# Monotonicity of Zeros of Laguerre-Sobolev-Type Orthogonal Polynomials

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**Abstract**

Denote by $x_{n,k}^{M,N}(\alpha)$, $k = 1, \ldots, n$, the zeros of the Laguerre-Sobolev-Type polynomials $L_n^{(\alpha,M,N)}(x)$ orthogonal with respect to the inner product

$$\langle p, q\rangle = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + Mp(0)q(0) + Np'(0)q'(0)$$

where $\alpha > -1$, $M \geq 0$ and $N \geq 0$. We prove that $x_{n,k}^{M,N}(\alpha)$ interlace with the zeros of Laguerre orthogonal polynomials $L_n^{(\alpha)}(x)$ and establish monotonicity with respect to the parameters $M$ and $N$ of $x_{n,k}^{M,0}(\alpha)$ and $x_{n,k}^{0,N}(\alpha)$. Moreover, we find $N_0$ such that $x_{n,n}^{M,N}(\alpha) < 0$ for all $N > N_0$, where $x_{n,n}^{M,N}(\alpha)$ is the smallest zero of $L_n^{(\alpha,M,N)}(x)$. Further, we present monotonicity and asymptotic relations of certain functions involving $x_{n,k}^{M,0}(\alpha)$ and $x_{n,k}^{0,N}(\alpha)$.

*Key words:* Orthogonal polynomials, Laguerre polynomials, Sobolev-Type orthogonal polynomials, zeros, monotonicity, asymptotics.

## 1 Introduction and statement of results

Consider the sequence of Laguerre-Sobolev-Type polynomials $\{L_n^{(\alpha,M,N)}(x)\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product

$$\langle p,q\rangle = \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx + Mp(0)q(0) + Np'(0)q'(0), \quad (1.1)$$

where $\alpha > -1$, $M \geq 0$ and $N \geq 0$. They were defined and studied first by Koekoek and Meijer [10]. Dueñas and Marcellán [7] considered the Laguerre-Sobolev-Type orthogonal polynomials $\widehat{L}_n^{(\alpha,\widehat{M},\widehat{N})}(x)$ generated by the inner product

$$\langle p,q\rangle = \int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx + \widehat{M}p(0)q(0) + \widehat{N}p'(0)q'(0),$$

where $\alpha > -1$, $\widehat{M} \geq 0$ and $\widehat{N} \geq 0$. It is clear that the sequences $\{L_n^{(\alpha,M,N)}(x)\}_{n=0}^{\infty}$ and $\{\widehat{L}_n^{(\alpha,\widehat{M},\widehat{N})}(x)\}_{n=0}^{\infty}$ coincide when $\widehat{M} = \Gamma(\alpha+1)M$ and $\widehat{N} = \Gamma(\alpha+1)N$. Hence all the results concerning the zeros of $L_n^{(\alpha,M,N)}(x)$ obtained in this paper can be rewritten in a obvious manner substituting $M$ by $\widehat{M}/\Gamma(\alpha+1)$ and $N$ by $\widehat{N}/\Gamma(\alpha+1)$.

Let $L_n^{(\alpha)}(x)$, $n = 0,1,\ldots$, be the classical Laguerre polynomial, orthogonal with respect to the inner product

$$\langle p,q\rangle = \int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx$$

and normalized by (2.4) below. In the sequel we denote by $x_{n,k}(\alpha)$ the zeros of the Laguerre polynomial $L_n^{(\alpha)}(x)$ and by $x_{n,k}^{M,N}(\alpha)$, $x_{n,k}^{M}(\alpha)$, and $x_{n,k}^{N}(\alpha)$ the zeros of $L_n^{(\alpha,M,N)}(x)$, $L_n^{(\alpha,M,0)}(x)$, and $L_n^{(\alpha,0,N)}(x)$, respectively, all arranged in decreasing order. We prove that the zeros $x_{n,k}^{M,N}(\alpha)$ interlace with the zeros $x_{n,k}(\alpha)$ when $M,N > 0$ and establish the monotonicity of the zeros $x_{n,k}^{M}(\alpha)$ and $x_{n,k}^{N}(\alpha)$ with respect to the parameters $M$ and $N$, respectively.

**Theorem 1** *The inequalities*

$$x_{n,k+1}^{M,N}(\alpha) < x_{n,k+1}(\alpha) < x_{n,k}^{M,N}(\alpha) < x_{n,k}(\alpha) \quad (1.2)$$

*hold for every $n \in \mathbb{N}$, $n \geq 2$, and each $k$ with $1 \leq k \leq n-1$. Moreover, for* *every fixed n the smallest zero $x_{n,n}^{M,N}(\alpha)$ satisfies*

$$x_{n,n}^{M,N}(\alpha) > 0, \quad \text{for} \quad N < N_0,$$

$$x_{n,n}^{M,N}(\alpha) = 0, \quad \text{for} \quad N = N_0,$$

$$x_{n,n}^{M,N}(\alpha) < 0, \quad \text{for} \quad N > N_0,$$

*where*

$$N_0 = \frac{(\alpha+1)\Gamma(n-1)\Gamma(\alpha+4)}{\Gamma(n+\alpha+2)}. \tag{1.3}$$

It is quite interesting that $N_0$ does not depend on $M$.

In the case $N = 0$ we obtain the following statement which was already derived by Dueñas and Marcellán [6].

**Corollary 1** *The inequalities*

$$0 < x_{n,k+1}^{M}(\alpha) < x_{n,k+1}(\alpha) < x_{n,k}^{M}(\alpha) < x_{n,k}(\alpha) \tag{1.4}$$

*hold for every $n \in \mathbb{N}$, $n \geq 2$, and each $k$ with $1 \leq k \leq n-1$. Moreover, the smallest zero $x_{n,n}^{M}(\alpha)$ behaves like $\mathcal{O}(1/M)$ as $M$ goes to infinity.*

When $M = 0$ Theorem 1 yields:

**Corollary 2** *The inequalities*

$$x_{n,k+1}^{N}(\alpha) < x_{n,k+1}(\alpha) < x_{n,k}^{N}(\alpha) < x_{n,k}(\alpha) \tag{1.5}$$

*hold for every $n \in \mathbb{N}$, $n \geq 2$, and each $k$ with $1 \leq k \leq n-1$. Moreover, the smallest zero $x_{n,n}^{N}(\alpha)$ satisfies*

$$x_{n,n}^{N}(\alpha) > 0, \quad \text{for} \quad N < N_0,$$

$$x_{n,n}^{N}(\alpha) = 0, \quad \text{for} \quad N = N_0,$$

$$x_{n,n}^{N}(\alpha) < 0, \quad \text{for} \quad N > N_0,$$

*where $N_0$ is given by (1.3).*

Setting $N_0 = \widehat{N_0}/\Gamma(\alpha+1)$, we conclude that the smallest zero $\widehat{x}^{\widehat{N}}_{n,n}(\alpha)$ of the $n$th Laguerre-Sobolev-Type orthogonal polynomial defined by Dueñas and Marcellán [7] satisfies

$$\widehat{x}^{\widehat{N}}_{n,n}(\alpha) > 0, \quad \text{for} \quad \widehat{N} < \widehat{N_0},$$

$$\widehat{x}^{\widehat{N}}_{n,n}(\alpha) = 0, \quad \text{for} \quad \widehat{N} = \widehat{N_0},$$

$$\widehat{x}^{\widehat{N}}_{n,n}(\alpha) < 0, \quad \text{for} \quad \widehat{N} > \widehat{N_0},$$

where

$$\widehat{N_0} = \Gamma(\alpha+1)N_0 = \frac{\Gamma(n-1)\Gamma(\alpha+2)\Gamma(\alpha+4)}{\Gamma(n+\alpha+2)}. \tag{1.6}$$

Observe that comprehensive numerical results were furnished in [7] in an attempt to determine the values of $\widehat{N}$ for which $\widehat{x}^{\widehat{N}}_{n,n}(\alpha)$ is positive (negative). Needless to say, those numerical values of $\widehat{N_0}$ coincide with our explicit expression (1.6).

It was proved in [6] that $x^M_{n,k}(\alpha) \to x_{n-1,k}(\alpha+2)$ as $M \to \infty$, for each $k = 1,\ldots,n-1$. We provide a sharp quantitative result concerning the asymptotics of the differences $x^M_{n,k}(\alpha) - x_{n-1,k}(\alpha+2)$.

**Theorem 2** *For every $n \geq 2$ and each $\alpha > -1$,*

$$0 < x^M_{n,n}(\alpha) < x_{n,n}(\alpha) < x_{n,n-1}(\alpha+2) < x^M_{n,n-1}(\alpha) < x_{n-1,n-1}(\alpha) < \cdots$$

$$\cdots < x_{n-1,1}(\alpha+2) < x^M_{n,1}(\alpha) < x_{n,1}(\alpha).$$

*Moreover, the zeros $x^M_{n,k}(\alpha)$ are decreasing functions of $M$, with $M > 0$, $x^M_{n,n}(\alpha) \to 0$, $x^M_{n,k}(\alpha) \to x_{n-1,k}(\alpha+2)$ as $M \to \infty$,*

$$\lim_{M\to\infty} M x^M_{n,n}(\alpha) = (\alpha+2)g_n(\alpha)$$

*and*

$$\lim_{M\to\infty} M[x^M_{n,k}(\alpha) - x_{n-1,k}(\alpha+2)] = g_n(\alpha), \quad k = 1,\ldots,n-1, \tag{1.7}$$

*where*

$$g_n(\alpha) = \frac{\alpha+1}{(\alpha+2)\binom{n+\alpha+1}{n-1}}. \tag{1.8}$$

It is quite surprising that the limit $g_n(\alpha)$ in (1.7) does not depend on $k$.

**Theorem 3** *For every $n \geq 2$ and each $\alpha > -1$, the quantities*

$$M[x_{n,1}^M(\alpha) - x_{n-1,1}(\alpha+2)]$$

*are increasing functions of $M > 0$.*

In fact, numerical experiments suggest that the quantities $Mx_{n,n}^M(\alpha)$ and

$$M[x_{n,k}^M(\alpha) - x_{n-1,k}(\alpha+2)]$$

are increasing functions of $M > 0$ for each $k$ with $1 \leq k \leq n-1$.

Having in mind the limit relation (1.7) in Theorem 2 and the monotonicity in Theorem 3, we obtain:

**Corollary 3** *The inequalities*

$$x_{n,1}^M(\alpha) \leq x_{n-1,1}(\alpha+2) + g_n(\alpha)/M$$

*hold for every $M > 0$, $\alpha > -1$ and $n \in \mathbb{N}$ with $n \geq 2$.*

In order to formulate the corresponding result about the asymptotic of $x_{n,k}^N$, we define the polynomial

$$\begin{aligned} F_{n,\alpha}(x) := &-\frac{n(\alpha+2)-(\alpha+1)}{(\alpha+1)(\alpha+3)}\binom{n+\alpha}{n-2}L_n^{(\alpha)}(x) \\ &-\frac{n-1}{\alpha+1}\binom{n+\alpha}{n-1}\frac{d}{dx}L_n^{(\alpha)}(x) - \frac{1}{\alpha+1}\binom{n+\alpha}{n-1}\frac{d^2}{dx^2}L_n^{(\alpha)}(x). \end{aligned} \tag{1.9}$$

**Theorem 4** *For every $n \geq 2$ and each $\alpha > -1$, the polynomial $F_{n,\alpha}(x)$ possesses only real distinct zeros that we denote by $\zeta_{n,k}(\alpha)$, arranged in decreasing order. Moreover,*

$$\zeta_{n,n}(\alpha) < x_{n,n}^N(\alpha) < x_{n,n}(\alpha) < \cdots < \zeta_{n,1}(\alpha) < x_{n,1}^N(\alpha) < x_{n,1}(\alpha).$$

*The zeros $x_{n,k}^N(\alpha)$ are decreasing functions of $N$, for $N > 0$, $x_{n,k}^N(\alpha) \to \zeta_{n,k}(\alpha)$ as $N \to \infty$, and*

$$\lim_{N\to\infty} N[x_{n,k}^N(\alpha) - \zeta_{n,k}(\alpha)] = g_{n,k}(\alpha), \quad k = 1, \ldots, n,$$

*where*

$$\begin{aligned} g_{n,k}(\alpha) = &\binom{n+\alpha}{n-1}\frac{(n-1)(n+\alpha+1)}{(\alpha+1)(\alpha+2)^2(\alpha+3)^2}\times \\ &\frac{a_n(\alpha)[\zeta_{n,k}(\alpha)]^2 + b_n(\alpha)\zeta_{n,k}(\alpha) + c_n(\alpha)}{n[\zeta_{n,k}(\alpha)]^2 - (\alpha+1)\zeta_{n,k}(\alpha)} \end{aligned} \tag{1.10}$$

*with*

$$a_n(\alpha) = (\alpha+2)^2 n^2 - \alpha(\alpha+2)n - (\alpha+1), \quad b_n(\alpha) = (\alpha+1)(\alpha+2)(\alpha+3)$$

*and*

$$c_n(\alpha) = -(\alpha+1)(\alpha+2)^2(\alpha+3).$$

The reader should observe that we make no claim about the monotonicity of the quantities $N[x_{n,k}^N(\alpha) - \zeta_{n,k}(\alpha)]$. Numerical experiments show that, curiously enough, all the quantities

$$N[x_{n,k}^N(\alpha) - \zeta_{n,k}(\alpha)], \quad k = 1, \ldots, n-2, n,$$

increase with $N > 0$ while $N[x_{n,n-1}^N(\alpha) - \zeta_{n,n-1}(\alpha)]$ decreases.

## 2 Preliminaries

We consider the polynomials $L_n^{(\alpha,M,N)}(x)$ which are orthogonal with respect to the inner product (1.1). Koekoek and Meijer [10] obtained the closed form representation

$$L_n^{(\alpha,M,N)}(x) = A_n L_n^{(\alpha)}(x) + B_n \frac{d}{dx} L_n^{(\alpha)}(x) + C_n \frac{d^2}{dx^2} L_n^{(\alpha)}(x), \tag{2.1}$$

where

$$\begin{aligned}
A_n = A_n^{(M,N)}(\alpha) &= 1 + M\binom{n+\alpha}{n-1} + \frac{n(\alpha+2)-(\alpha+1)}{(\alpha+1)(\alpha+3)} N \binom{n+\alpha}{n-2} \\
&\quad + \frac{MN}{(\alpha+1)(\alpha+2)} \binom{n+\alpha}{n-1}\binom{n+\alpha+1}{n-2}, \\
B_n = B_n^{(M,N)}(\alpha) &= M\binom{n+\alpha}{n} + \frac{n-1}{\alpha+1} N \binom{n+\alpha}{n-1} \\
&\quad + \frac{2MN}{(\alpha+1)^2} \binom{n+\alpha}{n}\binom{n+\alpha+1}{n-2}, \\
C_n = C_n^{(M,N)}(\alpha) &= \frac{N}{\alpha+1}\binom{n+\alpha}{n-1} + \frac{MN}{(\alpha+1)^2}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1}.
\end{aligned}$$

We shall need some technical results concerning Laguerre polynomials (see [12]).

1. Three term recurrence relation:

$$\begin{aligned} L_{-1}^{(\alpha)}(x) = 0, \quad L_0^{(\alpha)}(x) = 1, \\ nL_n^{(\alpha)}(x) = (-x + 2n + \alpha - 1)L_{n-1}^{(\alpha)}(x) - (n + \alpha - 1)L_{n-2}^{(\alpha)}(x) \end{aligned} \tag{2.2}$$

2. MacLaurin expansion:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} = \frac{(-1)^n}{n!} x^n + \cdots . \tag{2.3}$$

3. Value at the origin:

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}, \quad n \geq 0. \tag{2.4}$$

4. Formulae for the derivative:

$$\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x) = \frac{1}{x}\{nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)\}, \quad n \geq 1. \tag{2.5}$$

5. Second order linear differential equation:

$$x[L_n^{(\alpha)}(x)]'' + (\alpha + 1 - x)[L_n^{(\alpha)}(x)]' + nL_n^{(\alpha)}(x) = 0. \tag{2.6}$$

We also shall use a simple lemma concerning the behavior of the zeros of linear combinations of two polynomials with interlacing zeros.

**Lemma 1** *Let $h_n(x) = (-1)^n a(x - x_1) \cdots (x - x_n)$ and $g_n(x) = (-1)^{n-1} b(x - y_1) \cdots (x - y_n)$ be polynomials with real zeros, where $a$ and $b$ are real positive constants. If, for any real constant $c > 0$, the polynomial*

$$f(x) = h_n(x) - cg_n(x)$$

*has $n$ real zeros $\eta_n < \eta_{n-1} < \cdots < \eta_1$ which interlace with the zeros of $h_n(x)$ in the following form*

$$\eta_n < x_n < \cdots < \eta_1 < x_1,$$

*then*

$$y_n < \eta_n < x_n < \cdots < y_1 < \eta_1 < x_1.$$

*Moreover, each $\eta_k$ is a decreasing function of $c$ and*

$$\lim_{c \to \infty} \eta_k = y_k, \ k = 1, \ldots, n.$$

We omit the proof because similar results about interlacing of zeros are known in the literature, see for instance [13], [11, p. 117] and [9, Theorem 5]. A monotonicity property of linear combination of polynomials with interlacing zeros was proved in [2, Lemma 1]. The lemma can be generalized for linear combination where instead of a constant, $c$ is a continuous function of constant sign. Notice that, given $n \in \mathbb{N}$ and two polynomials $q_n(x)$ and $q_{n-1}(x)$ with exact degrees $n$ and $n-1$, whose zeros interlace, it is very well known (see [13]) that they belong to a sequence of monic polynomials orthogonal with respect to a nontrivial probability measure $\mu$ supported on the real line. The above lemma means that the polynomial $f(x)$ is orthogonal with respect to the measure $\mu$ to every polynomial of degree at most $n-2$, i.e. it is quasi-orthogonal in the sense of [4, Definition 5.1 on page 64]. However, it does not follow from the interlacing of the zeros of $q_n$ and and $q_{n-1}(x)$, for all $n = 2, 3 \ldots$ that the sequence $\{q_n\}_{n=1}^{\infty}$ is orthogonal, see [5].

Other recent results concerning zeros of linear combinations of orthogonal polynomials have been obtained in [1] and [3].

## 3 Proof of the Results

In order to prove Theorem 1 we shall need the following fact:

**Lemma 2** *If* $n \geq 2$ *then*

$$nx_{n,k}(\alpha) - (\alpha + 1) > 0$$

*for each* $k$, $k = 1, \ldots, n$, *and for every* $\alpha > -1$.

***Proof.*** Let $n \geq 2$, $1 \leq k \leq n$ and $\alpha > -1$. Then, using inequality (3.4) in Ismail and Li [8], we obtain

$$\begin{aligned} nx_{n,k}(\alpha) - (\alpha + 1) &\geq nx_{n,n}(\alpha) - (\alpha + 1) \\ &> (n-1)(2n + \alpha - 1) - n\sqrt{1 + a(n-1)(n + \alpha - 1)}, \end{aligned}$$

with $a = 4\cos^2(\pi/(n+1))$. Observe that

$$(n-1)(2n + \alpha - 1) - n\sqrt{1 + a(n-1)(n + \alpha - 1)} > 0$$

is equivalent to

$$a_n\alpha^2 + b_n\alpha + c_n > 0, \tag{3.1}$$

where

$$a_n = (n-1)^2,$$

$$b_n = -(n-1)[(a-4)n^2 + 4n + 2],$$

$$c_n = -(n-1)[(a-4)n^3 + (4-a)n^2 + 5n + 1].$$

Inequality (3.1) is true for $n = 2$ and $n = 3$ because it is equivalent to

$$(\alpha+1)^2 > 0$$

and

$$4[(\alpha+1)^2 + 9] > 0,$$

respectively.

Now, let $n \geq 4$. Obviously the leading coefficient $a_n$ of (3.1) is positive and straightforward calculations show that the discriminant $b_n^2 - 4a_n c_n$ is equal to

$$n^2(n-1)^2[4(1+2a) - (4-a)an^2].$$

We shall need to prove that the expression

$$\sigma(n) = 4(1+2a) - (4-a)an^2$$

is negative for $n \geq 4$, $\alpha > -1$, and $a = 4\cos^2(\pi/(n+1))$. In fact, $\sigma(n) < 0$ if and only if

$$\frac{4(1+2a)}{a(4-a)} < n^2.$$

Having in mind that

$$2\cos^2(\theta) = 1 + \cos(2\theta), \quad 4\cos^2(\theta)\sin^2(\theta) = \sin^2(2\theta),$$

$$\sin(\theta) \geq \frac{2\theta}{\pi}, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

and setting $\pi = (n+1)\theta$ we obtain

$$\frac{4(1+2a)}{a(4-a)} = \frac{5 + 4\cos 2\theta}{\sin^2(2\theta)} \leq \frac{9\pi^2}{16\theta^2} = \frac{9}{16}(n+1)^2.$$

Thus,

$$\frac{4(1+2a)}{a(4-a)} \leq \frac{9}{16}(n+1)^2 < n^2$$

for every $n \geq 4$.

**Proof of Theorem 1:** By (2.1)

$$L_n^{(\alpha,M,N)}(x) = A_n L_n^{(\alpha)}(x) - \Omega_{n-1}^{(\alpha)}(x), \tag{3.2}$$

where

$$\Omega_{n-1}^{(\alpha)}(x) = \Omega_{n-1}^{(\alpha,M,N)}(x) = -B_n \frac{d}{dx} L_n^{(\alpha)}(x) - C_n \frac{d^2}{dx^2} L_n^{(\alpha)}(x).$$

On the other hand, (2.2) and (2.5) yield

$$\begin{aligned} L_n^{(\alpha,M,N)}(x) = \frac{1}{x^2} \left[ A_n x^2 + nB_n x - n(\alpha+1)C_n \right] L_n^{(\alpha)}(x) \\ -\frac{(n+\alpha)}{x^2} \left[ (B_n + C_n)x - (\alpha+1)C_n \right] L_{n-1}^{(\alpha)}(x). \end{aligned} \tag{3.3}$$

Now, evaluating (3.2) and (3.3) for $x = x_{n,k}(\alpha)$ we obtain

$$\begin{aligned} & L_n^{(\alpha,M,N)}(x_{n,k}(\alpha)) = -\Omega_{n-1}^{(\alpha)}(x_{n,k}(\alpha)) \\ & = -\frac{(n+\alpha)}{[x_{n,k}(\alpha)]^2} \left[ (B_n + C_n)x_{n,k}(\alpha) - (\alpha+1)C_n \right] L_{n-1}^{(\alpha)}(x_{n,k}(\alpha)) \end{aligned} \tag{3.4}$$

for $1 \leq k \leq n$. Since $B_n = B_n^{(M,N)}(\alpha)$ and $C_n = C_n^{(M,N)}(\alpha)$ are increasing functions of $M$, by Lemma 2 we obtain

$$\begin{aligned} & \left( B_n^{(M,N)}(\alpha) + C_n^{(M,N)}(\alpha) \right) x_{n,k}(\alpha) - (\alpha+1)C_n^{(M,N)}(\alpha) \\ & \geq \left( B_n^{(0,N)}(\alpha) + C_n^{(0,N)}(\alpha) \right) x_{n,k}(\alpha) - (\alpha+1)C_n^{(0,N)}(\alpha) \\ & = C_n^{(0,N)}(\alpha) \left[ \left( \frac{B_n^{(0,N)}(\alpha)}{C_n^{(0,N)}(\alpha)} + 1 \right) x_{n,k}(\alpha) - (\alpha+1) \right] \\ & = C_n^{(0,N)}(\alpha) \left[ n x_{n,k}(\alpha) - (\alpha+1) \right] > 0. \end{aligned}$$

Hence,

$$sign\left( L_n^{(\alpha,M,N)}(x_{n,k}(\alpha)) \right) = -sign\left( L_{n-1}^{(\alpha)}(x_{n,k}(\alpha)) \right) = (-1)^{n+1-k}, \tag{3.5}$$

for $k = 1, \ldots, n$. This immediately implies that $n-1$ zeros of $L_n^{(\alpha,M,N)}(x)$ interlace with the $n$ zeros of $L_n^{(\alpha)}(x)$, i.e.,

$$x_{n,k+1}(\alpha) < x_{n,k}^{M,N}(\alpha) < x_{n,k}(\alpha), \quad n \geq 2, \quad 1 \leq k \leq n-1.$$

Let us show that $x_{n,n}^{M,N}(\alpha) < x_{n,n}(\alpha)$. It is clear, by (3.5), that

$$L_n^{(\alpha,M,N)}(x_{n,n}(\alpha)) < 0.$$

On the other hand, (2.1) and (2.3) implies

$$\lim_{x \to -\infty} L_n^{(\alpha,M,N)}(x) = +\infty.$$

Then $x_{n,n}^{M,N}(\alpha) < x_{n,n}(\alpha)$.

In order to investigate the location of $x_{n,n}^{M,N}(\alpha)$ with respect to the origin, it suffices to observe that $L_n^{(\alpha,M,N)}(0) = 0$ if and only if $N = N_0$.

It remains to show that, for the case $N = 0$, $x_{n,n}^{M}(\alpha) > 0$ and $x_{n,n}^{M}(\alpha) \sim \mathcal{O}(1/M)$ as $M \to \infty$. By (3.5)

$$L_n^{(\alpha,M,0)}(x_{n,n}(\alpha)) < 0$$

and, by (2.1) and (2.4),

$$L_n^{(\alpha,M,0)}(0) = 1.$$

Then $x_{n,n}^{M}(\alpha) > 0$.

It was proved in [6] that $x_{n,n}^{M}(\alpha) \sim \mathcal{O}(1/M)$ as $M \to \infty$. Here we furnish a straightforward proof of this fact. Let $\xi_n^M(\alpha)$ and $\eta_n^M(\alpha)$ be the real numbers that

$$L_n^{(\alpha,M,0)}(0) + \frac{d}{dt}L_n^{(\alpha,M,0)}(t)|_{t=0}\, \xi_n^M(\alpha) = 0 \tag{3.6}$$

and

$$L_n^{(\alpha,M,0)}(0) + \frac{L_n^{(\alpha,M,0)}(x_{n,n}(\alpha)) - L_n^{(\alpha,M,0)}(0)}{x_{n,n}(\alpha)}\, \eta_n^M(\alpha) = 0. \tag{3.7}$$

By (3.6), $\xi_n^M(\alpha)$ is the intersection of the tangent line to $L_n^{(\alpha,M,0)}(x)$ at the point $(0, L_n^{(\alpha,M,0)}(0))$ with the real axis, and by (3.7), $\eta_n^M(\alpha)$ is the intersection of the secant line of $L_n^{(\alpha,M,0)}(x)$ through $(0, L_n^{(\alpha,M,0)}(0))$ and $(x_{n,n}(\alpha), L_n^{(\alpha,M,0)}(x_{n,n}(\alpha)))$ with the real axis. Since $L_n^{(\alpha,M,0)}(x)$ is a convex function in $(0, x_{n,n}^{M}(\alpha))$ and $0 < x_{n,n}^{M}(\alpha) < x_{n,n}(\alpha)$, then

$$0 < \xi_n^M(\alpha) < x_{n,n}^{M}(\alpha) < \eta_n^M(\alpha) < x_{n,n}(\alpha). \tag{3.8}$$

Moreover, solving (3.6) and (3.7) we obtain

$$\xi_n^M(\alpha) = \frac{\alpha}{n + M\binom{n+\alpha}{n-1}(n-\alpha)}$$

and

$$\eta_n^M(\alpha) = \frac{x_{n,n}(\alpha)\binom{n+\alpha}{n}}{1 + M\binom{n+\alpha}{n}L_n^{(\alpha,M,0)}(x_{n,n}(\alpha))}.$$

Hence, $\xi_n^M(\alpha)$ and $\eta_n^M(\alpha)$ behave like $\mathcal{O}(1/M)$ when $M$ goes to infinity, and from inequalities (3.8), the smallest zero $x_{n,n}^{M}(\alpha)$ also behaves like $\mathcal{O}(1/M)$ as $M \to \infty$.

**Proof of Theorem 2:** By (2.1), $L_n^{(\alpha,M,0)}(x)$ can be written as

$$L_n^{(\alpha,M,0)}(x) = L_n^{(\alpha)}(x) - MG_{n,\alpha}(x), \tag{3.9}$$

with

$$\begin{aligned} G_{n,\alpha}(x) : &= -\binom{n+\alpha}{n-1} L_n^{(\alpha)}(x) - \binom{n+\alpha}{n} \frac{d}{dx} L_n^{(\alpha)}(x), \\ &= \frac{1}{n}\binom{n+\alpha}{n-1} x L_{n-1}^{(\alpha+2)}(x), \end{aligned} \tag{3.10}$$

where the latter equality follows from (2.3). Thus, (3.9), (1.4) and Lemma 1 immediately imply the interlacing property

$$0 < x_{n,n}^M(\alpha) < x_{n,n}(\alpha) < \cdots x_{n-1,1}(\alpha+2) < x_{n,1}^M(\alpha) < x_{n,1}(\alpha). \tag{3.11}$$

Moreover, each $x_{n,k}^M(\alpha)$ is an decreasing function of $M$, with $M > 0$. Also, we conclude that the zeros of $L_n^{(\alpha,M,0)}(x)$ go to the zeros of $G_{n,\alpha}(x)$, a result that was already obtained in [6]. In other words, we have that

$$\lim_{M\to\infty} x_{n,n}^M(\alpha) = 0, \quad \lim_{M\to\infty} x_{n,k}^M(\alpha) = x_{n-1,k}(\alpha+2), \; 1 \le k \le n-1.$$

It follows from (3.9) and (3.11) that, for $M$ sufficiently large,

$$MG'_{n,\alpha}(x_{n-1,k}(\alpha+2)) \sim \frac{L_n^{(\alpha)}(x_{n,k}^M(\alpha))}{x_{n,k}^M(\alpha) - x_{n-1,k}(\alpha+2)}, \quad 1 \le k \le n-1,$$

and

$$MG'_{n,\alpha}(0) \sim \frac{L_n^{(\alpha)}(x_{n,n}^M(\alpha))}{x_{n,n}^M(\alpha)}.$$

Therefore

$$\begin{aligned} \lim_{M\to\infty} M[x_{n,k}^M(\alpha) - x_{n-1,k}(\alpha+2)] &= \lim_{M\to\infty} \frac{L_n^{(\alpha)}(x_{n,k}^M(\alpha))}{G'_{n,\alpha}(x_{n-1,k}(\alpha+2))} \\ &= \frac{L_n^{(\alpha)}(x_{n-1,k}(\alpha+2))}{G'_{n,\alpha}(x_{n-1,k}(\alpha+2))} \end{aligned}$$

and

$$\lim_{M\to\infty} M[x_{n,n}^M(\alpha)] = \lim_{M\to\infty} \frac{L_n^{(\alpha)}(x_{n,n}^M(\alpha))}{G'_{n,\alpha}(0)} = \frac{L_n^{(\alpha)}(0)}{G'_{n,\alpha}(0)}.$$

Using (2.2), (2.4), (2.6), and (3.10) we obtain

$$\frac{G'_{n,\alpha}(x_{n-1,k}(\alpha+2))}{L_n^{(\alpha)}(x_{n-1,k}(\alpha+2))} = \frac{1}{g_n(\alpha)} \quad \text{and} \quad \frac{G'_{n,\alpha}(0)}{L_n^{(\alpha)}(0)} = \frac{1}{(\alpha+2)g_n(\alpha)},$$

where $g_n(\alpha)$ is given by (1.8).

**Proof of Theorem 3:** By (3.11),

$$x_{n-1,1}(\alpha+2) < x_{n,1}^M(\alpha) < x_{n,1}(\alpha).$$

Thus, by the Mean Value Theorem, there exist some real number $\eta \in (x_{n-1,1}(\alpha+2), x_{n,1}^M(\alpha))$ such that

$$[x_{n,1}^M(\alpha) - x_{n-1,1}(\alpha+2)] = \frac{-L_n^{(\alpha)}(x_{n-1,1}(\alpha+2))}{\left[L_n^{(\alpha,M,0)}(x)\right]'|_{x=\eta}}. \tag{3.12}$$

On the other hand, differentiation in (3.9) yields

$$[L_n^{(\alpha)}(x)]'|_{x=x_{n,k}^M(\alpha)}\frac{dx_{n,k}^M(\alpha)}{dM} - G_{n,\alpha}(x_{n,k}^M(\alpha)) - MG'_{n,\alpha}(x_{n,k}^M(\alpha))\frac{dx_{n,k}^M(\alpha)}{dM} = 0,$$

or equivalently,

$$\frac{dx_{n,k}^M(\alpha)}{dM} = \frac{L_n^{(\alpha)}(x_{n,k}^M(\alpha))}{M[L_n^{(\alpha,M,0)}(x)]'|_{x=x_{n,k}^M(\alpha)}}. \tag{3.13}$$

From (3.12) and (3.13) we obtain

$$\begin{aligned}\frac{d}{dM}\left\{M[x_{n,1}^M(\alpha) - x_{n-1,1}(\alpha+2)]\right\} &= M\frac{dx_{n,1}^M(\alpha)}{dM} + [x_{n,1}^M(\alpha) - x_{n-1,1}(\alpha+2)] \\ &= \frac{L_n^{(\alpha)}(x_{n,1}^M(\alpha))}{\left[L_n^{(\alpha,M,0)}(x)\right]'|_{x=x_{n,1}^M(\alpha)}} - \frac{L_n^{(\alpha)}(x_{n-1,1}(\alpha+2))}{\left[L_n^{(\alpha,M,0)}(x)\right]'|_{x=\eta}}.\end{aligned}$$

Since, depending on the parity of $n$, $L_n^{(\alpha,M,0)}(x)$ and $L_n^{(\alpha)}(x)$ both are either increasing and convex or decreasing and concave functions in $(x_{n-1,1}(\alpha+2), \infty)$, we conclude that

$$\left|L_n^{(\alpha)}(x_{n,1}^M(\alpha))\right| < \left|L_n^{(\alpha)}(x_{n-1,1}(\alpha+2))\right|$$

and

$$\left|\left[L_n^{(\alpha,M,0)}(x)\right]'|_{x=x_{n,1}^M(\alpha)}\right| > \left|\left[L_n^{(\alpha,M,0)}(x)\right]'|_{x=\eta}\right|,$$

which completes the proof of Theorem 3.

**Proof of Theorem 4:** Recall that

$$\begin{aligned}F_{n,\alpha}(x) = &-\frac{n(\alpha+2)-(\alpha+1)}{(\alpha+1)(\alpha+3)}\binom{n+\alpha}{n-2}L_n^{(\alpha)}(x) \\ &-\frac{n-1}{\alpha+1}\binom{n+\alpha}{n-1}\frac{d}{dx}L_n^{(\alpha)}(x) - \frac{1}{\alpha+1}\binom{n+\alpha}{n-1}\frac{d^2}{dx^2}L_n^{(\alpha)}(x).\end{aligned}$$

Then, by (2.1), we can rewrite $L_n^{(\alpha,0,N)}(x)$ as

$$L_n^{(\alpha,0,N)}(x) = L_n^{(\alpha)}(x) - NF_{n,\alpha}(x). \tag{3.14}$$

Thus, (3.14), (1.5) and Lemma 1 immediately imply the interlacing property

$$\zeta_{n,n}(\alpha) < x_{n,n}^N(\alpha) < x_{n,n}(\alpha) < \cdots < \zeta_{n,1}(\alpha) < x_{n,1}^N(\alpha) < x_{n,1}(\alpha), \tag{3.15}$$

where $\zeta_{n,n}(\alpha) < \cdots < \zeta_{n,1}(\alpha)$ are the zeros of $F_{n,\alpha}(x)$. Moreover, each $x_{n,k}^N(\alpha)$ is an decreasing function of $N$, with $N > 0$. Also, we conclude that the zeros of $L_n^{(\alpha,0,N)}(x)$ go to the zeros of $F_{n,\alpha}(x)$, that is,

$$\lim_{M\to\infty} x_{n,k}^N(\alpha) = \zeta_{n,k}(\alpha), \;\; 1 \leq k \leq n.$$

It follows from (3.14) and (3.15) that

$$NF'_{n,\alpha}(\zeta_{n,k}(\alpha)) \sim \frac{L_n^{(\alpha)}(x_{n,k}^N(\alpha))}{x_{n,k}^N(\alpha) - \zeta_{n,k}(\alpha)}, \quad 1 \leq k \leq n,$$

for all sufficiently large values of $N$. In other words, we have

$$\lim_{N\to\infty} N[x_{n,k}^N(\alpha) - \zeta_{n,k}(\alpha)] = \lim_{N\to\infty} \frac{L_n^{(\alpha)}(x_{n,k}^N(\alpha))}{F'_{n,\alpha}(\zeta_{n,k}(\alpha))} = \frac{L_n^{(\alpha)}(\zeta_{n,k}(\alpha))}{F'_{n,\alpha}(\zeta_{n,k}(\alpha))}.$$

Using (2.2), (2.4), (2.6), and (1.9) we obtain

$$\frac{F'_{n,\alpha}(\zeta_{n,k}(\alpha))}{L_n^{(\alpha)}(\zeta_{n,k}(\alpha))} = \frac{1}{g_{n,k}(\alpha)},$$

where $g_{n,k}(\alpha)$ is given by (1.10).

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