A Note on Monotonicity of Zeros of Generalized Hermite-Sobolev Type Orthogonal Polynomials

Francisco Marcellán [a] Fernando R. Rafaeli [b]

[a] *Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III, Leganés-Madrid, Spain*

[b] *Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Brazil*

Abstract

In this paper we analyze a Mehler-Heine formula and the behaviour of zeros of orthogonal polynomials associated with a symmetrization process of Laguerre-Sobolev type orthogonal polynomials. More precisely, the limit of these zeros, their monotonicity as well as their dependence of the masses is obtained. In such a way, we obtain a natural generalization of the results given in [3].

Key words: Generalized Hermite polynomials, Sobolev type orthogonal polynomials, Mehler-Heine type formulas, zeros, monotonicity, asymptotics.

AMS Mathematics Subject Classification: 42C05

1 Introduction

Let us consider the classical orthogonal polynomials of Jacobi, Laguerre and Hermite, usually denoted by $P_n^{(\alpha,\beta)}(x)$, $L_n^{(\alpha)}(x)$, and $H_n(x)$, respectively, which are orthogonal with respect to the measures $d\mu_0 = (1-x)^\alpha(1+x)^\beta dx$, $\alpha, \beta > -1$ (the beta distribution, supported on $[-1,1]$) $d\mu_1 = x^\alpha e^{-x}dx$, $\alpha > -1$,(the gamma distribution, supported on $[0, +\infty)$, and $d\mu_2 = e^{-x^2}dx$ (the normal distribution supported on $\mathbb{R}$). In the last twenty years, several authors (see [1]) have considered polynomials orthogonal with respect to Sobolev inner products

$$\langle p, q\rangle = \int_E p(x)q(x)d\mu_0 + Mp(c)q(c) + Np'(c)q'(c)$$

Preprint submitted to Elsevier *19 February 2010*

where $d\mu_0$ is a probability measure supported on an infinite subset E of the real line, M and N are nonnegative real numbers, and $c \in \mathbb{R}$. More precisely, estimates for these orthogonal polynomials in terms of the location of c with respect to E as well as the location of their zeros with respect to the convex hull of E have been analyzed. In particular, these questions have been considered in [2] when $d\mu_0$ is the Jacobi weight and $c = 1$. On the other hand, in [4] and [11] when $d\mu_0$ is the Laguerre weight and $c = 0$, some asymptotic properties of these polynomials like outer relative asymptotics and Mehler-Heine type formulas as well as their consequences in the behavior of their zeros have been studied. Finally, if $d\mu_0$ is the Hermite weight and $c = 0$ some analogue problems for Sobolev inner products involving higher order derivatives have been studied in [3].

More recently, the authors focused the attention in some new analytic properties of the zeros of the Jacobi-Sobolev type polynomials $P_n^{(\alpha,\beta,M,N)}(x)$ (see [7] and [9]) as well as for the Laguerre-Sobolev type polynomials $L_n^{(\alpha,M,N)}(x)$, (see [6], [8], and [10]), associated with Jacobi and Laguerre weights, respectively, which are orthogonal with respect to some special cases of Sobolev type inner products

$$\langle p, q\rangle = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} p(x)q(x)d\mu_0 + Mp(1)q(1) + Np'(1)q'(1)$$

and

$$\langle p, q\rangle = \frac{1}{\Gamma(\alpha+1)}\int_{0}^{\infty} p(x)q(x)d\mu_1 + Mp(0)q(0) + Np'(0)q'(0), \tag{1}$$

respectively.
More precisely, in [6] and [7] it was proved the monotonicity of the zeros of $P_n^{(\alpha,\beta,M,N)}(x)$ and $L_n^{(\alpha,M,N)}(x)$ with respect to parameters M and N.
In this paper, we consider the generalized Hermite-Sobolev type polynomials $Q_n(x) = H_n^{(\alpha,M_0,M_1,M_2,M_3)}(x)$, which are orthogonal with respect to the Sobolev inner product

$$\langle p, q\rangle = \int_{-\infty}^{\infty} p(x)q(x)|x|^{2\alpha}e^{-x^2}dx + \sum_{i=0}^{3} M_i p^{(i)}(0)q^{(i)}(0), \tag{2}$$

with $\alpha > -1/2$ and $M_i \geq 0$, $i = 0, 1, 2, 3$.
Notice that this inner product is symmetric, i.e, $\langle x^{2n}, x^{2m+1}\rangle = 0$, $n, m \in \mathbb{N}$.
Generalized Hermite polynomials ($M_i = 0, i = 0, 1, 2, 3$) appear in the symmetrization process of Laguerre polynomials (see [5]).

The structure of the manuscript is as follows. In Section 2 we obtain an explicit representation of the generalized Hermite-Sobolev type orthogonal polynomials in terms of Laguerre-Sobolev type orthogonal polynomials following a symmetrization process. As an immediate consequence, a Mehler-Heine type is deduced. Thus, we extend the results of [3] which correspond to the case $\alpha = 0$. In Section 3, as a consequence of the previous result, we study the limit of the zeros of the orthogonal polynomials in terms of the zeros of Bessel functions of the first kind. The behavior of these zeros in terms of the masses M_i, $i = 0, 1, 2, 3$ is also deduced.

2 Representation and asymptotic properties of generalized Hermite-Sobolev type polynomials

In the sequel, we will consider the n-th polynomial $L_n^{(\alpha,M,N)}(x)$ normalized in the same form as in [11], i.e, with leading coefficient $(-1)^n/n!$.

Proposition 1 *Let $\{Q_n\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the inner product* (2). *Then, for $n \in \mathbb{N}$, we get*

$$H_{2n}^{(\alpha,M_0,M_2)}(x) := Q_{2n}(x) = L_n^{(\alpha-1/2,M_0/\Gamma(\alpha+1/2),4M_2/\Gamma(\alpha+1/2))}(x^2) \tag{3}$$

and

$$H_{2n+1}^{(\alpha,M_1,M_3)}(x) := Q_{2n+1}(x) = x\, L_n^{(\alpha+1/2,M_1/\Gamma(\alpha+3/2),36M_3/\Gamma(\alpha+3/2))}(x^2). \tag{4}$$

In the case $\alpha = 0$, we obtain the Hermite-Sobolev type orthogonal polynomials

$$H_{2n}^{(M_0,M_2)}(x) := H_{2n}^{(0,M_0,M_2)}(x) = L_n^{(-1/2,M_0/\Gamma(1/2),4M_2/\Gamma(1/2))}(x^2)$$

and

$$H_{2n+1}^{(M_1,M_3)}(x) := H_{2n+1}^{(0,M_1,M_3)}(x) = x\, L_n^{(1/2,M_1/\Gamma(3/2),36M_3/\Gamma(3/2))}(x^2).$$

This situation was considered in [3].

Proof of Theorem 1: Let $Q_{2n}(x) = P_n(x^2)$ and $Q_{2n+1}(x) = xR_n(x^2)$. Then

$$\begin{aligned}
&\langle Q_{2n}, Q_{2m} \rangle \\
&= \int_{-\infty}^{\infty} P_n(x^2)P_m(x^2)|x|^{2\alpha}e^{-x^2}dx + M_0P_n(0)P_m(0) + 4M_2P'_n(0)P'_m(0) \\
&= 2\int_{0}^{\infty} P_n(x^2)P_m(x^2)x^{2\alpha}e^{-x^2}dx + M_0P_n(0)P_m(0) + 4M_2P'_n(0)P'_m(0).
\end{aligned}$$

Taking $x^2 = t$, we get

$$\begin{aligned}
&\langle Q_{2n}, Q_{2m} \rangle \\
&= \int_{0}^{\infty} P_n(t)P_m(t)t^{\alpha-1/2}e^{-t}dt + M_0P_n(0)P_m(0) + 4M_2P'_n(0)P'_m(0) \\
&= K_{2n}\delta_{n,m}.
\end{aligned}$$

Thus,

$$H_{2n}^{(\alpha,M_0,M_2)}(t) = Q_{2n}(t) = P_n(t^2) = L_n^{(\alpha-1/2,M_0/\Gamma(\alpha+1/2),4M_2/\Gamma(\alpha+1/2))}(t^2),$$

where $M_0 = \Gamma(\alpha + 1/2)M$ and $M_2 = \Gamma(\alpha + 1/2)N/4$ in (1). On the other hand,

$$\begin{aligned}
&\langle Q_{2n+1}, Q_{2m+1} \rangle \\
&= \int_{-\infty}^{\infty} R_n(x^2)R_m(x^2)x^2|x|^{2\alpha}e^{-x^2}dx + M_1R_n(0)R_m(0) + 36M_3R'_n(0)R'_m(0) \\
&= 2\int_{0}^{\infty} R_n(x^2)R_m(x^2)x^{2(\alpha+1)}e^{-x^2}dx + M_1R_n(0)R_m(0) + 36M_3R'_n(0)R'_m(0).
\end{aligned}$$

Taking $x^2 = t$, we obtain

$$\begin{aligned}
&\langle Q_{2n+1}, Q_{2m+1} \rangle \\
&= \int_{0}^{\infty} R_n(t)R_m(t)t^{(\alpha+1/2)}e^{-t}dt + M_1R_n(0)R_m(0) + 36M_3R'_n(0)R'_m(0) \\
&= K_{2n+1}\delta_{n,m}.
\end{aligned}$$

Thus,

$$H_{2n+1}^{(\alpha,M_1,M_3)}(t) = Q_{2n+1}(t) = t\,R_n(t^2) = t\,L_n^{(\alpha+1/2,M_1/\Gamma(\alpha+3/2),36M_3/\Gamma(\alpha+3/2))}(t^2),$$

where $M_1 = \Gamma(\alpha+3/2)M$ and $M_3 = \Gamma(\alpha+3/2)N/36$ in (1).

Proposition 2 *If we denote*

$$g_{\alpha,i}(x) = x^{-\alpha/2} J_{\alpha+2i}(2\sqrt{x}), \tag{5}$$

where $J_\beta(x)$ is the Bessel function of the first kind, then the following Mehler-Heine type formulas hold uniformly on compact subsets of $\mathbb{C}$:

(a) For generalized Hermite-Sobolev type polynomials of even degree

$$\lim_{n\to\infty} \frac{H_{2n}^{(\alpha,M_0,M_2)}(\sqrt{x/n})}{n^{\alpha-1/2}} = \begin{cases} -g_{\alpha-1/2,1}(x), & M_0>0, M_2=0; \\ \frac{1}{\alpha+3/2}(g_{\alpha-1/2,2}(x) - (\alpha+3/2)g_{\alpha-1/2,1}(x) - g_{\alpha-1/2,0}(x)), & M_0=0, M_2>0; \\ g_{\alpha-1/2,2}(x), & M_0, M_2>0. \end{cases}$$

(b) For generalized Hermite-Sobolev type polynomials of odd degree

$$\lim_{n\to\infty} \frac{H_{2n+1}^{(\alpha,M_1,M_3)}(\sqrt{x/n})}{n^{\alpha}} = \begin{cases} -x^{1/2}g_{\alpha+1/2,1}(x), & M_1>0, M_3=0; \\ \frac{x^{1/2}}{\alpha+5/2}(g_{\alpha+1/2,2}(x) - (\alpha+5/2)g_{\alpha+1/2,1}(x) - g_{\alpha+1/2,0}(x)), & M_1=0, M_3>0; \\ x^{1/2}g_{\alpha+1/2,2}(x), & M_1, M_3>0. \end{cases}$$

Proof. This is an immediate consequence of the Proposition 1 and [11, Proposition 2.10] (see also [4]).

3 Zeros of generalized Hermite-Sobolev type polynomials

Let denote by $h_{2n,k}^{M_0,M_2}(\alpha)$, $1 \le k \le 2n$, and $h_{2n+1,k}^{M_1,M_3}(\alpha)$, $1 \le k \le 2n+1$, the zeros of generalized Hermite-Sobolev type polynomials $H_{2n}^{(\alpha,M_0,M_2)}(x)$ and $H_{2n+1}^{(\alpha,M_1,M_3)}(x)$, respectively, arranged in the following form: $[h_{2n,n}^{M_0,M_2}(\alpha)]^2 <$

$\cdots < [h_{2n,1}^{M_0,\,M_2}(\alpha)]^2$ and $[h_{2n+1,n}^{M_1,\,M_3}(\alpha)]^2 < \cdots < [h_{2n+1,1}^{M_1,\,M_3}(\alpha)]^2$.

Notice that, because of the symmetry property of the inner product, we get that $h_{2n,k}^{M_0,\,M_2}(\alpha) = -h_{2n,2n-k+1}^{M_0,\,M_2}(\alpha)$ and $h_{2n+1,k}^{M_1,\,M_3}(\alpha) = -h_{2n+1,2n+2-k}^{M_1,\,M_3}(\alpha)$, $k = 1, \ldots, n$. Next, we will analyze the limit behaviour, interlacing, and monotonicity properties of these zeros.

Let $x_{n,k}^{M,\,N}(\alpha)$ and $x_{n,k}(\alpha)$ the zeros of $L_n^{(\alpha,\,M,\,N)}(x)$ and $L_n^{(\alpha)}(x)$, respectively, all arranged in an increasing order. Then, from (3) and (4), we have that

$$\left[h_{2n,k}^{M_0,\,M_2}(\alpha)\right]^2 = x_{n,k}^{M_0/\Gamma(\alpha+1/2),\,4M_2/\Gamma(\alpha+1/2)}(\alpha - 1/2)$$

and

$$\left[h_{2n+1,k}^{M_1,\,M_3}(\alpha)\right]^2 = x_{n,k}^{M_1/\Gamma(\alpha+3/2),\,36M_3/\Gamma(\alpha+3/2)}(\alpha + 1/2).$$

Since, depending of the values of M_2 and M_3, the least zero of the Laguerre-Sobolev type orthogonal polynomials $x_{n,n}^{M_0/\Gamma(\alpha+1/2),\,4M_2/\Gamma(\alpha+1/2)}(\alpha - 1/2)$ and $x_{n,n}^{M_1/\Gamma(\alpha+3/2),\,36M_3/\Gamma(\alpha+3/2)}(\alpha+1/2)$, respectively can be negative, then the zeros $h_{2n,n}^{M_0,\,M_2}(\alpha)$ and $h_{2n+1,n}^{M_1,\,M_3}(\alpha)$ can be complex, i.e, the polynomials $H_{2n}^{(\alpha,\,M_0,\,M_2)}(x)$ and $H_{2n+1}^{(\alpha,\,M_1,\,M_3)}(x)$ can have at most one pair of complex zeros located in the imaginary axis. Thus, at most two complex zeros appear in our situation.

Proposition 3 *Let $j_{\alpha,i}$ be the i-th positive zero of the Bessel function $J_\alpha(x)$. Then,*

(a) If $M_0, M_1 > 0$ and $M_2 = M_3 = 0$, then

$$\lim_{n\to\infty} n\left[h_{2n,n}^{M_0,0}(\alpha)\right]^2 = 0$$

$$\lim_{n\to\infty} n\left[h_{2n,n-i+1}^{M_0,0}(\alpha)\right]^2 = \frac{j_{\alpha+3/2,i-1}^2}{4},\quad i \geq 2,$$

$$\lim_{n\to\infty} n\left[h_{2n+1,n}^{M_1,0}(\alpha)\right]^2 = 0$$

$$\lim_{n\to\infty} n\left[h_{2n+1,n-i+1}^{M_1,0}(\alpha)\right]^2 = \frac{j_{\alpha+5/2,i-1}^2}{4},\quad i \geq 2.$$

(b) If $M_0, M_1 = 0$ and $M_2, M_3 > 0$, then

$$\lim_{n\to\infty} n\left[h_{2n,n-i+1}^{0,\,M_2}(\alpha)\right]^2 = z_{\alpha-1/2,i}^2,$$

$$\lim_{n\to\infty} n\left[h_{2n+1,n-i+1}^{0,\,M_3}(\alpha)\right]^2 = z_{\alpha+1/2,i}^2,$$

where $z_{\alpha,i}$ denotes the i-th real zero of the function

$$Z_\alpha(x) = \frac{1}{\alpha+2}(g_{\alpha,2}(x) - (\alpha+2)g_{\alpha,1}(x) - g_{\alpha,0}(x)).$$

(c) If $M_0, M_1, M_2, M_3 > 0$, then

$$\lim_{n\to\infty} n\left[h^{M_0,\,M_2}_{2n,n-i+1}(\alpha)\right]^2 = 0,\ i = 1,2$$

$$\lim_{n\to\infty} n\left[h^{M_0,\,M_2}_{2n,n-i+1}(\alpha)\right]^2 = \frac{j^2_{\alpha+7/2,i-2}}{4},\ i \geq 3.$$

$$\lim_{n\to\infty} n\left[h^{M_1,\,M_3}_{2n+1,n-i+1}(\alpha)\right]^2 = 0,\ i = 1,2$$

$$\lim_{n\to\infty} n\left[h^{M_1,\,M_3}_{2n+1,n-i+1}(\alpha)\right]^2 = \frac{j^2_{\alpha+9/2,i-2}}{4},\ i \geq 3.$$

Proof. This is an immediate consequence of Proposition 2.

Proposition 4 *Let $\alpha > -1/2$. Then,*

(a) The inequalities

$$\left[h^{M_0,M_2}_{2n,k+1}(\alpha)\right]^2 < x_{n,k+1}(\alpha-1/2) < \left[h^{M_0,M_2}_{2n,k}(\alpha)\right]^2 < x_{n,k}(\alpha-1/2)$$

hold for every $n \in \mathbb{N}$, $n \geq 2$, and each k with $1 \leq k \leq n-1$. Moreover, for every fixed n the zero $h^{M_0,M_2}_{2n,n}(\alpha)$ satisfies

$$\left[h^{M_0,M_2}_{2n,n}(\alpha)\right]^2 > 0, \quad \text{for} \quad M_2 < \widetilde{M}_2,$$

$$\left[h^{M_0,M_2}_{2n,n}(\alpha)\right]^2 = 0, \quad \text{for} \quad M_2 = \widetilde{M}_2,$$

$$\left[h^{M_0,M_2}_{2n,n}(\alpha)\right]^2 < 0, \quad \text{for} \quad M_2 > \widetilde{M}_2,$$

where

$$\widetilde{M}_2 = \frac{\Gamma(n-1)\Gamma(\alpha+3/2)}{4(\alpha+7/2)_{n-2}}.$$

(b) The inequalities

$$\left[h^{M_1,M_3}_{2n+1,k+1}(\alpha)\right]^2 < x_{n,k+1}(\alpha+1/2) < \left[h^{M_1,M_3}_{2n+1,k}(\alpha)\right]^2 < x_{n,k}(\alpha+1/2)$$

hold for every $n \in \mathbb{N}$, $n \geq 2$, and each k with $1 \leq k \leq n-1$. Moreover, for every fixed n the zero $h^{M_1,M_3}_{2n+1,n}(\alpha)$ satisfies

$$\left[h^{M_1,M_3}_{2n+1,n}(\alpha)\right]^2 > 0, \quad \text{for} \quad M_3 < \widetilde{M}_3,$$

$$\left[h^{M_1,M_3}_{2n+1,n}(\alpha)\right]^2 = 0, \quad \text{for} \quad M_3 = \widetilde{M}_3,$$

$$\left[h^{M_1,M_3}_{2n+1,n}(\alpha)\right]^2 < 0, \quad \text{for} \quad M_3 > \widetilde{M}_3,$$

where

$$\widetilde{M}_3 = \frac{\Gamma(n-1)\Gamma(\alpha+5/2)}{36(\alpha+9/2)_{n-2}}.$$

Proof. This is an immediate consequence of Theorem 1 in [6].

Proposition 5 *Let $\alpha > -1/2$. Then,*

(a) For $M_2 = 0$, the inequalities

$$0 < \left[h_{2n,n}^{M_0,0}(\alpha)\right]^2 < x_{n,n}(\alpha-1/2) < x_{n-1,n-1}(\alpha+3/2) < \cdots$$
$$\cdots < x_{n-1,1}(\alpha+3/2) < \left[h_{2n,1}^{M_0,0}(\alpha)\right]^2 < x_{n,1}(\alpha-1/2)$$

hold for every $n \in \mathbb{N}$, $n \geq 2$. Moreover, the zeros $h_{2n,k}^{M_0,0}(\alpha)$ are decreasing functions of M_0,

$$\left[h_{2n,n}^{M_0,0}(\alpha)\right]^2 \to 0, \quad \left[h_{2n,k}^{M_0,0}(\alpha)\right]^2 \to x_{n-1,k}(\alpha+3/2) \text{ as } M_0 \to \infty,$$

$$\lim_{M_0\to\infty} M_0 \left[h_{2n,n}^{M_0,0}(\alpha)\right]^2 = (\alpha+3/2)g_n(\alpha-1/2)$$

and

$$\lim_{M_0\to\infty} M_0 \left\{\left[h_{2n,k}^{M_0,0}(\alpha)\right]^2 - x_{n-1,k}(\alpha+3/2)\right\} = g_n(\alpha-1/2), \quad k = 1,\ldots,n-1.$$

(b) For $M_3 = 0$, the inequalities

$$0 < \left[h_{2n+1,n}^{M_1,0}(\alpha)\right]^2 < x_{n,n}(\alpha+1/2) < x_{n-1,n-1}(\alpha+5/2) < \cdots$$
$$\cdots < x_{n-1,1}(\alpha+5/2) < \left[h_{2n+1,1}^{M_1,0}(\alpha)\right]^2 < x_{n,1}(\alpha+1/2)$$

hold for every $n \in \mathbb{N}$, $n \geq 2$. Moreover, the zeros $h_{2n+1,k}^{M_1,0}(\alpha)$ are decreasing functions of M_1,

$$\left[h_{2n+1,n}^{M_1,0}(\alpha)\right]^2 \to 0, \quad \left[h_{2n+1,k}^{M_1,0}(\alpha)\right]^2 \to x_{n-1,k}(\alpha+5/2) \text{ as } M_1 \to \infty,$$

$$\lim_{M_1\to\infty} M_1 \left[h_{2n+1,n}^{M_1,0}(\alpha)\right]^2 = (\alpha+5/2)g_n(\alpha+1/2)$$

and

$$\lim_{M_1\to\infty} M_1 \left\{\left[h_{2n+1,k}^{M_1,0}(\alpha)\right]^2 - x_{n-1,k}(\alpha+5/2)\right\} = g_n(\alpha+1/2), \; k = 1,\ldots,n-1.$$

The function $g_n(\alpha)$ given above is defined in [6, (1.8)].

Proof of Proposition 5: This is an immediate consequence of the Proposition 1 and Theorem 2 in [6].

Proposition 6 *Let $\alpha > -1/2$ and $\zeta_{n,k}(\alpha)$ the zeros of the polynomial $F_{n,\alpha}(x)$ defined in* [6, (1.9)]. *Then,*

(a) *the inequalities, for $M_0 = 0$,*

$$\zeta_{n,n}(\alpha - 1/2) < \left[h_{2n,n}^{0,M_2}(\alpha)\right]^2 < x_{n,n}(\alpha - 1/2) < \cdots$$
$$\cdots < \zeta_{n,1}(\alpha - 1/2) < \left[h_{2n,1}^{0,M_2}(\alpha)\right]^2 < x_{n,1}(\alpha - 1/2)$$

hold for every $n \in \mathbb{N}$, $n \geq 2$. Moreover, the zeros $\left[h_{2n,k}^{0,M_2}(\alpha)\right]^2$ are decreasing functions of M_2, $\left[h_{2n,k}^{0,M_2}(\alpha)\right]^2 \to \zeta_{n,k}(\alpha - 1/2)$ as $M_2 \to \infty$, and

$$\lim_{M_2 \to \infty} M_2 \left\{ \left[h_{2n,k}^{0,M_2}(\alpha)\right]^2 - \zeta_{n,k}(\alpha - 1/2) \right\} = g_{n,k}(\alpha - 1/2), \quad k = 1, \ldots, n.$$

(b) *the inequalities, for $M_1 = 0$,*

$$\zeta_{n,n}(\alpha + 1/2) < \left[h_{2n+1,n}^{0,M_3}(\alpha)\right]^2 < x_{n,n}(\alpha + 1/2) < \cdots$$
$$\cdots < \zeta_{n,1}(\alpha + 1/2) < \left[h_{2n+1,1}^{0,M_3}(\alpha)\right]^2 < x_{n,1}(\alpha + 1/2)$$

hold for every $n \in \mathbb{N}$, $n \geq 2$. Moreover, the zeros $\left[h_{2n+1,k}^{0,M_3}(\alpha)\right]^2$ are decreasing functions of M_3, $\left[h_{2n+1,k}^{0,M_3}(\alpha)\right]^2 \to \zeta_{n,k}(\alpha + 1/2)$ as $M_3 \to \infty$, and

$$\lim_{M_3 \to \infty} M_3 \left\{ \left[h_{2n+1,k}^{0,M_3}(\alpha)\right]^2 - \zeta_{n,k}(\alpha + 1/2) \right\} = g_{n,k}(\alpha + 1/2), \quad k = 1, \ldots, n.$$

The function $g_{n,k}(\alpha)$ above is given in [6, eq. (1.10)].

Proof of Proposition 6: This is an immediate consequence of the Proposition 1 and Theorem 4 in [6].

4 Acknowledgements

The work of the first author has been supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain, grant MTM2009-12740-C03-01. The work of the second author has been supported by FAPESP. This manuscript was finished during a stay of the second author in the Departmento de Matemáticas, Universidad Carlos III de Madrid, in the framework of a joint project of Dirección General de Investigación, Ministerio de Educación y Ciencia of Spain and the Brazilian Science Foundation CAPES, Project CAPES/DGU 160/08.

References

[1] M. Alfaro, F. Marcellán, M. L. Rezola, and A. Ronveaux, *On orthogonal polynomials of Sobolev type: Algebraic properties and zeros*, SIAM J. Math. Anal. **23**(1992), 737–757.

[2] M. Alfaro, F. Marcellán, and M. L. Rezola, *Estimates for Jacobi-Sobolev type orthogonal polynomials of Sobolev type*, Appl. Anal. **67**(1997), 157–174.

[3] M. Alfaro, J. J. Moreno-Balcázar, A. Peña, and M. L. Rezola, *Asymptotic for a generalization of Hermite polynomials*, Asympt. Anal. **66** (2010), 103–117.

[4] R. Álvarez-Nodarse and J. J. Moreno-Balcázar, *Asymptotic properties of generalized Laguerre orthogonal polynomials*, Indag. Mathem. N. S. **15** (2) (2004), 151–165.

[5] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach. 1978.

[6] D. K. Dimitrov, F. Marcellán, and F. R. Rafaeli, *Monotonicity of zeros of Laguerre-Sobolev-type orthogonal polynomials*. Submitted.

[7] D. K. Dimitrov, M. V. Mello, and F. R. Rafaeli, *Monotonicity of zeros of Jacobi-Sobolev-type orthogonal polynomials*, Appl. Numer. Math. In press.

[8] H. Dueñas and F. Marcellán, *Laguerre-type orthogonal polynomials. Electrostatic interpretation*, Int. J. Pure Appl. Math. **38** (2007), 345–358.

[9] H. Dueñas and F. Marcellán, *Jacobi-type orthogonal polynomials: holonomic equation and electrostatic interpretation*, Comm. Anal. Theory Cont. Frac. **15** (2008), 4-19.

[10] H. Dueñas and F. Marcellán, *The Laguerre-Sobolev-type orthogonal polynomials. Holonomic equation and electrostatic interpretation*, Rocky Mount. J. Math. In press.

[11] F. Marcellán and J. J. Moreno-Balcázar, *Asymptotics and zeros of Sobolev orthogonal polynomials on unbounded supports*, Acta Appl. Math. **94** (2) (2006), 163–192.