A Finite Class of Orthogonal Functions Generated by Routh-Romanovski Polynomials

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Abstract

It is known that some orthogonal systems are mapped onto other orthogonal system by the Fourier transform. In this paper we introduce a finite class of orthogonal functions, which is the Fourier transform of Routh-Romanovski orthogonal polynomials, and obtain its orthogonality relation using Parseval identity.

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1 Introduction

It is well known that Hermite, Laguerre, and Jacobi orthogonal polynomials are solutions of a second order linear differential operator $\mathfrak{L} = a_2(x)D^2 + a_1(x)D$ where D is the standard derivative operator, a_2 is a polynomial of degree at most 2, and a_1 is a polynomial of degree 1. Some characterizations of these three sequences are given in [1], [4], [5], and [24].

Three other sequences of classical orthogonal polynomials [12], [19] are associated with a positive-semi definite linear functional, which are finitely orthogonal in the sense that the support of the corresponding linear functional, considered as a distribution in the dual space of polynomials with real coefficients, is a finite subset of the real line. Some parametric constraints must appear in these sequences in order to have such a finite orthogonality. One of them is known in the literature as Routh-Romanovski orthogonal polynomials, introduced first by E. J. Routh ([24]) and then independently by V. I. Romanovski [23]. They have attracted some attention due to their potential application to trigonometric quark confinement potential of QCD traits. There exists some criticism about these "finite" orthogonal polynomials since they can be reduced to Jacobi polynomials. In the contributions by P. A. Lesky (see [15] as well as the recent monograph [13]) they are deeply analyzed in the framework of the spectral analysis of second order linear differential operators with polynomial coefficients as the same form as classical Hermite, Laguerre and Jacobi cases.

The following table shows the main characteristics of six sequences of classical orthogonal polynomials.

Definition	Weight function	Kind & Interval	Param. constraint
$P_n^{(u,v)}(x)$	$(1-x)^u(1+x)^v$	Infinite, $[-1,1]$	$\forall n,\ u>-1, v>-1$
$L_n^{(u)}(x)$	$x^u\exp(-x)$	Infinite, $[0,\infty)$	$\forall n,\ u>-1$
$H_n(x)$	$\exp(-x^2)$	Infinite, $(-\infty,\infty)$	-
$J_n^{(u,v)}(x;a,b,c,d)$	$(1-x)^u(1-x)^v$ $\times\exp\left(v\arctan\frac{ax+b}{cx+d}\right)$	Finite, $(-\infty,\infty)$	$\max n > u-\frac{1}{2}$, $ad-bc\neq 0$
$M_n^{(u,v)}(x)$	$x^v(1+x)^{-(u+v)}$	Finite, $[0,\infty)$	$\max n<(u-1)/2$, $v>-1$
$N_n^{(u)}(x)$	$x^{-u}\exp(-1/x)$	Finite, $[0,\infty)$	$\max n<(u-1)/2$

Table 1: Characteristics of classical orthogonal polynomials

On the other side, if the linear functional u satisfies a general Pearson equation $D(A(x)u)=B(x)u$ such that A is a non-zero polynomial and B is a polynomial of degree at least 1, then the semi-classical linear functionals (introduced by J. Shohat [25]) appear where u is a positive definite linear functional associated with a weight function supported on the real line. In this sense, if there exists a sequence $(P_n)_{n\geq 0}$ of monic polynomials orthogonal with respect to u, where its support as a distribution is an infinite subset of the real line, then it satisfies a holonomic second order linear differential equation $A(x;n)P_n''(x)+B(x;n)P_n'(x)+C(x;n)P_n(x)=0$, where A, B, and C are polynomials of degree independent of n but its coefficients dependent on n.

In [16] all sequences of monic polynomials orthogonal with respect to such a linear functional u of infinite support are obtained by assuming that A, a monic polynomial, and B are independent of n and the degree of C is uniformly bounded. Indeed, up to a linear change of variable, they are the Hermite, Laguerre, Jacobi, and Bessel orthogonal polynomials as well as the corresponding symmetrized orthogonal polynomial sequences for Laguerre, Jacobi and Bessel cases. All of the mentioned cases are, in fact, illustrative examples of semi-classical sequences of orthogonal polynomials.

Another interesting example is the incomplete symmetric monic sequence [20], orthogonal with respect to a linear functional, satisfying a holonomic equation with $A(x;n)=x^2(1-x^{2m})$, $B(x;n)=-2x((a+mb+1)x^{2m}-a+m-1)$ i.e. polynomials of degree and coefficients independent of n, and $C(x;n)=\alpha_n x^{2m}+\beta+\frac{1-(-1)^n}{2}\gamma$. Note that they can also be obtained via Jacobi polynomials using a change of the variable $y=x^{2m}$.

In the sequel, let v be a symmetric linear functional, i.e. $< v, x^{2n+1} >= 0$, and consider the linear functional u such that $< u, x^n >=< v, x^{2n} >$ where $< ., . >$ denotes the duality bracket. It is well known that if $(P_n)_{n\geq 0}$ is a sequence of polynomials orthogonal with respect to u and $(Q_n)_{n\geq 0}$ is a sequence of polynomials orthogonal with respect to v, then $Q_{2n}(x) = P_n(x^2)$ and $Q_{2n+1}(x) = xP_n^*(x^2)$, where $(P_n^*)_{n\geq 0}$ denotes the sequence of polynomials orthogonal with respect to the linear functional xu. The linear functional v is said to be the symmetrized linear functional of u.

By using a symmetrization process for families of classical orthogonal polynomials described in Table 1, four families of symmetric orthogonal polynomials can be derived [17], which are explicitly expressible in terms of a symmetric class of polynomials $S_n(x; p, q, r, s)$ [17] defined by

$$S_n\left(\begin{array}{cc} r, & s \\ p, & q \end{array}\middle| x\right) = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{\left(2i + (-1)^{n+1} + 2[n/2]\right) p + r}{\left(2i + (-1)^{n+1} + 2\right) q + s}\right) x^{n-2k}.$$

Definition	Weight function	Kind & Interval	Param. constraint
$S_n\left(\begin{array}{cc} -2u-2v-2, & 2u \\ -1, & 1 \end{array}\middle| x\right)$	$x^{2u}(1-x^2)^v$	Infinite, $[-1, 1]$	$u > -1/2$, $v > -1$
$S_n\left(\begin{array}{cc} -2, & 2u \\ 0, & 1 \end{array}\middle| x\right)$	$x^{2u}(1-x^2)^v$	Infinite, $(-\infty, \infty)$	$u > -1/2$
$S_n\left(\begin{array}{cc} -2u-2v-2, & 2u \\ 1, & 1 \end{array}\middle| x\right)$	$x^{-2u}(1+x^2)^{-v}$	Finite, $(-\infty, \infty)$	$\max n < u + v - 1/2$, $u < 1/2,\ v > 0$
$S_n\left(\begin{array}{cc} -2u+2, & 2 \\ 1, & 0 \end{array}\middle| x\right)$	$x^{-2u}\exp(-1/x^2)$	Finite, $(-\infty, \infty)$	$\max n > u - 1/2$

Table 2: Characteristics of symmetrized classical orthogonal polynomials

The family of $S_n(x; p, q, r, s)$ satisfies the holonomic equation

$$x^2(px^2 + q)\Phi_n''(x) + x(rx^2 + s)\Phi_n'(x) \\ - \left(n(r + (n-1)p)x^2 + (1 - (-1)^n)s/2\right)\Phi_n(x) = 0.$$

Tables 1 and 2 show that there are totally ten sequences of classical and symmetrized of classical orthogonal polynomials. Except the Routh-Romanovski polynomials $J_n^{(u,v)}(x; a, b, c, d)$, the Fourier transforms of all ten sequences have been found. Indeed, in [21] the Fourier transforms of generalized Ultraspherical polynomials $S_n\left(\begin{array}{cc} -2u-2v-2, & 2u \\ -1, & 1 \end{array}\middle| x\right)$ and the generalized Hermite polynomials $S_n\left(\begin{array}{cc} -2, & 2u \\ 0, & 1 \end{array}\middle| x\right)$ are derived. In [22], the Fourier transforms of the finite orthogonal polynomials $S_n\left(\begin{array}{cc} -2u-2v+2, & -2u \\ 1, & 1 \end{array}\middle| x\right)$ and

$S_n\left(\begin{array}{cc} -2u+2, & 2 \\ 1, & 0 \end{array}\middle| x\right)$ are obtained and, finally, in [17] we get the Fourier transforms of finite orthogonal polynomials $M_n^{(u,v)}(x)$ and $N_n^{(u)}(x)$. Notice that the Fourier transforms of classical Jacobi, Laguerre and Hermite polynomials are already known in the literature, see e.g. [9] and [14]. Hence, to complete the analysis of families of orthogonal polynomials of Tables 1 and 2, only the Fourier transform of Routh-Romanovski polynomials should be calculated. For this purpose, we first review the general properties of $J_n^{(u,v)}(x;a,b,c,d)$ and begin our treatment with the differential equation

$$\begin{aligned}
&((ax+b)^2+(cx+d)^2)y_n''(x)\\
&\quad+\big(2(1-p)(a^2+c^2)x+q(ad-bc)+2(1-p)(ab+cd)\big)\,y_n'(x)\\
&\qquad-n(n+1-2p)(a^2+c^2)y_n(x)=0,
\end{aligned} \tag{1}$$

where $p,q\in\mathbb{R}$, $n\in\mathbb{Z}^+$, and a,b,c,d are all real parameters such that $ad-bc>0$.

According to [12], one of the solutions of the equation (1) is the real polynomial

$$\begin{aligned}
y_n(x) &= J_n^{(p,q)}(x;a,b,c,d)\\
&= (-1)^n((ab+cd)+i(ad-bc))^n(n+1-2p)_n\\
&\quad\times\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a^2+c^2}{(ab+cd)+i(ad-bc)}\right)^k\\
&\quad\times{}_2F_1\left(\begin{array}{c} k-n,\quad p-n-iq/2 \\ 2p-2n \end{array}\middle|\frac{2(ad-bc)}{(ad-bc)-i(ab+cd)}\right)x^k,
\end{aligned} \tag{2}$$

in which $i=\sqrt{-1}$ and ${}_2F_1(\cdot)$ is a special case of the generalized hypergeometric function [3] of order $(p,q)=(2,1)$ defined by

$${}_pF_q\left(\begin{array}{cccc} a_1, & a_2, & \ldots & a_p \\ b_1, & b_2, & \ldots & b_q \end{array}\middle| x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\ldots(a_p)_k}{(b_1)_k(b_2)_k\ldots(b_q)_k}\frac{x^k}{k!}, \tag{3}$$

with $(r)_k=\prod\limits_{i=0}^{k-1}(r+i)$.

The polynomials (2) can also be represented by the Rodrigues formula [18]

$$\begin{aligned}
&J_n^{(p,q)}(x;a,b,c,d)\\
&= (-1)^n\left((ax+b)^2+(cx+d)^2\right)^p\exp\left(-q\arctan\frac{ax+b}{cx+d}\right)\\
&\quad\times\frac{d^n}{dx^n}\left(\left((ax+b)^2+(cx+d)^2\right)^{n-p}\exp\left(q\arctan\frac{ax+b}{cx+d}\right)\right).
\end{aligned} \tag{4}$$

Using Sturm-Liouville theory for continuous functions, it is shown in [18] that the polynomials (2) are finitely orthogonal with respect to the weight function

$$W^{(p,q)}(x;a,b,c,d)=\left((ax+b)^2+(cx+d)^2\right)^{-p}\exp\left(q\arctan\frac{ax+b}{cx+d}\right), \tag{5}$$

on the real line as follows

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left((ax+b)^2+(cx+d)^2\right)^{-p}\exp\left(q\arctan\frac{ax+b}{cx+d}\right)\\
&\qquad\qquad\times J_n^{(p,q)}(x;a,b,c,d)J_m^{(p,q)}(x;a,b,c,d)dx \qquad (6)\\
&= \left(\int_{-\infty}^{\infty}W^{(p,q)}(x;a,b,c,d)\left(J_n^{(p,q)}(x;a,b,c,d)\right)^2dx\right)\delta_{m,n}\\
&= \|\cdot\|_2^2\ \begin{cases}0 & (n\neq m)\\ 1 & (n=m)\end{cases},
\end{aligned}$$

where $m,n=0,1,...,N\leq p-1/2$ with $N=\max\{m,n\}$, $a,b,c,d,q\in\mathbb{R}$ and $ad-bc>0$.

In this paper, we give the explicit form of the norm square$\|\cdot\|_2^2$ in (6) to be able to obtain the Fourier transform of the standard form of Routh-Romanovski polynomials and then introduce a new set of finite orthogonal functions via Parseval's identity.

2 Computation of the Norm Square of Routh-Romanovski polynomials

To calculate the norm square value $\|\cdot\|_2^2$ of Routh-Romanovski polynomials, if the Rodrigues formula (4) is replaced in (6), then we get

$$\begin{aligned}
\|\cdot\|_2^2 &= \frac{n!(a^2+c^2)^n\Gamma(2p-n)}{\Gamma(2p-2n)}\\
&\quad\times\int_{-\infty}^{\infty}\left((ax+b)^2+(cx+d)^2\right)^{n-p}\exp\left(q\arctan\frac{ax+b}{cx+d}\right)dx, \qquad (7)
\end{aligned}$$

where

$$\Gamma(z)=\int_0^{\infty}x^{z-1}e^{-x}dx,\quad \mathrm{Re}(z)>0, \qquad (8)$$

is the well-known Gamma function.

Relation (7) can still be simplified via Cauchy beta integral formula ([2], [6]), which says that if $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$ and $\mathrm{Re}(c+d)>1$, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dt}{(a+it)^c(b-it)^d}=\frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)}(a+b)^{1-(c+d)}. \qquad (9)$$

In particular, taking into account that

$$(a-it)^{p+iq}(a+it)^{p-iq}=(a^2+t^2)^p\exp(2q\arctan t/a), \qquad (10)$$

for such a choice of the parameters, from (9) we get

$$\int_{-\pi/2}^{\pi/2} e^{st}\cos^r t dt = \frac{2^{-r}\Gamma(r+1)\pi}{\Gamma\left(1+\frac{r+is}{2}\right)\Gamma\left(1+\frac{r-is}{2}\right)}. \tag{11}$$

By using (11) we can now obtain the explicit form of the integral

$$I^* = \int_{-\infty}^{\infty} \left((ax+b)^2+(cx+d)^2\right)^{n-p} \exp\left(q\arctan\frac{ax+b}{cx+d}\right)dx, \tag{12}$$

and the norm square value afterwards. Indeed, it is enough to set $x = \frac{(ad-bc)\tan t-(ab+cd)}{a^2+c^2}$ in (12) and then use (11) to finally get

$$\begin{aligned} I^* &= \int_{-\pi/2}^{\pi/2} \left(\frac{(ad-bc)^2}{a^2+c^2}\right)^{n-p} \left(1+\tan^2 t\right)^{n-p+1} \\ &\qquad \times\frac{ad-bc}{a^2+c^2}\exp\left(q\arctan\frac{a\tan t-c}{c\tan t+a}\right)dt \\ &= \frac{(ad-bc)^{2n-2p+1}}{(a^2+c^2)^{n-p+1}}\int_{-\pi/2}^{\pi/2}\cos^{2p-2n-2} t\exp\left(q\left(t-\arctan\frac{c}{a}\right)\right)dt \\ &= \frac{(ad-bc)^{2n-2p+1}}{(a^2+c^2)^{n-p+1}}\exp\left(-q\arctan\frac{c}{a}\right)\int_{-\pi/2}^{\pi/2} e^{qt}\cos^{2p-2n-2} t dt \\ &= \frac{(ad-bc)^{2n-2p+1}}{(a^2+c^2)^{n-p+1}}\exp\left(-q\arctan\frac{c}{a}\right)\frac{2^{2n+2-2p}\Gamma(2p-2n-1)\pi}{\Gamma(p-n+iq/2)\Gamma(p-n-iq/2)}. \end{aligned} \tag{13}$$

This gives us the norm square value of the Routh-Romanovski polynomials in (7) as follows.

Corollary 1 *We have*

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \left((ax+b)^2+(cx+d)^2\right)^{-p}\exp\left(q\arctan\frac{ax+b}{cx+d}\right) \\ &\qquad \times J_n^{(p,q)}(x;a,b,c,d)J_m^{(p,q)}(x;a,b,c,d)dx \\ &= \left(\frac{2^{2n+1-2p}(ad-bc)^{2n-2p+1}\exp(-q\arctan(c/a))}{(2p-2n-1)(a^2+c^2)^{-p+1}}\right. \\ &\qquad \left.\times\frac{n!\Gamma(2p-n)}{\Gamma(p-n+iq/2)\Gamma(p-n-iq/2)}\right)\delta_{m,n}, \end{aligned} \tag{14}$$

where $m,n=0,1,...,N=\max\{m,n\}\le p-1/2,\ a,b,c,d,q\in\mathbb{R}$ *and* $ad-bc>0$.

On the other hand, since the weight function of orthogonality relation (14) can be simplified as

$$\begin{aligned}&\left((ax+b)^2+(cx+d)^2\right)^{-p}\exp(q\arctan\tfrac{ax+b}{cx+d})\\&=(ax+b)+i(cx+d))^{-p-iq/2}\,(ax+b)-i(cx+d))^{-p+iq/2}\\&=\left|(a+ic)^{-p-iq/2}\right|^2\left(x+\tfrac{b+id}{a+ic}\right)^{-p-iq/2}\left(x+\tfrac{b-id}{a-ic}\right)^{-p+iq/2},\end{aligned} \tag{15}$$

and the corresponding orthogonality interval is $(-\infty,\infty)$, after a suitable linear change of variable the standard form of polynomials (4) can be considered as

$$\begin{aligned}I_n^{(p,q)}(x) &= J_n^{(p,q)}(x;1,0,0,1)\\&= (-i)^n(n+1-2p)_n\sum_{k=0}^{n}\begin{pmatrix}n\\k\end{pmatrix}\\&\qquad\times{}_2F_1\left(\begin{array}{c}k-n,\quad p-n-iq/2\\2p-2n\end{array}\bigg|\,2\right)(-ix)^k.\end{aligned} \tag{16}$$

Taking into account the identity (see [12])

$$\begin{aligned}&\sum_{k=0}^{n}\begin{pmatrix}n\\k\end{pmatrix}{}_2F_1\left(\begin{array}{c}k-n,\quad 1-q^*-n\\1-p^*-n\end{array}\bigg|\,\frac{1}{s^*}\right)\left(\frac{r^*}{s^*}x\right)^k\\&=\frac{(q^*)_n}{(-s^*)^n(p^*)_n}\,{}_2F_1\left(\begin{array}{c}-n,\quad p^*\\q^*\end{array}\bigg|\,r^*x+s^*\right),\end{aligned} \tag{17}$$

the polynomials in (16) take the form

$$\begin{aligned}I_n^{(p,q)}(x) &= (2i)^n(1-p+iq/2)_n\,{}_2F_1\left(\begin{array}{c}-n,\quad n+1-2p\\1-p+iq/2\end{array}\bigg|\,\frac{1-ix}{2}\right)\\&= n!(2i)^nP_n^{(-p+iq/2,-p-iq/2)}(ix),\end{aligned} \tag{18}$$

where $P_n^{(\alpha,\beta)}(x)$ are the well-known Jacobi orthogonal polynomials [2].

Corollary 2 *For the standard polynomials* $I_n^{(p,q)}(x)$ *we have*

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+x^2)^{-p}\exp(q\arctan x)I_n^{(p,q)}(x)I_m^{(p,q)}(x)dx\\&=\frac{n!2^{2n+1-2p}\Gamma(2p-n)\delta_{m,n}}{(2p-2n-1)\Gamma(p-n+iq/2)\Gamma(p-n-iq/2)},\end{aligned} \tag{19}$$

where $m,n=0,1,...,N=\max\{m,n\}\le p-1/2$ *and* $q\in\mathbb{R}$.

It is well known that some orthogonal systems are mapped onto each other by some integral transforms such as Fourier, Mellin and Hankel transforms, see e.g. [7]. Following this approach, in the next section we obtain the Fourier transform of the standard form of Routh-Romanovski polynomials to introduce a finite class of orthogonal functions by using the Parseval's identity.

3 Fourier transform of polynomials $I_n^{(p,q)}(x)$ and its orthogonality relation

The Fourier transform of a function $g \in L^2(\mathbb{R})$, is defined as [7]

$$G(s) = \mathbf{F}(g(x)) = \int_{-\infty}^{\infty} e^{-isx} g(x)dx, \tag{20}$$

and for its inverse transform we have

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} G(s)ds. \tag{21}$$

For $g, h \in L^2(\mathbb{R})$, the Parseval's identity related to the Fourier transform is (see [7])

$$\int_{-\infty}^{\infty} g(x)h(x)dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{F}(g(x))\overline{\mathbf{F}(h(x))}ds. \tag{22}$$

Now, we define the following specific functions

$$\begin{cases} g(x) = (1+x^2)^{-\beta} \exp(\alpha \arctan x) I_n^{(c,d)}(x), \\ h(x) = (1+x^2)^{-u} \exp(l \arctan x) I_m^{(v,w)}(x), \end{cases} \tag{23}$$

where α, β, c, d, and l, u, v, w are real parameters.

Note that if the Fourier transform exists for the functions defined in (23), then the computation of the Fourier transform of one of them is sufficient. For instance, for the function g defined in (23) we have

$$\begin{aligned} \mathbf{F}(g(x)) &= \int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta} \exp(\alpha \arctan x) I_n^{(c,d)}(x)dx \\ &= (2i)^n (1-c+id/2)_n \int_{-\infty}^{\infty} e^{-isx}(1-ix)^{-\beta+i\frac{\alpha}{2}}(1+ix)^{-\beta-i\frac{\alpha}{2}} \\ &\qquad \times \left(\sum_{k=0}^{n} \frac{(-n)_k(n+1-2c)_k}{(1-c+id/2)_k k! 2^k} (1-ix)^k \right) dx \\ &= (2i)^n (1-c+id/2)_n \sum_{k=0}^{n} \frac{(-n)_k(n+1-2c)_k}{(1-c+id/2)_k k! 2^k} \\ &\qquad \times \int_{-\infty}^{\infty} e^{-isx}(1-ix)^{-\beta+k+i\frac{\alpha}{2}}(1+ix)^{-\beta-i\frac{\alpha}{2}} dx. \end{aligned} \tag{24}$$

Now, it remains in (24) to evaluate

$$A_k^*(s;\alpha,\beta) = \int\limits_{-\infty}^{\infty} e^{-isx}(1-ix)^{-\beta+k+i\frac{\alpha}{2}}(1+ix)^{-\beta-i\frac{\alpha}{2}}\,dx. \tag{25}$$

If $\mathrm{Re}(p+q) > 1$, then (see [7] p. 119, formula 12)

$$\int\limits_{-\infty}^{\infty} e^{-isx}(1-ix)^{-p}(1+ix)^{-q}dx = \begin{cases} \frac{\pi}{\Gamma(p)}\ (s/2)^{\frac{p+q}{2}-1}\,W_{\frac{p-q}{2},\frac{1-p-q}{2}}(2s), & s>0, \\ \\ \frac{\pi}{\Gamma(q)}\ (-s/2)^{\frac{p+q}{2}-1}\,W_{\frac{q-p}{2},\frac{1-p-q}{2}}(-2s), & s<0, \end{cases} \tag{26}$$

where $W_{a,b}(s)$ denotes the second kind Whittaker functions (see [7] p. 386) defined by

$$W_{a,b}(s) = s^{1/2}e^{-s/2}\left(\frac{\Gamma(-2b)}{\Gamma(1/2-b-a)}\,s^b\,{}_1F_1\left(\begin{array}{c} 1/2+b-a \\ 2b+1 \end{array}\middle|\, s\right)\right.$$
$$\left. +\frac{\Gamma(2b)}{\Gamma(1/2+b-a)}\,s^{-b}\,{}_1F_1\left(\begin{array}{c} 1/2-b-a \\ -2b+1 \end{array}\middle|\, s\right)\right), \quad 2b\notin \mathbf{Z}\,. \tag{27}$$

Hence, for $Re(2\beta-k)<1$ and $k+1-2\beta\notin\mathbf{Z}$, (25) would be equal to

$$A_k^*(s;\alpha,\beta) = \begin{cases} \frac{\pi}{\Gamma(\beta-k-i\alpha/2)}\ (s/2)^{\beta-1-k/2}\,W_{-\frac{k+i\alpha}{2},\frac{k+1}{2}-\beta}(2s), & s>0, \\ \\ \frac{\pi}{\Gamma(\beta+i\alpha/2)}\ (-s/2)^{\beta-1-k/2}\,W_{\frac{k+i\alpha}{2},\frac{k+1}{2}-\beta}(-2s), & s<0. \end{cases} \tag{28}$$

Thus relation (24) becomes

$$\mathbf{F}(g(x)) = 2^n i^n \frac{\Gamma(1-c+n+id/2)}{\Gamma(1-c+id/2)}\sum_{k=0}^{n}\frac{(-n)_k(n+1-2c)_k}{(1-c+id/2)_k k!2^k}A_k^*(s;\alpha,\beta)\,. \tag{29}$$

For simplicity if we here introduce the function

$$B_n(s;p_1,p_2,p_3,p_4) = \sum_{k=0}^{n}\frac{(-n)_k(n+1-2p_3)_k}{(1-p_3+ip_4/2)_k k!2^k}A_k^*(s;p_1,p_2), \tag{30}$$

then it is clear from (29) that

$$\mathbf{F}(g(x)) = 2^n i^n \frac{\Gamma(1-c+n+id/2)}{\Gamma(1-c+id/2)}\,B_n(s;\alpha,\beta,c,d)\,. \tag{31}$$

As a further consequence, by referring to (22) and (23) we have

$$\overline{\mathbf{F}(h(x))} = \frac{(-i)^m 2^m\Gamma(1-v+m-iw/2)}{\Gamma(1-v-iw/2)}\,B_m(s;-l,u,v,-w)\,. \tag{32}$$

By substituting (31) and (32) in Parseval's identity (22) one gets

$$\int_{-\infty}^{\infty} (1+x^2)^{-(\beta+u)} \exp((\alpha+l)\arctan x)\, I_n^{(c,d)}(x)\, I_m^{(v,w)}(x)\, dx$$
$$= \frac{(-1)^m i^{n+m} 2^{m+n} \Gamma(1-c+n+id/2)\,\Gamma(1-v+m-iw/2)}{\Gamma(1-c+id/2)\,\Gamma(1-v-iw/2)} \int_{-\infty}^{\infty} B_n(s;\alpha,\beta,c,d)\, B_m(s;-l,u,v,-w)\, ds\,. \tag{33}$$

Now, if in the left hand side of (33) we take

$$c = v = \beta + u \quad \texttt{and} \quad d = w = \alpha + l,$$

then according to the orthogonality relation (19) and the constraints about the parameters mentioned above, the following theorem, which is the main result of this paper, will be finally deduced.

Main Theorem. *The sequence of functions* $\{B_n(s;p_1,p_2,p_3,p_4)\}_{n\geq 0}$ *defined in (30) satisfies the finite orthogonality relation*

$$\int_{-\infty}^{\infty} B_n(s;\alpha,\beta,\nu,\omega)\, B_m(s;\alpha-\omega,\nu-\beta,\nu,-\omega)\, ds$$
$$= \frac{\pi\, 2^{2-2\nu} n!}{(2\nu-2n-1)} \frac{\Gamma(2\nu-n)\,\Gamma^2(1-\nu-i\omega/2)}{\Gamma(\nu-n+iw/2)\Gamma(\nu-n-iw/2)\Gamma^2(1-\nu+n+i\omega/2)}\, \delta_{m,n}\,, \tag{34}$$

for $m,n = 0,1,...,N = \max\{m,n\} < 2\beta-1 \leq \nu-1/2, n+1-2\beta \notin \mathbf{Z}$, *and* $\alpha,\ \omega \in \mathbb{R}$.

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