# SOME NEW IDENTITIES INVOLVING SHEFFER-APPELL POLYNOMIAL SEQUENCES VIA MATRIX APPROACH

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Abstract. In this contribution some new identities involving Sheffer-Appell polynomial sequences using generalized Pascal functional and Wronskian matrices are deduced. As a direct application of them, identities involving families of polynomials as Euler, Bernoulli, Miller-Lee and Apostol-Euler polynomials, among others, are given.

## 1. introduction

Sequences of polynomials play an important role in many problems of pure and applied mathematics in the framework of approximation theory, statistics, combinatorics and classical analysis (see, for example, [19, 22–25]). The sequence of Sheffer polynomials constitutes one of the most important family of polynomial sequences. A polynomial sequence $\{s_n(x)\}_{n\geq 0}$ is said to be a Sheffer polynomial sequence [6, 9, 24, 27] if its generating function has the following form:

$$A(y)e^{xH(y)} = \sum_{n=0}^{\infty} s_n(x)\, \frac{y^n}{n!}, \tag{1.1}$$

where

$$A(y) = A_0 + A_1 y + \cdots ,$$

and

$$H(y) = H_1 y + H_2 y^2 + \cdots ,$$

with $A_0 \neq 0$ and $H_1 \neq 0$.

Let us recall an alternative definition of the Sheffer polynomial sequences [24, Pg. 17]. Indeed, let $h(y) = \sum_{n=1}^{\infty} h_n \frac{y^n}{n!}, h_1 \neq 0$, and $l(y) = \sum_{n=0}^{\infty} l_n \frac{y^n}{n!}$, $l_0 \neq 0$, be, respectively, delta series and invertible series with complex coefficients. Then there exists a unique sequence of Sheffer polynomials $\{s_n(x)\}_{n\geq 0}$ satisfying the orthogonality conditions

$$\langle l(y)h(y)^k | s_n(x)\rangle = n!\delta_{n,k} \qquad \forall\, n, k \geqq 0, \tag{1.2}$$

where $\delta_{n,k}$ is the Kronecker delta. Notice that the above orthogonality is defined as follows. Given a formal power series $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ we can introduce a linear functional in the linear space of polynomials associated with $f$ such that $\langle f(x)|x^n\rangle = a_n$, $n \geq 0$, and extended by linearity to every polynomial. See [24, p.6]. The polynomial sequence $\{s_n(x)\}_{n\geq 0}$ is said to be polynomial Sheffer sequence for the pair $(l(y), h(y))$.

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Notice that an algebraic approach to Sheffer polynomial sequences has been done in [8]. On the other hand, a perspective about Sheffer polynomials and the monomiality principle by using algebraic methods appears in [9].

Roman [24, p. 18, Theorem 2.3.4] introduced the exponential generating function of $s_n(x)$ as follows

$$\frac{1}{l(\tilde{h}(y))}\, e^{x\tilde{h}(y)} = \sum_{n=0}^{\infty} s_n(x)\, \frac{y^n}{n!}. \tag{1.3}$$

where $\tilde{h}$ is the compositional inverse of $h$.

The Sheffer sequence for the pair $(l(y), y)$ is called an Appell sequence for $l(y)$. In fact, Roman [24] characterized Appell sequences as follows

$\{\alpha_n(x)\}_{n\geq 0}$ is an Appell polynomial sequence if either

$$\frac{d}{dx}\big(\alpha_n(x)\big) = n\alpha_{n-1}(x), \qquad n \in \mathbb{N},$$

or if there exists an exponential generating function of the form (see also the recent works [20, 28])

$$A(y)e^{xy} = \sum_{n=0}^{\infty} \alpha_n(x)\, \frac{y^n}{n!}, \tag{1.4}$$

where $\mathbb{N}$ denotes the set of positive integer numbers and

$$A(y) = \frac{1}{l(y)}.$$

We also note that for $H(y) = y$, the generating function (1.1) of the Sheffer polynomials $s_n(x)$ reduces to the generating function (1.4) of the Appell polynomials $\alpha_n(x)$.

A determinantal approach to Appell polynomials has been given in [7]. In [13] (see also [26]) He and Ricci derived some recurrence relations and differential equations for the Appell polynomial sequence. Further, in [32] (see also [1]) Youn and Yang obtained some identities and differential equation for the Sheffer polynomial sequence by using matrix algebra.

Now, in order to recall the definition of the generalized Pascal functional matrix of an analytic function (see [30]), let

$$\mathcal{F} = \left\{ h(y) = \sum_{r=0}^{\infty} \alpha_r\, \frac{y^r}{r!}, \alpha_r \in \mathbb{C} \right\}.$$

Then the generalized Pascal functional matrix $[P_n(h(y))]$, which is a lower triangular matrix of order $(n+1)\times(n+1)$ for $h(y) \in \mathcal{F}$, is defined by

$$P_n[h(y)]_{ij} = \begin{cases} \dbinom{i}{j} h^{(i-j)}(y), & i \geqq j, \\ \\ 0, & \text{otherwise,} \end{cases} \tag{1.5}$$

for all $i, j = 0, 1, 2, \cdots, n$. Here $h^{(i)}(y)$ denotes the $i$th order derivative of $h(y)$.

We next recall the $n$th order Wronskian matrix of analytic functions $h_1(y), h_2(y), \cdots, h_m(y)$, with size $(n+1) \times m$, as follows:

$$W_n[h_1(y), h_2(y), \cdots, h_m(y)] = \begin{bmatrix} h_1(y) & h_2(y) & h_3(y) & \cdots & h_m(y) \\ h_1'(y) & h_2'(y) & h_3'(y) & \cdots & h_m'(y) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1^{(n)}(y) & h_2^{(n)}(y) & h_3^{(n)}(y) & \cdots & h_m^{(n)}(y) \end{bmatrix}. \quad (1.6)$$

Next we summarize some properties and relations between the Wronskian matrices and the generalized Pascal functional matrices since they constitute a basic tool of our work (see, for example, [31, 32]).

**Property I.** For $h(y), l(y) \in \mathbb{C}[y]$, $P_n[h(y)]$ and $W_n[h(y)]$ satisfy that is,

$$P_n[uh(y) + vl(y)] = uP_n[h(y)] + vP_n[l(y)]$$

and

$$W_n[uh(y) + vl(y)] = uW_n[h(y)] + vW_n[l(y)],$$

where $u, v \in \mathbb{C}$.

**Property II.** For $h(y), l(y) \in \mathbb{C}[y]$,

$$P_n[h(y)l(y)] = P_n[h(y)]P_n[l(y)] = P_n[l(y)]P_n[h(y)].$$

**Property III.** For $h(y), l(y) \in \mathbb{C}[y]$,

$$W_n[h(y)l(y)] = P_n[h(y)]W_n[l(y)] = P_n[l(y)]W_n[h(y)].$$

**Property IV.** For $h(y), l(y) \in \mathbb{C}[y]$, with $h(0) = 0$ and $h'(0) \neq 0$,

$$W_n[l(h(y))]\big|_{y=0} = W_n\left[1, h(y), h^2(y), \cdots, h^n(y)\right]\big|_{y=0} \Omega_n^{-1} W_n[l(y)]\big|_{y=0},$$

where

$$\Omega_n = \mathrm{diag}[0!, 1!, 2!, \cdots, n!].$$

In [15] the authors introduced the Sheffer-Appell polynomials as a discrete Appell convolution of Sheffer polynomials. Its generating function, series definition as well as a determinantal definition are deduced. These polynomials are strongly related to the monomiality principle and some of their properties are presented. Results for the Sheffer-Bernoulli and Sheffer-Euler polynomials are obtained. In particular, differential equations satisfied by these polynomials are given.

The Sheffer-Appell polynomial sequences are umbral composition ([24], pg. 41) of Appell and Sheffer polynomial sequences. Hence they are particular Sheffer sequences. The generating function of the Sheffer-Appell polynomial sequence $\{{}_sA_n(x)\}_{n\geq 0}$ is defined as

$$\frac{1}{l(h^{-1}(y))l(y)} e^{xh^{-1}(y)} = \sum_{n=0}^{\infty} {}_sA_n(x) \frac{y^n}{n!}, \quad (1.7)$$

where $\frac{1}{l(h^{-1}(y))l(y)}e^{xh^{-1}(y)}$ is analytic. Then by using Taylor's theorem, we obtain

$$ {}_sA_k(x) = \frac{d^k}{dy^k}\left(\frac{1}{l(h^{-1}(y))l(y)}e^{xh^{-1}(y)}\right)\Bigg|_{y=0}, \quad k \geq 0. \tag{1.8}$$

For the Sheffer-Appell polynomial sequence $\{{}_sA_n(x)\}_{n\geq 0}$ associated to the pair $(l(y), h(y))$ we introduce the vector

$$ {}_s\vec{\mathbf{A}}_n(x) = [{}_sA_0(x), {}_sA_1(x), \dots, {}_sA_n(x)]^T, \tag{1.9}$$

which can also be expressed as

$$ {}_s\vec{\mathbf{A}}_n(x) = [{}_sA_0(x), {}_sA_1(x), \dots, {}_sA_n(x)]^T = W_n\left[\frac{1}{l(h^{-1}(y))l(y)}e^{xh^{-1}(y)}\right]\Bigg|_{y=0}. \tag{1.10}$$

As an auxiliary result we deduce an expression for the Wronskian matrix of the vector ${}_s\vec{\mathbf{A}}_n(x)$.

**Lemma 1.1.** *Let $\{{}_sA_n(x)\}_{n\geq 0}$ be the Sheffer-Appell polynomial sequence associated with the pair $(l(y), h(y))$. Then*

$$\begin{aligned} &W_n[{}_s\vec{\mathbf{A}}_n(x)]^T\Omega_n^{-1} \\ &\quad = W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \cdots, (h^{-1}(y))^n]\big|_{y=0} \\ &\qquad \cdot \Omega_n^{-1}P_n\left[\frac{1}{l(y)}\right]\Bigg|_{y=0} P_n\left[\frac{1}{l(h(y))}\right]\Bigg|_{y=0} P_n\left[e^{xy}\right]\Bigg|_{y=0}. \end{aligned} \tag{1.11}$$

*Proof.* Let us begin with (1.10), that is,

$$ {}_s\vec{A}_n(x) = W_n\left[\frac{1}{l(h^{-1}(y))l(y)}\ e^{xh^{-1}(y)}\right]\Bigg|_{y=0}. \tag{1.12}$$

Applying Property IV in (1.12), we get

$$ {}_s\vec{A}_n(x) = W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \cdots, (h^{-1}(y))^n]\Bigg|_{y=0} \Omega_n^{-1}W_n\left[\frac{1}{l(y)l(h(y))}e^{xy}\right]\Bigg|_{y=0} \tag{1.13}$$

Taking into account

$$ W_n[e^{xy}]\Bigg|_{y=0} = [1,\ \ x,\ \ x^2, \cdots, x^n]^T, \tag{1.14}$$

then (1.13) becomes

$$\begin{aligned} {}_s\vec{A}_n(x) &= W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \cdots, (h^{-1}(y))^n]\Bigg|_{y=0} \\ &\quad \cdot \Omega_n^{-1}P_n\left[\frac{1}{l(y)}\right]\Bigg|_{y=0} P_n\left[\frac{1}{l(h(y))}\right]\Bigg|_{y=0} [1,\ \ x,\ \ x^2, \cdots, x^n]^T. \end{aligned} \tag{1.15}$$

Now, by taking the $k$th order derivative with respect to $x$ in both hand sides of (1.15) and dividing the resulting equation by $k!$, we obtain

$$\begin{aligned}
&\frac{1}{k!}[{}_sA_0^{(k)}(x), {}_sA_1^{(k)}(x), \cdots, {}_sA_n^{(k)}(x)]^T \\
&\quad = W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \cdots, (h^{-1}(y))^n]\Big|_{y=0} \\
&\qquad \cdot \Omega_n^{-1} P_n\left[\frac{1}{l(y)}\right]\Bigg|_{y=0} P_n\left[\frac{1}{l(h(y))}\right]\Bigg|_{y=0} \\
&\qquad \cdot \left[0, \ \cdots \ ,0, \ 1, \ \binom{k+1}{k}x, \ \binom{k+2}{k}x^2, \cdots, \binom{n}{k}x^{n-k}\right]^T. \qquad (1.16)
\end{aligned}$$

Hence, the right-hand side and left-hand side of (1.16) are the $k$th columns of

$$W_n[1, (h^{-1}(y)), (h^{-1}(y))^2, \cdots, (h^{-1}(y))^n]\Big|_{y=0} \Omega_n^{-1} P_n\left[\frac{1}{l(y)}\right]\Bigg|_{y=0} P_n\left[\frac{1}{l(h(y))}\right]\Bigg|_{y=0} P_n[e^{xy}]\Big|_{y=0}$$

and

$$W_n[{}_sA_0(x), {}_sA_1(x), \cdots, {}_sA_n(x)]^T \Omega^{-1},$$

respectively. Thus, the statement of Lemma 1.1 follows. □

Once we have this basic background, we describe the structure of the paper. In Section 2, we derive some identities involving Sheffer-Appell polynomials. A relation between two different sequences of such polynomials is also deduced in Theorem 6. In Section 3 we present some applications of the preceding results to particular families as Laguerre type Appell polynomials, Apostol-Euler-Appell polynomials and Miller-Lee type Appell polynomials.

## 2. Some identities involving Sheffer-Appell polynomial sequences via matrix approach

Identities involving orthogonal polynomials are obtained in the literature by using different approaches (see, e.g. [5, 11, 12, 14]). In this section, we derive some results for Sheffer-Appell polynomial sequences by using the generalized Pascal functional and Wronskian matrices.

First, we obtain a representation of the polynomial of degree $n+1$ in terms of the previous ones.

**Theorem 1.** *The Sheffer-Appell polynomial sequence ${}_sA_n(x) \sim (l(y), h(y))$ satisfies the following relation*

$$a_0 {}_sA_{n+1}(x) = \sum_{k=0}^{n} \frac{{}_sA_n^{(k)}(x)}{k!}(b_k x - b_{k+1} - c_k) - \sum_{k=1}^{n} \binom{n}{k-1} a_{n-k+1} {}_sA_k(x). \qquad (2.1)$$

*Here*

$$a_k = \left(h'(h^{-1}(y)) l(h^{-1}(y))\right)^{(k)}\Big|_{y=0}, k \geqq 0,$$

$$b_k = \big(l'(y)\big)^{(k)}\Big|_{y=0}, k \geqq 0,$$

*and*

$$c_k = \left(\frac{h'(y)l'(h(y))l(y)}{l(h(y))}\right)^{(k)}\Bigg|_{y=0}, k \geqq 0.$$

*Proof.* Let consider

$$W_n\left[h'(h^{-1}(y))l(h^{-1}(y))\frac{d}{dy}\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Bigg|_{y=0}. \tag{2.2}$$

On one hand, applying property III, we get

$$P_n\left[h'(h^{-1}(y))l(h^{-1}(y))\right]\Big|_{y=0} W_n\left[\frac{d}{dy}\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Bigg|_{y=0}$$

or, equivalently,

$$= \begin{bmatrix} a_0 & 0 & 0 & \dots & 0 \\ a_0 & a_1 & 0 & \dots & 0 \\ a_2 & \binom{2}{1}a_1 & a_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & \binom{n}{1}a_{n-1} & \dots & \dots & a_0 \end{bmatrix} \begin{bmatrix} {}_sA_1(x) \\ {}_sA_2(x) \\ {}_sA_3(x) \\ \vdots \\ {}_sA_{n+1}(x) \end{bmatrix}. \tag{2.3}$$

On the other hand, we can write (2.2) as

$$W_n\left[\left(xl(h^{-1}(y)) - \frac{h'(h^{-1}(y))l'(y)l(h^{-1}(y))}{l(y)} - l'(h^{-1}(y))\right)\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right]\Bigg|_{y=0}. \tag{2.4}$$

Applying property IV in (2.4), we get

$$= W_n\left[1, h^{-1}(y), (h^{-1}(y))^2, \dots, (h^{-1}(y))^n\right]\Big|_{y=0} \Omega_n^{-1}$$

$$W_n\left[\left(xl(y) - \frac{h'(y)l(y)l'(h(y))}{l(h(y))} - l'(y)\right)\frac{e^{xy}}{l(h(y))l(y)}\right]\Bigg|_{y=0} \tag{2.5}$$

According to property III, (2.5) reads

$$= W_n\left[1, h^{-1}(y), (h^{-1}(y))^2, \dots, (h^{-1}(y))^n\right]\Big|_{y=0} \Omega_n^{-1} P_n\left[\frac{1}{l(y)}\right]\Bigg|_{y=0} P_n\left[\frac{1}{l(h(y))}\right]\Bigg|_{y=0}$$

$$.P_n\left[e^{xy}\right]\Big|_{y=0} W_n\left[xl(y) - \frac{h'(y)l(y)l'(h(y))}{l(h(y))} - l'(y)\right]\Bigg|_{y=0}. \tag{2.6}$$

From Lemma 1.1 we get

$$= W_n\left[{}_sA_0(x), {}_sA_1(x), {}_sA_2(x), \dots, {}_sA_n(x)\right]^T \Omega_n^{-1}$$

$$\left[xW_n[l(y)]\Big|_{y=0} - W_n\left[\frac{h'(y)l(y)l'(h(y))}{l(h(y))}\right]\Bigg|_{y=0} - W_n[l'(y)]\Big|_{y=0}\right]$$

or, equivalently,

$$= \begin{bmatrix} {}_sA_0(x) & 0 & 0 & \dots & 0 \\ {}_sA_1(x) & \frac{{}_sA_1'(x)}{1!} & 0 & \dots & 0 \\ {}_sA_2(x) & \frac{{}_sA_2'(x)}{1!} & \frac{{}_sA_2''(x)}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_sA_n(x) & \frac{{}_sA_n'(x)}{1!} & \frac{{}_sA_n''(x)}{2!} & \dots & \frac{{}_sA_n^{(n)}(x)}{n!} \end{bmatrix} \begin{bmatrix} xb_0 - b_1 - c_0 \\ xb_1 - b_2 - c_1 \\ xb_2 - b_3 - c_2 \\ \vdots \\ xb_n - b_{n+1} - c_n \end{bmatrix}. \tag{2.7}$$

Finally, identifying the $n$th rows of (2.3) and (2.7), the statement follows. □

Next, we obtain an interesting relation satisfied by the polynomial of degree $n$ and their derivatives in terms of the polynomials of degree at most $n-1$.

**Theorem 2.** *Let ${}_sA_n(x) \sim (l(y), h(y))$ be the Sheffer-Appell polynomial sequence. Then,*

$$n \sum_{k=1}^{n} \binom{n-1}{k-1} a_{n-k\,s}A_k(x) = \sum_{k=0}^{n} (xb_k + c_k + d_k) \frac{{}_sA_n^{(k)}(x)}{k!}, \tag{2.8}$$

*where*

$$a_k = \left(h'(h^{-1}(y))\right)^{(k)} \Big|_{y=0}, k \geqq 0,$$

$$b_k = (h(y))^{(k)} \Big|_{y=0}, k \geqq 0,$$

$$c_k = \left( \frac{-h(y)l'(h(y)))h'(y)}{l(h(y))} \right)^{(k)} \Bigg|_{y=0}, k \geqq 0,$$

*and*

$$d_k = \left( \frac{-l'(y)h(y)}{l(y)} \right)^{(k)} \Bigg|_{y=0}, k \geqq 0.$$

*Proof.* From

$$W_n \left[ yh'(h^{-1}(y)) \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \Bigg|_{y=0} \tag{2.9}$$

and applying property III in (2.9), we get

$$P_n [y] \Big|_{y=0} P_n \left[h'(h^{-1}(y)\right] \Big|_{y=0} W_n \left[ \frac{d}{dy} \left( \frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)} \right) \right] \Bigg|_{y=0}$$

or, equivalently,

$$= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 0 \end{bmatrix} \begin{bmatrix} a_0 & 0 & 0 & \dots & 0 \\ a_1 & a_0 & 0 & \dots & 0 \\ a_2 & \binom{2}{1}a_1 & a_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & \binom{n}{1}a_{n-1} & \binom{n}{2}a_{n-2} & \dots & a_0 \end{bmatrix} \begin{bmatrix} {}_sA_1(x) \\ {}_sA_2(x) \\ \vdots \\ {}_sA_n(x) \\ {}_sA_{n+1}(x) \end{bmatrix}. \tag{2.10}$$

On the other hand, we can rewrite (2.9) as

$$W_n\left[\left(xh(h^{-1}(y))-\frac{h'(h^{-1}(y))l'(y)h(h^{-1}(y))}{l(y)}-\frac{l'(h^{-1}(y))h(h^{-1}(y))}{l(h^{-1}(y))}\right)\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right]\Bigg|_{y=0}. \tag{2.11}$$

From property IV we deduce

$$\begin{aligned}&= W_n[1,(h^{-1}(y)),(h^{-1}(y))^2,\dots,(h^{-1}(y))^n]\Big|_{y=0}\Omega_n^{-1}\\ &.W_n\left[\left(xh(y)-\frac{h'(y)l'(h(y))h(y)}{l(h(y))}-\frac{l'(y)h(y)}{l(y)}\right)\frac{e^{xy}}{l(y)l(h(y))}\right]\Bigg|_{y=0}.\end{aligned} \tag{2.12}$$

Next, using property III in (2.12), we get

$$\begin{aligned}&= W_n[1,(h^{-1}(y)),(h^{-1}(y))^2,\dots,(h^{-1}(y))^n]\Big|_{y=0}P_n\left[\frac{1}{l(y)}\right]\Bigg|_{y=0}P_n\left[\frac{1}{l(h(y))}\right]\Bigg|_{y=0}\\ &.P_n\left[e^{xy}\right]\Big|_{y=0}\Omega_n^{-1}W_n\left[xh(y)-\frac{h'(y)l'(h(y))h(y)}{l(h(y))}-\frac{l'(y)h(y)}{l(y)}\right]\Bigg|_{y=0}.\end{aligned} \tag{2.13}$$

As a consequence of Lemma 1.1,

$$\begin{aligned}&= W_n\left[{}_sA_0(x),{}_sA_1(x),{}_sA_2(x),\dots,{}_sA_n(x)\right]^T\Omega_n^{-1}\\ &.\left[xW_n\left[h(y)\right]\Big|_{y=0}+W_n\left[-\frac{h'(y)l'(h(y))h(y)}{l(h(y))}\right]\Bigg|_{y=0}+W_n\left[-\frac{l'(y)h(y)}{l(y)}\right]\Bigg|_{y=0}\right]\end{aligned}$$

or, equivalently,

$$\begin{aligned}&= W_n\left[{}_sA_0(x),{}_sA_1(x),{}_sA_2(x),\dots,{}_sA_n(x)\right]^T\Omega_n^{-1}\\ &.[xb_0+c_0+d_0,xb_1+c_1+d_1,\dots,xb_n+c_n+d_n]^T.\end{aligned} \tag{2.14}$$

Equating the $n$th rows of (2.10) and (2.14), our statement follows. □

On the other hand, we get an algebraic relation for a Sheffer-Appell polynomial of degree $n$ and their derivatives.

**Theorem 3.** *The Sheffer-Appell polynomial sequence ${}_sA_n(x)\sim(l(y),h(y))$ satisfies*

$$a_n=\sum_{k=0}^{n}(-1)^k\,{}_sA_n^{(k)}(x)\frac{x^k}{k!}, \tag{2.15}$$

*where*

$$a_k=\left(\frac{1}{l(y)l(h^{-1}(y))}\right)^{(k)}\Bigg|_{y=0},k\geqq 0.$$

*Proof.* Let consider

$$W_n\left[\frac{1}{l(h^{-1}(y))l(y)}\right]\Bigg|_{y=0}. \tag{2.16}$$

On one hand, from (1.6), we get

$$W_n\left[\frac{1}{l(h^{-1}(y))l(y)}\right]\Bigg|_{y=0}=[a_0,a_1,a_2,\dots,a_n]^T. \tag{2.17}$$

On the other hand, (2.16) can be written as

$$W_n\left[\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}e^{-xh^{-1}(y)}\right]\Big|_{y=0}. \tag{2.18}$$

By using property IV in (2.18), we get

$$= W_n\left[1, h^{-1}(y), (h^{-1}(y))^2, \dots, (h^{-1}(y))^n\right]\Big|_{y=0} \Omega_n^{-1} W_n\left[\frac{e^{xy}}{l(y)l(h(y))}e^{-xy}\right]\Big|_{y=0}. \tag{2.19}$$

Thus, according to property III, (2.19) becomes

$$\begin{aligned} &= W_n\left[1, h^{-1}(y), (h^{-1}(y))^2, \dots, (h^{-1}(y))^n\right]\Big|_{y=0} \Omega_n^{-1} P_n\left[\frac{1}{l(y)}\right]\Big|_{y=0} \\ &.P_n\left[\frac{1}{l(h(y))}\right]\Big|_{y=0} P_n\left[e^{xy}\right]\Big|_{y=0} W_n\left[e^{-xy}\right]\Big|_{y=0}, \end{aligned} \tag{2.20}$$

and applying Lemma 1.1, we obtain

$$= W_n\left[{}_sA_0(x), {}_sA_1(x), {}_sA_2(x), \dots, {}_sA_n(x)\right]^T \Omega_n^{-1}\left[1, -x, (-x)^2, \dots, (-x)^n\right]^T. \tag{2.21}$$

Identifying the $n$th rows of (2.17) and (2.21), we get the desired result. $\square$

Next, we obtain a representation of the Sheffer-Appell polynomial of degree $n-1$ in terms of the polynomial of degree $n$ and their derivatives.

**Theorem 4.** *Let ${}_sA_n(x) \sim (l(y), h(y))$ be the Sheffer-Appell polynomial sequence. Then,*

$$\sum_{k=1}^{n} a_k \frac{{}_sA_n^{(k)}(x)}{k!} = n\,{}_sA_{n-1}(x), \tag{2.22}$$

*where*

$$a_k = h^{(k)}(0), 1 \le k \le n.$$

*Proof.* Let consider

$$W_n\left[y\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Big|_{y=0}. \tag{2.23}$$

On one hand, from property III, (2.23) reads

$$W_n\left[y\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Big|_{y=0} = P_n[y]\Big|_{y=0} W_n\left[\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right]\Big|_{y=0}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & n-1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & n & 0 \end{bmatrix} \begin{bmatrix} {}_sA_0(x) \\ {}_sA_1(x) \\ {}_sA_2(x) \\ \vdots \\ {}_sA_{n-1}(x) \\ {}_sA_n(x) \end{bmatrix}. \tag{2.24}$$

According to property IV (2.24) becomes

$$W_n\left[y\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Bigg|_{y=0} = W_n[1,(h^{-1}(y)),(h^{-1}(y))^2,\ldots,(h^{-1}(y))^n]\Big|_{y=0}\Omega_n^{-1} \times W_n\left[h(y)\frac{e^{xy}}{l(y)l(h(y))}\right]\Bigg|_{y=0}. \tag{2.25}$$

By applying property III, it yields

$$W_n\left[y\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Bigg|_{y=0} = W_n[1,(h^{-1}(y)),(h^{-1}(y))^2,\ldots,(h^{-1}(y))^n]\Big|_{y=0}\Omega_n^{-1} .P_n\left[e^{xy}\right]\Big|_{y=0} P_n\left[\frac{1}{l(y)}\right]\Bigg|_{y=0} P_n\left[\frac{1}{l(h(y))}\right]\Bigg|_{y=0} W_n\left[h(y)\right]\Big|_{y=0}. \tag{2.26}$$

Finally, as a consequence of Lemma 1.1, we get

$$= W_n[{}_sA_0(x),{}_sA_1(x),\ldots,{}_sA_n(x)]^T\Omega^{-1}W_n\left[h(y)\right]\Bigg|_{y=0}$$

or, equivalently,

$$= \begin{bmatrix} {}_sA_0(x) & 0 & 0 & \ldots & 0 \\ {}_sA_1(x) & \frac{{}_sA_1'(x)}{1!} & 0 & \ldots & 0 \\ {}_sA_2(x) & \frac{{}_sA_2'(x)}{1!} & \frac{{}_sA_2''(x)}{2!} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_sA_n(x) & \frac{{}_sA_n'(x)}{1!} & \frac{{}_sA_n''(x)}{2!} & \ldots & \frac{{}_sA_n^{(n)}(x)}{n!} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}. \tag{2.27}$$

Equating $n$th rows of (2.24) and (2.27), we get the statement. □

Next, a connection between the Sheffer-Appell polynomials and the function $h$ is deduced.

**Theorem 5.** *The Sheffer-Appell polynomial sequence ${}_sA_n(x) \sim (l(y),h(y))$ satisfies*

$$\sum_{k=0}^{n}\binom{n}{k}a_{n-k}\,{}_sA_k(x) = \sum_{k=1}^{n}\left((h^{-1})^k\right)^{(n)}(0)\frac{x^k}{k!}, \tag{2.28}$$

*where*

$$a_k = \left(l(y)l(h^{-1}(y))\right)^{(k)}\Bigg|_{y=0}, k \geqq 0.$$

*Proof.* Let consider the expression

$$W_n\left[e^{xh^{-1}(y)}\right]\Bigg|_{y=0} \tag{2.29}$$

or, equivalently,

$$W_n\left[l(y)l(h^{-1}(y))\frac{e^{xh^{-1}(y)}}{l(y)l(h^{-1}(y))}\right]\Bigg|_{y=0}. \tag{2.30}$$

On one hand, from property III we get

$$W_n\left[l(y)l(h^{-1}(y))\frac{e^{xh^{-1}(y)}}{l(y)l(h^{-1}(y))}\right]\Bigg|_{y=0} = P_n[l(y)l(h^{-1}(y))]\Big|_{y=0}.W_n\left[\frac{e^{xh^{-1}(y)}}{l(y)l(h^{-1}(y))}\right]\Bigg|_{y=0}$$

$$= \begin{bmatrix} a_0 & 0 & 0 & \dots & 0 \\ a_1 & a_0 & 0 & \dots & 0 \\ a_2 & \binom{2}{1}a_1 & a_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & \binom{n}{1}a_{n-1} & \binom{n}{2}a_{n-2} & \dots & a_0 \end{bmatrix} \begin{bmatrix} {}_sA_0(x) \\ {}_sA_1(x) \\ {}_sA_2(x) \\ \vdots \\ {}_sA_n(x) \end{bmatrix}. \tag{2.31}$$

On the other hand, by applying property IV in (2.30), we deduce

$$W_n\left[l(y)l(h^{-1}(y))\frac{e^{xh^{-1}(y)}}{l(y)l(h^{-1}(y))}\right]\Bigg|_{y=0}$$

$$= W_n[1,(h^{-1}(y)),(h^{-1}(y))^2,\dots,(h^{-1}(y))^n]\Bigg|_{y=0}\Omega_n^{-1}.W_n\left[e^{xy}\right]\Bigg|_{y=0},$$

or, equivalently,

$$= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{(h^{-1})^{(1)}(0)}{1!} & \frac{((h^{-1})^2)^{(1)}(0)}{2!} & \dots & \frac{((h^{-1})^n)^{(1)}(0)}{n!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{(h^{-1})^{(n)}(0)}{1!} & \frac{((h^{-1})^2)^{(n)}(0)}{2!} & \dots & \frac{((h^{-1})^n)^{(n)}(0)}{n!} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{bmatrix}. \tag{2.32}$$

Equating $n$th rows of (2.31) and (2.32), we get the result stated in the Theorem 6. □

Our next step is two find a relation between two different sequences of Sheffer-Appell poynomial sequences.

**Theorem 6.** *Let ${}_sA_n(x) \sim (l(y), h(y))$ and ${}_rA_n(x) \sim (h'(y), h(y))$ be the two Sheffer-Appell polynomial sequences. Then,*

$$a_{0s}A_{n+1}(x) + \sum_{k=1}^{n}\binom{n}{k-1}a_{n-k+1s}A_k(x)$$

$$= (xb_0 + c_0 + d_0)_rA_n(x) + \sum_{k=1}^{n-1}\binom{n}{k}(xb_{n-k} + c_{n-k} + d_{n-k})_rA_k(x), \tag{2.33}$$

*where*

$$a_k = \left(l(y)l(h^{-1}(y))\right)^{(k)}\Bigg|_{y=0}, k \geqq 0,$$

$$b_k = \left(h'(y)\right)^{(k)}\Bigg|_{y=0}, k \geqq 0,$$

$$c_k = \left(\frac{-h'(h^{-1}(y))l'(y)h'(y)}{l(y)}\right)^{(k)}\Bigg|_{y=0}, k \geqq 0,$$

*and*

$$d_k = \left(\frac{-l'(h^{-1}(y))h'(y)}{l(h^{-1}(y))}\right)^{(k)}\Bigg|_{y=0}, k \geqq 0.$$

*Proof.* Let consider

$$W_n\left[l(h^{-1}(y))l(y)\frac{d}{dy}\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Big|_{y=0}. \tag{2.34}$$

On one hand, from property III (2.34) reads as

$$P_n\left[\frac{d}{dy}\left(\frac{e^{xh^{-1}(y)}}{l(h^{-1}(y))l(y)}\right)\right]\Big|_{y=0} W_n\left[l(h^{-1}(y))l(y)\right]\Big|_{y=0}$$

or, equivalently,

$$= \begin{bmatrix} {}_sA_1(x) & 0 & 0 & \dots & 0 \\ {}_sA_2(x) & {}_sA_1(x) & 0 & \dots & 0 \\ {}_sA_3(x) & \binom{2}{1}{}_sA_2(x) & {}_sA_1(x) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_sA_{n+1}(x) & \binom{n}{1}{}_sA_n(x) & \dots & \dots & n\,{}_sA_1(x) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{2.35}$$

On the other hand, (2.34) can be written as

$$W_n\left[\left(xh'(y) - \frac{h'(h^{-1}(y))l'(y)h'(y)}{l(y)} - \frac{l'(h^{-1}(y))h'(y)}{l(h^{-1}(y))}\right)\frac{e^{xh^{-1}(y)}}{h'(h^{-1}(y))h'(y)}\right]\Big|_{y=0}. \tag{2.36}$$

By using property III, we get

$$= P_n\left[xh'(y) - \frac{h'(h^{-1}(y))l'(y)h'(y)}{l(y)} - \frac{l'(h^{-1}(y))h'(y)}{l(h^{-1}(y))}\right]\Big|_{y=0} W_n\left[\frac{e^{xh^{-1}(y)}}{h'(h^{-1}(y))h'(y)}\right]\Big|_{y=0}.$$

$$= \left[xP_n\left[h'(y)\right]\Big|_{y=0} + P_n\left[\frac{-h'(h^{-1}(y))l'(y)h'(y)}{l(y)}\right]\Big|_{y=0}\right.$$
$$\left. + P_n\left[\frac{-l'(h^{-1}(y))h'(y)}{l(h^{-1}(y))}\right]\Big|_{y=0}\right][r_0(x), r_1(x), \dots, r_n(x)]^T$$

or, equivalently,

$$= \begin{bmatrix} xb_0+c_0+d_0 & 0 & 0 & \dots & 0 \\ xb_1+c_1+d_1 & xb_0+c_0+d_0 & 0 & \dots & 0 \\ xb_2+c_2+d_2 & x\binom{2}{1}b_1+\binom{2}{1}c_1+\binom{2}{1}d_1 & xb_0+c_0+d_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ xb_n+c_n+d_n & x\binom{n}{1}b_{n-1}+\binom{n}{1}c_{n-1}+\binom{n}{1}d_{n-1} & \dots & \dots & xb_0+c_0+d_0 \end{bmatrix}$$
$$\cdot \begin{bmatrix} {}_rA_0(x) \\ {}_rA_1(x) \\ {}_rA_2(x) \\ \vdots \\ {}_rA_n(x) \end{bmatrix}. \tag{2.37}$$

Equating the $n^{th}$ rows of (2.35) and (2.37), we get the statement. □

## 3. Examples

**Examples of Theorem 1.** By applying Theorem 1 to Euler type polynomials [21]

$$E_n(x) \sim \left(\frac{e^y+1}{2}, y\right),$$

we have

$$a_k = b_k = c_k = \frac{1}{2}, k \geqq 0.$$

Hence, we get the following expression for the Euler type Appell polynomials ${}_EA_n(x)$:

$${}_EA_{n+1}(x) = \left(\frac{x}{2}-1\right)\sum_{k=0}^{n}\binom{n}{k}{}_EA_{n-k}(x) - \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k-1}{}_EA_k(x). \tag{3.1}$$

If we apply Theorem 1 to the Miller-Lee type polynomials [2]

$$G_n^{(m)}(x) \sim \left((1-y)^{m+1}, y\right),$$

we have

$$a_k = b_k = c_k = (-m-1)_k, k \geqq 0.$$

Thus the following expression for the Miller-Lee type Appell polynomials ${}_GA_n(x)$ holds

$$\begin{aligned} {}_GA_{n+1}(x) &= \sum_{k=0}^{n}\binom{n}{k}(x+m-k)(-m-1)_k{}_GA_{n-k}(x) \\ &\quad - \sum_{k=1}^{n}\binom{n}{k-1}(-m-1)_{n-k+1}{}_GA_k(x). \end{aligned} \tag{3.2}$$

For the generalized Laguerre polynomials [21]

$$L_n^{\lambda}(x) \sim \left((1-y)^{-\lambda-1}, \frac{y}{y-1}\right),$$

from the statement of Theorem 1 we get

$$a_k = (-\lambda-1)_k, k \geqq 0,$$
$$b_k = (\lambda+1)_k, k \geqq 0,$$

and

$$c_k = (\lambda+1)(\lambda+4)_k, k \geqq 0.$$

As a consequence, for the Laguerre type Appell polynomials ${}_LA_n(x)$ we get

$$\begin{aligned} {}_LA_{n+1}(x) &= \sum_{k=0}^{n}\binom{n}{k}{}_LA_{n-k}(x)[(\lambda+1)_k x - (\lambda+1)_{k+1} \\ &-(\lambda+1)(\lambda+4)_k] - \sum_{k=1}^{n}\binom{n}{k-1}(-\lambda-1)_{n-k+1}{}_LA_k(x). \end{aligned} \tag{3.3}$$

Let consider the Apostol-Euler type polynomials, see [17, 18]

$$\epsilon_n(x;\lambda) \sim \left(\frac{\lambda e^y+1}{y}, y\right).$$

Then,

$$a_k = b_k = \begin{cases} \frac{\lambda+1}{2}, & k=0, \\ \frac{\lambda}{2}, & k>0, \end{cases}$$

and

$$c_k = \frac{\lambda}{2}, k \geqq 0, .$$

Hence, for the Apostol-Euler-Appell polynomials ${}_{\epsilon}A_n(x;\lambda)$ the statement of Theorem 1 reads as

$$(\lambda+1)_{\epsilon}A_{n+1}(x;\lambda) - [(\lambda+1)x-\lambda]_{\epsilon}A_n(x;\lambda) = \lambda(x-2)\sum_{k=1}^{n}\binom{n}{k}{}_{\epsilon}A_{n-k}(x;\lambda) - \lambda\sum_{k=1}^{n}\binom{n}{k-1}{}_{\epsilon}A_k(x;\lambda). \quad (3.4)$$

**Example of Theorem 2.** For the generalized Laguerre polynomials $L_n^{\lambda}(x) \sim \left((1-y)^{-\lambda-1}, \frac{y}{y-1}\right)$, we have

$$a_k = \begin{cases} -1, & k = 1, \\ 0, & otherwise, \end{cases}$$

$$b_k = \begin{cases} 0, & k = 0, \\ -(k)!, & k > 0, \end{cases}$$

$$c_k = (\lambda+1)[(4)_k - (3)_k],$$

$$d_k = (\lambda+1)[(1)_k - (2)_k].$$

As a consequence, for the Laguerre type Appell polynomials ${}_{L}A_n(x)$ we obtain

$$-n(n-1)_{L}A_{n-1}(x) = \sum_{k=1}^{n}\binom{n}{k}{}_{L}A_{n-k}(x)[(\lambda+1)[(4)_k - (3)_k - (2)_k + (1)_k - x(1)_k]. \quad (3.5)$$

**Example of Theorem 3.** By applying Theorem 3 to Miller-Lee type polynomials given by

$$G_n^{(m)}(x) \sim \left((1-y)^{m+1}, y\right),$$

we have

$$a_k = (2m+2)_k, k \geqq 0.$$

Hence, for Miller-Lee-Appell polynomials ${}_{G}A_n(x)$ we have the following algebraic relation

$$(2m+2)_n = \sum_{k=0}^{n}(-1)^k{}_{G}A_n^{(k)}(x)\frac{x^k}{k!}. \quad (3.6)$$

**Examples of Theorem 4.** For the Bernoulli type polynomials $B_n(x) \sim \left(\frac{y}{e^y-1}, e^y - 1\right)$, [21] in Theorem 4 we have

$$a_k = \begin{cases} 0, & k = 0, \\ 1, & k > 0. \end{cases}$$

As a consequence, for the Bernoulli type Appell polynomials ${}_{B}A_n(x)$ we get

$$n_{B}A_{n-1}(x) = \sum_{k=1}^{n}\binom{n}{k}{}_{B}A_{n-k}(x). \quad (3.7)$$

For the generalized Laguerre polynomials $L_n^{\lambda}(x) \sim \left((1-y)^{-\lambda-1}, \frac{y}{y-1}\right)$, we have

$$a_k = \begin{cases} 0, & k = 0, \\ -(k)!, & k > 0. \end{cases}$$

Thus, for the Laguerre type Appell polynomials ${}_LA_n(x)$ we obtain

$$n{}_LA_{n-1}(x) = -\sum_{k=1}^{n} \binom{n}{k} {}_LA_{n-k}(x)k!. \tag{3.8}$$

**Examples of Theorem 5.** For the Euler type polynomials $E_n(x) \sim \left(\frac{e^y+1}{2}, y\right)$, we have

$$a_k = \begin{cases} 1, & k = 0, \\ \frac{2^k+2}{4}, & k > 0. \end{cases}$$

Thus, for the Euler type Appell polynomials ${}_EA_k(x)$ we get

$${}_EA_k(x) = \sum_{k=1}^{n} ((f^{-1})^k)^{(n)}(0) - \frac{1}{4}\sum_{k=0}^{n-1} \binom{n}{k} (2^{n-k}+2){}_EA_k(x). \tag{3.9}$$

When we deal with the Miller-Lee type polynomials $G_n^{(m)}(x) \sim \left((1-y)^{m+1}, y\right)$, we have

$$a_k = (-2m-2)_k, k \geqq 0.$$

Then, for the Miller-Lee type Appell polynomials ${}_GA_n(x)$

$$\sum_{k=0}^{n} \binom{n}{k} (-2m-2)_{n-k}{}_GA_k(x) = \sum_{k=1}^{n} \frac{x^k}{k!}((f^{-1})^k)^{(n)}(0). \tag{3.10}$$

**Example of Theorem 6.** Applying Theorem 6 to the Miller-Lee type polynomials $G_n^{(m)}(x) \sim \left((1-y)^{m+1}, y\right)$, we have

$$a_k = c_k = d_k = -(m+1)k!, k \geqq 0,$$

and

$$b_k = \begin{cases} 1, & k = 0, \\ 0, & k > 0. \end{cases}$$

Hence,

$$\begin{aligned} {}_GA_{n+1}(x) + \sum_{k=1}^{n} \binom{n}{k-1} (n-k+1)!{}_GA_k(x) &= \left(2 - \frac{x}{m+1}\right) {}_rA_n(x) \\ &\quad + 2\sum_{k=1}^{n-1} \binom{n}{k} (n-k)!{}_rA_k(x). \end{aligned} \tag{3.11}$$

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