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Linear spectral transformations, Hessenberg matrices, and orthogonal polynomials.

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Abstract In this manuscript we analyze some linear spectral transformations of a positive definite Hermitian linear functional. In particular we focus our attention in the behavior of the Verblunsky parameters of the perturbed linear functional when we add the linear functional defined by the Lebesgue measure on the unit circle. Some illustrative examples are pointed out.

Keywords Quasi-definite linear functionals · Toeplitz matrices · orthogonal polynomials on the unit circle · linear spectral transformations · Hessenberg matrices · Verblunsky parameters.

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1 Introduction.

Orthogonal polynomials with respect to nontrivial probability measures supported on the real line have attracted the interest of researchers long time ago. Their relations with continued fractions and rational approximation, their role in the spectral theory of linear differential operators and Sturm-Liouville problems, as well as their applications in the theory of integrable systems (Toda lattices and Lax pairs, among others) constitute an illustrative

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sample of their impact in many domains of Mathematics and Physics.

Surprisingly, the theory of orthogonal polynomials with respect to nontrivial probability measures supported on the unit circle has not been so popular until the mid 80's. The monographs by G. Szegő (see [22] and [13]) and Ya. L. Geronimus (see [10] and [9]) were the main (and the few) major contributions to the subject despite the fact that people working in linear prediction theory and digital signal processing used as a basic background orthogonal polynomials on the unit circle (see [7] and references therein). The very recent monograph by B. Simon ([20]) constitutes an updated overview of the most remarkable directions of research in the theory, both from a theoretical approach (extensions of the Szegő theory from an analytic point of view), as well as from their applications in the spectral analysis of unitary operators and the (GGT and CMV) matrix representations of the multiplication operator, quadrature formulas, and integrable systems (Schur flows and Ablowitz-Ladik equations), among others.

In the last years, some attention has been paid to the connections between some canonical perturbations of measures supported on the unit circle, the corresponding sequences of orthogonal polynomials, the GGT matrix representations, and Carathéodory functions (see [6], [8], [11], [12], [15], [17], [18], [19] among others) in the framework of the theory of the so called rational spectral transformations. It is the natural counterpart of the theory developed in [21] and [24] for measures supported on the real line.

The aim of our contribution is twofold. First, we present the state of the art in the theory of linear spectral transformations (Christoffel, Uvarov, and Geronimus as particular examples) and the connection with the LU and QR factorization of Jacobi and Hessenberg matrices, respectively. Second, we study an interesting example of a linear spectral transformation on the unit circle far away of the above canonical cases. The key idea is the analysis of the perturbation of a measure supported on the unit circle by a new measure such that the corresponding Carathéodory functions differ in a polynomial. In particular, when we add a constant, i.e. the corresponding Toeplitz matrix is the result of the addition of a constant diagonal matrix to the initial Toeplitz matrix, we study some new analytic propositionerties of the corresponding sequences of orthogonal polynomials in a different way to the work done in [3] and [4].

The structure of the paper is as follows. In Section 2 we present a basic background about orthogonal polynomials with respect to measures supported on the real line. We emphasize the role of the Jacobi matrices associated with such polynomials. The connection between canonical spectral linear transforms of the measures and LU and UL factorizations of the corresponding Jacobi matrices is stated. In Section 3 we deal with some analog problems for polynomials orthogonal with respect to measures supported on the unit circle. In this case, the connection between canonical linear spectral transforms of the measures and QR factorizations of the corresponding Hessenberg matrices is given. In Section 4 we focus our attention in a perturbation of the Toeplitz matrices associated with such measures. In particular, we analyze the perturbation of a Toeplitz matrix by the addition of a scalar matrix. The expression of the resulting orthogonal polynomials in terms of the original sequence is deduced. Finally, Section 5 is devoted to the study of such a kind of perturbation when the initial Toeplitz matrix is associated with the Bernstein-Szegő measure and the Tchebychev measure, respectively.

2 Orthogonal polynomials on the real line and Jacobi matrices.

Given a nontrivial probability measure μ supported on some infinite subset E of the real line, a (unique) sequence of polynomials $\{p_n\}_{n\geqslant 0}$ can be defined as

$$\int_E p_m(x)p_n(x)d\mu(x) = \delta_{m,n}, \quad n,m \geqslant 0, \tag{1}$$

where $\delta_{m,n}$ is the Kronecker delta. $\{p_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to μ and

$$p_n(x) = \gamma_n x^n + \zeta_n x^{n-1} + \text{lower degree terms}, \tag{2}$$

with $\gamma_n > 0$ for every $n \geqslant 0$. Starting from the initial conditions $p_0(x) = 1$ and $p_{-1}(x) = 0$, one can obtain this family of polynomials by means of the so-called three term recurrence relation [5]

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \tag{3}$$

where the recurrence coefficients are

$$a_n = \int_E xp_{n-1}(x)p_n(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0, \quad n \geqslant 1,$$

and

$$b_n = \int_E xp_n^2(x)d\mu(x) = \frac{\zeta_n}{\gamma_n} - \frac{\zeta_{n+1}}{\gamma_{n+1}}, \quad n \geqslant 0.$$

(3) has a matrix representation of the form

$$xp(x) = \tilde{\mathbf{J}}p(x),$$

where $p(x) = [p_0(x), p_1(x), \ldots]^t$ and $\tilde{\mathbf{J}}$ is the symmetric tridiagonal matrix

$$\tilde{\mathbf{J}} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \ddots \\ 0 & 0 & a_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

that is known in the literature as Jacobi matrix [5]. It represents the multiplication operator with respect to the basis of orthonormal polynomials defined as above.

On the other hand, the monic orthogonal polynomials with respect to μ are given by $P_n(x) = p_n(x)/\gamma_n$, $n \geqslant 0$. In such a case, (3) becomes

$$P_{n+1}(x) = (x - b_n)P_n(x) - d_n P_{n-1}(x), \quad n \geqslant 0, \tag{4}$$

with $d_n = a_n^2$, and its corresponding matrix representation is

$$\mathbf{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ d_1 & b_1 & 1 & 0 & \cdots \\ 0 & d_2 & b_2 & 1 & \ddots \\ 0 & 0 & d_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

that is known as monic Jacobi matrix.

2.1 Spectral transformations

Spectral transformations of nontrivial probability measures supported on the real line have been extensively studied during the last years and within a wide variety of frameworks including bispectral problems, integrable systems, matrix theory, group theory, and Stieltjes functions (see [1], [14], [16], [21] among others). We are interested here in the effect of such transformations on the corresponding Jacobi matrices (see [2], [23], among others).

We consider three canonical transformations as follows

(i) Christoffel transformation

$$d\tilde{\mu} = (x-\beta)d\mu, \quad \beta \notin supp(\mu).$$

(ii) Uvarov transformation

$$d\tilde{\mu} = d\mu + M_r\delta(x-\beta), \quad M_r \in \mathbb{R}.$$

(iii) Geronimus transformation

$$d\tilde{\mu} = \frac{d\mu}{x-\beta} + M_r\delta(x-\beta), \quad \beta \notin supp(\mu), M_r \in \mathbb{R}.$$

We will denote these transformations by $\mathcal{R}_C(\beta)$, $\mathcal{R}_U(\beta, M_r)$, and $\mathcal{R}_C(\beta, M_r)$, respectively. They are related as follows

$$\mathcal{R}_C(\beta) \circ \mathcal{R}_G(\beta, M_r) = \mathcal{I} \quad \text{(Identity transformation)},$$
$$\mathcal{R}_G(\beta, M_r) \circ \mathcal{R}_C(\beta) = \mathcal{R}_U(\beta, M_r).$$

If we denote by $S(x)$ the Stieltjes function associated with μ, i.e. $S(x) = \frac{\mu_0}{x} + \sum_{k=1}^{\infty} \frac{\mu_k}{x^{k+1}}$, where $\mu_k = \int_E x^k d\mu(x)$ are the moments of μ, then these transformations can be expressed in terms of the corresponding Stieltjes functions as

$$\widetilde{S}(x) = \frac{A(x)S(x) + B(x)}{D(x)}, \tag{5}$$

where $\widetilde{S}(x)$ is the Stieltjes function associated with $\tilde{\mu}$, and $A(x)$, $B(x)$, $D(x)$ are polynomials in the variable x, which are explicitly given (see [24]) for each of the above transformations. Furthermore, [24] shows that the linear spectral transformations (5), constitute an non-commutative group whose generators are transformations (i) and (iii).

On the other hand, it has been shown that the Jacobi matrices associated with the perturbed measures $\tilde{\mu}$ for the above transformations can be obtained from the original Jacobi matrix via LU and UL factorizations using the Darboux transformation, defined as follows.

Consider the monic Jacobi matrix associated with some nontrivial probability measure μ. Notice that, if all of its principal leading submatrices are nonsingular, then $\mathbf{J}$ has a unique LU factorization where $\mathbf{L}$ and $\mathbf{U}$ are bidiagonal matrices

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ l_1 & 1 & 0 & 0 & \cdots \\ 0 & l_2 & 1 & 0 & \ddots \\ 0 & 0 & l_3 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_2 & 1 & 0 & \cdots \\ 0 & 0 & u_3 & 1 & \ddots \\ 0 & 0 & 0 & u_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{6}$$

The matrix $\mathbf{J}_p := \mathbf{UL}$ is said to be the *Darboux transformation of* $\mathbf{J}$ *without parameter* . Notice that $\mathbf{J}_p$ is again a tridiagonal matrix with ones as entries on the upper diagonal and, according to Favard's theorem, it is a monic Jacobi matrix associated with some nontrivial measure $\tilde{\mu}$.

In a similar way, factorizing $\mathbf{J} = \mathbf{UL}$, with $\mathbf{L}$ and $\mathbf{U}$ as in (6), and defining $\mathbf{J}_d := \mathbf{LU}$, we obtain again a monic Jacobi matrix. In this situation, $\mathbf{J}_d$ is said to be the *Darboux transformation* of $\mathbf{J}$. Contrary to the LU factorization, the UL factorization is not unique and depends on a free parameter.

Remark 1 A necessary and sufficient condition for the existence of the LU factorization of $\mathbf{J}$ is $P_n(0) \neq 0$, $n \geqslant 1$. Furthermore, assuming the LU factorization exists, the elements of $\mathbf{L}$ and $\mathbf{U}$ are given by

$$l_1 = \frac{d_1}{b_0}, \quad l_n = \frac{d_n}{b_{n-1} - l_{n-1}}, \quad n \geqslant 2, \tag{7}$$

$$u_1 = b_0, \quad u_n = b_{n-1} - l_{n-1}, \quad n \geqslant 2. \tag{8}$$

Next, we describe transformations $(i) - (iii)$ in terms of the corresponding monic Jacobi matrices.

Proposition 1 *[2] Let μ be a nontrivial probability measure and denote by $\{P_n\}_{n\geqslant 0}$ and $\mathbf{J}$ its corresponding sequence of monic orthogonal polynomials and monic Jacobi matrix, respectively. Let $\beta \in \mathbb{R}$ such that $P_n(\beta) \neq 0$, $n \geqslant 1$. Then, if we apply the transformation*

$$\mathbf{J} - \beta\mathbf{I} = \mathbf{LU}, \quad \tilde{\mathbf{J}} := \mathbf{UL} + \beta\mathbf{I},$$

then $\tilde{\mathbf{J}}$ is the monic Jacobi matrix associated with $d\tilde{\mu} = (x - \beta)d\mu$, i.e. the Christoffel transformation.

Notice that if β does not belong to the convex hull of $supp(\mu)$ then $P_n(\beta) \neq 0$ for every $n \geq 1$. It is not difficult to show that we can iterate the previous result and extend it to describe the monic Jacobi matrix associated with m consecutive Christoffel transformations. Indeed,

Proposition 2 *[2] Let μ, $\{P_n\}_{n\geqslant 0}$, and $\mathbf{J}$ be as in the previous Proposition. Consider the following transformations of $\mathbf{J}$*

$$\begin{aligned}
\mathbf{C}_1 &:= \mathbf{J} - \beta_1\mathbf{I} = \mathbf{L_1U_1}, & \tilde{\mathbf{C}}_1 &:= \mathbf{U_1L_1} + \beta_1\mathbf{I},\\
\mathbf{C}_2 &:= \tilde{\mathbf{C}}_1 - \beta_2\mathbf{I} = \mathbf{L_2U_2}, & \tilde{\mathbf{C}}_2 &:= \mathbf{U_2L_2} + \beta_2\mathbf{I},\\
&\vdots & &\\
\mathbf{C}_m &:= \tilde{\mathbf{C}}_{m-1} - \beta_m\mathbf{I} = \mathbf{L_mU_m}, & \tilde{\mathbf{C}}_m &:= \mathbf{U_mL_m} + \beta_m\mathbf{I},
\end{aligned}$$

with $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}$. If we denote by $\{P_{n,i}\}$ the sequence of monic orthogonal polynomials associated with $\tilde{\mathbf{C}}_i$, $1 \leqslant i \leqslant m-1$, and assuming that $P_n(\beta) \neq 0$, $P_{n,i}(\beta_{i+1}) \neq 0$, $n \geqslant 1$, $1 \leqslant i \leqslant m-1$, then $\tilde{\mathbf{C}}_m$ is the monic Jacobi matrix associated with the measure

$$d\tilde{\mu} = (x - \beta_1)(x - \beta_2)\ldots(x - \beta_m)d\mu.$$

Notice that in the previous results for the Christoffel transformation, only the Darboux transformation without parameter was used. On the other hand, for transformations (ii) and (iii) there are analogous results that use both the Darboux transformation and the Darboux transformation without parameter, as follows.

Proposition 3 *[2] Let* $\mathbf{J}_0$ *be the monic Jacobi matrix associated with the nontrivial probability measure* μ*. Consider the following transformations of* $\mathbf{J}_0$

$$\begin{aligned} \mathbf{J}_0 - \beta\mathbf{I} &= \mathbf{L_1 U_1}, & \mathbf{J}_1 &:= \mathbf{U_1 L_1}, \\ \mathbf{J}_1 &= \mathbf{U_2 L_2}, & \mathbf{J}_2 &:= \mathbf{L_2 U_2} + \beta\mathbf{I}. \end{aligned}$$

Then $\mathbf{J}_2$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = d\mu + M_r \delta(x-\beta),$$

i.e. the Uvarov transformation of μ*, where*

$$M_r = \frac{\mu_0(b_0 - \beta - s)}{s},$$

with $\mu_0 = \int_E d\mu(x)$ *and* s *is the free parameter associated with the UL factorization of* $\mathbf{J}_1$*.*

Proposition 4 *[2] Let* $\mathbf{J}_1$ *be the monic Jacobi matrix associated with the measure* $\hat{\mu}$*. Suppose there exists a measure* μ *such that* $d\hat{\mu} = (x-\beta)d\mu$*. If we apply the following transformation to* $\mathbf{J}_1$

$$\mathbf{J}_1 - \beta\mathbf{I} = \mathbf{U_1 L_1}, \quad \mathbf{J}_2 := \mathbf{L_1 U_1} + \beta\mathbf{I},$$

then $\mathbf{J}_2$ *is the monic Jacobi matrix associated with the measure*

$$d\tilde{\mu} = \frac{d\hat{\mu}}{x-\beta} + M_r \delta(x-\beta),$$

i.e. the Geronimus transformation of $\hat{\mu}$*, where* $M_r = \frac{\int_E d\hat{\mu}}{s}$ *and* s *is the free parameter associated with the UL factorization of* $\mathbf{J}_1$*.*

3 Orthogonal polynomials on the unit circle and Hessenberg matrices.

3.1 Background

Let $\mathcal{L}$ be a linear functional in the linear space of Laurent polynomials satisfying $c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$, i.e. $\mathcal{L}$ is Hermitian. The complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ are said to be the *moments* associated with $\mathcal{L}$ and the matrix

$$\mathbf{T} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \tag{9}$$

is known in the literature as a Toeplitz matrix [13]. $\mathbb{P}$ will denote the linear space of polynomials with complex coefficients. If $\mathbf{T}_n$, the $(n+1)\times(n+1)$ principal leading submatrix of $\mathbf{T}$, is non-singular for every $n \geqslant 0$, then $\mathcal{L}$ is said to be quasi-definite and the existence of a sequence of monic polynomials, orthogonal with respect to $\mathcal{L}$, is guaranteed. On the other hand, if $\det \mathbf{T}_n > 0$, for every $n \geqslant 0$, then $\mathcal{L}$ is said to be positive definite and it has the following integral representation

$$\langle \mathcal{L}, p(z) \rangle = \int_{\mathbb{T}} p(z) d\sigma(z), \quad p \in \mathbb{P}, \tag{10}$$

where σ is a nontrivial positive Borel measure supported on the unit circle $\mathbb{T} = \{z : |z| = 1\}$.

The concepts presented in the previous Section concerning measures supported on the real line have a counterpart for measures supported on the unit circle. If σ is a nontrivial probability measure supported on $\mathbb{T}$, then there exists a (unique) family of polynomials $\{\varphi_n\}_{n\geqslant 0}$, with $deg\, \varphi_n = n$ and positive leading coefficient, such that

$$\int_{\mathbb{T}} \varphi_n(z) \overline{\varphi_m(z)} d\sigma(z) = \delta_{m,n}. \tag{11}$$

$\{\varphi_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to σ. Denoting by κ_n the leading coefficient of $\varphi_n(z)$, $\Phi_n(z) = \varphi_n(z)/\kappa_n$ is the corresponding sequence of monic orthogonal polynomials. These polynomials satisfy the following forward and backward recurrence relations (see [10], [13], [20], [22])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \tag{12}$$

$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \tag{13}$$

where $\Phi_n^*(z) = z^n \bar{\Phi}_n(z^{-1})$ is the so-called reversed polynomial and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$ are known as Verblunsky or reflection parameters. It is important to notice that in the positive definite case we get $|\Phi_n(0)| < 1$, $n \geqslant 1$.

The multiplication operator with respect to $\{\varphi_n\}_{n\geqslant 0}$ is represented in a matrix form by

$$z\varphi(z) = \mathbf{H}_\varphi \varphi(z), \tag{14}$$

where $\varphi(z) = [\varphi_0(z), \varphi_1(z), \ldots, \varphi_n(z), \ldots]^t$ and $\mathbf{H}_\varphi$ is a lower Hessenberg matrix whose entries are

$$h_{n,j} = \begin{cases} \frac{\kappa_n}{\kappa_{n+1}} & \text{if } j = n+1, \\ -\frac{\kappa_j}{\kappa_n} \Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if } j \leqslant n, \\ 0 & \text{if } j > n+1. \end{cases} \tag{15}$$

Proposition 5 $\mathbf{H}_\varphi$ *satisfies*

(i) $\mathbf{H}_\varphi \mathbf{H}_\varphi^* = \mathbf{I}$,

(ii) $\mathbf{H}_\varphi^* \mathbf{H}_\varphi = \mathbf{I} - \lambda_\infty(0)\varphi(0)\varphi(0)^*$,

where $\mathbf{I}$ *is the semi-infinite identity matrix and* $\lambda_\infty(0) = \prod_{n=0}^{\infty}(1 - |\Phi_{n+1}(0)|^2)$.

Remark 2 Proposition 5 states that the infinite matrix $\mathbf{H}_\varphi$ is unitary if and only if $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 = +\infty$. In terms of the measure σ this fact is equivalent to $\log \sigma' \notin L^1\left(\frac{d\theta}{2\pi}\right)$. In other words, σ does not belong to the Szegő class (see [20]).

Remark 3 From (14), is not difficult to show that $\mathbf{H}_\Phi$, the Hessenberg matrix associated with $\{\Phi_n\}_{n\geqslant 0}$, has as entries

$$h_{n,j} = \begin{cases} 1 & \text{if } j = n+1, \\ -\frac{\mathbf{k}_n}{\mathbf{k}_j} \Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if } j \leqslant n, \\ 0 & \text{if } j > n+1, \end{cases} \tag{16}$$

where

$$\mathbf{k}_n = \frac{1}{\kappa_n^2}$$

3.2 Spectral transformations.

Transformations $(i)-(iii)$ defined in the previous Section for measures supported on the real line have also a counterpart for measures supported on the unit circle. They are defined as follows

(i) Christoffel transformation

$$d\sigma_C = |z-\alpha|^2 d\sigma, \quad \alpha \in \mathbb{C}, |z| = 1.$$

(ii) Uvarov transformation with one mass

$$d\sigma_U = d\sigma + M_c\delta(z-\alpha), \quad |\alpha| = 1, M_c \in \mathbb{R}_+.$$

(iii) Uvarov transformation with two masses

$$d\sigma_U = d\sigma + M_c\delta(z-\alpha) + \bar{M}_c\delta(z-\bar{\alpha}^{-1}), \quad |\alpha| \in \mathbb{R}_+ \setminus \{0,1\}, M_c \in \mathbb{C}.$$

(iv) Geronimus transformation

$$d\sigma_G = \frac{1}{|z-\alpha|^2} d\sigma + M_c\delta(z-\alpha) + \bar{M}_c\delta(z-\bar{\alpha}^{-1}), \quad |\alpha| > 1, M_c \in \mathbb{C}, |z| = 1.$$

We will denote these transformations by $\mathcal{F}_C(\alpha)$, $\mathcal{F}_U(\alpha, M_c)$, and $\mathcal{F}_G(\alpha, M_c)$, respectively. As in the case of the real line, these transformations are related by

$$\begin{aligned} &\mathcal{F}_C(\alpha) \circ \mathcal{F}_G(\alpha, M_c) = \mathcal{I} \quad \text{(Identity transformation)}, \\ &\mathcal{F}_G(\alpha, M_c) \circ \mathcal{F}_C(\alpha) = \mathcal{F}_U(\alpha, M_c). \end{aligned}$$

Furthermore, introducing the Carathéodory function associated with σ

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k,$$

these transformations can be expressed in terms of the corresponding Carathéodory functions as

$$\widetilde{F}(z) = \frac{A(z)F(z) + B(z)}{D(z)}, \tag{17}$$

where $\widetilde{F}(z)$ is the Carathéodory function associated with the perturbed measure and $A(z)$, $B(z)$, $D(z)$ are polynomials that are explicitly given (see [18]) for the above transformations.

Next, we focus our attention in the study of these transformations from the point of view of the corresponding Hessenberg matrices. We state a relation between the Hessenberg matrices associated with the original and perturbed measures by means of a QR factorization. Next, we describe some results in this direction.

We begin with the Christoffel transformation. Let denote by $\{\psi_n\}_{n\geqslant 0}$ the sequence of orthonormal polynomials with respect to σ_C. The relation between both families of polynomials is (see [11])

$$(z-\alpha)\psi_n(z) = \sqrt{\frac{K_n(\alpha,\alpha)}{K_{n+1}(\alpha,\alpha)}}\varphi_{n+1}(z) - \sum_{j=0}^{n} \frac{\varphi_{n+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{n+1}(\alpha,\alpha)K_n(\alpha,\alpha)}}\varphi_j(z), \tag{18}$$

where $K_n(z,y) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(y)}$.

If

$$\varphi(z) = [\varphi_0(z),\ \varphi_1(z),\ \ldots, \varphi_n(z), \ldots]^t \ \text{ and } \ \psi(z) = [\psi_0(z),\ \psi_1(z),\ \ldots, \psi_n(z), \ldots]^t,$$

then the matrix representation of (18) is

$$(z-\alpha)\psi(z) = \mathbf{M}_C\varphi(z), \tag{19}$$

where $\mathbf{M}_C$ is a lower Hessenberg matrix with entries

$$m_{i,j} = \begin{cases} -\frac{\varphi_{i+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{i+1}(\alpha,\alpha)K_i(\alpha,\alpha)}}, & \text{if } j \leqslant i, \\ \sqrt{\frac{K_i(\alpha,\alpha)}{K_{i+1}(\alpha,\alpha)}}, & \text{if } j = i+1, \\ 0, & \text{if } j > i+1. \end{cases} \tag{20}$$

Notice that $(\mathbf{M}_C)_n$, the $n \times n$ leading submatrix of $\mathbf{M}_C$, is a quasi-unitary matrix, i.e. its first $n-1$ rows constitute an orthonormal set, and the last row is orthogonal with respect to this set, but is not normalized.

Furthermore, if we denote by $\mathbf{L}_{\varphi\psi}$ the lower triangular matrix such that $\varphi(z) = \mathbf{L}_{\varphi\psi}\psi(z)$, then $\mathbf{L}_{\varphi\psi}$ can be computed from $\mathbf{H}_\varphi$ and $\mathbf{M}_C$ as follows

Proposition 6 *[6],[19]*

$$\mathbf{L}_{\varphi\psi} = (\mathbf{H}_\varphi - \alpha\mathbf{I})\mathbf{M}_C^*. \tag{21}$$

Now it is possible to determine the relation between $\mathbf{H}_\psi$, the Hessenberg matrix associated with σ_C, and $\mathbf{H}_\varphi$.

Proposition 7 *[6],[19]*

$$\mathbf{H}_\varphi - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_C, \tag{22}$$

$$\mathbf{H}_\psi - \alpha\mathbf{I} = \mathbf{M}_C\mathbf{L}_{\varphi\psi}. \tag{23}$$

Notice that an "almost" *QR* factorization appears, since $\mathbf{M}_C$ is not a unitary matrix (but it is very close). The iteration of the canonical Christoffel transformation has been analyzed in [11], [15], and [17].

The results for the Christoffel transformation can be used to obtain a similar result for the Uvarov transformation. If we assume that there exists a sequence of polynomials $\{\upsilon_n\}_{n\geqslant 0}$ orthonormal with respect to σ_U as defined in (*ii*), then the relation between the Hessenberg matrices $\mathbf{H}_\varphi$ and $\mathbf{H}_\upsilon$ associated with σ and σ_U, respectively, is (see [6], [18])

Proposition 8

$$\mathbf{H}_\varphi - \alpha\mathbf{I} = \mathbf{L}_{\varphi\psi}\mathbf{M}_C, \tag{24}$$

$$\mathbf{H}_\upsilon - \alpha\mathbf{I} = \mathbf{L}_U\mathbf{M}_U, \tag{25}$$

where $\mathbf{L}_U = \mathbf{L}_{\upsilon\varphi}\mathbf{L}_{\varphi\psi}$, $\mathbf{M}_U = \mathbf{M}_C\mathbf{L}_{\upsilon\varphi}^{-1}$, *and* $\mathbf{L}$ *are the matrices of change of bases for the orthonormal polynomial families denoted by their subindices.*

The iteration of the canonical Uvarov transformation, with $|\alpha| = 1$, has been studied in [10] and [17].

Finally, let us consider the Geronimus transformation defined in (*iii*). Necessary and sufficient conditions for the existence of a sequence of monic polynomials orthogonal with respect to σ_G were studied in [8], as well as the relation between the corresponding families of monic orthogonal polynomials and their associated Hessenberg matrices. If we denote by $\{G_n\}_{n\geqslant 0}$ the sequence of monic polynomials orthogonal with respect to σ_G and by $\mathbf{M}_G$ the corresponding Hessenberg matrix, from the relation between $\{G_n\}_{n\geqslant 0}$ and $\{\Phi_n\}_{n\geqslant 0}$, then we get

Proposition 9 *[8] Let $\mathbf{L}_G$ be the lower triangular matrix with 1 on the diagonal elements such that $G(z) = \mathbf{L}_G\Phi(z)$ and denote by $\mathbf{H}_G$ the Hessenberg matrix associated with $\{G_n\}_{n\geqslant 0}$. Then,*

$$\mathbf{H}_\Phi - \alpha\mathbf{I} = \mathbf{M}_G\mathbf{L}_G \tag{26}$$

and

$$\mathbf{H}_G - \alpha\mathbf{I} = \mathbf{L}_G\mathbf{M}_G. \tag{27}$$

4 A diagonal perturbation of a Toeplitz matrix

Let $\mathcal{L}$ be a linear functional in the linear space Λ of Laurent polynomials. Define a linear functional $\widetilde{\mathcal{L}}$ such that its associated bilinear functional satisfies

$$\langle p(z), q(z)\rangle_{\widetilde{\mathcal{L}}} := \langle p(z), q(z)\rangle_{\mathcal{L}} + \boldsymbol{m}\int_{\mathbb{T}} p(z)\overline{q(z)}\frac{dz}{2\pi z}, \tag{28}$$

with $\boldsymbol{m} \in \mathbb{R}$ and $p, q \in \mathbb{P}$. Assume that $\mathcal{L}$ is a positive definite linear functional. Then, this transformation, that will be denoted by $\mathcal{F}_D(\boldsymbol{m})$, can be expressed in terms of the corresponding measure σ as

$$d\tilde{\sigma} = d\sigma + \boldsymbol{m}\frac{d\theta}{2\pi}, \tag{29}$$

i.e. the addition of a Lebesgue measure to σ. We will assume $\boldsymbol{m} \in \mathbb{R}_+$ in order for $\tilde{\sigma}$ to be a positive Borel measure supported in $\mathbb{T}$. On the other hand, the moments $\{\tilde{c}_k\}_{k\in\mathbb{Z}}$ associated with $\widetilde{\mathcal{L}}$ are given by

$$\tilde{c}_0 = c_0 + \boldsymbol{m}, \quad \tilde{c}_k = c_k, \quad k = \pm 1, 2, \ldots. \tag{30}$$

As a consequence, the Carathéodory function of the linear functional $\widetilde{\mathcal{L}}$ is

$$\widetilde{F}(z) = F(z) + \boldsymbol{m}. \tag{31}$$

Notice that $\widetilde{\mathbf{T}}$, the Toeplitz matrix associated with $\widetilde{\mathcal{L}}$, is

$$\widetilde{\mathbf{T}} = \mathbf{T} + \boldsymbol{m}\mathbf{I}, \tag{32}$$

i.e. a constant is added to the main diagonal of $\mathbf{T}$. Now we proceed to obtain the sequence of monic orthogonal polynomials with respect to $\widetilde{\mathcal{L}}$.

Proposition 10 *Let $\mathcal{L}$ be a positive definite linear functional and denote by $\{\Phi_n\}_{n\geqslant 0}$ its associated sequence of monic orthogonal polynomials. Then, $\{\Psi_n\}_{n\geqslant 0}$, the sequence of monic polynomials orthogonal with respect to $\widetilde{\mathcal{L}}$ defined by* (28)*, is given by*

$$\Psi_n(z) = \Phi_n(z) - \mathbf{K}^t_{n-1}(z,0)(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}^t_{n-1})^{-1}\mathbf{\Phi}_n(0), \tag{33}$$

with $\mathbf{K}_{n-1}(z,0) = [K_{n-1}(z,0), K^{(0,1)}_{n-1}(z,0), \ldots, K^{(0,n-1)}_{n-1}(z,0)]^t$, $\mathbf{D}_n = diag\{\frac{1}{0!}, \ldots, \frac{1}{(n-1)!}\}$, $\mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi'_n(0), \ldots, \Phi^{(n-1)}_n(0)]^t$*, and*

$$\mathbf{P}_{n-1} = \begin{pmatrix} \varphi_0(0) & \varphi_1(0) & \cdots & \varphi_{n-1}(0) \\ 0 & \varphi'_1(0) & \cdots & \varphi'_{n-1}(0) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \varphi^{(n-1)}_{n-1}(0) \end{pmatrix}.$$

Proof Set

$$\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z), \tag{34}$$

where, for $0 \leqslant k \leqslant n-1$,

$$\begin{aligned} \lambda_{n,k} &= \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}}}{\mathbf{k}_k} = \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\widetilde{\mathcal{L}}} - \boldsymbol{m}\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi y}}{\mathbf{k}_k}, \\ &= -\frac{\boldsymbol{m}}{\mathbf{k}_k}\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi y}. \end{aligned}$$

Thus,

$$\Psi_n(z) = \Phi_n(z) - \boldsymbol{m}\int_{\mathbb{T}} \Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi y}, \tag{35}$$

$$= \Phi_n(z) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi^{(j)}_n(0)}{j!}\frac{K^{(0,j)}_{n-1}(z,0)}{j!}, \tag{36}$$

and $K^{(i,j)}_n(z,y)$ denotes the $i-th$ ($resp.$ $j-th$) partial derivative of $K_n(z,y)$ with respect to the variable z ($resp.$ y). In particular, for $0 \leqslant i \leqslant n-1$ we get

$$\Psi^{(i)}_n(0) = \Phi^{(i)}_n(0) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi^{(j)}_n(0)}{j!}\frac{K^{(i,j)}_{n-1}(0,0)}{j!}. \tag{37}$$

So, we have the following linear system of n equations and n unknowns

$$\begin{aligned} \Psi_n(0) &= \Phi_n(0) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi_n(0)}{j!}\frac{K^{(0,j)}_{n-1}(0,0)}{j!}, \\ \Psi'_n(0) &= \Phi'_n(0) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi^{(j)}_n(0)}{j!}\frac{K^{(1,j)}_{n-1}(0,0)}{j!}, \\ \Psi''_n(0) &= \Phi''_n(0) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi^{(j)}_n(0)}{j!}\frac{K^{(2,j)}_{n-1}(0,0)}{j!}, \\ &\vdots \\ \Psi^{(n-1)}_n(0) &= \Phi^{(n-1)}_n(0) - \boldsymbol{m}\sum_{j=0}^{n-1} \frac{\Psi^{(j)}_n(0)}{j!}\frac{K^{(n-1,j)}_{n-1}(0,0)}{j!}, \end{aligned}$$

which reads as

$$\begin{pmatrix} 1+\boldsymbol{m}\frac{K_{n-1}^{(0,0)}(0,0)}{(0!)^2} & \boldsymbol{m}\frac{K_{n-1}^{(0,1)}(0,0)}{(1!)^2} & \cdots & \boldsymbol{m}\frac{K_{n-1}^{(0,n-1)}(0,0)}{(n-1!)^2} \\ \boldsymbol{m}\frac{K_{n-1}^{(1,0)}(0,0)}{(0!)^2} & 1+\boldsymbol{m}\frac{K_{n-1}^{(1,1)}(0,0)}{(1!)^2} & \cdots & \boldsymbol{m}\frac{K_{n-1}^{(1,n-1)}(0,0)}{(n-1!)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{m}\frac{K_{n-1}^{(n-1,0)}(0,0)}{(0!)^2} & \boldsymbol{m}\frac{K_{n-1}^{(n-1,1)}(0,0)}{(1!)^2} & \cdots & 1+\boldsymbol{m}\frac{K_{n-1}^{(n-1,n-1)}(0,0)}{(n-1!)^2} \end{pmatrix} \boldsymbol{\Psi}_n(0) = \boldsymbol{\Phi}_n(0), \tag{38}$$

where

$$\boldsymbol{\Psi}_n(0) = [\Psi_n(0), \Psi_n^{'}(0), \ldots, \Psi_n^{(n-1)}(0)]^t, \quad \boldsymbol{\Phi}_n(0) = [\Phi_n(0), \Phi_n^{'}(0), \ldots, \Phi_n^{(n-1)}(0)]^t.$$

Denoting

$$\mathbf{R}_{n-1} = \begin{pmatrix} K_{n-1}^{(0,0)}(0,0) & K_{n-1}^{(0,1)}(0,0) & \cdots & K_{n-1}^{(0,n-1)}(0,0) \\ K_{n-1}^{(1,0)}(0,0) & K_{n-1}^{(1,1)}(0,0) & \cdots & K_{n-1}^{(1,n-1)}(0,0) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(n-1,0)}(0,0) & K_{n-1}^{(n-1,1)}(0,0) & \cdots & K_{n-1}^{(n-1,n-1)}(0,0) \end{pmatrix}, \tag{39}$$

and $\mathbf{D}_n = diag\{\frac{1}{0!}, \frac{1}{1!}, \ldots, \frac{1}{(n-1)!}\}$, (38) becomes

$$\boldsymbol{\Psi}_n(0) = \boldsymbol{m}^{-1}(\boldsymbol{m}^{-1}\mathbf{I}_n + \mathbf{R}_{n-1}\mathbf{D}_n^2)^{-1}\boldsymbol{\Phi}_n(0). \tag{40}$$

Thus, if $\mathbf{K}_{n-1}(z,0) = [K_{n-1}(z,0), K_{n-1}^{(0,1)}(z,0), \ldots, K_{n-1}^{(0,n-1)}(z,0)]^t$, then (36) can be written

$$\begin{aligned} \Psi_n(z) &= \Phi_n(z) - \boldsymbol{m}\mathbf{K}_{n-1}^t(z,0)\mathbf{D}_n^2\boldsymbol{\Psi}_n(0), \\ &= \Phi_n(z) - \mathbf{K}_{n-1}^t(z,0)(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_{n-1})^{-1}\boldsymbol{\Phi}_n(0), \end{aligned}$$

which is (33), since $\mathbf{R}_{n-1} = \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t$.

In particular, for the corresponding Verblunsky parameters we get

$$\Psi_n(0) = \Phi_n(0) - \mathbf{K}_{n-1}^t(0,0)(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}\boldsymbol{\Phi}_n(0).$$

On the other hand, considering the derivatives of order j with respect to the variable z in the Christoffel-Darboux formula, (see [10],[20], and [22])

$$K_{n-1}(z,y) = \frac{\overline{\Phi_n^*(y)}\Phi_n^*(z) - \overline{\Phi_n(y)}\Phi_n(z)}{\mathbf{k}_n(1-\bar{y}z)},$$

we get

$$K_{n-1}^{(j,0)}(z,y) = \frac{\overline{\Phi_n^*(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n^*(z)}{1-\bar{y}z}\right)^{(j)} - \frac{\overline{\Phi_n(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n(z)}{1-\bar{y}z}\right)^{(j)}.$$

Thus

$$K_{n-1}^{(j,0)}(0,y) = \frac{\overline{\Phi_n^*(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n^*(z)}{1-\bar{y}z}\right)^{(j)}(0) - \frac{\overline{\Phi_n(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n(z)}{1-\bar{y}z}\right)^{(j)}(0).$$

Now, taking into account the Leibniz rule

$$\left(\frac{\Phi_n(z)}{1-\bar{y}z}\right)^{(j)} = \sum_{k=0}^{j}\binom{j}{k}\Phi_n^{(j-k)}(z)\frac{k!\bar{y}^k}{(1-\bar{y}z)^{k+1}}.$$

the evaluation at $z = 0$ yields

$$j! \sum_{k=0}^{j} \frac{\Phi_n^{(j-k)}(0)}{(j-k)!} \bar{y}^k = j! \overline{\sum_{k=0}^{j} \frac{\overline{\Phi_n^{(j-k)}(0)}}{(j-k)!} y^k} = j! \overline{T_j^*(\Phi_n(y); 0)},$$

where $T_j(p(y); 0)$ denotes the $j-th$ Taylor polynomial of $p(y)$ around $y = 0$. In an analog way

$$j! \sum_{k=0}^{j} \frac{\Phi_n^{*(j-k)}(0)}{(j-k)!} \bar{y}^k = j! \overline{T_j^*(\Phi_n^*(y); 0)}.$$

Thus, we have proved

Proposition 11

$$K_{n-1}^{(0,j)}(z, 0) = j! \left[\frac{\Phi_n^*(z)}{\mathbf{k}_n} T_j^*(\Phi_n^*(z); 0) - \frac{\Phi_n(z)}{\mathbf{k}_n} T_j^*(\Phi_n(z); 0) \right].$$

From the previous Proposition, if we denote

$$\mathbf{T}(\Phi_n(z); 0) = [T_0^*(\Phi_n(z); 0), T_1^*(\Phi_n(z); 0), \ldots, T_{n-1}^*(\Phi_n(z); 0)]^t,$$

then (33) becomes

$$\Psi_n(z) = \Phi_n(z) - \frac{\Phi_n(z)}{\mathbf{k}_n} \mathbf{T}^t(\Phi_n(z); 0) \mathbf{D}_n^{-1} (\boldsymbol{m}^{-1} \mathbf{D}_n^{-2} + \mathbf{P}_{n-1} \mathbf{P}_{n-1}^t)^{-1} \mathbf{\Phi}_n(0) \tag{41}$$

$$- \frac{\Phi_n^*(z)}{\mathbf{k}_n} \mathbf{T}^t(\Phi_n^*(z); 0) \mathbf{D}_n^{-1} (\boldsymbol{m}^{-1} \mathbf{D}_n^{-2} + \mathbf{P}_{n-1} \mathbf{P}_{n-1}^t)^{-1} \mathbf{\Phi}_n(0)$$

$$= a_n(z) \Phi_n(z) + b_n(z) \Phi_n^*(z), \tag{42}$$

with

$$a_n(z) = 1 - \frac{1}{\mathbf{k}_n} \mathbf{T}^t(\Phi_n(z); 0) \mathbf{D}_n^{-1} (\boldsymbol{m}^{-1} \mathbf{D}_n^{-2} + \mathbf{P}_{n-1} \mathbf{P}_{n-1}^t)^{-1} \mathbf{\Phi}_n(0),$$

$$b_n(z) = -\frac{1}{\mathbf{k}_n} \mathbf{T}^t(\Phi_n^*(z); 0) \mathbf{D}_n^{-1} (\boldsymbol{m}^{-1} \mathbf{D}_n^{-2} + \mathbf{P}_{n-1} \mathbf{P}_{n-1}^t)^{-1} \mathbf{\Phi}_n(0).$$

5 Examples

5.1 A Bernstein-Szegő case.

Let us consider the measure σ such that

$$d\sigma = \frac{1 - |\beta|^2}{|e^{i\theta} - \beta|^2} \frac{d\theta}{2\pi}, \quad |\beta| < 1, \tag{43}$$

and apply to it the transformation considered in the previous Section, i.e. let us define a new measure $\tilde{\sigma}$

$$d\tilde{\sigma} = \frac{1 - |\beta|^2}{|e^{i\theta} - \beta|^2} \frac{d\theta}{2\pi} + \boldsymbol{m} \frac{d\theta}{2\pi}, \quad \boldsymbol{m} \in \mathbb{R}_+. \tag{44}$$

Our goal is to find the sequence of monic polynomials orthogonal with respect to (44), that will be denoted by $\{\Psi_n\}_{n\geqslant 0}$. (43) is known in the literature as Bernstein-Szegő measure, and its corresponding sequence of monic orthogonal polynomials is (see [20])

$$\Phi_n(z) = z^{n-1}(z-\beta), \quad n \geqslant 1.$$

Furthermore, we have $\Phi_n^*(z) = 1-\bar{\beta}z$, $n \geqslant 1$. Notice that, for $n = 1$, from (42) we get

$$\begin{aligned}\Psi_1(z) &= a_1(z)\Phi_1(z) + b_1(z)\Phi_1^*(z),\\ &= [1-\beta^2(1-|\beta|^2)(\boldsymbol{m}+(1-|\beta|^2)^{-1}](z-\beta)\\ &- [\beta(1-|\beta|^2)(\boldsymbol{m}+(1-|\beta|^2)^{-1}](1-\bar{\beta}z).\end{aligned} \tag{45}$$

For $n \geqslant 2$, $T_j(\Phi_n(0);0) = 0$, $0 \leqslant j \leqslant n-2$, and

$$\begin{aligned}T_{n-1}(\Phi_n(z);0) &= \Phi_n(0) + \Phi_n^{'}(0)z + \Phi_n^{''}(0)\frac{z^2}{2!} + \ldots + \Phi_n^{(n-1)}(0)\frac{z^{n-1}}{(n-1)!},\\ &= -\beta z^{n-1}.\end{aligned}$$

Thus, $\mathbf{T}(\Phi_n(z);0) = [0,0,\ldots,-\bar{\beta}]^t$. On the other hand, $T_0(\Phi_n^*(0);0) = 1$, and $T_j(\Phi_n^*(z);0) = 1-\bar{\beta}z$, $1 \leqslant j \leqslant n-1$.

Therefore, $\mathbf{T}(\Phi_n^*(z);0) = [1, z-\beta, z(z-\beta),\ldots,z^{n-2}(z-\beta)]^t$. Furthermore, in this case

$$\mathbf{P}_{n-1} = \frac{1}{\sqrt{1-|\beta|^2}}\mathbf{D}_n^{-1}\mathbf{B}_n,$$

where

$$\mathbf{B}_n = \begin{pmatrix} 1 & \beta & 0 & \cdots & 0\\ 0 & 1 & \beta & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & 1 & \beta\\ 0 & \cdots & \cdots & 0 & 1\end{pmatrix}. \tag{46}$$

On the other hand, denoting by

$$\begin{aligned}\mathbf{N}_n &= \mathbf{D}_n^{-1}(\boldsymbol{m}^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^t)^{-1}, &(47)\\ &= [\boldsymbol{m}^{-1}\mathbf{I}_n + \frac{1}{1-|\beta|^2}\mathbf{B}_n\mathbf{B}_n^t]^{-1}\mathbf{D}_n, &(48)\end{aligned}$$

and taking into account that $\mathbf{\Phi}_n(0) = [0,0,\cdots,-(n-1)!\beta]^t$, we get

$$\begin{aligned}a_n(z) &= 1 - \frac{|\beta|^2}{1-|\beta|^2}(n-1)!n_{n-1,n-1},\\ b_n(z) &= -\frac{\beta}{1-|\beta|^2}(n-1)!\left[n_{0,n-1} + (z-\beta)\sum_{i=0}^{n-2} n_{i+1,n-1}z^i\right],\end{aligned} \tag{49}$$

where $n_{i,j}$, $0 \leqslant i,j \leqslant n-1$ are the entries of $\mathbf{N}_n$. Therefore, all elements on (42) are known and we can compute $\Psi_n(z)$. The corresponding Verblunsky parameters, for $n \geqslant 2$, are

$$\begin{aligned}\Psi_n(0) &= b_n(0),\\ &= \frac{\beta}{1-|\beta|^2}(n-1)!(\beta n_{1,n-1} - n_{0,n-1}).\end{aligned}$$

The first 20 Verblunsky coefficients for different values of $\boldsymbol{m}$ and β are shown below.

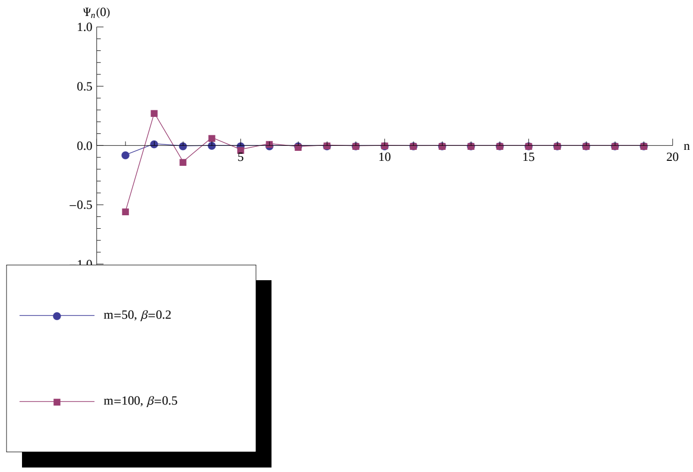

Fig. 1 Verblunsky coefficients for Example 5.1

5.2 Tchebychev case.

Let us consider the measure

$$d\sigma = |e^{i\theta} - 1|^2 \frac{d\theta}{2\pi}, \tag{50}$$

and its corresponding sequence of monic orthogonal polynomials, given by (see [20])

$$\Phi_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} (k+1) z^k, \qquad n \geqslant 0. \tag{51}$$

We introduce the perturbation defined in the previous Section and obtain

$$d\tilde{\sigma} = d\sigma + \boldsymbol{m} \frac{d\theta}{2\pi}. \tag{52}$$

We now proceed to get an explicit expression for the sequence of monic polynomials orthogonal with respect to $\tilde{\sigma}$. Notice that

$$T_j(\Phi_n(z); 0) = \frac{1}{n+1} \sum_{k=0}^{j} (k+1) z^k, \quad 0 \leqslant j \leqslant n-1,$$

and, as a consequence,

$$T_j^*(\Phi_n(z); 0) = \frac{1}{n+1} \sum_{k=0}^{j} (k+1) z^{j-k}, \quad 0 \leqslant j \leqslant n-1.$$

On the other hand, since

$$\Phi_n^*(z) = \frac{1}{n+1} \sum_{k=0}^{n} (k+1) z^{n-k}, \quad n \geqslant 0,$$

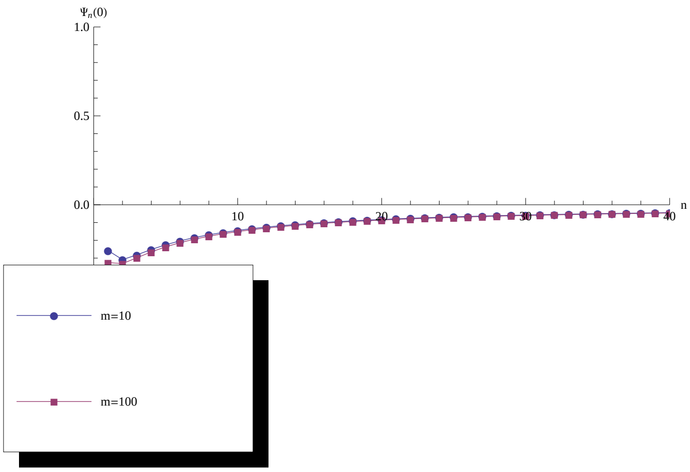

Fig. 2 Verblunsky coefficients for Example 5.2

we get

$$T_j(\Phi_n^*(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(n+1-k)z^k, \quad 1 \leqslant j \leqslant n-1,$$

and, thus,

$$T_j^*(\Phi_n^*(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(n+1-k)z^{j-k}, \quad 1 \leqslant j \leqslant n-1.$$

Therefore, $\mathbf{T}(\Phi_n(0);0) = \frac{1}{n+1}[1,2,\ldots,n]^t$ and $\mathbf{T}(\Phi_n^*(0);0) = \frac{1}{n+1}[n+1,n,n-1,\ldots,2]^t$. Furthermore, since

$$\varphi_n(z) = \sqrt{\frac{2}{(n+1)(n+2)}}\sum_{k=0}^{n}(k+1)z^k, \qquad n \geqslant 0, \tag{53}$$

are the orthonormal polynomials with respect to σ, we obtain

$$\mathbf{P}_{n-1} = \mathbf{D}_n^{-1}\mathbf{A}_n\boldsymbol{\Lambda}_n \tag{54}$$

where

$$\mathbf{A}_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\Lambda}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_{n-1} \end{pmatrix},$$

and $p_j = \sqrt{\frac{2}{(j+1)(j+2)}}$.

Finally, we have $\mathbf{\Phi}_n(0) = \frac{1}{n+1}[1,2!,\cdots,n!]^t$ and we can obtain $\Psi_n(z)$ explicitly from (42). The following figure shows the behavior of the corresponding Verblunsky coefficients for different values of $\boldsymbol{m}$.

Remark 4 Notice that (50) is a Christoffel transformation of the Lebesgue measure with $\alpha = 1$ (see Section 2). This example can be generalized as follows. Let σ be an absolutely continuous measure whose Radon-Nikodyn derivative with respect to the Lebesgue measure is $\sigma' = |z-\alpha|^2$, i.e. a positive trigonometric polynomial of degree 1. Applying the transformation in Section 3, with $\boldsymbol{m} \in \mathbb{R}_+$, and assuming $\alpha \in \mathbb{C}$, we obtain

$$\boldsymbol{m} + |z-\alpha|^2 = \boldsymbol{m} + 1 - \alpha z^{-1} - \bar{\alpha} z + |\alpha|^2, \quad (55)$$

i.e. another positive trigonometric polynomial that can be represented by $|\delta z - \gamma|^2$, where $\delta \in \mathbb{R}, \gamma \in \mathbb{C}$. Indeed, as

$$|\delta z - \gamma|^2 = \delta^2 - \delta\gamma z^{-1} - \delta\bar{\gamma} z + |\gamma|^2$$

the comparison of the coefficients with (55) yields $1 + |\alpha|^2 + \boldsymbol{m} = \delta^2 + |\gamma|^2$ and $\alpha = \delta\gamma$. Thus,

$$1 + |\alpha|^2 + \boldsymbol{m} = \frac{|\alpha|^2}{\delta^2} + \delta^2,$$

so we can get δ and γ in terms of $\boldsymbol{m}$ and α. In other words, in this case we can express the addition to a Tchebychev measure of a Lebesgue measure (multiplied by a constant $\boldsymbol{m}$) as a Christoffel transformation of the Lebesgue measure.

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References

1. M. Adler and P. van Moerbeke, *Darboux Transforms on Band Matrices, Weights, and Associated Polynomials*, Int. Math. Res. Not. **18** , 935–984 (2001).
2. M. I. Bueno and F. Marcellán, *Darboux transformations and perturbations of linear functionals*, Linear Alg. and Appl. **384** , 215–242 (2004).
3. A. Cachafeiro, F. Marcellán, and C. Pérez, *Lebesgue perturbation of a quasi-definite Hermitian functional. The positive definite case*, Linear Alg. Appl. **369**, 235–250 (2003).
4. A. Cachafeiro, F. Marcellán, and C. Pérez, *Orthogonal polynomials with respect to a sum of a arbitrary measure and a Bernstein-Szegő measure*, Adv. Comput. Math. **26**, 81–104 (2007).
5. T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
6. L. Daruis, J. Hernández, and F. Marcellán, *Spectral transformations for Hermitian Toeplitz matrices*, J. Comput. Appl. Math. **202**, 155–176 (2007).
7. P. Delsarte, Y. Genin, *On the Role of Orthogonal Polynomials on the Unit circle in Digital Signal Processing Applications*, in Orthogonal Polynomials: Theory and Practice, P. Nevai Editor, Kluwer Academic Publishers, Dordrecht, 115–133 (1990).
8. L. Garza, J. Hernández, and F. Marcellán, *Orthogonal polynomials and measures on the unit circle. The Geronimus transformation*, J. Comput. Appl. Math. Published on line doi:101016/j.cam2007.11.023.
9. Ya. L. Geronimus, *Orthogonal polynomials*, English translation of the appendix to the Russian translation of Szegő's book [22], Fizmatzig, Moscow, 1961, in "Two papers on Special Functions", Amer. Math. Soc. Transl., series 2, vol. **108**. Amer. Math. Soc. Providence, Rhode Island, 1977.
10. Ya. L. Geronimus, *Polynomials orthogonal on a circle and their applications*, Amer. Math. Soc. Transl. Series **1**. Amer. Math. Soc. Providence, Rhode Island. 1–78 (1962).
11. E. Godoy and F. Marcellán, *An analogue of the Christoffel formula for polynomial modification of a measure on the unit circle*, Boll. Un. Mat. Ital. **5**-A, 1–12 (1991).

12. E. Godoy and F. Marcellán, *Orthogonal polynomials and rational modifications of measures*, Canad. J. Math. **45**, 930–943 (1993).
13. U. Grenander and G. Szegő, *Toeplitz Forms and their Applications*, University of California Press, Berkeley 1958, Chelsea, New York, 2^{nd} edition, 1984.
14. F. A. Grunbaum and L. Haine, *Bispectral Darboux Transformations: An extension of the Krall polynomials*, Int. Math. Res. Not. **8**, 359–392 (1997).
15. M. E. H. Ismail and R. W. Ruedemann, *Relation between polynomials orthogonal on the unit circle with respect to different weights*, J. Approx. Theory **71**, 39–60 (1992).
16. W. B. Jones, O. Njåstad, and W. J. Thron, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc. **21**, 113–152 (1989).
17. X. Li and F. Marcellán, *Representation of Orthogonal Polynomials for modified measures*, Comm. Anal. Theory of Cont. Fract. **7**, 9–22 (1999).
18. F. Marcellán, *Polinomios ortogonales no estándar. Aplicaciones en Análisis Numérico y Teoría de Aproximación*, Rev. Acad. Colomb. Ciencias Exactas, Físicas y Naturales, **30** (117), 563-579 (2006). (In Spanish).
19. F. Marcellán and J. Hernández, *Christoffel transforms and Hermitian linear functionals*, Mediterr. J. Math. **2**, 451–458 (2005).
20. B. Simon, *Orthogonal polynomials on the unit circle*, 2 vols. Amer. Math. Soc. Coll. Publ. Series, vol. **54**, Amer. Math. Soc. Providence, Rhode Island, 2005.
21. V. Spiridonov, L. Vinet, and A. Zhedanov, *Spectral transformations, self-similar reductions and orthogonal polynomials*, J. Phys. A: Math. Gen. **30**, 7621–7637 (1997).
22. G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Series. vol 23, Amer. Math. Soc., Providence, Rhode Island, 4^{th} edition, 1975.
23. G. J. Yoon, *Darboux transforms and orthogonal polynomials*, Bull. Korean Math. Soc. **39**, 359–376 (2002).
24. A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. **85**, 67–83 (1997).