QUADRATIC DECOMPOSITION OF A FAMILY of H_q-SEMICLASSICAL ORTHOGONAL POLYNOMIAL SEQUENCES

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Abstract

We deal with monic orthogonal polynomial sequences, $\{B_n\}_{n\geq 0}$, satisfying the three-term recurrence relation $B_{n+2}(x) = (x-\beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x)$ $n = 0, 1, 2, ...,$ with initial conditions $B_0(x) = 1$ and $B_1(x) = x - \beta_0$, where $\beta_n = (-1)^n\beta_0$ and $\gamma_n \neq 0$ for all $n \geq 1$. These sequences are characterized by the relation $B_{2n}(x) = P_n(x^2), n \geq 0$, $B_1(x) = x-\beta_0$, where $\{P_n\}_{n\geq 0}$ is a monic orthogonal polynomial sequence. In this paper we show that the sequence $\{B_n\}_{n\geq 0}$ is H_q-semiclassical if and only if the sequence $\{P_n\}_{n\geq 0}$ is H_{q^2}-semiclassical. Then, we express the characteristic elements of the H_q-semiclassical sequence $\{B_n\}_{n\geq 0}$, such as the q-Pearson equation satisfied by the corresponding linear functional, the class of the linear functional, the first order linear q-difference equation satisfied by the Stieltjes function, and the coefficients of the structure relation for such a sequence of polynomials, in terms of the characteristic elements of the sequence $\{P_n\}_{n\geq 0}$. In particular, if the sequence $\{P_n\}_{n\geq 0}$ is H_{q^2}-semiclassical of class zero, then we obtain a new non-symmetric H_q-semiclassical sequence of polynomials $\{B_n\}_{n\geq 0}$ of class $s = 1$.

1 Introduction

A sequence of polynomials $\{B_n\}_{n\geq 0}$, $degB_n = n,\ n \geq 0$, is said to be a D-semiclassical sequence if it is orthogonal with respect to a linear functional w satisfying the so called Pearson equation

$$D(\Phi w) + \Psi w = 0, \tag{1.1}$$

where D is the standard derivative operator, Φ and Ψ are polynomials such that Φ is monic and degree $\Psi > 0$.
Let us consider a monic orthogonal polynomial sequence (MOPS, in short) $\{B_n\}_{n\geq 0}$ satisfying the three-term recurrence relation (TTRR)

$$\begin{gathered} B_0(x) = 1,\ B_1(x) = x - \beta_0, \\ B_{n+2}(x) = (x - (-1)^{n+1}\beta_0)B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geq 0, \end{gathered} \tag{1.2}$$

where $\beta_0 \in \mathbb{C}$ and $\gamma_{n+1} \neq 0,\ n = 0, 1, 2, ...$
The family of sequences $\{B_n\}_{n\geq 0}$ satisfying (1.2) is wide enough to accommodate all symmetric sequences of orthogonal polynomials ($\beta_0 = 0$) as well as non-symmetric sequences ($\beta_0 \neq 0$). For these sequences the following quadratic decomposition holds (see [11])

$$\begin{gathered} B_{2n}(x) = P_n(x^2),\ \ n \geq 0, \\ B_{2n+1}(x) = (x - \beta_0)P_n^*(x^2),\ \ n \geq 0, \end{gathered} \tag{1.3}$$

where $\{P_n\}_{n\geq 0}$ is a MOPS and $\{P_n^*\}_{n\geq 0}$ is the sequence of monic kernel polynomials of K-parameter β_0^2 associated with $\{P_n\}_{n\geq 0}$ such that (see [6])

$$P_n^*(x) = \frac{1}{x - \beta_0^2}[P_{n+1}(x) - \frac{P_{n+1}(\beta_0^2)}{P_n(\beta_0^2)}P_n(x)],\ n \geq 0.$$

From (1.2), (1.3) and the fact that $\gamma_{n+1} \neq 0,\ n \geq 0$, we have

$$P_n(\beta_0^2) \neq 0, n \geq 0. \tag{1.4}$$

Conversely, if $\beta_0 \in \mathbb{C}$ and $\{P_n\}_{n\geq 0}$ is a MOPS satisfying (1.4) then there exists a unique MOPS $\{B_n\}_{n\geq 0}$ such that $B_1(x) = x - \beta_0,\ B_{2n}(x) = P_n(x^2),\ \ n \geq 0$. In this case $B_{2n+1}(x)$ is given by (1.3) and $\{B_n\}_{n\geq 0}$ satisfies (1.2) (see [11]).

Recently, the above quadratic decomposition becomes an interesting process to construct new D-semiclassical MOPS. In collaboration with J. Alaya, the first author started in [2] [3] with some families of symmetric D-semiclassical MOPS $\{B_n\}_{n\geq 0}$ of class $s \leq 2$ and they proved that the sequences $\{P_n\}_{n\geq 0}$ in their quadratic decomposition (1.3) are D-semiclassical of class one. Later on, in [4] [5] it is showed that this result and its converse remain valid for all sequences $\{B_n\}_{n\geq 0}$ satisfying (1.2). In particular, by analyzing the case when $\{P_n\}_{n\geq 0}$ is D-classical (Hermite, Laguerre, Bessel, Jacobi), they determine all D-semiclassical MOPS $\{B_n\}_{n\geq 0}$ of class one satisfying (1.2).

Then, it is a natural question to consider what happens if we replace the derivative operator D by the q-operator H_q introduced by W. Hahn (see [7] as well as [9]). In other words, let assume that either $\{B_n\}_{n\geq 0}$ or $\{P_n\}_{n\geq 0}$ is a H_q-semiclassical MOPS (see [8]). What can we say about the other one?. The aim of our contribution is to give a satisfactory answer to this question.

The paper is organized as follows. In Section 2 we introduce the basic background and some preliminary results to be used in the sequel. In Section 3 we show that the sequence $\{B_n\}_{n\geq 0}$ is H_q-semiclassical if and only if the sequence $\{P_n\}_{n\geq 0}$ is H_{q^2}-semiclassical. In Section 4 we express the polynomial coefficients of the first order linear q-difference equation satisfied by the Stieltjes function associated with $\{B_n\}_{n\geq 0}$ in terms of those corresponding to $\{P_n\}_{n\geq 0}$ and, as a consequence, we show that $2s' \leq s \leq 2s'+3$, where s and s' are, respectively, the classes of $\{B_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$. In Section 5 we find explicit relations between the coefficients of the structure relations of $\{B_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$. As an application of the previous results, in Section 6 we obtain a new non-symmetric H_q-semiclassical MOPS of class one. The coefficients of their TTRR and structure relation are deduced.

2 Notations and preliminary results

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and $\mathcal{P}'$ be its dual. We denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Let $\langle u, f\rangle$ be the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ and $S(u)(z) = -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}$ is the formal Stieltjes function of u where $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, are the moments of u.
Let us introduce the following operations on $\mathcal{P}'$.
The left multiplication of a linear functional u by a polynomial g

$$\langle gu, f\rangle = \langle u, gf\rangle, f \in \mathcal{P}. \tag{2.1}$$

The right multiplication of a linear functional by a polynomial f

$$(uf)(x) = \langle u, \frac{xf(x) - \xi f(\xi)}{x-\xi}\rangle. \tag{2.2}$$

The dilation of a linear functional u

$$\langle h_a u, f\rangle = \langle u, h_a f\rangle, a \in \mathbb{C}^*, f \in \mathcal{P}, \tag{2.3}$$

where

$$(h_a f)(x) = f(ax). \tag{2.4}$$

The even part of a linear functional

$$\langle \sigma u, f\rangle = \langle u, \sigma f\rangle, \tag{2.5}$$

where

$$\sigma f(x) = f(x^2). \tag{2.6}$$

The division of a linear functional u by a polynomial of first degree

$$\left\langle (x-c)^{-1}u, f\right\rangle = \langle u, \theta_c f\rangle, c \in \mathbb{C}, f \in \mathcal{P}, \tag{2.7}$$

where

$$(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}. \tag{2.8}$$

The q-derivative of a linear functional u

$$\langle H_q u, f\rangle = -\langle u, H_q f\rangle, q \in \widetilde{\mathbb{C}}, f \in \mathcal{P}, \tag{2.9}$$

where

$$(H_q f)(x) = \frac{f(qx)-f(x)}{(q-1)x}, \tag{2.10}$$

and

$$\widetilde{\mathbb{C}} = \mathbb{C} - \big(\{0\} \cup \bigcup_{n\geq 0} \{z \in \mathbb{C},\ z^n = 1\}\big).$$

Using (2.9) and (2.10) it is easy to see that

$$(H_q u)_n = -[n]_q (u)_{n-1}, n \geq 0$$

where $[n]_q,\ n \geq 0$, denotes the basic q-number defined by

$$[n]_q = \frac{q^n - 1}{q-1} = 1 + q + ... + q^{n-1},\ n > 0,\ [0]_q = 0.$$

Remark.
If $q \to 1$, then we get the standard derivative D.

Definition 2.1 *(See [6]) A sequence of polynomials $\{B_n\}_{n\geq 0}$ is said to be a monic orthogonal polynomial sequence (MOPS) with respect to a linear functional w if*
i) $\deg B_n = n$ *and the leading coefficient of $B_n(x)$ is equal to 1.*
ii) $\langle w, B_n B_m\rangle = r_n \delta_{n,m}, n, m \geq 0, r_n \neq 0, n \geq 0.$
In such conditions, the linear functional w is said to be regular or quasi-definite [6].

In the sequel we consider regular linear functionals w such that $(w)_0 = 1$.
It is well known that a MOPS satisfies a TTRR (see [6])

$$\begin{gathered} B_0(x) = 1, B_1(x) = x - \beta_0, \\ B_{n+2}(x) = (x - \beta_{n+1})\ B_{n+1}(x) - \gamma_{n+1} B_n(x), n \geq 0 \end{gathered} \tag{2.11}$$

with

$$(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C}^*, n \geq 0.$$

The orthogonality is preserved by dilation. Indeed, the sequence $\{\widehat{B}_n\}_{n\geq 0}$ defined by $\widehat{B}_n(x) = a^{-n}B_n(ax),\ n \geq 0$, satisfies the TTRR ([6])

$$\begin{gathered}\widehat{B}_0(x) = 1, \widehat{B}_1(x) = x - \tfrac{\beta_0}{a},\\ \widehat{B}_{n+2}(x) = (x - \tfrac{\beta_{n+1}}{a})\widehat{B}_{n+1}(x) - \tfrac{\gamma_{n+1}}{a^2}\widehat{B}_n(x), n \geq 0.\end{gathered} \tag{2.12}$$

Such a sequence of monic polynomials is orthogonal with respect to the linear functional $\widehat{w} = (h_{a^{-1}})w$.

A linear functional w is said to be symmetric if and only if $(w)_{2n+1} = 0,\ n \geq 0$, Or, equivalently, in (2.11) $\beta_n = 0,\ n \geq 0$.

If the sequence $\{B_n\}_{n\geq 0}$ satisfies (1.2), then for the sequence $\{P_n\}_{n\geq 0}$ given in (1.3) we get (see [11])

$$\begin{gathered}P_0(x) = 1,\ P_1(x) = x - \beta_0^P\\ P_{n+2}(x) = (x - \beta_{n+1}^P)\ P_{n+1}(x) - \gamma_{n+1}^P P_n(x), n \geq 0,\end{gathered} \tag{2.13}$$

where

$$\begin{cases} \beta_0^P = \gamma_1 + \beta_0^2 \\ \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3} + \beta_0^2, n \geq 0, \\ \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2},\ n \geq 0, \end{cases} \tag{2.14}$$

Furthermore, if we denote by w, u, and v the linear functionals associated with $\{B_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, and $\{P_n^*\}_{n\geq 0}$, respectively, then [11]

$$u = \sigma w \tag{2.15}$$

$$\beta_0 u = \sigma(xw) \tag{2.16}$$

$$v = \gamma_1^{-1}(x - \beta_0^2)u. \tag{2.17}$$

Definition 2.2 *[8] Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional w. $\{B_n\}_{n\geq 0}$ is said to be a H_q-semiclassical MOPS (respectively, w is said to be a H_q-semiclassical linear functional) of class s if the following conditions hold. There exist two polynomials Φ, a monic polynomial of degree t, and Ψ of degree $p > 0$, such that*

$$H_q(\Phi w) + \Psi w = 0 \tag{2.18}$$

(q-Pearson equation) as well as

$$\prod_{c\in Z_\Phi} \{|q(h_q\Psi)(c) + (H_q\Phi)(c)| + \langle w, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \theta_c\Phi)\rangle\} \neq 0 \tag{2.19}$$

where Z_Φ denotes the set of zeros of Φ.
The class s of $\{B_n\}_{n\geq 0}$ is given by $s = \max(p - 1, t - 2)$.

If w is a H_q-semiclassical linear functional of class s fulfilling (2.18), then $\widehat{w} = h_{a^{-1}}w$ is also a H_q-semiclassical linear functional of the same class and satisfies the following q-Pearson equation [8]

$$H_q(\widehat{\Phi}\widehat{w}) + \widehat{\Psi}\widehat{w} = 0 \tag{2.20}$$

where

$$\widehat{\Phi}(z) = a^{-t}\Phi(az),\ \widehat{\Psi}(z) = a^{1-t}\Psi(az). \tag{2.21}$$

Proposition 2.3 *([1], [8]) Let w be a symmetric H_q-semiclassical linear functional of class s satisfying* (2.18) *and* (2.19). *If s is a non negative even number, then Φ and Ψ are even and odd functions, respectively. If s is a non negative odd number, then Φ and Ψ are odd and even functions, respectively.*

In the sequel, we assume that $\{B_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, u, v, and w satisfy $(1.2) - (1.4), (2.13) - (2.17)$.

3 Case when either $\{P_n\}_{n\geq 0}$ or $\{B_n\}_{n\geq 0}$ is H_q-semiclassical

First of all, we will prove the following

Lemma 3.1 *Let $\widetilde{u}$ be a linear functional, Φ, Ψ, and B three polynomials. Let us split up these polynomials according to their even and odd parts*

$$\Phi(x) = \Phi^e(x^2) + x\Phi^o(x^2), \Psi(x) = \Psi^e(x^2) + x\Psi^o(x^2). \tag{3.1}$$

Then,

$$\begin{aligned} \sigma\Big(H_q(\Phi\widetilde{u}) + \Psi\widetilde{u}\Big) \quad &= \quad (q+1)H_{q^2}(\Phi^e(x)\sigma(x\widetilde{u})) + (q+1)H_{q^2}(x\Phi^o(x)\sigma\widetilde{u}) + \\ &\quad\quad \Psi^e(x)\sigma\widetilde{u} + \Psi^o(x)\sigma(x\widetilde{u}). \end{aligned} \tag{3.2}$$

Proof.
From the linearity of the operator σ we get

$$\sigma(H_q(\Phi\widetilde{u}) + \Psi\widetilde{u}) = \sigma(H_q(\Phi\widetilde{u})) + \sigma(\Psi\widetilde{u}). \tag{3.3}$$

On the other hand, using (2.5) and (2.9), it is easy to see that for a polynomial f and a linear functional Ω we have

$$\sigma(H_q\Omega) = (q+1)H_{q^2}(\sigma(x\Omega)), \tag{3.4}$$

$$\sigma(f(x^2)\Omega) = f(x)\sigma\Omega. \tag{3.5}$$

Hence

$$\begin{aligned}\sigma(H_q(\Phi\widetilde{u})) &= (q+1)H_{q^2}\big(\sigma(x\Phi\widetilde{u})\big)\\ &= (q+1)H_{q^2}\Big(\sigma\big((x\Phi^e(x^2)+x^2\Phi^o(x^2))\widetilde{u}\big)\Big)\\ &= (q+1)H_{q^2}\big(\Phi^e(x)\sigma(x\widetilde{u}) + (q+1)H_{q^2}\big(x\Phi^o(x)\sigma\widetilde{u}\big)\end{aligned}$$

as well as

$$\begin{aligned}\sigma(\Psi\widetilde{u}) &= \sigma\Big(\big(\Psi^e(x^2)+x\Psi^o(x^2)\big)\widetilde{u}\Big)\\ &= \Psi^e(x)\sigma\widetilde{u} + \Psi^o(x)\sigma(x\widetilde{u}).\end{aligned}$$

So, the statement follows. □

Proposition 3.2 *If $\{P_n\}_{n\geq 0}$ is a H_{q^2}-semiclassical MOPS of class s' then $\{B_n\}_{n\geq 0}$ is a H_q-semiclassical MOPS of class $s \leq 2s'+3$. Furthermore, if u satisfies*

$$H_{q^2}(\Phi^P u) + \Psi^P u = 0 \tag{3.6,}$$

then w satisfies (2.18) *with*

$$\Phi(x) = (x+q^{-1}\beta_0)\Phi^P(x^2), \tag{3.7}$$

and

$$\Psi(x) = (q^{-1}+1)x(q^{-1}x+\beta_0)\Psi^P(x^2) - q^{-1}(q^{-1}+1)\Phi^P(x^2). \tag{3.8}$$

Proof.
Let $\Omega = H_q(\Phi w) + \Psi w$. To prove that $\Omega = 0$ it is enough to prove that $\sigma\Omega = 0$ and $\sigma(x\Omega) = 0$.
First of all, let us recall that for a polynomial f and a linear functional Ω we have

$$H_q(f\Omega) = (h_{q^{-1}}f)H_q\Omega + q^{-1}(H_{q^{-1}}f)\Omega. \tag{3.9}$$

The even and odd parts of the polynomials Φ and Ψ in (3.7) and (3.8) are
$\Phi^e(x) = q^{-1}\beta_0\Phi^P(x), \Phi^o(x) = \Phi^P(x)$,
$\Psi^e(x) = (q^{-1}+1)q^{-1}x\Psi^P(x) - q^{-1}(q^{-1}+1)\Phi^P(x), \Psi^o(x) = (q^{-1}+1)\beta_0\Psi^P(x)$.
Then, from Lemma 3.1 we get

$$\begin{aligned}\sigma\Omega &= (q+1)H_{q^2}(q^{-1}\beta_0\Phi^P(x)\sigma(xw)) + (q+1)H_{q^2}(x\Phi^P(x)\sigma w)\\ &\quad +(q^{-1}+1)q^{-1}(x\Psi^P(x) - \Phi^P(x))\sigma w + (q^{-1}+1)\beta_0\Psi^P(x)\sigma(xw).\end{aligned}$$

From (2.15) and (2.16) the previous relation becomes

$$\sigma\Omega = (q+1)H_{q^2}\big((x+q^{-1}\beta_0^2)\Phi^P(x)u\big) + (q^{-1}+1)\big(q^{-1}x\Psi^P(x) - q^{-1}\Phi^P(x) + \beta_0^2\Psi^P(x)\big)u.$$

On the other hand, from (3.9)

$$H_{q^2}\Big((x+q^{-1}\beta_0^2)\Phi^P(x)w\Big) = \Big(q^{-2}x + q^{-1}\beta_0^2\Big)H_{q^2}(\Phi^P u) + q^{-2}\Phi^P(x)u.$$

Therefore,

$$\sigma\Omega = (q^{-1}+1)(q^{-1}x+\beta_0^2)\Big(H_{q^2}(\Phi^P(x)u) + \Psi(x)^P u\Big) = 0.$$

In a similar way,

$$\begin{aligned}
\sigma(x\Omega) &= \sigma(xH_q(\Phi w) + x\Psi w) \\
&= \sigma\Big(H_q(qx\Phi w) + (x\Psi - \Phi)w\Big) \\
&= (q^{-1}+1)^2\beta_0 x\Big(H_{q^2}(\Phi^P(x)u) + \Psi^P(x)u\Big) \\
&= 0.
\end{aligned}$$

So, $\Omega = 0$. Hence w satisfies (2.18) with (3.7), and (3.9). Thus, $\{B_n\}_{n\geq 0}$ is a H_q-semiclassical MOPS.
Next we will prove that $s \leq 2s' + 3$. Without loss of generality, let assume that the q-distributional equation (3.6) is canonical for u, so that $s' = \max\{deg\Phi^P - 2,\, deg\Psi^P - 1\}$. Let write $deg\Psi = p$, $deg\Phi = t$, $deg\Psi^P = p'$, and $deg\Phi^P = t'$. From (3.7), and (3.8) we have successively $t = 2t' + 1$ and $p \leq max(2p' + 2, 2t')$. Therefore, $s \leq max(p-1, t-2) \leq max(2p'+1, 2t'-1) = 2s' + 3$. □

Remark.
1) When $q \to 1$, we recover the same result for the $D-$semiclassical case [4].
2) Although Proposition 3.2 proves the H_q-semiclassical character of the MOPS $\{B_n\}_{n\geq 0}$, its class s is not explicitly given. The best way to determine it is to analyze the first order linear q-difference equation that the corresponding formal Stieltjes function, $S(w)(z) = -\sum_{n\geq 0}\frac{(w)_n}{z^{n+1}}$, satisfies. This will be done in the following section.

Proposition 3.3 *If $\{B_n\}_{n\geq 0}$ is a H_q-semiclassical MOPS, then $\{P_n\}_{n\geq 0}$ is a H_{q^2}-semiclassical MOPS. Furthermore, if w satisfies* (2.18) *and we split these polynomials according to their even and odd parts as in* (3.1)*, then u satisfies*

$$H_{q^2}(\Phi_1^P u) + \Psi_1^P u = 0 \tag{3.10}$$

where

$$\begin{cases} \Phi_1^P(x) = (q+1)(x\Phi^o(x) + \beta_0\Phi^e(x)), \\ \Psi_1^P(x) = \Psi^e(x) + \beta_0\Psi^o(x), \end{cases} \tag{3.11}$$

and

$$H_{q^2}(\Phi_2^P u) + \Psi_2^P u = 0 \tag{3.12}$$

where

$$\begin{cases} \Phi_2^P(x) = q(q+1)(x\Phi^e(x) + \beta_0 x\Phi^o(x)), \\ \Psi_2^P(x) = x\Psi^o(x) - \Phi^e(x) + \beta_0(\Psi^e(x) - \Phi^o(x)). \end{cases} \tag{3.13}$$

Proof.
Applying Lemma 3.1 in (2.18) and taking into account (2.15) and (2.16) we get (3.10) and (3.11).
On the other hand, the multiplication by x in (2.18) yields

$$qH_q(x\Phi(x)w) + (x\Psi(x) - \Phi(x))w = 0. \tag{3.14}$$

Applying, again, Lemma 3.1 to (3.14) and taking into account (2.15) and (2.16) we get (3.12) and (3.13).

Notice that from (3.10) we can not conclude that u is a H_{q^2}-semiclassical linear functional since we have not proved that at least one of the polynomials Φ_1^P and Ψ_1^P is not zero, which is not always true. For example if w is symmetric ($\beta_0 = 0$) of even class s, then, according to Proposition 2.3, we have $\Phi_1^P = \Psi_1^P = 0$. This is the justification of (3.12).
Next, let us assume $\Phi_1^P = \Psi_1^P = 0$ and $\Phi_2^P = \Psi_2^P = 0$. Then, from (3.11) and (3.13),

$$\begin{cases} x\Phi^o(x) + \beta_0\Phi^e(x) = 0, \\ x\Phi^e(x) + \beta_0 x\Phi^o(x) = 0 \end{cases}$$

and, as a consequence,

$$\begin{cases} (x - \beta_0^2)\Phi^e(x) = 0, \\ (x^2 - \beta_0^2 x)\Phi^o(x) = 0. \end{cases}$$

Hence

$$\Phi^e(x) = \Phi^o(x) = 0.$$

Therefore $\Phi(x) = 0$.
In a similar way, using (3.11) and (3.13), we prove that $\Psi(x) = 0$, a contradiction. Thus, at least one of the polynomials Φ_1^P, Ψ_1^P, Φ_2^P, and Ψ_2^P is not zero. Therefore, u is a H_{q^2}-semiclassical linear functional. □

4 First order linear q-difference equation

It is well known that the linear functional w satisfies (2.18) if and only if its formal Stieltjes function $S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}$ satisfies the following first order linear q-difference equation [8]

$$A(z)H_{q^{-1}}(S(w))(z) = C(z)S(w)(z) + D(z) \tag{4.1}$$

with

$$\begin{aligned} A(z) &= q^{deg\Phi}(h_{q^{-1}}\Phi)(z), \\ C(z) &= -q^{deg\Phi}((H_{q^{-1}}\Phi)(z) + q\Psi(z)), \\ D(z) &= -q^{deg\Phi}(H_{q^{-1}}(w\theta_0\Phi)(z) + q(w\theta_0\Psi)(z)). \end{aligned} \tag{4.2}$$

The condition (2.19) is equivalent to the fact that polynomials A, C, and D are coprime. In this case, from (4.2) and Definition 2.2, the class s of w is given by

$$s = max(degA - 2, degC - 1). \tag{4.3}$$

Assume that $\{P_n\}_{n\geq 0}$ is a H_{q^2}-semiclassical sequence. Then, according to Proposition 3.2, $\{B_n\}_{n\geq 0}$ is a H_q-semiclassical sequence. Therefore, its formal Stieltjes function satisfies a first order linear q-difference equation. Next we will give the polynomial coefficients A, C, and D of the first order linear q-difference equation associated with the Stieltjes function of w in terms of those of u, denoted by A^P, C^P, and D^P. We need the following

Lemma 4.1 *([5])*

$$S(w)(z) = (z + \beta_0)S(u)(z^2). \tag{4.4}$$

Proposition 4.2 *If $S(u)(z)$ satisfies the first order linear q-difference equation*

$$A^P(z)H_{q^{-2}}(S(u))(z) = C^P(z)S(u)(z) + D^P(z), \tag{4.5}$$

then $S(w)(z)$ satisfies (4.1) *with*

$$A(z) = (z + \beta_0)A^P(z^2), \tag{4.6}$$

$$C(z) = A^P(z^2) + (q^{-1} + 1)z(q^{-1}z + \beta_0)C^P(z^2), \tag{4.7}$$

$$D(z) = (q^{-1} + 1)z(z + \beta_0)(q^{-1}z + \beta_0)D^P(z^2). \tag{4.8}$$

Proof.
Using the definition of the operator H_q, it is easy to see that for polynomials f and g we have

$$H_{q^{-1}}(f(z^2)) = (q^{-1} + 1)z(H_{q^{-2}}f)(z^2), \tag{4.9}$$

and

$$(H_{q^{-1}}(\frac{f}{g}))(z) = \frac{(H_{q^{-1}}f)(z)g(z) - (H_{q^{-1}}g)(z)f(z)}{g(z)(h_{q^{-1}}g)(z)}. \tag{4.10}$$

On the other hand, from (4.4)

$$S(u)(z^2) = \frac{S(w)(z)}{z + \beta_0}. \tag{4.11}$$

Applying the operator $H_{q^{-1}}$ to both hand sides of the last relation and taking into account (4.9) and (4.10) we get

$$(H_{q^{-2}}S(u))(z^2) = \frac{(z + \beta_0)H_{q^{-1}}(S(w))(z) - S(w)(z)}{(q^{-1} + 1)z(z + \beta_0)(q^{-1}z + \beta_0)}. \tag{4.12}$$

On the other hand, the change of variable $z \to z^2$ in (4.5) yields

$$A^P(z^2)H_{q^{-2}}(S(u))(z^2) = C^P(z^2)S(u)(z^2) + D^P(z^2). \tag{4.13}$$

Replacing (4.11) and (4.12), (4.13) becomes

$$A^P(z^2)\left(\frac{(z+\beta_0)H_{q^{-1}}(S(w))(z) - S(w)(z)}{(q^{-1}+1)z(z+\beta_0)(q^{-1}z+\beta_0)}\right) = C^P(z^2)\frac{S(w)(z)}{(z+\beta_0)} + D^P(z^2).$$

Hence,

$$\begin{array}{rcl}(z+\beta_0)\ A^P(z^2)H_{q^{-1}}(S(w))(z) & = & (A^P(z^2) + (q^{-1}+1)(q^{-1}z+\beta_0)zC^P(z^2))S(w)(z) \\ & & +(q^{-1}+1)z(z+\beta_0)(q^{-1}z+\beta_0)D^P(z^2).\end{array}$$

□

Notice that if (4.1) can not be simplified, then $s = 2s'+3$, where s' is the class of $\{P_n\}_{n\geq 0}$. Indeed, assuming that A^P, C^P, and D^P are coprime, then, from (4.3), we have $s' = max(degA^P - 2, degC^P - 1)$. We must analyze two cases

(i) $degA^P = s'+2,\ degC^P \leq s'+1$

(ii) $degA^P \leq s'+1,\ degC^P = s'+1$.

Using (4.6) and (4.7) in the first case we get $degA = 2s'+5,\ degC \leq 2s'+4$. In the second one, $degA \leq 2s'+3,\ degC = 2s'+4$ holds. In both cases we have $s = 2s'+3$.

Next, if (4.1) can be simplified, then we will find the class s in terms of s'.

Proposition 4.3 *We will analyze the following two cases*

A $\beta_0 = 0$.

(A$_1$) *If* $A^P(0) \neq 0$, *then* $s = 2s'+3$.
(A$_2$) *If* $A^P(0) = 0$ *and* $(A^P)'(0) + (q^{-1}+1)q^{-1}C^P(0) \neq 0$, *then* $s = 2s'+1$.
(A$_3$) *If* $A^P(0) = 0$ *and* $(A^P)'(0) + (q^{-1}+1)q^{-1}C^P(0) = 0$, *then* $s = 2s'$.

B $\beta_0 \neq 0$.

(B$_1$) *If* $A^P(0) \neq 0,\ A^P(q^2\beta_0^2) \neq 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) \neq 0$, *then* $s = 2s'+3$.
(B$_2$) *If* ($A^P(0) \neq 0,\ A^P(q^2\beta_0^2) \neq 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) = 0$) *or* ($A^P(0) \neq 0,\ A^P(q^2\beta_0^2) = 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) \neq 0$) *or* ($A^P(0) = 0,\ A^P(q^2\beta_0^2) \neq 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) \neq 0$), *then* $s = 2s'+2$.
(B$_3$) *If* ($A^P(0) \neq 0,\ A^P(q^2\beta_0^2) = 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) = 0$) *or* ($A^P(0) = 0,\ A^P(q^2\beta_0^2) = 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) \neq 0$) *or* ($A^P(0) = 0,\ A^P(q^2\beta_0^2) \neq 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) = 0$), *then* $s = 2s'+1$.
(B$_4$) *If* ($A^P(0) = 0,\ A^P(q^2\beta_0^2) = 0,\ A^P(\beta_0^2) - (1-q^{-2})\beta_0^2C^P(\beta_0^2) = 0$), *then* $s = 2s'+1$.

Proof.
First, notice that A, C and D have not a common zero other than $-\beta_0,\ 0,\ -q\beta_0$. Indeed, if c is a common zero of A, C, and D not equal to $-\beta_0, 0$ and $-q\beta_0$ then, according to (4.6) – (4.8), c^2 is a common zero of A^P, C^P and D^P, which yields a contradiction.

Now, let us assume that $-\beta_0,\ 0,\ -q\beta_0$ are common zeros of A, C and D. Two situations appear.

1- First case $\beta_0 = 0$.

According to Proposition 4.2, $S(w)(z)$ satisfies (4.1) with

$$A(z) = zA^P(z^2), \tag{4.14}$$

$$C(z) = A^P(z^2) + (q^{-1}+1)q^{-1}z^2C^P(z^2), \tag{4.15}$$

$$D(z) = (q^{-1}+1)q^{-1}z^3D^P(z^2). \tag{4.16}$$

1-1 If $A^P(0) \neq 0$, then $A, C,$ and D are coprime and, therefore, (4.1) cannot be simplified and $s = 2s'+3$.

1-2 If $A^P(0) = 0$, then (4.1) can be divided by z^2 and $S(w)(z)$ satisfies

$$A(z)H_{q^{-1}}(S(w))(z) = C(z)S(w)(z) + D(z)$$

with

$$\begin{cases} A(z) &= \dfrac{A^P(z^2)}{z}, \\ C(z) &= \dfrac{A^P(z^2)}{z^2} + (q^{-1}+1)q^{-1}C^P(z^2), \\ D(z) &= (q^{-1}+1)q^{-1}zD^P(z^2). \end{cases} \tag{4.17}$$

1-2-1 If $(A^P)'(0) + (q^{-1}+1)q^{-1}C^P(0) \neq 0$, then $s = 2s'+1$.

1-2-2 If $(A^P)'(0) + (q^{-1}+1)q^{-1}C^P(0) = 0$, then $S(w)(z)$ satisfies (4.1) with

$$\begin{cases} A(z) &= \dfrac{A^P(z^2)}{z^2}, \\ C(z) &= \dfrac{A^P(z^2)}{z^3} + (q^{-1}+1)q^{-1}\dfrac{C^P(z^2)}{z}, \\ D(z) &= (q^{-1}+1)q^{-1}D^P(z^2). \end{cases} \tag{4.18}$$

Therefore, $s = 2s'$.

2- Second case $\beta_0 \neq 0$.

2-1 If $A^P(0) \neq 0$, then (4.1) cannot be simplified by z. We must analyze the possibilities of simplification by either $z+\beta_0$ or $q^{-1}z+\beta_0$.

2-1-1 If $A^P(\beta_0^2)-(1-q^{-2})\beta_0^2 C^P(\beta_0^2) \neq 0$, then (4.1) cannot be divided by $z+\beta_0$.

2-1-1-1 If $A^P(q^2\beta_0^2) \neq 0$, then (4.1) cannot be simplified by $q^{-1}z+\beta_0$ and we have $s=2s'+3$.

2-1-1-2 If $A^P(q^2\beta_0^2)=0$, then $s=2s'+2$.

2-1-2 If $A^P(\beta_0^2)-(1-q^{-2})\beta_0^2 C^P(\beta_0^2)=0$, then (4.1) can be simplified by $z+\beta_0$.

2-1-2-1 If $A^P(q^2\beta_0^2) \neq 0$, then $s=2s'+2$.

2-1-2-2 If $A^P(q^2\beta_0^2)=0$, then $s=2s'+1$ and $S(w)(z)$ satisfies (4.1) with

$$\left\{\begin{array}{rcl} A(z) & = & \dfrac{A^P(z^2)}{(q^{-1}z+\beta_0)}, \\ C(z) & = & \dfrac{A^P(z^2)}{(z+\beta_0)(q^{-1}z+\beta_0)}+(q^{-1}+1)\dfrac{zC^P(z^2)}{z+\beta_0}, \\ D(z) & = & (q^{-1}+1)zD^P(z^2). \end{array}\right. \tag{4.19}$$

2-2 If $A^P(0)=0$, then (4.1) can be simplified by z.

2-2-1 If $A^P(\beta_0^2)-(1-q^{-2})\beta_0^2 C^P(\beta_0^2) \neq 0$, then (4.1) cannot be simplified by $z+\beta_0$.

2-2-1-1 If $A^P(q^2\beta_0^2) \neq 0$, then $s=2s'+2$.

2-2-1-2 If $A^P(q^2\beta_0^2)=0$, then $s=2s'+1$.

2-2-2 If $A^P(\beta_0^2)-(1-q^{-2})\beta_0^2 C^P(\beta_0^2)=0$, then (4.1) can be simplified by $z+\beta_0$.

2-2-2-1 If $A^P(q^2\beta_0^2) \neq 0$, then $s=2s'+1$.

2-2-2-2 If $A^P(q^2\beta_0^2)=0$, then $s=2s'$. $\square$

Notice that in all cases we have

$$2s' \leq s \leq 2s'+3. \tag{4.20}$$

5 The structure relation

Let us assume that $\{P_n\}_{n\geq 0}$ is a H_{q^2}-semiclassical polynomial sequence. Then, it satisfies the following so-called structure relation (see [8])

(5.1)

$$\Phi^P(x)(H_{q^2}P_{n+1})(x) = \frac{C^P_{n+1}(x) - C^P_0(x)}{2}P_{n+1}(x) - \gamma^P_{n+1}D^P_{n+1}(x)P_n(x),\ n \geq 0,$$

where, for $n \geq 0$,

(5.2) $$C^P_{n+1}(x) = -C^P_n(x) + 2(x - \beta^P_n)D^P_n(x) + 2x(q^2 - 1)\Sigma_n(x),$$

(5.3)

$$\begin{array}{rcl} \gamma^P_{n+1}D^P_{n+1}(x) & = & -\Phi^P(x) + \gamma^P_n D^P_{n-1}(x) + (x - \beta^P_n)^2 D^P_n(x) - (\frac{q^2+1}{2}x - \beta^P_n)C^P_n(x) \\ & & +x(q^2 - 1)(\frac{1}{2}C^P_0(x) + (x - \beta^P_n)\Sigma^P_n(x), \end{array}$$

with

(5.4) $$C^P_0(x) = q^{-2deg\Phi^P}C^P(q^2x),\ D^P_0(x) = q^{-2deg\Phi^P}D^P(q^2x),\ D^P_{-1}(x) = 0,$$

and

$$\Sigma^P_n(x) = \sum_{k=0}^{n} D^P_k(x),\ n \geq 0.$$

The structure relation plays an important role in the analysis of some properties of orthogonal polynomials like the location and the electrostatic interpretation of their zeros (see [8]). Indeed, they represent the ladder operators for such polynomials. On the other hand, from their coefficients $C^P_n(x)$ and $D^P_n(x)$ we get a second order linear q^2-difference equation (a discrete q-holonomic equation), satisfied by H_{q^2}-semiclassical MOPS. It reads as the natural extension of the Bochner one (see [7] and [8])

(5.5)

$$J_{q^2}(x,n)(H_{q^2}\circ H_{q^{-2}}P_{n+1})(x) + K_{q^2}(x,n)(H_{q^{-2}}P_{n+1})(x) + L_{q^2}(x,n)P_{n+1}(x) = 0,\ n \geq 0,$$

where, for $n \geq 0$,

(5.6)

$$\left\{\begin{array}{rcl} J_{q^2}(x,n) & = & q^2\Phi^P(x)D^P_{n+1}(x), \\ K_{q^2}(x,n) & = & D^P_{n+1}(q^{-2}x)(H_{q^{-2}}\Phi^P)(x) - (H_{q^{-2}}D^P_{n+1})(x)\Phi^P(q^{-2}x) \\ & & +C^P_0(q^{-2}x)D^P_{n+1}(x), \\ L_{q^2}(x,n) & = & \frac{1}{2}\Big(C^P_{n+1}(q^{-2}x) - C^P_0(q^{-2}x)\Big)(H_{q^{-2}}D^P_{n+1})(x) \\ & & -\frac{1}{2}\Big(H_{q^{-2}}(C^P_{n+1} - C^P_0)\Big)(x)D^P_{n+1}(q^{-2}x) - D^P_{n+1}(x)\Sigma^P_n(q^{-2}x). \end{array}\right.$$

According to Proposition 3.2, $\{B_n\}_{n\geq 0}$ is a H_q-semiclassical polynomial sequence. Then, it satisfies a structure relation. Our aim is to express $C_n(x)$ and $D_n(x)$, $n \geq 0$, the coefficients of the structure relation of the sequence $\{B_n\}_{n\geq 0}$, in terms of those of $\{P_n\}_{n\geq 0}$ which we will denote by $C^P_n(x)$ and $D^P_n(x)$, $n \geq 0$.

Proposition 5.1 *If the sequence* $\{P_n\}_{n\geq 0}$ *satisfies* (5.1), *then* $\{B_n\}_{n\geq 0}$ *satisfies*
(5.7)
$$\Phi(x)(H_qB_{n+1})(x)=\frac{C_{n+1}(x)-C_0(x)}{2}B_{n+1}(x)-\gamma_{n+1}D_{n+1}(x)B_n(x),\ n\geq 0$$

where Φ *is defined by* (3.7) *and*

$$\begin{aligned}C_{2n}(x) &= q^{-1}\Phi^P(x^2)+(1-q)(q^{-1}+1)x^2C_0^P(x^2)+\\ &+(q^{-1}+1)x(qx+\beta_0)(C_n^P(x^2)+2\tfrac{\gamma_n^PD_n^P(x^2)}{\gamma_{2n-1}}),\ n\geq s'+2,\end{aligned}\tag{5.8}$$

$$D_{2n}(x)=(q^{-1}+1)x(qx+\beta_0)(x+\beta_0)D_n^P(x^2),\ n\geq s'+2.\tag{5.9}$$

Proof.
For $n\geq 0$ the change of variable $x\to x^2$ in (5.1) yields
$\Phi^P(x^2)(H_{q^2}P_{n+1})(x^2)=\frac{C_{n+1}^P(x^2)-C_0^P(x^2)}{2}P_{n+1}(x^2)-\gamma_{n+1}^PD_{n+1}^P(x^2)P_n(x^2).$

The multiplication in both sides by $(q^{-1}+1)x(qx+\beta_0)$ together with (3.7) gives
(5.10)
$$\begin{aligned}(q+1)x\Phi(x)(H_{q^2}P_{n+1})(x^2) &= (q^{-1}+1)x(qx+\beta_0)\tfrac{C_{n+1}^P(x^2)-C_0^P(x^2)}{2}P_{n+1}(x^2)\\ &-\gamma_{n+1}^P(q^{-1}+1)x(qx+\beta_0)D_{n+1}^P(x^2)P_n(x^2),\ n\geq 0.\end{aligned}$$

On the other hand, applying the operator H_q to both sides of (1.3) and taking into account (4.9), we get

$$(H_qB_{2n+2})(x)=(q+1)x(H_{q^2}P_{n+1})(x^2),\ n\geq 0.\tag{5.11}$$

Replacing (1.3) and (5.11) in (5.10)
(5.12)
$$\begin{aligned}\Phi(x)(H_qB_{2n+2})(x) &= (q^{-1}+1)x(qx+\beta_0)\tfrac{C_{n+1}^P(x^2)-C_0^P(x^2)}{2}B_{2n+2}(x)\\ &-\gamma_{n+1}^P(q^{-1}+1)x(qx+\beta_0)D_{n+1}^P(x^2)B_{2n}(x),\ n\geq 0.\end{aligned}$$

The change of indices $n\to 2n$ in (1.2) gives

$$B_{2n}(x)=\frac{1}{\gamma_{2n+1}}(-B_{2n+2}(x)+(x+\beta_0)B_{2n+1}(x)),\ n\geq 0.\tag{5.13}$$

Therefore, for $n\geq 0$, (5.12) becomes
(5.14)
$$\begin{aligned}\Phi(x)(H_qB_{2n+2})(x) &= (q^{-1}+1)x(qx+\beta_0)(\tfrac{C_{n+1}^P(x^2)-C_0^P(x^2)}{2}+\tfrac{\gamma_{n+1}^PD_{n+1}^P(x^2)}{\gamma_{2n+1}})B_{2n+2}(x)\\ &-(q^{-1}+1)x(qx+\beta_0)(x+\beta_0)\tfrac{\gamma_{n+1}^PD_{n+1}^P(x^2)}{\gamma_{2n+1}}B_{2n+1}(x).\end{aligned}$$

The identification with (5.7), where $n \to 2n+1$, leads to

$$M(x,n)B_{2n+2}(x) = N(x,n)B_{2n+1}(x),\ n \geq 0,$$

where, for $n \geq 0$,
$M(x,n) = (q^{-1}+1)x(qx+\beta_0)(\frac{C^P_{n+1}(x^2)-C^P_0(x^2)}{2}+\frac{\gamma^P_{n+1}D^P_{n+1}(x^2)}{\gamma_{2n+1}})-\frac{C_{2n+2}(x)-C_0(x)}{2}$,
$N(x,n) = (q^{-1}+1)x(qx+\beta_0)(x+\beta_0)\frac{\gamma^P_{n+1}D^P_{n+1}(x^2)}{\gamma_{2n+1}} - \gamma_{2n+2}D_{2n+2}(x)$.
Because B_{2n+1} and B_{2n+2} are coprime, then B_{2n+2} divides $N(x,n)$, which is a polynomial of degree at most $2s'+3$. As a consequence, $M(x,n) = N(x,n) = 0$, $n \geq s'+1$. Therefore

$$D_{2n+2}(x) = (q^{-1}+1)\frac{\gamma^P_{n+1}}{\gamma_{2n+1}\gamma_{2n+2}}x(qx+\beta_0)(x+\beta_0)D^P_{n+1}(x^2),\ n \geq s'+1, \tag{5.15}$$

and
(5.16)
$$C_{2n+2}(x) = C_0(x)+2(q^{-1}+1)x(qx+\beta_0)(\frac{C^P_{n+1}(x^2)-C^P_0(x^2)}{2}+\frac{\gamma^P_{n+1}D^P_{n+1}(x^2)}{\gamma_{2n+1}}),\ n \geq s'+1.$$

Taking into account (2.14) and changing $n \to n-1$ in (5.15), we get (5.9).
Using (4.2), (4.7), and (5.4) we have

$$C_0(x) = q^{-1}\Phi^P(x^2) + (q^{-1}+1)x(x+\beta_0)C^P_0(x^2). \tag{5.17}$$

Replacing (5.17) in (5.16), we get (5.8). □

6 Example

Using the previous results, n this section we will deduce the coefficients of the TTRR and the structure relation of a non-symmetric H_q-semiclassical MOPS $\{B_n\}_{n\geq 0}$ of class $s = 1$ satisfying (1.1). According to Proposition 4.3 the component $\{P_n\}_{n\geq 0}$ of the sequence $\{B_n\}_{n\geq 0}$ in the quadratic decomposition must be H_{q^2}-classical and satisfies one of the following conditions

(A_1) $A^P(0) = 0$, $A^P(q^2\beta_0^2) = 0$, $A^P(\beta_0^2) - (1-q^{-2})\beta_0^2 C^P(\beta_0^2) \neq 0$.

(A_2) $A^P(0) \neq 0$, $A^P(q^2\beta_0^2) = 0$, $A^P(\beta_0^2) - (1-q^{-2})\beta_0^2 C^P(\beta_0^2) = 0$.

(A_3) $A^P(0) = 0$, $A^P(q^2\beta_0^2) \neq 0$, $A^P(\beta_0^2) - (1-q^{-2})\beta_0^2 C^P(\beta_0^2) = 0$.

In the sequel, we will assume that conditions in (A_1) hold and $\beta_0 \neq 0$. We will proceed with a suitable dilation in the variable that, for a sake of simplicity, yields $\beta_0 = 1$.
Then, for $q \to q^2$, (4.2) leads to $\Phi^P(0) = \Phi^P(1) = 0$. Therefore,

$$\Phi^P(x) = x^2 - x. \tag{6.1}$$

If

$$\Psi^P(x) = \hat{b}x + \bar{b} \tag{6.2}$$

and using again (4.2), where $q \to q^2$, then we deduce

$$A^P(x) = x^2 - q^2 x, \tag{6.3}$$

$$C^P(x) = -q^4\Big((q^2\hat{b} + q^{-2} + 1)x + q^2\bar{b} - 1\Big),\ D^P(x) = -q^4(1 + q^2\hat{b}). \tag{6.4}$$

Using Proposition 4.2, after division by $x(x+q)$, we get

$$\begin{cases} A(x) = x(x+1)(x-q), \\ C(x) = \Big(1 - (q+1)(q^4\hat{b} + q^2 + 1)\Big)x^2 - qx - q^2(q^{-1}+1)(q^2\bar{b} - 1), \\ D(x) = -q^3(q^{-1}+1)(1+q^2\hat{b})(x+1). \end{cases} \tag{6.5}$$

Hence, from (4.2), we obtain

$$\Phi(x) = x(x-1)(x+q^{-1}),$$

and

$$\Psi(x) = \Big(-q^{-4}[4]_q + q^{-9}(q^{-1}+1)(q^2\hat{b}+q^{-2}+1)\Big)x^2 - q^{-1}x + q^{-2} + q^{-9}(q^{-1}+1)(q^2\bar{b}-1).$$

The coefficients of the TTRR of H_q-classical polynomial sequences are given in [9] and [10]. Indeed, in a unified way can be represented in terms of the coefficients of Φ and Ψ. Here, we will use these expressions but we will replace q, $\hat{b}$, and $\bar{b}$ by q^2, $-\hat{b}$, and $-\bar{b}$, respectively. In [9] and [10] all H_q-classical linear functionals fulfilling $H_q(\Phi u) = \Psi u$ are described. Now, we need all H_{q^2}-classical linear functionals fulfilling $H_{q^2}(\Phi u) + \Psi u = 0$. Thus,

$$\gamma_n^P = \frac{q^{4n-4}[n]_{q^2}([n-2]_{q^2} - \hat{b})([n-1]_{q^2} + \bar{b})(-q^{2n-2}\bar{b} + [n-1]_{q^2} - \hat{b})}{([2n-1]_{q^2} - \hat{b})([2n-2]_{q^2} - \hat{b})^2([2n-3]_{q^2} - \hat{b})},\ n \geq 2, \tag{6.6}$$

$$\gamma_1^P = -\frac{\bar{b}(\bar{b} + \hat{b}) + \hat{b}^2}{\hat{b}^2(1 - \hat{b})}, \tag{6.7}$$

$$\beta_n^P = \frac{-[n]_{q^2}([n-1]_{q^2} + \bar{b})([2n]_{q^2} - \hat{b}) + [n+1]_{q^2}([n]_{q^2} + \bar{b})([2n-2]_{q^2} - \hat{b})}{([2n-2]_{q^2} - \hat{b})([2n]_{q^2} - \hat{b})},\ n \geq 0, \tag{6.8}$$

where

$$[n]_{q^2} - \hat{b} \neq 0,\ q^{4n}(\frac{[n]_{q^2} + \bar{b}}{[2n]_{q^2} - \hat{b}})^2 + q^{2n}(\frac{[n]_{q^2} + \bar{b}}{[2n]_{q^2} - \hat{b}}) \neq 0,\ n \geq 0.$$

The MOPS $\{P_n\}_{n\geq 0}$ satisfies the following structure relation ([10])

(6.9) $\Phi^P(x)(H_{q^2}P_{n+1})(x) = a_{n+1}P_{n+2}(x) + b_{n+1}P_{n+1}(x) + c_{n+1}P_n(x), \; n \geq 0,$

where, for $n \geq 1$,

$$a_n = [n]_{q^2}, \tag{6.10}$$

$$b_n = -\frac{[n]_{q^2}([n-1]_{q^2} - \hat{b})(-\bar{b}q^{2n-2}[2]_{q^2} - \hat{b} + [n]_{q^2}(1 - q^{2n-2}))}{([2n]_{q^2} - \hat{b})([2n-2]_{q^2} - \hat{b})}, \tag{6.11}$$

(6.12)
$$c_n = \frac{-[n]_{q^2}q^{2n-2}([n-1]_{q^2} + \bar{b})(-q^{2n-2}\bar{b} + [n-1]_{q^2}\hat{a} - \hat{b})([n-2]_{q^2}\hat{a} - \hat{b})([n-1]_{q^2} - \hat{b})}{([2n-1]_{q^2} - \hat{b})([2n-2]_{q^2} - \hat{b})^2([2n-3]_{q^2} - \hat{b})}.$$

Using the TTRR fulfilled by $\{P_n\}_{n\geq 0}$, (6.9) can be written as follows
(6.13)
$$\Phi^P(x)(H_{q^2}P_{n+1})(x) = \frac{C^P_{n+1}(x) - C^P_0(x)}{2}P_{n+1}(x) - \gamma^P_{n+1}D^P_{n+1}(x)P_n(x), \; n \geq 0,$$

where

$$C^P_0(x) = -(q^4\hat{b} + q^2 + 1)x - (q^2\bar{b} - 1), \tag{6.14}$$

$$C^P_{n+1}(x) = C^P_0(x) + 2b_{n+1} + 2a_{n+1}(x - \beta^P_{n+1}), \tag{6.15}$$

$$D^P_0(x) = -(1 + q^2\hat{b}), \tag{6.16}$$

$$D^P_{n+1}(x) = a_{n+1} - \frac{c_{n+1}}{\gamma^P_{n+1}}, \; n \geq 0. \tag{6.17}$$

Next, we will find the coefficients $\gamma_n, \; n \geq 1$, of the TTRR. First of all, because it is very difficult to explicitly determine these coefficients by solving directly the system (2.14), we will give another expression of them. Changing the indices $n \to 2n$ in (1.2) and evaluating such an expression at $x = 1$, we obtain

$$B_{2n+2}(1) = -\gamma_{2n+1}B_{2n}(1), \; n \geq 0.$$

Taking into account the first expression of (1.3) we have

$$P_{n+1}(1) = -\gamma_{2n+1}P_n(1), \; n \geq 0.$$

Hence

$$\gamma_{2n+1} = -\frac{P_{n+1}(1)}{P_n(1)}, \; n \geq 0. \tag{6.18}$$

On the other hand, evaluating (6.13) at $x = 1$ we get

$$0 = \frac{C_{n+1}^P(1) - C_0^P(1)}{2} P_{n+1}(1) - \gamma_{n+1}^P D_{n+1}^P(1) P_n(1),\ n \geq 0.$$

Therefore,

$$\frac{P_{n+1}(1)}{P_n(1)} = \frac{2\gamma_{n+1}^P D_{n+1}^P(1)}{C_{n+1}^P(1) - C_0^P(1)},\ n \geq 0. \tag{6.19}$$

From (6.18) and (6.19)

$$\gamma_{2n+1} = -\frac{2\gamma_{n+1}^P D_{n+1}^P(1)}{C_{n+1}^P(1) - C_0^P(1)},\ n \geq 0. \tag{6.20}$$

Using (6.15) and (6.17), (6.20) becomes

$$\gamma_{2n+1} = \frac{c_{n+1} - a_{n+1}\gamma_{n+1}^P}{b_{n+1} + a_{n+1}(1 - \beta_{n+1}^P)},\ n \geq 0. \tag{6.21}$$

The third relation in (2.14) gives

$$\gamma_{2n+2} = \gamma_{n+1}^P \frac{b_{n+1} + a_{n+1}(1 - \beta_{n+1}^P)}{c_{n+1} - a_{n+1}\gamma_{n+1}^P},\ n \geq 0. \tag{6.22}$$

On the other hand, using the relations (see [10])

$c_{n+1} = -\frac{\widehat{\lambda}_{n+1}}{[n+1]_{q^2}} \gamma_{n+1}^P$, $b_{n+1} = [n]_{q^2} s_{n+1} - [n+1]_{q^2} s_n - [n]_{q^2}$,

$\beta_{n+1}^P = s_n - s_{n+1}$, $a_{n+1} = [n+1]_{q^2}$,

where

$s_n = \frac{\bar{\lambda}_n}{\widehat{\lambda}_n - \widehat{\lambda}_{n-1}}$, $\bar{\lambda}_n = -q^{2-2n}[n]_{q^2}([n-1]_{q^2} + \bar{b})$, $\widehat{\lambda}_n = q^{2-2n}[n]_{q^2}([n-1]_{q^2} - \hat{b})$,
(6.21) and (6.22) become

$$\gamma_{2n+1} = \frac{q^{4-4n}([2n-1]_{q^2} - \hat{b})\gamma_{n+1}^P}{s_n}, \tag{6.23}$$

$$\gamma_{2n+2} = \frac{q^{4n-4} s_n}{[2n-1]_{q^2} - \hat{b}}. \tag{6.24}$$

Taking into account (6.6) and replacing s_n by its expression in (6.23) and (6.24) we get, respectively,

$$\gamma_{2n+1} = -\frac{([n-1]_{q^2} - \hat{b})(-q^{2n}\bar{b} + [n]_{q^2} - \hat{b})}{([2n]_{q^2} - \hat{b})([2n-1]_{q^2} - \hat{b})},\ n \geq 1, \tag{6.25}$$

(6.26)
$$\gamma_{2n+2}=-\frac{q^{4n}[n+1]_{q^2}([n]_{q^2}+\bar{b})}{([2n+1]_{q^2}-\hat{b})([2n]_{q^2}\hat{a}-\hat{b})},\ n\geq 0.$$

Next, we will find the polynomial coefficients of the structure relation and the corresponding second order linear q-difference equation.
Replacing (6.1), (6.6), (6.14) – (6.17), in (5.8) and (5.9) , after the division by $x(x+1)$,

(6.27)
$$\begin{aligned} C_{2n}(x) &= \Big(-(1+q)q^3\hat{b}-1-q-q^2+2[2n]_q\Big)x^2+q^{-1}(1-2q^{2n})x \\ &\quad -(1+q^{-1})(q^2\bar{b}-1+2[n]_{q^2}),\ n\geq 2, \end{aligned}$$

(6.28)
$$D_{2n}(x)=(q+1)q^{1-2n}([2n-1]_{q^2}-\hat{b})(qx+1),\ n\geq 2.$$

From (5.4) and (6.5), where $q^2\to q$, we obtain

(6.29)
$$C_0(x)=\Big(-(1+q)q^3\hat{b}-1-q-q^2\Big)x^2-q^{-1}x-(1+q^{-1})(q^2\bar{b}-1),$$

(6.30)
$$D_0(x)=(q+1)q([-1]_{q^2}-\hat{b})(qx+1),$$

where, by convention $[-1]_{q^2}=q^{-2}$.
On the other hand, from (5.2) and (5.3), where $q^2\to q$, we can easily see that (6.27) and (6.28) remain valid for $n=1$.
Finally, from (5.2) and (5.3), where $q^2\to q$, we can easily prove by induction that

(6.31)
$$\begin{aligned} C_{2n+1}(x) &= \Big(-(1+q)q^3\hat{b}-1-q-q^2+2[2n+1]_q\Big)x^2+q^{-1}x \\ &\quad +(1+q)(q\bar{b}-q^{-1}-2q^{1-2n}([n-1]_{q^2}-\hat{b})),\ n\geq 0, \end{aligned}$$

(6.32)
$$D_{2n+1}(x)=(q+1)q^{1-2n}([2n]_{q^2}-\hat{b})(x-1),\ n\geq 0.$$

As a conclusion, (6.27), (6.28), (6.31), and (6.32) can be written as follows

(6.33)
$$\begin{aligned} C_n(x) &= \Big(-(1+q)q^3\hat{b}-1-q-q^2+2[n]_q\Big)x^2+q^{-1}(1-(1+(-1)^n)q^n)x \\ &\quad +(1+q)(q\bar{b}-q^{-1})-(1-(-1)^n)q^{2-n}([n-3]_q-(q+1)\hat{b}) \\ &\quad -(1+(-1)^n)q^{-1}[n]_q,\ n\geq 0, \end{aligned}$$

(6.34)
$$D_n(x)=(q+1)q^{\frac{3-(-1)^n}{2}-n}([n-1]_{q^2}-\hat{b})\Big((\frac{1+q}{2}-(-1)^n\frac{1-q}{2})x+(-1)^n\Big),\ n\geq 0.$$

The basic information of this H_q-semiclassical MOPS of class one is summarized in Table 1.

Table 1

Basic elements of the sequence $\{B_n\}_{n\geq 0}$
$\Phi(x) = x(x-1)(x+q^{-1})$,
$\Psi(x) = \Big(-q^{-4}[4]_q + q^{-9}(q^{-1}+1)(q^2\hat{b}+q^{-2}+1)\Big)x^2 - q^{-1}x + q^{-2} + q^{-9}(q^{-1}+1)(q^2\bar{b}-1)$,
$\gamma_{2n+1} = -\frac{([n-1]_{q^2}-\hat{b})(-q^{2n}\bar{b}+[n]_{q^2}-\hat{b})}{([2n]_{q^2}-\hat{b})([2n-1]_{q^2}-\hat{b})},\ n\geq 0$,
$\gamma_{2n+2} = -\frac{q^{4n}[n+1]_{q^2}([n]_{q^2}+\bar{b})}{([2n+1]_{q^2}-\hat{b})([2n]_{q^2}\hat{a}-\hat{b})},\ n\geq 0$,
$C_n(x) = \Big(-(1+q)q^3\hat{b} - 1 - q - q^2 + 2[n]_q\Big)x^2 + q^{-1}(1-(1+(-1)^n)q^n)x + (1+q)(q\bar{b}-q^{-1}) - (1-(-1)^n)q^{2-n}([n-3]_q-(q+1)\hat{b}) - (1+(-1)^n)q^{-1}[n]_q,\ n\geq 0$,
$D_n(x) = (q+1)q^{\frac{3-(-1)^n}{2}-n}([n-1]_{q^2}-\hat{b})\Big((\frac{1+q}{2}-(-1)^n\frac{1-q}{2})x+(-1)^n\Big)\ n\geq 0.$

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