionals such that (45) holds. Among such pairs, we will identify all the symmetric $(1,1)-$ coherent ones later on.

Lemma 26. ***i).*** *For $m=2n$, (34) implies*

$$xD\left(r_{2n}^E(x)\widetilde{v}\right)=-\frac{1}{2}r_{2n}^E(x)\widetilde{v}-\frac{1}{2}x\widetilde{\phi}_{2n}^E(x)\widetilde{u},$$

where $\phi_{2n+1}(x):=x\widetilde{\phi}_{2n}(x)$.

ii). *For $m=2n+1$, (34) yields*

$$D\left(x\widetilde{r}_{2n}^E(x)\widetilde{v}\right)=-\frac{1}{2}x\phi_{2n+2}^E(x)\widetilde{u},$$

where $r_{2n+1}(x)=x\widetilde{r}_{2n}(x)$.

Proof. We will prove ii). The proof of i) is similar. The Pearson type relation is equivalent to

$$D(x\widetilde{r}_{2n}(x)v)=-\phi_{2n+2}(x)u. \tag{46}$$

For every polynomial p,

$$\begin{aligned}\left\langle D\left(x\widetilde{r}_{2n}^E(x)\widetilde{v}\right),p(x)\right\rangle &= -\left\langle \widetilde{v},x\widetilde{r}_{2n}^E(x)p'(x)\right\rangle\\ &= -\left\langle v,x^2\widetilde{r}_{2n}(x)p'(x^2)\right\rangle\\ &= \frac{1}{2}\left\langle D\left(x\widetilde{r}_{2n}(x)v\right),p(x^2)\right\rangle,\end{aligned}$$

and from (46)

$$\begin{aligned}\left\langle D\left(x\widetilde{r}_{2n}^E(x)\widetilde{v}\right),p(x)\right\rangle &= -\frac{1}{2}\left\langle \phi_{2n+2}(x)u,p(x^2)\right\rangle\\ &= -\frac{1}{2}\left\langle \phi_{2n+2}^E(x)\widetilde{u},p(x)\right\rangle.\end{aligned}$$

Thus, our statement follows. □

From the previous lemma, we get that $D(r_2(x)v) = -\phi_3(x)u$ implies

$$xD\left(r_2^E(x)\widetilde{v}\right) = -\frac{1}{2}r_2^E(x)\widetilde{v} - \frac{1}{2}x\widetilde{\phi}_2^E(x)\widetilde{u}. \tag{47}$$

On the other hand, $D(r_3(x)v) = -\phi_4(x)u$ is equivalent to

$$D(x\widetilde{r}_2^E(x)\widetilde{v}) = -\frac{1}{2}\phi_4^E(x)\widetilde{u}, \tag{48}$$

and $D(r_{4(x)}v) = -\phi_5 u = -x\widetilde{\phi}_4(x)u$ yields

$$xD(r_4^E(x)\widetilde{v}) = -\frac{1}{2}r_4^E(x)\widetilde{v} - \frac{1}{2}x\widetilde{\phi}_4^E(x)\widetilde{u}. \tag{49}$$

On the other hand, let u and v be the symmetrizations of $\widetilde{u}$ and $\widetilde{v}$, respectively. Then,

Lemma 27. *u and v satisfy (45) if and only if $\widetilde{u}$ and $\widetilde{v}$*

$$x\sigma_4^E(x)\widetilde{u} = 2xr_2^E(x)\widetilde{v}. \tag{50}$$

Proof. We assume that $x^2\sigma_4(x)u = 2x^2r_2(x)v$. Let p be any polynomial. Then

$$\left\langle x\sigma_4^E(x)\widetilde{u}, p(x)\right\rangle = \left\langle u,)x^2p(x^2\sigma_4^E(x^2)\right\rangle = \left\langle 2xr_2^E(x)\widetilde{v}, p(x)\right\rangle.$$

On the other hand, assume that $x\sigma_4^E(x)\widetilde{u} = 2xr_2^E(x)\widetilde{v}$. If $p(x) = \sum_{k=0}^{n} a_k x^k$, then $p^E(x^2) = \sum_{k=0}^{[n/2]} a_{2k}x^{2k}$. As a consequence, $\left\langle x^2\sigma_4(x)u, p(x)\right\rangle = \left\langle 2x^2r_2(x)v, p(x)\right\rangle$. □

Taking derivatives in both hand sides and by using (47) we get

$$D(x\sigma_4^E(x)\widetilde{u}) = 2D(xr_2^E\widetilde{v}) = r_2^E(x)\widetilde{v} - x\widetilde{\phi}_2^E(x)\widetilde{u}.$$

If we multiply by x, then

$$xD(x\sigma_4^E(x)\widetilde{u}) = xr_2^E(x)\widetilde{v} - x^2\widetilde{\phi}_2^E(x)\widetilde{u} = \left(\frac{1}{2}x\sigma_4^E(x) - x^2\widetilde{\phi}_2^E(x)\right)\widetilde{u}$$

and, equivalently,

$$D(x^2\sigma_4^E(x)\widetilde{u}) = \left(\frac{1}{2}x\sigma_4^E(x) + x\sigma_4^E(x) - x^2\widetilde{\phi}_2^E(x)\right)\widetilde{u} = \left(\frac{3}{2}x\sigma_4^E(x) - x^2\widetilde{\phi}_2^E(x)\right)\widetilde{u}.$$

Next we summarize the above results.

Proposition 28. *If $A(x) = 2\left(x^2 - \xi^2\right)^2$ and (u, v) is a symmetric $(1,1)-$coherent pair, then $(\widetilde{u}, \widetilde{v})$ satisfy (50) and*

$$D(\widetilde{\phi}\widetilde{u}) + \widetilde{\psi}\widetilde{u} = 0, \tag{51}$$

where

$$\widetilde{\phi}(x) = x^2\sigma_4^E(x),$$

$$\sigma_4^E(x) = x\widetilde{\phi}_2^E(x) - \widetilde{\phi}_4^E(x),$$

and

$$\widetilde{\psi}(x) = x^2\widetilde{\phi}_2^E(x) - \frac{3}{2}x\sigma_4^E(x) = -\frac{1}{2}x^2\widetilde{\phi}_2^E(x) + \frac{3}{2}x\widetilde{\phi}_4^E(x).$$

Moreover $\deg\widetilde{\psi} \leq 3$ *and* $\deg\widetilde{\phi} \leq 4$. *As a consequence, $\widetilde{u}$ is a semiclassical linear functional of class at most* 2.

In the sequel, given a linear functional $\widetilde{U}$ and its symmetrized U, $\{\widetilde{\mu}_n^U\}_{n\geq 0}$ and $\{\mu_n^U\}_{n\geq 0}$ will denote the corresponding moment sequences. From (36) we get $\phi_3(x) = \frac{\lambda_{1,0}}{\langle u,P_1^2\rangle}x + \frac{3\lambda_{1,2}}{\langle u,P_3^2\rangle}P_3(x)$, with $\lambda_{1,0} = \langle v, r_2\rangle = \mu_2^v - \xi^2$. After some straightforward calculations, we get

$$\lambda_{1,2} = \mu_4^v - \left(\xi^2 + \frac{1}{3}\gamma_1^u + \frac{1}{3}\gamma_2^u\right)\mu_2^v + \frac{\xi^2}{3}\left(\gamma_1^u + \gamma_2^u\right),$$

where $\{\gamma_n^u\}_{n\geq 1}$ are the coefficients of the three term recurrence relation that the MOPS $\{P_n\}_{n\geq 0}$ satisfies. Then

$$\widetilde{\phi}_2^E(x) = \frac{3\lambda_{1,2}}{\langle u, P_3^2\rangle}x + \left(\frac{\lambda_{1,0}}{\langle u, P_1^2\rangle} - \frac{3\lambda_{1,2}}{\langle u, P_3^2\rangle}\left(\gamma_1^u + \gamma_2^u\right)\right).$$

In particular,

$$\widetilde{\phi}_2^E(0) = \frac{\mu_2^v - \xi^2}{\langle u, P_1^2\rangle} - \frac{3\mu_4^v - 3\left(\xi^2 + \frac{1}{3}\gamma_1^u + \frac{1}{3}\gamma_2^u\right)\mu_2^v + \xi^2\left(\gamma_1^u + \gamma_2^u\right)}{\langle u, P_3^2\rangle}\left(\gamma_1^u + \gamma_2^u\right). \tag{52}$$

From (50) and taking into account $\widetilde{u}$ is a linear functional of class $s \leq 2$, according to the above classification we can find its companion $\widetilde{v}$. As a consequence, we can deduce all the candidates (u, v) to be symmetric $(1,1)-$coherent pairs. From (51) we get

$$x^2\sigma_4^E(x)D(\widetilde{u}) = -\left(\widetilde{\psi}(x) + \left(x^2\sigma_4^E(x)\right)'\right)\widetilde{u}. \tag{53}$$

In the sequel we consider $\widetilde{s} \leq 1$. The case $\widetilde{s} = 2$ will not be considered. From the classification of the semiclassical linear functionals of class $\widetilde{s} \leq 1$, we will analyze the semiclassical character of $\widetilde{u}$ taking into account the algebraic structure of $\sigma_4^E(x)$.

3.1.1 $\widetilde{u}$ of class $\widetilde{s} = 0$

In order to arrive to a classical case, we start the discussion by considering the following situations

i). $\sigma_4^E(x) = x^2$, $\widetilde{\psi}(x) = x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}x\right)$. From (51) we get $D(x^4\widetilde{u}) = -x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}x\right)\widetilde{u}$ or, equivalently,

$$D(x^2\widetilde{u}) = -\left(\widetilde{\phi}_2^E(x) + \frac{1}{2}x\right)\widetilde{u} + N_1\delta(x) + N_2\delta'(x).$$

It is easy to see that $N_1 = \left\langle \widetilde{u}, \widetilde{\phi}_2^E(x) + \frac{1}{2}x\right\rangle$, and $N_2 = \frac{1}{2}\left\langle \widetilde{u}, x^2\right\rangle - \left\langle \widetilde{u}, x\widetilde{\phi}_2^{E(x)}\right\rangle$. Thus, if $\left\langle \widetilde{u}, \widetilde{\phi}_2^E(x)\right\rangle + \frac{1}{2}\widetilde{\mu}_1^u = \frac{1}{2}\widetilde{\mu}_2^u - \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle = 0$, then $\widetilde{u}$ is the Bessel classical functional since

$$D(x^2\widetilde{u}) = -\left(\widetilde{\phi}_2^E(x) + \frac{1}{2}x\right)\widetilde{u}.$$

ii). $\sigma_4^E(x) = x(x-1)$, $\widetilde{\psi}(x) = x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}(x-1)\right)$. Then $D(x^3(x-1)\widetilde{u}) = -x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}(x-1)\right)\widetilde{u}$.

This yields

$$D(x(x-1)\widetilde{u}) = -\left(\widetilde{\phi}_2^E(x) + \frac{1}{2}(x-1)\right)\widetilde{u} + N_1\delta(x) + N_2\delta'(x),$$

where $N_1 = \left\langle \widetilde{u}, \widetilde{\phi}_2^E(x) + \frac{1}{2}(x-1)\right\rangle$ and $N_2 = \frac{1}{2}\left\langle \widetilde{u}, x(x-1)\right\rangle - \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle$. If $\left\langle \widetilde{u}, \widetilde{\phi}_2^E(x)\right\rangle + \frac{1}{2}(\widetilde{\mu}_1^u - 1) = \frac{1}{2}(\widetilde{\mu}_2^u - \widetilde{\mu}_1^u) - \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle = 0$, then $\widetilde{u} = \mathcal{J}_{[0,1]}^{(\alpha,\beta)}$, i.e. the Jacobi classical functional on $[0,1]$, such that

$$D(x(x-1)\widetilde{u}) = -\left(\widetilde{\phi}_2^E(x) + \frac{1}{2}(x-1)\right)\widetilde{u}.$$

iii). $\sigma_4^E(x) = x$, $\widetilde{\psi}(x) = x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}\right)$. From (51) we get $x^3D(\widetilde{u}) = -\left(x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}\right) + 3x^2\right)\widetilde{u}$ and

$$xD(\widetilde{u}) = -\left(\widetilde{\phi}_2^E(x) + \frac{3}{2}\right)\widetilde{u} + N_1\delta(x) + N_2\delta'(x).$$

Then $\langle xD(\widetilde{u}), 1\rangle = -\left\langle \widetilde{u}, \left(\widetilde{\phi}_2^E(x) + \frac{3}{2}\right)\right\rangle + N_1$ and $\langle xD(\widetilde{u}), x\rangle = -\left\langle \widetilde{u}, \widetilde{\phi}_2^E(x) + \frac{3}{2}x\right\rangle - N_2$. If $N_1 = \left\langle \widetilde{u}, \widetilde{\phi}_2^E(x)\right\rangle + \frac{1}{2} = 0$ and $N_2 = -\left\langle \widetilde{u}, \widetilde{\phi}_2^{E(}x)\right\rangle + \frac{1}{2}\widetilde{\mu}_1^u = 0$ we get $D(x\widetilde{u}) = -\left(\widetilde{\phi}_2^E(x) + \frac{1}{2}\right)\widetilde{u}$, i.e. $\widetilde{u}$ is the classical Laguerre linear functional.

Remark 29. *We do not consider $\sigma_4^E(x) = 1$, since in such a case $\widetilde{u}$ is the classical Hermite functional.*

3.1.2 $\widetilde{u}$ of class $\widetilde{s} = 1$

In order to get a semiclassical case with $\widetilde{s} = 1$ we will discuss two possible situations in order to reduce the degrees of the polynomials involved in the starting Pearson equation.

a).

i). $\sigma_4^E(x) = x^2$, $\widetilde{\psi}(x) = x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}x\right)$. From (51)

$$x^3D(\widetilde{u}) = -\left(\widetilde{\psi}(x) + 4x^3\right)\widetilde{u} = -x\left(\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}x\right) + 4x\right)\widetilde{u} + M\delta(x).$$

If

$$M = \langle x^3D(\widetilde{u}), 1\rangle + \left\langle x\left(\widetilde{\phi}_2^E(x) + \frac{5}{2}x\right)\widetilde{u}, 1\right\rangle = \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle - \frac{1}{2}\widetilde{\mu}_2^u = 0,$$

then you can reduce the Pearson equation to

$$D(x^3\widetilde{u}) = -x\left(\widetilde{\phi}_2^E(x) + \frac{5}{2}x\right)\widetilde{u} + 3x^2\widetilde{u} = \left(-x\widetilde{\phi}_2^E(x) + \frac{1}{2}x^2\right)\widetilde{u}.$$

and you have Here $\widetilde{\psi}(x) = x\widetilde{\phi}_2^E(x) - \frac{1}{2}x^2$, $\widetilde{\psi}(0) = 0$ and $\widetilde{\psi}'(0) = \widetilde{\phi}_2^E(0)$. If $\widetilde{\phi}_2^E(0) \neq 0$, then $\widetilde{u}$ corresponds to the case $A_{3,2}$ of the Belmehdi's classification in [5], and, as a consequence, $\widetilde{u} = x^{-1}\mathcal{B}^{(\alpha)} + M\delta(x)$.

ii). $\sigma_4^E(x) = x(x-1)$, $\widetilde{\psi}(x) = x^2\left(\widetilde{\phi}_2^E(x) - \frac{3}{2}(x-1)\right)$. In this case

$$x^2(x-1)D(\widetilde{u}) = -\left(x\widetilde{\phi}_2^E(x) - \frac{3}{2}x^2 + \frac{3}{2}x + 4x^2 - 3x\right)\widetilde{u} + M_1\delta(x).$$

Then

$$\begin{aligned} M_1 &= -3\langle \widetilde{u}, x^2\rangle + 2\langle \widetilde{u}, x\rangle + \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle + \frac{5}{2}\langle \widetilde{u}, x^2\rangle - \frac{3}{2}\langle \widetilde{u}, x\rangle \\ &= \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle - \frac{1}{2}\widetilde{\mu}_2^u + \frac{1}{2}\widetilde{\mu}_1^u. \end{aligned}$$

If $M_1 = 0$, then $D(x^2(x-1)\widetilde{u}) = -\left(x\widetilde{\phi}_2^E(x) + \frac{1}{2}x - \frac{1}{2}x^2\right)\widetilde{u}$, and, according to the case A_2 in [5], $\widetilde{u}$ has an integral representation with weight function $w(x) = (1-x)^{\alpha}x^{\beta}e^{-\frac{\gamma}{x}}$, on $[0,1]$, with $\alpha\gamma \neq 0$, $\gamma > 0$, $\alpha > -1$.

iii). $\sigma_4^E(x) = (x-1)(x-\zeta)$, $\zeta \neq 0, 1$, $\widetilde{\psi}(x) = x\left(x\widetilde{\phi}_2^E(x) - \frac{3}{2}(x-1)(x-\zeta)\right)$. Then,

$$D(x(x-1)(x-\zeta)\widetilde{u}) = -\left(x\widetilde{\phi}_2^E(x) - \frac{1}{2}(x-1)(x-\zeta)\right)\widetilde{u} + M\delta(x).$$

If

$$M = \left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle - \frac{1}{2}\widetilde{\mu}_2^u + \left(\frac{1}{2}\zeta + \frac{1}{2}\right)\widetilde{\mu}_1^u - \frac{1}{2}\zeta = 0,$$

this corresponds to the case A_1 in [5] with $\widetilde{\omega}(x) = (1-x)^{\alpha}x^{\beta}|x-\zeta|^{\gamma}$ on $[0,1]$ with the conditions $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > -1$, $\zeta \in (0,1)$.

b).

i). $\sigma_4^E(x)=x$, $\widetilde{\psi}(x)=x^2\left(\widetilde{\phi}_2^E(x)-\frac{3}{2}\right)$. As above, if $M=\left\langle \widetilde{u}, x\widetilde{\phi}_2^E\right\rangle-\frac{1}{2}\widetilde{\mu}_1^u=0$, then

$$D(x^2\widetilde{u})=-x\left(\widetilde{\phi}_2^E(x)-\frac{1}{2}\right)\widetilde{u},$$

and, according to the case B_2 in [5], we obtain an integral representation of $\widetilde{u}$ in terms of the weight function

$$w(x)=x^{\alpha}\left(1+x\right)^{\beta+1}e^{-x+\frac{\beta}{x}},$$

on $[0,\infty)$, with $\beta<0$, $\alpha,\beta>-1$.

ii).$\sigma_4^E(x)=x-1$, $\widetilde{\psi}(x)=x\left(x\widetilde{\phi}_2^E(x)-\frac{3}{2}\left(x-1\right)\right)$. Then, $D(x(x-1)\widetilde{u})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}\left(x-1\right)\right)\widetilde{u}$, when $M=\left\langle \widetilde{u}, x\widetilde{\phi}_2^E(x)\right\rangle-\frac{1}{2}\widetilde{\mu}_1^u+\frac{1}{2}=0$. This is the case B_1 in [5] with $\widetilde{\omega}(x)=\left(1-x\right)^{\alpha+1}x^{\beta+1}e^{-\lambda x}$ on $[0,1]$ and the conditions $\alpha\beta\neq0$, $\alpha,\beta>-1$.

iii). $\sigma_4^E(x)=1$, $\widetilde{\psi}(x)=x\left(x\widetilde{\phi}_2^E(x)-\frac{3}{2}\right)$. If $M=\left\langle \widetilde{u}, x\widetilde{\phi}_2^E\right\rangle-\frac{1}{2}=0$, then $D(x\widetilde{u})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}\right)\widetilde{u}$, and, according to the case B_3 in [5], we get that $\widetilde{u}$ is represented in terms of the weight function $w(x)=x^{2\mu}e^{-x^2-\lambda x}$, on $\mathbb{R}^+$, $\mu>-1/2$, $\lambda\in\mathbb{R}$.

As the classical case, it is possible to reduce (50). Indeed, the general form of the Pearson equation is

$$D(x\sigma_4^E(x)\widetilde{u})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}\sigma_4^E(x)\right)\widetilde{u}.$$

Taking derivatives in (50) and using (47), we get $2D(xr_2^E(x)\widetilde{v})=-\left(x\widetilde{\phi}_2^E(x)-\frac{1}{2}\sigma_4^E(x)\right)\widetilde{u}$. In other words,

$$r_2^E(x)\widetilde{v}-x\widetilde{\phi}_2^E(x)\widetilde{u}=-x\widetilde{\phi}_2^E(x)\widetilde{u}+\frac{1}{2}\sigma_4^E(x)\widetilde{u},$$

and, as a consequence, $2r_2^E\widetilde{v}=\sigma_4^E(x)\widetilde{u}$.

Remark 30. *Notice that according to Theorem 17 we get* $\widehat{\Psi}(x)=x\left(\left(\sigma_4^E\right)'(x)+2\widetilde{\phi}_2^E(x)\right)$ *and, as a consequence, the class of* u *is* $s=2$.

3.2 Case $A(x)=2(x^2-\xi_1^2)(x-\xi_2^2)$, $\xi_1^2\neq\xi_2^2$

In this case, the following result is obtained in [12].

Theorem 31. *Suppose that* $A(x)=2(x^2-\xi_1^2)(x-\xi_2^2)$, $\xi_1^2\neq\xi_2^2$. *Then there exist odd and even polynomials* ψ *and* ϕ, *respectively, with* $\deg\psi\leq3$ *and* $\deg\phi\leq4$ *such that*

$$D\left(\phi v\right)+\psi v=0. \tag{54}$$

As a consequence, v *is a semiclassical linear functional of class at most* 2. *Besides*

$$x\phi(x)u=x(x^2-\xi^2)v \tag{55}$$

holds, where $\xi\in\{\xi_1,\xi_2\}$. *Also,* $(x^2-\xi^2)Dv=-\left(\phi'(x)+\psi(x)\right)u$.

Multiplying in (55) by x, if we define $\psi(x):=x\widetilde{\psi}(x)$, where $\widetilde{\psi}$ is an even polynomial of degree ≤2, and using the symmetrization process, after straightforward calculations you get

$$D\left(x\phi^E(x)\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-\phi^E(x)\right)\widetilde{v}, \tag{56}$$

$$x\phi^E(x)D(\widetilde{u})=\frac{1}{2}\left(\rho^E(x)+2x\right)\widetilde{v}-\left(2x(\phi^E)'(x)+\frac{1}{2}x\widetilde{\psi}^E(x)+\phi^E(x)\right)\widetilde{u}, \tag{57}$$

$$x\rho^E(x)D(\widetilde{v})=-\frac{1}{2}\rho^E(x)\widetilde{v}-\frac{1}{2}x\left(2(\phi^E)'(x)+\widetilde{\psi}^E(x)\right)\widetilde{u}, \tag{58}$$

and

$$x\phi^E(x)\widetilde{u}=x(x-\xi^2)\widetilde{v}. \tag{59}$$

Notice that $\widetilde{v}$ is semiclassical of class $\widetilde{s}\leq1$. Next, the class of $\widetilde{v}$ will be analyzed according to the zeros of ϕ^E.

3.2.1 $\widetilde{v}$ semiclassical of class $\widetilde{s}=0$

A1. $\phi^E(x)=x^2$. In this case (57) can be written as $D\left(x^3\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x^2\right)\widetilde{v}$. Since $\widetilde{v}$ is classical, we can reduce the degree of the polynomials involved in this relation in one degree, namely $D\left(x^2\widetilde{v}\right)=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+x\right)\widetilde{v}+N\delta(x)$. Since

$$0=\left\langle D\left(x^2\widetilde{v}\right),1\right\rangle=-\frac{1}{2}\left\langle\left(\widetilde{\psi}^E(x)+x\right)\widetilde{v},1\right\rangle+N,$$

if $N=0$, equivalently, $\left\langle\widetilde{v},\widetilde{\psi}^E(x)+x\right\rangle=0$, then $D\left(x^2\widetilde{v}\right)=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+x\right)\widetilde{v}$. In such a way, it is well known that $\widetilde{v}=\mathcal{B}^{(\alpha)}$.

A2. $\phi^E(x)=x(x-1)$. (57) reads $D\left(x(x-1)\widetilde{v}\right)=-\frac{1}{2}\left[\widetilde{\psi}^E(x)+(x-1)\right]\widetilde{v}+N\delta(x)$. Since

$$0=\left\langle D\left(x(x-1)\widetilde{v}\right),1\right\rangle=-\frac{1}{2}\left\langle\widetilde{v},\widetilde{\psi}^E(x)+x-1\right\rangle+N,$$

if $\left\langle\widetilde{v},\widetilde{\psi}^E(x)+x-1\right\rangle=0$, then $D\left(x(x-1)\widetilde{v}\right)=-\frac{1}{2}\left[\widetilde{\psi}^E(x)+(x-1)\right]\widetilde{v}$. This means that $\widetilde{v}=\mathcal{J}_{(0,1)}^{(\alpha,\beta)}$ as well as the integral representation

$$\left\langle\widetilde{v},p(x)\right\rangle=\int_0^1 p(x)(1-x)^\alpha x^\beta dx.$$

A3. $\phi^E(x)=x$. If $N=\left\langle\widetilde{v},\widetilde{\psi}^E(x)+1\right\rangle=0$, then $D\left(x\widetilde{v}\right)=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+1\right)\widetilde{v}$. As consequence, $\widetilde{v}=\mathcal{L}^{(\alpha)}$.

On one hand, from the symmetrization process and since the class of $\widetilde{v}$ is 0, the class s of v is determined by the polynomial $\Psi(x):=\left(\phi^E\right)'(x)+\widetilde{\psi}^E(x)-\frac{\phi^E(x)}{x}$. Indeed, if $\Psi(0)=0$, then $s=0$. If $\Psi(0)\neq 0$, then $s=1$. In Table 3 we describe the conditions leading to $\Psi(0)=0$.

$\widetilde{v}$	$\widetilde{\psi}^E$	$\Psi(x)$	Conditions for $\Psi(0)=0$
$\mathcal{B}^{(\alpha)}$	$-(2\alpha+5)x-4$	$(-2\alpha-4)x-4$	$\Psi(0)\neq 0$ always
$\mathcal{J}_{(0,1)}^{(\alpha,\beta)}$	$(2\alpha+2\beta+3)x-(2\beta+1)$	$(2\alpha+2\beta+4)x-(2\beta+1)$	$\beta=-1/2$
$\mathcal{L}^{(\alpha)}$	$2x-(2\alpha+3)$	$2x-(2\alpha+3)$	$\alpha=-3/2$

Table 3: Conditions for v to be classical

Next, we will prove that we can reduce (60) in order to obtain

$$\phi^E(x)\widetilde{u}=\rho^E(x)\widetilde{v},$$

where $\rho^E(x):=x-\xi^2$. In general, the Pearson equation is

$$D(\phi^E(x)\widetilde{v})=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+\frac{\phi^E(x)}{x}\right)\widetilde{v},\tag{60}$$

or, equivalently,

$$\phi^E(x)D\widetilde{v}=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+\frac{\phi^E(x)}{x}+2\left(\phi^E\right)'(x)\right)\widetilde{v},\tag{61}$$

under the condition $\left\langle\widetilde{v},\widetilde{\psi}^E(x)+\frac{\phi^E(x)}{x}\right\rangle=0$.

The case A1, where $\widetilde{v}$ is the classical Bessel functional, reads as

$$D(x^2\widetilde{v})=\left((\alpha+2)x+2\right)\widetilde{v}=-\frac{1}{2}\left(\widetilde{\psi}^E(x)+x\right)\widetilde{v}.$$

Then $\widetilde{\psi}^E(x)=(-2\alpha-5)x-4$, and the above differential relation can be written as

$$x^2D(\widetilde{v})=(\alpha x+2)\widetilde{v},\tag{62}$$

with the condition $\langle \widetilde{v}, (\alpha+2)x+2\rangle = 0$. Also, in this case, the linear functionals $\widetilde{u}$ and $\widetilde{v}$ are related by $x^3\widetilde{u} = x\rho^E(x)\widetilde{v}$ and, as a consequence,

$$\widetilde{u} = \frac{\rho^E(x)}{x^2}\widetilde{v} + K_1\delta(x) + K_2\delta'(x) + K_3\delta''(x).$$

From (59) and (63) we get

$$\left(\alpha x + 2 + \frac{1}{2}\rho^E(x)\right)\widetilde{v} - \xi^2 x D(\widetilde{v}) = -\frac{1}{2}x\left(\left(-2\alpha-1\right)x - 4\right)\widetilde{u}.$$

The action of the linear functionals of both hand sides on $p(x) = x$ yields

$$\begin{aligned}\left\langle \widetilde{v}, \alpha x^2 + 2x + \frac{1}{2}x\rho^E(x) + 2x\xi^2\right\rangle &= \frac{1}{2}\left\langle \widetilde{u}, \left(2\alpha+1\right)x^3 + 4x^2\right\rangle \\ &= \frac{1}{2}\left\langle \rho^E(x)\widetilde{v}, \left(2\alpha+1\right)x + 4\right\rangle + 8K_3.\end{aligned}$$

As a consequence,

$$\begin{aligned}8K_3 &= \left\langle \widetilde{v}, \alpha x^2 + 2x + \frac{1}{2}x\rho^E(x) + 2x\xi^2 - \frac{1}{2}\left(2\alpha+1\right)x\rho^E(x) - 2\rho^E(x)\right\rangle \\ &= \left\langle \widetilde{v}, \alpha x^2 + 2x + 2x\xi^2 + \left(-\alpha x - 2\right)\rho^E(x)\right\rangle \\ &= \xi^2\left\langle \widetilde{v}, \left(2+\alpha\right)x + 2\right\rangle.\end{aligned}$$

Thus, $K_3 = 0$. In a similar way, in the case A2 we get

$$\widetilde{u} = \frac{\rho^E(x)}{x(x-1)}\widetilde{v} + K_1\delta(x) + K_2\delta'(x) + K_3\delta(x-1).$$

The action of the linear functional of both hand sides on $p(x) = x-1$ yields

$$\begin{aligned}&-\frac{1}{2}\left\langle\left(\widetilde{\psi}^E(x) + x - 1 + 4x - 2 - x + \xi^2\right)\widetilde{v}, (x-1)\right\rangle - (1-\xi^2)\left\langle xD\widetilde{v}, x-1\right\rangle \\ = &-\frac{1}{2}\left\langle \widetilde{u}, x(x-1)\left(4x - 2 + \widetilde{\psi}^E(x)\right)\right\rangle,\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\left\langle \widetilde{v}, -\frac{1}{2}(x-1)\left(\widetilde{\psi}^E(x) + 4x - 3 + \xi^2\right) - (1-\xi^2)\left(2x-1\right)\right\rangle \\ = &-\frac{1}{2}\left\langle \widetilde{v}, \rho^E(x)\left(4x - 2 + \widetilde{\psi}^E(x)\right)\right\rangle + \frac{1}{2}K_2\left(-2 + \widetilde{\psi}^E(0)\right).\end{aligned}$$

Then

$$\frac{1}{2}K_2\left(-2 + \widetilde{\psi}^E(0)\right) = -\frac{1}{2}\left(\xi - 1\right)\left\langle \widetilde{v}, \widetilde{\psi}^E(x) + x - 1\right\rangle.$$

In this case, since $\widetilde{v} = \mathcal{J}^{(\alpha,\beta)}_{(0,1)}$, it is well known that

$$\frac{1}{2}\left(\widetilde{\psi}^E(x) + (x-1)\right) = \left(\alpha+\beta+2\right)x - \left(\beta+1\right).$$

In other words,

$$\widetilde{\psi}^E(x) = \left(2\alpha + 2\beta + 3\right)x - \left(2\beta+1\right).$$

If $\widetilde{\psi}^E(0) = 2$, then $\beta = -3/2$. Up to for this value $K_2 = 0$. In the same way, for the case A3 (60) becomes

$$x\widetilde{u} = (x - \xi^2)\widetilde{v},$$

when $2 + \widetilde{\psi}^E(0) \neq 0$. This means that $\alpha \neq -\frac{1}{2}$.

3.2.2 $\widetilde{v}$ semiclassical of class $\widetilde{s}=1$

From (57) the following situations appear.

A. deg $\left(x\phi^E(x)\right)=3$, $1\le$ deg $\left(x\widetilde{\psi}^E(x)-\phi^E(x)\right)\le 2$.

A1. $\phi^E(x)=x^2$, $\Psi(x)=\frac{1}{2}(x\widetilde{\psi}^E(x)-x^2)$ and $D\left(x^3\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x^2\right)\widetilde{v}$. This corresponds to the case A_{32} in [5], where

$$D\left(x^3\widetilde{v}\right)=x\left((\alpha+2)\,x+2\right)\widetilde{v}$$

with the condition $\Psi'(x)\neq 0$. Since $\Psi'(x)=\frac{1}{2}\left(\widetilde{\psi}^E(x)+x\left(\widetilde{\psi}^E\right)'(x)-2x\right)$, the above condition means $\Psi'(0)=\frac{1}{2}\widetilde{\psi}^E(0)$ and $\widetilde{\psi}^E(0)\neq 0$. In addition, $\widetilde{v}=x^{-1}\mathcal{B}^{(\alpha)}+M\delta(x)$.

A2. $\phi^E(x)=x(x-1)$, $\Psi(x)=\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x(x-1)\right)$ and $D\left(x^2(x-1)\widetilde{v}\right)=-\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x(x-1)\right)\widetilde{v}$. It corresponds to the case A_2 in [5], where $\widetilde{v}$ satisfies $D\left(x^2(x-1)\widetilde{v}\right)=-x\left(-(\alpha+\beta+3)\,x+\beta+2\right)\widetilde{v}$, and

$$\widetilde{v}=x^{-1}\left(\tau_{1/2}\circ h_{1/2}\right)\mathcal{J}^{(\alpha,\beta+1)}+s\delta(x),\quad s\neq 0,$$

taking into account that for every polynomial p and $\alpha,\beta+1>-1$,

$$\left\langle\mathcal{J}^{(\alpha,\beta+1)},p(x)\right\rangle=\int_{-1}^{1}p(x)(1-x)^{\alpha}(x+1)^{\beta+1}dx.$$

The affine transformation $2t=x+1$ yields

$$\begin{aligned}\left\langle\left(\tau_{1/2}\circ h_{1/2}\right)\mathcal{J}^{(\alpha,\beta+1)},p(x)\right\rangle &= \int_{-1}^{1}p\left(\frac{1}{2}x+\frac{1}{2}\right)(1-x)^{\alpha}(x+1)^{\beta+1}dx,\\ &= \left\langle\mathcal{J}_{[0,1]}^{(\alpha,\beta+1)},p(x)\right\rangle.\end{aligned}$$

As a consequence, $\widetilde{v}=\mathcal{J}_{[0,1]}^{(\alpha,\beta)}+s\delta(x)$, $s\neq 0$.

A3. $\phi^E(x)=(x-1)(x-\zeta)$, $\Psi(x)=\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x(x-1)\right)$. It corresponds to the case A_1 in [5], where $\widetilde{v}$ satisfies

$$\begin{aligned}&D\left(x(x-1)(x-\zeta)\widetilde{v}\right)\\ =\;&-\left[-\left(\alpha+\beta+\gamma+3\right)x^2+\left(\left(\alpha+\beta+2\right)\zeta+\alpha+\gamma+2\right)x-\zeta(\alpha+1)\right],\end{aligned}$$

and it has the integral representation

$$\langle\widetilde{v},p(x)\rangle=\int_{0}^{1}p(x)\left(1-x\right)^{\alpha}x^{\beta}\left|x-\zeta\right|^{\gamma}dx,$$

with the conditions $\alpha\beta\gamma\neq 0$, $\alpha,\beta,\gamma>0$, $\zeta\in(0,1)$.

B. deg $\left(x\phi^E(x)\right)<3$, deg $\left(x\widetilde{\psi}^E(x)-\phi^E(x)\right)=2$.

B1. $\phi^E(x)=x-1$, $\Psi(x)=\frac{1}{2}\left(x\widetilde{\psi}^E(x)-(x-1)\right)$. It corresponds to the case B_1 in [5], where $\widetilde{v}$ satisfies $D\left(x(x-1)\widetilde{v}\right)=-\left(2\lambda x^2+\left(-\alpha-\beta-2\lambda-2\right)x+\beta+1\right)\widetilde{v}$, and has the integral representation

$$\langle\widetilde{v},p(x)\rangle=\int_{0}^{1}p(x)\left(1-x\right)^{\alpha+1}x^{\beta+1}e^{-\lambda x}dx,$$

with the conditions $\alpha\beta\neq 0$, $\alpha,\beta>-1$, and deg $\widetilde{\psi}^E=1$.

B2. $\phi^E(x)=x$, $\Psi(x)=\frac{1}{2}\left(x\widetilde{\psi}^E(x)-x\right)$. This is the case B_2 in [5], where $\widetilde{v}$ satisfies

$$D\left(x^2\widetilde{v}\right)=-x\left(x-\alpha-2\right)\widetilde{v}.$$

Besides, for $\alpha>-1$

$$\langle\widetilde{v},p(x)\rangle=\int_{0}^{\infty}p(x)x^{\alpha}e^{-x}dx+sp(0),$$

$s \neq 0$.

B3. $\phi^E(x) = 1$, $\Psi(x) = \frac{1}{2}\left(x\widetilde{\psi}^E(x) - 1\right)$. It corresponds to the case B_3 in [5], where $\widetilde{v}$ satisfies

$$D\left(x\widetilde{v}\right) = -\left(2x^2 - \lambda x - 2\mu - 1\right)\widetilde{v},$$

and it has the integral representation

$$\langle \widetilde{v}, p(x) \rangle = \int_0^\infty p(x) x^{2\mu} e^{-x^2 - \lambda x} dx,$$

with the conditions $\mu > -1/2$, $\lambda \in \mathbb{R}$ and $\deg \widetilde{\psi}^E = 1$.

Now, we will analyze the reduction of (60) in the positive-definite case in order to get integral representations of such linear functionals. Then, we assume that $\widetilde{v}$ has an integral representation in terms of a weight function $\omega_{\widetilde{v}}$ on an interval $[a, b]$ with $a \geq 0$, that is

$$\langle \widetilde{v}, p(x) \rangle = \int_a^b p(x) \omega_{\widetilde{v}} dx.$$

First, we analyze the $A2$ and $B2$ cases. We get the rational relation $x^2 \sigma_1(x)\widetilde{u} = x(x - \xi^2)\widetilde{v}$ with $\sigma_1(x) = x - 1$ in $A2$ and $\sigma_1(x) = 1$ in $B2$. Besides

$$\langle \widetilde{u}, p(x) \rangle = \int_a^b p(x) \frac{\rho(x)\omega_{\widetilde{v}}}{\phi^E(x)} dx + M_1 p(0) + M_2 p'(0) + N p(1),$$

where $N = 0$ in B2. By using (59) and (60) we get

$$\begin{aligned}
\left\langle x\rho^E(x) D\widetilde{v}, p(x) \right\rangle &= -\left\langle \widetilde{v}, \left(x\rho^E p\right)' \right\rangle \\
&= \int_a^b p\rho^E \frac{x\phi^E(x)\omega'_{\widetilde{v}}}{\phi^E(x)} dx, \\
&= -\frac{1}{2}\int_a^b p(x)\left(x\widetilde{\psi}^E(x) + 2x\left(\phi^E\right)'\right)\frac{\rho^E(x)\omega_{\widetilde{v}}(x)}{\phi^E(x)} dx - \frac{1}{2}\int_a^b p(x)\omega_{\widetilde{v}}(x) dx.
\end{aligned}$$

Since

$$\begin{aligned}
&-\frac{1}{2}\int_a^b p(x)\left(x\widetilde{\psi}^E(x) + 2x\left(\phi^E\right)'\right)\frac{\rho^E(x)\omega_{\widetilde{v}}(x)}{\phi^E(x)} dx \\
= &-\frac{1}{2}\left\langle \widetilde{u}, p(x)\left(x\widetilde{\psi}^E(x) + 2x\left(\phi^E\right)'\right)\right\rangle \\
&+\frac{1}{2}M_2 p(0)\left(\widetilde{\psi}^E(0) + 2\left(\phi^E\right)'(0)\right) + \frac{1}{2}N p(1)\left(\widetilde{\psi}^E(1) + 2\left(\phi^E\right)'(1)\right),
\end{aligned}$$

you get

$$M_2 p(0)\left(\widetilde{\psi}^E(0) + 2\left(\phi^E\right)'(0)\right) + N p(1)\left(\widetilde{\psi}^E(1) + 2\left(\phi^E\right)'(1)\right) = 0,$$

for every polynomial p. In particular, for $p(x) = x - 1$

$$M_2\left(\widetilde{\psi}^E(0) + 2\left(\phi^E\right)'(0)\right) = 0.$$

Next we deal with $\widetilde{\psi}^E(0) + 2\left(\phi^E\right)'(0) \neq 0$. When $\phi^E(x) = x(x-1)$, $\widetilde{v}$ is positive definite if $\alpha, (\beta+1) > -1$. Taking into account that in this case $\widetilde{\psi}^E(x) = -\left(2\alpha + 2\beta + 5\right)x + 2\beta + 3$, then

$$\widetilde{\psi}^E(0) + 2\left(\phi^E\right)'(0) = 2\beta + 1,$$

and $M_2 = 0$ if $\beta \neq -1/2$. In a similar way, we get $\widetilde{\psi}^E(x) = 2x - 2\alpha - 3$ and $\widetilde{v}$ is positive definite if $\alpha > -1$. After straightforward calculations

$$\widetilde{\psi}^E(0) + 2\left(\phi^E\right)'(0) = -2\alpha - 1.$$

Thus $M_2 = 0$ if $\alpha \neq -1/2$.

In $A3$ and $B1$ we get

$$\langle \widetilde{u}, p(x) \rangle = \int_a^b p(x) \frac{\rho(x)\omega_{\widetilde{v}}}{\phi^E(x)} dx + M_1 p(0) + M_2 p(1) + M_3 p(\zeta),$$

where $M_3 = 0$ in $B1$. An iteration of the above procedure yields

$$\begin{aligned} & -\frac{1}{2}\int_a^b p(x)\left(x\widetilde{\psi}^E(x) + 2x\left(\phi^E\right)'\right)\frac{\rho^E(x)\omega_{\widetilde{v}}(x)}{\phi^E(x)}dx \\ = & -\frac{1}{2}\left\langle \widetilde{u}, p(x)\left(x\widetilde{\psi}^E(x) + 2x\left(\phi^E\right)'\right)\right\rangle \\ & +\frac{1}{2}M_2 p(1)\left(\widetilde{\psi}^E(1) + 2\left(\phi^E\right)'(1)\right) + \frac{1}{2}M_3 p(\zeta)\left(\zeta\widetilde{\psi}^E(\zeta) + 2\zeta\left(\phi^E\right)'(\zeta)\right). \end{aligned}$$

Then

$$\frac{1}{2}M_2 p(1)\left(\widetilde{\psi}^E(1) + 2\left(\phi^E\right)'(1)\right) + \frac{1}{2}M_3 p(\zeta)\left(\zeta\widetilde{\psi}^E(\zeta) + 2\zeta\left(\phi^E\right)'(\zeta)\right) = 0. \tag{63}$$

On one hand, in A3

$$\begin{aligned} & x\widetilde{\psi}^E(x) \\ = & -(2\alpha + 2\beta + 2\gamma + 5)x^2 \\ & +2\left((\alpha+\beta+2)\zeta + \alpha + \gamma + 2 - \frac{1}{2}(1+\zeta)\right)x - 2\zeta(\alpha+1) + \zeta. \end{aligned}$$

Then $\alpha = -\frac{1}{2}$. In this way the case A3 will not be considered. On the other hand, in the case B1

$$x\widetilde{\psi}^E(x) = 4\lambda x^2 + (-2\alpha - 2\beta - 4\lambda - 3)x + (2\beta + 1),$$

and, thus, $\beta = -1/2$ and $\widetilde{\psi}^E(x) = 4\lambda x - 2\alpha - 4\lambda - 2$. Then $\widetilde{\psi}^E(1) = -2\alpha - 2$ and

$$\widetilde{\psi}^E(1) + 2\left(\phi^E\right)'(1) = -2\alpha.$$

So, $M_2 = 0$ if $\alpha \neq 0$.

In the case B3 we can not simplify the factor x. However, we get

$$x\widetilde{\psi}^E(x) = 4x^2 - 2\lambda x - 4\mu - 1,$$

and, as a consequence, $\mu = -\frac{1}{4}$. Then $\widetilde{\psi}^E(x) = 4x - 2\lambda$ and $\widetilde{v}$ satisfies $D(x\widetilde{v}) = -\frac{1}{2}\left(4x^2 - 2\lambda x - 1\right)\widetilde{v}$, as well as

$$\langle \widetilde{v}, p(x) \rangle = \int_0^\infty p(x) x^{-1/2} e^{-x^2 - \lambda x} dx.$$

4 Positive-definite symmetric $(1,1)-$coherent pairs (u, v)

According to the functionals $\widetilde{u}$ and $\widetilde{v}$ obtained in the previous section when $A(x) = 2(x^2 - \xi_1^2)(x - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$, or $A(x) = 2\left(x^2 - \xi^2\right)^2$, respectively, the symmetrization process allows us to recover the original symmetric functionals u and v and, as a consequence, we get a classification of symmetric $(1,1)-$coherent pairs. Of course, if we recover one pair (u, v) also we must prove that it is symmetric $(1,1)-$coherent one. For this purpose we state the next results.

Theorem 32. *([12]). Let u be a symmetric, semiclassical and quasi-definite linear functional of odd class s satisfying*

$$D(\phi u) + \psi u = 0,$$

where $\deg\phi \leq s+2$, and $\deg\psi \leq s+1$. Notice that, ϕ and ψ are even and odd polynomials, respectively. $\{P_n\}_{n\geq 0}$ will denote the corresponding MOPS, We assume that the linear functional $w = x\phi(x)u$ is quasi-definite, with $\{W_n\}_{n\geq 0}$ as the corresponding MOPS. Then

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \sum_{k=1}^{(s+1)/2} \eta_{n,n-2k} W_{n-2k}(x), \quad n \geq s+1,$$

with $\eta_{n,n-(s+1)} \neq 0$.

Theorem 33. *Let u be a symmetric, semiclassical and quasi-definite linear functional of even class s satisfying*

$$D(\phi u) + \psi u = 0,$$

where $\deg \phi \leq s+2$, and $\deg \psi \leq s+1$. Notice that ϕ and ψ are odd and even polynomials, respectively. $\{P_n\}_{n\geq 0}$ will denote the corresponding MOPS. We assume that the linear functional $w = \phi(x)u$ is quasi-definite, with $\{W_n\}_{n\geq 0}$ as the corresponding MOPS. Then

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \sum_{k=1}^{s/2} \eta_{n,n-2k} W_{n-2k}(x), \quad n \geq s,$$

with $\eta_{n,n-s} \neq 0$.

Proof. It is enough to expand the sequence $\left\{\frac{P'_{n+1}(x)}{n+1}\right\}_{n\geq 0}$ in terms of the basis $\{W_n\}_{n\geq 0}$ and to consider its quasi-orthogonal character described in Theorem 10, B). □

As a consequence of above theorems we get the next result.

Corollary 34. *Let u be as above with class s either 1 or 2. Let v denote a symmetric and quasi-definite linear functional such that there exist even polynomials p and q, with $0 \leq \deg p \leq 4$ and $\deg q = 2$ such that*

$$p(x)u = q(x)v,$$

holds. In addition, let $\{Q_n\}_{n\geq 0}$ be the MOPS associated with v. Then (u, v) is a symmetric $(1,1)-$coherent pair.

Proof. We consider the above theorems with $s = 1$ and $s = 2$, respectively. In both cases we get

$$Q_n(x) = W_n(x) + \beta_n W_{n-2}(x)$$

and

$$\frac{P'_{n+1}(x)}{n+1} = W_n(x) + \lambda_n W_{n-2}(x),$$

where $\beta_n \lambda_n \neq 0$. From the above equations we obtain

$$\frac{P'_{n+1}(x)}{n+1} + \beta_{n-2} \frac{(\lambda_n - \beta_n)}{(\lambda_{n-2} - \beta_{n-2})} \frac{P'_{n-1}(x)}{n-1} = Q_n(x) + \lambda_{n-2} \frac{(\lambda_n - \beta_n)}{(\lambda_{n-2} - \beta_{n-2})} Q_{n-2}(x),$$

where $\beta_n \neq \lambda_n$ for every n. □

4.0.1 Case $A(x) = 2\left(x^2 - \xi^2\right)^2$

According to Theorem 17, if the class of $\widetilde{u}$ is $\widetilde{s} = 0$, then the class of u is either 0 or 1. The classical cases (Gegenbauer, Hermite) have been analyzed in [13]. We suppose that $s = 1$, i.e.

$$\widehat{\Psi}(0) = \left(\sigma_4^E\right)'(0) + \lim_{x\to 0} \frac{\sigma_4^E(x)}{x} + 2\widetilde{\phi}_2^E(0) \neq 0.$$

i). If $\sigma_4^E(x) = x^2$, assuming that $\widetilde{\phi}_2^E(0) \neq 0$, then $u = \overline{\mathcal{B}}^{(\alpha)}$ satisfies

$$D(x^3 u) = -2\left(\widetilde{\phi}_2(x) + x^2\right) u.$$

ii). If $\sigma_4^E(x) = x(x-1)$, assuming that $\widetilde{\phi}_2^E(0) \neq 1$, then

$$D(x\left(x^2 - 1\right) u) = -\left(2\widetilde{\phi}_2(x) + \frac{1}{2}(x^2 - 1)\right) u.$$

Notice that $u = \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)}$.

iii). If $\sigma_4^E(x) = x$, assuming $\widetilde{\phi}_2^E(0) \neq -1$, then $u = \overline{\mathcal{L}}^{(\alpha)}$ and satisfies

$$D(xu) = -\left(2\widetilde{\phi}_2(x) + \frac{1}{2}\right) u,$$

On the other and, if $\widetilde{u}$ is of class $\widetilde{s} = 1$, then from the symmetrization theorem we deduce that the class of u is $s = 2$. Next we will describe u according to σ_4^E.

i). If $\sigma_4^E(x) = x^2$ and $\widetilde{\phi}_2^E(0) \neq 0$, then $D(x^4 u) = -2x\widetilde{\phi}_2(x)u$. Thus $u = x^{-2}\overline{\mathcal{B}}^{(\alpha)} + M\delta(x)$.

ii). If $\sigma_4^E(x) = x(x-1)$, then u satisfies $D(x^2(x^2-1)u) = -2x\widetilde{\phi}_2(x)u$ and it has the integral representation

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha} |x|^{2\beta+1} e^{-\frac{\gamma}{x^2}}\, dx$$

with the conditions $\alpha\gamma \neq 0$, $\gamma > 0$, $\alpha > -1$.

iii). If $\sigma_4^E(x) = (x-1)(x-\zeta)$, with $\zeta \in (0,1)$, then u satisfies $D((x^2-1)(x^2-\zeta)u) = -2x\widetilde{\phi}_2(x)u$. Moreover,

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha} |x|^{2\beta+1} \left|x^2-\zeta\right|^{\gamma} dx,$$

with the conditions $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > -1$.

iv). If $\sigma_4^E(x) = x$, then u satisfies $D(x^2 u) = -2x\widetilde{\phi}_2(x)u$ as well as

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\, |x|^{2\alpha+1} \left(1+x^2\right)^{\beta+1} e^{-x^2+\frac{\beta}{x^2}}\, dx$$

with $\beta < 0$, $\alpha, \beta > -1$.

v). If $\sigma_4^E(x) = x-1$, then u satisfies $D((x^2-1)u) = -2x\widetilde{\phi}_2(x)u$. Moreover,

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha+1} |x|^{2\beta+3} e^{-\lambda x^2}\, dx$$

with the conditions $\alpha\beta \neq 0$, $\alpha, \beta > -1$.

vi). If $\sigma_4^E(x) = 1$, then u satisfies $Du = -2x\widetilde{\phi}_2(x)u$ and it has the integral representation

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\, |x|^{4\mu+1} e^{-x^4-\lambda x^2}\, dx,$$

under the conditions $\mu > -1/2$, $\lambda \in \mathbb{R}$.

Since in the previous cases u and v are related by

$$\sigma_4(x)u = 2r_2(x)v,$$

then according to Corollary 35, in each case the pair (u,v) is a symmetric $(1,1)-$coherent pair. Next, the corresponding symmetric $(1,1)-$coherent pairs are described in the positive-definite framework.

Theorem 35. *Let (u,v) be a symmetric $(1,1)-$coherent pair satisfying*

$$\sigma_4(x)u = 2(x^2-\xi^2)v,$$

such that σ_4 is an even polynomial with $\deg \sigma_4 \leq 4$ and u is a semiclassical linear functional of class at most 2. In addition, u and v are positive-definite and $A(x) = (x^2-\xi^2)^2$ in (39).

A. u of class $s = 1$.

$S_{1,1}$. *If* $\sigma_4(x) = x^2(x^2-1)$ *and either* $\xi^2 = 0$ *or* $\xi^2 = 1$, *then*

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha} |x|^{2\beta+1} dx$$

and

$$\begin{aligned}\langle v, p(x)\rangle &= \int_{-1}^{1} p(x)\frac{(1-x^2)^{\alpha+1}|x|^{2\beta+3}}{(x^2-\xi^2)}dx\\ &+\frac{M}{2}\left(\delta(x+|\xi|)+\delta(x-|\xi|)\right).\end{aligned}$$

$S_{1,2}$. *If* $\sigma_4(x) = x^2$ *and* $\xi^2 = 0$ *then* $u = \overline{\mathcal{L}}^{(\alpha)}$ *and* $v = \overline{\mathcal{L}}^{(\alpha)} + M\delta(x)$.

B. u *of class* $s = 2$.

$S_{1,3}$. *If* $\sigma_4(x) = x^2(x^2-1)$, $\alpha\gamma \neq 0$, $\gamma > 0$ $\alpha > -1$, *and either* $\xi^2 = 0$ *or* $\xi^2 = 1$, *then*

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha} |x|^{2\beta+1} e^{-\frac{\gamma}{x^2}} dx$$

and

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)\frac{\left(1-x^2\right)^{\alpha+1}}{(x^2-\xi^2)}|x|^{2\beta+3} e^{-\frac{\gamma}{x^2}} dx + \frac{M}{2}\left(p(|\xi|)+p(-|\xi|)\right).$$

$S_{1,4}$. *If* $\sigma_4(x) = (x^2-1)(x^2-\zeta)$, *with* $\zeta \in (0,1)$, $\alpha\beta\gamma \neq 0$, $\alpha, \beta, \gamma > 0$, *and either* $\xi^2 = 0$ *or* $\xi^2 = 1$, *then*

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha} |x|^{2\beta+1} \left|x^2-\zeta\right|^{\gamma} dx$$

and

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)\frac{\left(1-x^2\right)^{\alpha+1}}{(x^2-\xi^2)}|x|^{2\beta+1} \left|x^2-\zeta\right|^{\gamma+1} dx + \frac{M}{2}\left(p(|\xi|)+p(-|\xi|)\right).$$

$S_{1,5}$. *If* $\sigma_4(x) = x^2$, $\beta \in (-1, 0)$, $\alpha > -1$ *and* $\xi^2 = 0$, *then*

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)|x|^{2\alpha+1}\left(1+x^2\right)^{\beta+1} e^{-x^2+\frac{\beta}{x^2}} dx$$

and

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)|x|^{2\alpha+1}\left(1+x^2\right)^{\beta+1} e^{-x^2+\frac{\beta}{x^2}} dx + Mp(0).$$

$S_{1,6}$. *If* $\sigma_4(x) = 1$, $\mu > 0$, $\lambda \in \mathbb{R}$ *and* $\xi^2 = 0$, *then*

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)|x|^{4\mu+1} e^{-x^4-\lambda x^2} dx$$

and

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)|x|^{4\mu-1} e^{-x^4-\lambda x^2} dx + Mp(0).$$

$S_{1,7}$. *If* $\sigma_4(x) = x^2-1$, $\alpha\beta \neq 0$, $\alpha, \beta > -1$ *and either* $\xi^2 = 0$ *or* $\xi^2 = 1$, *then*

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha+1} |x|^{2\beta+3} e^{-\lambda x^2} dx$$

and

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)\frac{\left(1-x^2\right)^{\alpha+2}}{(x^2-\xi^2)}|x|^{2\beta+3} e^{-\lambda x^2} dx + \frac{M}{2}\left(p(|\xi|)+p(-|\xi|)\right).$$

4.1 Case $A(x) = 2(x^2 - \xi_1^2)(x - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$.

When $\widetilde{v}$ is a semiclassical linear functional of class $\widetilde{s} = 0$, the class of v is either 0 or 1. When the class of v is $s = 0$, we get $\mathcal{J}_{(0,1)}^{(\alpha,-1/2)}$ and $\mathcal{L}^{(-3/2)}$, which are non positive-definite linear functionals. Next, we describe the cases when the class of v is $s = 1$, and according to the expression of ϕ^E.

i). If $\phi^E(x) = x^2$, then $v = \overline{\mathcal{B}}^{(\alpha)}$ and it satisfies $D\left(x^3 v\right) = -\left(\widetilde{\psi}(x) + x^2\right) v$. Notice that this is not a positive-definite case.

ii). If $\phi^E(x) = x(x-1)$ and $\beta \neq -1/2$, then $v = \overline{\mathcal{J}}_{(0,1)}^{(\alpha,\beta)}$, moreover $D\left(x(x^2-1)v\right) = -\left(\widetilde{\psi}(x) + (x^2-1)\right) v$. Notice that

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha} |x|^{2\beta+1}\, dx.$$

iii). If $\phi^E(x) = x$, then v satisfies $D(xv) = -\left(\widetilde{\psi}(x) + 1\right) v$ and as a consequence $v = \overline{\mathcal{L}}^{(\alpha)}$. Thus,

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\, |x|^{2\alpha+1}\, e^{-x^2} dx.$$

If $\widetilde{v}$ is a semiclassical linear functional of class $\widetilde{s} = 1$, notice that, according to Theorem 17, v must be semiclassical of class $s = 2$. Next we describe the possible choices for v.

iv). If $\phi^E(x) = x^2$, then v satisfies $D\left(x^4 v\right) = -x\widetilde{\psi}(x)v$, i.e. $v = x^{-2}\overline{\mathcal{B}}^{(\alpha)} + M\delta(x)$. Notice that this is not a positive definite case.

v). If $\phi^E(x) = x(x-1)$, then $v = \overline{\mathcal{J}}_{[0,1]}^{(\alpha,\beta)} + s\delta(x)$, $s \neq 0$, and v satisfies

$$D\left(x^2(x^2-1)v\right) = -x\widetilde{\psi}(x)v = -x\left(-\left(2\alpha + 2\beta + 5\right)x + 2\beta + 3\right)v$$

with $\beta \neq -1/2$.

vi). If $\phi^E(x) = x - 1$, $\alpha \neq 0$, $\alpha > -1$, $\lambda \neq 0$, then v satisfies

$$D\left((x^2-1)v\right) = -2x\left(2\lambda x^2 - \alpha - 2\lambda - 1\right)v,$$

i.e.

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)\left(1-x^2\right)^{\alpha+1} x^2 e^{-\lambda x^2} dx.$$

vii). If $\phi^E(x) = x$, $\alpha > -1$, $\alpha \neq -1/2$ and $s \neq 0$, then

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\, |x|^{2\alpha+1}\, e^{-x^2} dx + sp(0)$$

and $D\left(x^2 v\right) = -x\left(2x^2 - 2\alpha - 3\right)v$.

viii). If $\phi^E(x) = 1$, then v satisfies $Dv = -x(4x^2 - 2\lambda)v$ and

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)e^{-x^4-\lambda x^2} dx.$$

Moreover,

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\left(x^2 - \zeta^2\right)e^{-x^4-\lambda x^2} dx + Mp(0).$$

In cases from i) to v) and vii) we will assume that $\xi^2 = 0$. From (58), we get

$$x\phi^E(x)D(\widetilde{u}) = \frac{3}{2}x\widetilde{v} - \left(2x(\phi^E)'(x) + \frac{1}{2}x\widetilde{\psi}^E(x) + \phi^E(x)\right)\widetilde{u}.$$

Taking into account $\phi^E(x)\widetilde{u} = x\widetilde{v}$, then

$$D(x\phi^E(x)\widetilde{u}) = \left(\frac{5}{2}\phi^E(x) - (x\phi^E(x))' - \frac{1}{2}x\widetilde{\psi}^E(x)\right)\widetilde{u}.$$

As a consequence, $\widetilde{u}$ is semiclassical of class at most 1. According to Theorem 17 and Corollary 35, since $\phi^E(0) = 0$, then the class of u must be at most 2 and the pairs (u, v) are symmetric $(1,1)-$coherent. For cases vi) and viii) we get $x^2u = x^2(x^2 - \xi^2)v$, and $x^2u = x^2v$, respectively. Then it is enough to apply the arguments of the above lemma but by using the fact that v is of class $s \leq 2$.

For the positive-definite case the previous analysis is summarized in the next

Theorem 36. *Let (u, v) be a symmetric $(1,1)-$coherent pair satisfying*

$$x\phi(x)u = x(x^2 - \xi^2)v$$

such that ϕ is an even polynomial with $\deg\phi(x) \leq 4$ and v is semiclassical of class at most 2. In addition, let assume that u and v are positive-definite as well as in (39) $A(x) = (x^2 - \xi_1^2)(x^2 - \xi_2^2)$, $\xi_1^2 \neq \xi_2^2$.

A. v classical.

$S_{2,1}$. If $\phi(x) = x^2(x^2 - 1)$, then $v = \overline{\mathcal{J}}_{(0,1)}^{(\alpha,-1/2)} = \mathcal{G}^{(\lambda)}$, $\lambda > -1$, $\lambda \neq 0$, i.e. the classical Gegenbauer functional. Thus

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\lambda-1/2}dx + M_1p(0) + \frac{M_2}{2}\left(p(1) + p(-1)\right).$$

B. v of class 1.

$S_{2,2}$. If $\phi(x) = x^2(x^2 - 1)$, $\beta \neq -1/2$, then

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha}\left|x\right|^{2\beta+1}dx$$

and

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha-1}\left|x\right|^{2\beta+1}dx + M_1p(0) + \frac{M_2}{2}\left(p(1) + p(-1)\right).$$

$S_{2,3}$. If $\phi(x) = x^2$ then

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\left|x\right|^{2\alpha+1}e^{-x^2}dx,$$

and

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\left|x\right|^{2\alpha+1}e^{-x^2}dx + Mp(0).$$

C. v of class 2.

$S_{2,4}$. If $\phi(x) = x^2(x^2 - 1)$, $\beta \neq -1/2$, then

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha}\left|x\right|^{2\beta+1}dx + sp(0)$$

and

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^{\alpha-1}\left|x\right|^{2\beta+1}dx + Mp(0) + \frac{N}{2}\left(p(1) + p(-1)\right).$$

$S_{2,5}$. If $\phi(x) = x^2$, $\alpha > -1$, $\alpha \neq -1/2$ and $s \neq 0$, then

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\left|x\right|^{2\alpha+1}e^{-x^2}dx + sp(0)$$

and

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)\,|x|^{2\alpha+1}\,e^{-x^2}dx + Mp(0).$$

$S_{2,6}$. *If* $\phi(x) = x^2 - 1$, $\xi^2 = 1$, $\alpha \neq 0$, $\alpha > -1$, $\lambda \neq 0$, *then*

$$\langle v, p(x)\rangle = \int_{-1}^{1} p(x)\left(1 - x^2\right)^{\alpha+1} x^2 e^{-\lambda x^2}dx$$

and

$$\langle u, p(x)\rangle = \int_{-1}^{1} p(x)\left(1 - x^2\right)^{\alpha} e^{-\lambda x^2}dx + Mp(0).$$

$S_{2,9}$. *If* $\phi(x) = 1$, *then*

$$\langle v, p(x)\rangle = \int_{-\infty}^{\infty} p(x)e^{-x^4-\lambda x^2}dx,$$

as well as

$$\langle u, p(x)\rangle = \int_{-\infty}^{\infty} p(x)|x^2 - \xi^2|e^{-x^4-\lambda x^2}dx + Mp(0).$$

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