Classical orthogonal polynomials with respect to a lowering operator generalizing the Laguerre Operator

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Abstract

We introduce the lowering operator $\widehat{\Omega}_{1,c} = \widehat{\Omega}_1 - cD^2$, where c is an arbitrary complex number and $\widehat{\Omega}_1$ is *the Generalized Laguerre Operator* introduced by G. Dattoli and P. E. Ricci. Then, we establish an intertwining relation between the operators $\widehat{\Omega}_{1,c}$ and the standard derivative D. On the other hand, an analogue of the Hahn characterization of D- classical orthogonal polynomials is given for the operator $\widehat{\Omega}_{1,c}$. As a consequence, some integral relations between the corresponding polynomials are deduced. Finally, some expansions in series of Laguerre polynomials are studied.

Keywords: Laguerre polynomials, Classical polynomials, Appell's property, Lowering, Transfer and Raising operators, Integral formulas.

1. Introduction

Let $\{L_n^{(1)}\}_{n\geq 0}$ be the monic (normalized) Laguerre polynomial sequence with parameter $\alpha = 1$. It is a sequence of polynomials orthogonal with respect to the weight function xe^{-x} on $(0, +\infty)$, i.e., $\int_0^{+\infty} xe^{-x} L_n^{(1)}(x) L_m^{(1)}(x)\, dx = (m+1)\delta_{n,m},\ n, m \geq 0$, where $\delta_{n,m}$ is the Kronecker delta. These polynomials

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can be represented as an hypergeometric function [11]

$$L_n^{(1)}(x) = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} \frac{(n+1)!}{(\nu+1)!}\, x^{\nu}. \tag{1}$$

It is well-known that $\{L_n^{(1)}\}_{n\geq 0}$ can be characterized taking into account its orthogonality as well as the following differential properties (see [14])
-*The Second-order linear differential equation*

$$x{L_{n+1}^{(1)}}''(x) - (x-2){L_{n+1}^{(1)}}'(x) + (n+1)L_{n+1}^{(1)}(x) = 0,\ n \geq 0. \tag{2}$$

-*The first structure relation*

$$x{L_{n+1}^{(1)}}'(x) = (n+1)L_{n+1}^{(1)}(x) + (n+1)(n+2)L_n^{(1)}(x),\ n \geq 0. \tag{3}$$

Substituting (3) in (2), we get

$$L_n^{(1)}(x) = \frac{\widehat{\Omega}_1 L_{n+1}^{(1)}(x)}{(n+1)(n+2)},\ n \geq 0, \tag{4}$$

where $\widehat{\Omega}_m = DxD + mD$. This means that the above family of standard orthogonal polynomials is an Appell sequence with respect to the operator $\widehat{\Omega}_1$. Polynomial Appell sequences with respect to the standard derivative operator D were introduced in [3]. For instance, in [7] the authors have considered this family of operators with parameter $m \neq -n$, $n \geq 1$. In [11] and [12], polynomial Appell sequence with respect to the second order linear differential operator $\mathcal{F}_\epsilon = 2xD^2 + (\epsilon+2)D$ have studied. They deduce that the orthogonal polynomial Appell sequences with respect to such an operator are the Laguerre sequences with parameter $\alpha = \frac{\epsilon}{2}$. Notice that $\widehat{\Omega}_1 = 2\mathcal{F}_2$.
Moreover, $\{L_n^{(1)}\}_{n\geq 0}$ is an $\widehat{\Omega}_1-$classical polynomial sequence, since it satisfies the *Hahn property* for the lowering operator $\widehat{\Omega}_1$, i.e., it is an orthogonal polynomial sequence whose sequence of $\widehat{\Omega}_1-$derivatives is also orthogonal, [8]. This Hahn's property can de considered for other differential operators. In Section 4.2 of the Doctoral Dissertation by A. F. Loureiro, for the second order (Laguerre) differential $\mathcal{F}_\epsilon$ the characterization and the description of the $\mathcal{F}_\epsilon$-classical orthogonal polynomials is given (see [11], Theorem 4.2.4, and subsection 4.2.3)). More precisely, Laguerre sequences with parameter $\alpha = \frac{\epsilon}{2}$ and some particular Jacobi sequences with parameters $(\frac{\epsilon}{2}, -\frac{1}{2} - \frac{\epsilon}{4} + b)$

are the only $\mathcal{F}_\epsilon$-classical orthogonal polynomial sequences. This result points out that these $\mathcal{F}_\epsilon$-classical orthogonal polynomial constitute a more restricted family than those D- classical families in the Hahn's sense.

For a given $c \in \mathbb{C}$, let consider $\widehat{\Omega}_{1,c} : \mathbb{P} \to \mathbb{P}$, the linear operator defined by in the linear space $\mathbb{P}$ of polynomials with complex coefficients

$$\widehat{\Omega}_{1,c} := \widehat{\Omega}_1 - cD^2, \quad (\widehat{\Omega}_{1,0} = \widehat{\Omega}_1).$$

Notice that our new operator is different of $\mathcal{F}_\epsilon$ but they share the fact that they are lowering operators, i.e., the image of a polynomial of degree n, $n \geq 1$ is a polynomial of degree $n-1$. For the canonical basis $\{x^n\}_{n\geq 0}$ we have

$\widehat{\Omega}_{1,c}(x^n) = n(n+1)x^{n-1} - cn(n-1)x^{n-2} \quad ; \quad \mathcal{F}_\epsilon(x^n) = n(2n+\epsilon)x^{n-1}.$

In this contribution we will analyze the $\widehat{\Omega}_{1,c}-$ classical polynomials and then we will provide a full description of them. We will emphasize two basic facts. The first is that the $\widehat{\Omega}_{1,c}$-classical character is preserved by shifting. The second is that the $\widehat{\Omega}_{1,c}$- classical polynomial sequences constitute a subfamily of the well-known $D-$classical ones (Hermite, Laguerre, Bessel, and Jacobi). More precisely, the $\widehat{\Omega}_{1,c}$-classical polynomial sequences are, after suitable shifting, the Laguerre polynomial sequence $\{L_n^{(1)}\}_{n\geq 0}$, when $c = 0$, or the Jacobi polynomial sequence $\{P_n^{(\alpha-2,1)}\}_{n\geq 0}$ with parameter $\alpha \neq -n+2$, $n \geq 1$, when $c = 1$. Notice that they are related to particular cases of the operator $\mathcal{F}_\epsilon$ for different values of ϵ. Our proof approach is completely different of the proof provided in [11] and, even more, shorter than it.

The paper is organized as follows. In Section 2, we introduce the basic background to be used throughout the paper. Section 3 gives the description of the $\widehat{\Omega}_{1,c}-$classical sequences. As a consequence, we establish a new integral relation between the monic Bessel and Jacobi polynomials. Finally, in Section 4 some expansions in series of Laguerre polynomials are presented.

2. Preliminaries and notations

2.1. Basic definitions

Let $\mathbb{P}$ be the linear space of polynomials with complex coefficients and let $\mathbb{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u. Let us define the following operations on $\mathbb{P}'$ [13]. For any $a \in \mathbb{C}\backslash\{0\}$, $b, c \in \mathbb{C}$, $f,\ p \in \mathbb{P}$, and $u \in \mathbb{P}'$,

$$\begin{aligned}
&\langle fu, p\rangle = \langle u, fp\rangle, \quad \langle Du, f\rangle = -\langle u, f'\rangle, \\
&\langle h_a u, p\rangle = \langle u, h_a p\rangle = \langle u, p(ax)\rangle, \quad \langle \tau_b u, p\rangle = \langle u, \tau_{-b} p\rangle = \langle u, p(x+b)\rangle, \\
&\langle (x-c)^{-1}u, p\rangle = \langle u, \theta_c(p)\rangle = \langle u, \frac{p(x)-p(c)}{x-c}\rangle, \quad \langle \delta_c, p\rangle = p(c),\ p \in \mathbb{P}.
\end{aligned}$$

Here δ_c denotes the Dirac linear functional at $c \in \mathbb{C}$.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathbb{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$. Recall that any $u \in \mathbb{P}'$ can be represented as $u = \sum_{n=0}^{+\infty} \langle u, P_n\rangle u_n$. So, if $\{u_n^{[1]}\}_{n\geq 0}$ denotes the dual sequence of the MPS $\{P_n^{[1]}\}_{n\geq 0}$ where $P_n^{[1]} := (n+1)^{-1}P'_{n+1}$, $n \geq 0$, then $Du_n^{[1]} = -(n+1)u_{n+1}$, $n \geq 0$. Likewise, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of the shifted MPS $\{\widetilde{P}_n\}_{n\geq 0}$, where $\widetilde{P}_n(x) := a^{-n}P_n(ax+b)$ with $(a, b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, is given by $\tilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n,\ n \geq 0$.

An MPS $\{P_n\}_{n\geq 0}$ is said to be orthogonal (MOPS) with respect to $u \in \mathbb{P}'$ if $\langle u, P_n P_m\rangle = 0,\ n \neq m,$ and $\langle u, P_n^2\rangle \neq 0,\ n \geq 0$. In this case, u is said to be quasi-definite (regular) [6]. Notice that $u = (u)_0 u_0$, with $(u)_0 \neq 0$.

Proposition 2.1. *[13]. An MPS $\{P_n\}_{n\geq 0}$ with dual sequence $\{u_n\}_{n\geq 0}$, is orthogonal if and only if one of the following statements hold.*

(i) $u_n = \langle u_0, P_n^2\rangle^{-1} P_n u_0,\ n \geq 0.$

(ii) $\{P_n\}_{n\geq 0}$ *satisfies a Three-Term Recurrence Relation*

$$\text{(TTRR)} \quad \begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0, \end{cases} \tag{5}$$

where $\beta_n = \langle u_0, xP_n^2\rangle / \langle u_0, P_n^2\rangle \in \mathbb{C}$ *and* $\gamma_{n+1} = \langle u_0, P_{n+1}^2\rangle / \langle u_0, P_n^2\rangle \in \mathbb{C}\backslash\{0\}$.

When $\{P_n\}_{n\geq 0}$ is an MOPS with respect to u_0, then $\{\widetilde{P}_n\}_{n\geq 0}$ is also orthogonal with respect to $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$ and satisfies [6]

$$\text{(TTRR)} \quad \begin{cases} \widetilde{P}_0(x) = 1, \quad \widetilde{P}_1(x) = x - \tilde{\beta}_0, \\ \widetilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\widetilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\widetilde{P}_n(x),\ n \geq 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}(\beta_n - b)$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

A linear functional u is said to be $D-$classical when it is quasi-definite and there exist two polynomials Φ and Ψ, Φ monic, $\deg \Phi = t \leq 2$, and

$\deg\Psi = 1$, such that u satisfies a Pearson equation, (PE): $(\Phi u)' + \Psi u = 0$, (see [5, 13, 14]). In such a case, the corresponding MOPS $\{P_n\}_{n\geq 0}$ is said to be $D-$classical. Any shift leaves invariant the $D-$classical character. Indeed, the shifted linear functional $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ fulfils $(\widetilde{\Phi}\tilde{u})' + \widetilde{\Psi}\tilde{u} = 0$, where $\widetilde{\Phi}(x) = a^{-t}\Phi(ax+b)$ and $\widetilde{\Psi}(x) = a^{1-t}\Psi(ax+b)$. Any $D-$classical polynomial sequence $\{P_n\}_{n\geq 0}$ can be characterized taking into account its orthogonality as well as a First Structure Relation (FSR), or a Second Structure Relation (SSR), or Second-Order Differential Equation (SODE) as follows.

$$\text{(FSR)} \qquad \Phi(x)P'_{n+1}(x) = r(x;n)P_{n+1}(x) + s_n P_n(x),\ n \geq 0, \tag{6}$$

$$\text{(SSR)} \qquad P_n(x) = P^{[1]}_n(x) + a_n P^{[1]}_{n-1}(x) + b_n P^{[1]}_{n-2}(x),\ n \geq 0, \tag{7}$$

$$\text{(SODE)} \quad \Phi(x)P''_{n+1}(x) - \Psi(x)P'_{n+1}(x) = \omega_n P_{n+1}(x),\ n \geq 0, \tag{8}$$

where for the fourth canonical situations, we have (see [1, 5, 13, 14]).

Table 1

(C_1) Hermite: $P_n(x) = H_n(x),\ n \geq 0.$

$\beta_n = 0,\ n \geq 0,\ \gamma_{n+1} = \frac{n+1}{2},\ n \geq 0, \quad \Phi(x) = 1, \quad \Psi(x) = 2x.$

$r(x;n) = 0,\ s_n = n+1,\ n \geq 0, \quad a_n = b_n = 0, n \geq 0, \quad \omega_n = -2(n+1), n \geq 0.$

(C_2) Laguerre: $P_n(x) = L^{(\alpha)}_n(x),\ n \geq 0, \quad (\alpha \neq -n,\ n \geq 1).$

$\beta_n = 2n+\alpha+1,\ n \geq 0, \gamma_{n+1} = (n+1)(n+\alpha+1),\ n \geq 0, \Phi(x) = x, \Psi(x) = x-\alpha-1,$

$r(x;n) = n+1,\ s_n = \gamma_{n+1},\ n \geq 0, \quad a_n = n, b_n = 0, n \geq 0, \quad \omega_n = -(n+1), n \geq 0.$

(C_3) Bessel $P_n(x) = B^{(\alpha)}_n(x),\ n \geq 0, \quad (\alpha \neq -\frac{n}{2},\ n \geq 0).$

$\beta_0 = -\frac{1}{\alpha},\ \beta_n = \frac{1-\alpha}{(n+\alpha-1)(n+\alpha)},\ n \geq 0,\ \gamma_n = -\frac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)},\ n \geq 1.$

$\Phi(x) = x^2, \quad \Psi(x) = -2(\alpha x+1).$

$r(x;n) = (n+1)(x - \frac{1}{n+\alpha}),\ s_n = -(2n+2\alpha+1)\gamma_{n+1},\ n \geq 0, \quad a_n = \frac{n}{(n+\alpha-1)(n+\alpha)}, n \geq 0,$

$b_n = \frac{(n-1)n}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)},\ n \geq 2,\ b_0 = b_1 = 0, \quad \omega_n = (n+1)(n+2\alpha), n \geq 0.$

(C_4) Jacobi $P_n(x) = P^{(\alpha,\beta)}_n(x),\ n \geq 0, \quad (\alpha,\ \beta \neq -n,\ \alpha+\beta \neq -n-1,\ n \geq 1).$

$\beta_0 = \frac{\alpha-\beta}{\alpha+\beta+2},\ \ \beta_n = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},\ \gamma_n = \frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\ n \geq 1.$

$\Phi(x) = x^2-1, \quad \Psi(x) = -(\alpha+\beta+2)x + \alpha - \beta.$

$r(x;n) = (n+1)(x - \frac{\alpha-\beta}{2n+\alpha+\beta+2}),\ s_n = -(2n+\alpha+\beta+3)\gamma_{n+1},\ n \geq 0.$

$a_n = -\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, n \geq 1, a_0 = 0, b_n = -\frac{4(n-1)n(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, n \geq 2,$

$b_0 = b_1 = 0, \quad \omega_n = (n+1)(n+\alpha+\beta+2), n \geq 0.$

Further, we need the following properties [6, 9].

$$B^{(\alpha)}_n(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{2^{n-\nu}\Gamma(n+2\alpha+\nu-1)}{\Gamma(2n+2\alpha-1)} x^\nu,\ n \geq 0. \tag{9}$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{2^{n-\nu}\Gamma(n+\alpha+\beta+\nu+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(\nu+\beta+1)}(x-1)^{\nu}, \quad (10)$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \quad n \geq 0, \quad \text{(symmetry)}. \quad (11)$$

2.2. Some properties of the operator $\widehat{\Omega}_{1,c}$

In the sequel, we will denote by "$\circ$" the composition law between linear operators on the linear space of polynomials. The following formulas are a straightforward consequence of the definition of the operators.

Lemma 2.1. *For any c in $\mathbb{C}$, we have*

(i) $\widehat{\Omega}_{1,c}(x-c)^{n+1} = (n+1)(n+2)(x-c)^n$, $n \geq 0$, *and* $\widehat{\Omega}_{1,c}(1) = 0$.

(ii) $\widehat{\Omega}_{1,c} \circ \Pi_{a,b} = a\Pi_{a,b} \circ \widehat{\Omega}_{1,ac+b}$ *and* $\Pi_{a,b} \circ \widehat{\Omega}_{1,c} = a^{-1}\widehat{\Omega}_{1,\frac{c-b}{a}} \circ \Pi_{a,b}$, *where* $\Pi_{a,b}f(x) = f(ax+b)$, *for every* $f \in \mathbb{P}$ *and where* $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$.

(iii) $\widehat{\Omega}_{1,c}(pq) = q\widehat{\Omega}_{1,c}(p) + p\widehat{\Omega}_{1,c}(q) + 2(x-c)p'q'$, *for every p and q in* $\mathbb{P}$.

Through an appropriate linear isomorphism called intertwining operator, we can establish a relationship between the operators $\widehat{\Omega}_{1,c}$ and the standard derivative D. We first recall the following formulas [9]:

$$n! = \int_0^{+\infty} t^n e^{-t} dt, \quad n = 0, 1, 2, ..., \quad (12)$$

$$\frac{1}{n!} = \frac{-1}{2\pi i} \int_{\mathscr{C}} (-z)^{-n-1} e^{-z} dz, \quad n = 0, 1, 2, ..., \quad (13)$$

where $\mathscr{C}$ is the following contour in the complex plane:

$\mathscr{C}$

Proposition 2.2. *For any $c \in \mathbb{C}$, we have $\mathfrak{S}_c \circ \widehat{\Omega}_{1,c} = D \circ \mathfrak{S}_c$, where the operator $\mathfrak{S}_c : \mathbb{P} \to \mathbb{P}$ and its reciprocal operator $\mathfrak{S}_c^{-1}$ are linear isomorphisms,*

$$\mathfrak{S}_c(p)(x) = \int_0^{+\infty} te^{-t} p\big(t(x-c)+c\big) dt, \quad p \in \mathbb{P},$$

$$\mathfrak{S}_c^{-1}(p)(x) = \int_{\mathscr{C}} z^{-2} e^{-z} p\big(-z^{-1}(x-c)+c\big) dz, \quad p \in \mathbb{P},$$

and $\mathscr{C}$ is the same contour as above.

Proof. The proof is a consequence of Lemma 2.1 (i), (12) and (13). ■

Notice that the operator $\mathfrak{S}_c$ can be characterized taking into account its linearity as well as the fact

$$\mathfrak{S}_c\big((x-c)^n\big) = (n+1)!(x-c)^n, \quad n \geq 0. \tag{14}$$

By transposition of the operator $\widehat{\Omega}_{1,c}$, we get ${}^t\widehat{\Omega}_{1,c} = \widehat{\Omega}_{-1} - cD^2$. So, $\widehat{\Omega}_{-1,c} : \mathbb{P}' \to \mathbb{P}'$ is the operator defined by

$$\widehat{\Omega}_{-1,c} = \widehat{\Omega}_{-1} - cD^2 = (x-c)D^2. \tag{15}$$

For any MPS $\{P_n\}_{n\geq 0}$ we define

$$Q_n(x;c) := \frac{\widehat{\Omega}_{1,c}\ P_{n+1}(x)}{(n+1)(n+2)},\ n \geq 0. \tag{16}$$

Clearly, $\{Q_n(\ ;c)\}_{n\geq 0}$ is an MPS, $\deg Q_n(\ ;c) = n$. If $\{v_n(c)\}_{n\geq 0}$ denotes the dual sequence of $\{Q_n(\ ;c)\}_{n\geq 0}$, then we have

$$\widehat{\Omega}_{-1,c}(v_n(c)) = (n+1)(n+2)u_{n+1},\ n \geq 0. \tag{17}$$

3. The $\widehat{\Omega}_{1,c}$−classical orthogonal polynomials

Definition 3.1. *An MOPS $\{P_n\}_{n\geq 0}$ (orthogonal with respect to u_0) is said to be $\widehat{\Omega}_{1,c}$−classical, when it satisfies the Hahn's property with respect to the operator $\widehat{\Omega}_{1,c}$, i.e., the MPS $\{Q_n(\ ;c)\}_{n\geq 0}$ given by (16) is also orthogonal. In this case u_0 is also said to be an $\widehat{\Omega}_{1,c}$−classical linear functional.*

Any shift leaves invariant the $\widehat{\Omega}_{1,c}$−classical character.

Lemma 3.1. *When $\{P_n\}_{n\geq 0}$ is $\widehat{\Omega}_{1,c}$−classical, then for any $(a,b) \in \mathbb{C}^* \times \mathbb{C}$ the shifted polynomial sequence $\{\widetilde{P}_n\}_{n\geq 0}$ given by $\widetilde{P}_n(x) = a^{-n}P_n(ax+b)$, $n \geq 0$, is $\widehat{\Omega}_{1,\widetilde{c}}$−classical, where $\widetilde{c} = a^{-1}(c-b)$.*

Proof. Assume that $\{P_n\}_{n\geq 0}$ is $\widehat{\Omega}_{1,c}$−classical. By Definition 3.1, the MPS $\{Q_n(\ ;c)\}_{n\geq 0}$ given by (16) is orthogonal. Notice that, we can write

$$(n+1)(n+2)Q_n(x;c) = (x-c)P''_{n+1}(x) + 2P'_{n+1}(x),\ n \geq 0, \tag{18}$$

For any fixed $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, let $\{\widetilde{P}_n\}_{n\geq 0}$ and $\{\widetilde{Q}_n\}_{n\geq 0}$ be the shifted MOPS given by $\widetilde{P}_n(x) = a^{-n}P_n(ax+b)$ and $\widetilde{Q}_n(x) = a^{-n}Q_n(ax+b;c)$.

Replacing x by $ax+b$ in (18), we get $(n+1)(n+2)\widetilde{Q}_n(x) = (x-\widetilde{c})\widetilde{P}''_{n+1}(x) + 2\widetilde{P}'_{n+1}(x)$, where $\widetilde{c} = a^{-1}(c-b)$, i.e., $(n+1)(n+2)\widetilde{Q}_n(x) = \widehat{\Omega}_{1,\widetilde{c}}\ \widetilde{P}_{n+1}(x)$, $n \geq 0$. Hence, $\{\widetilde{P}_n\}_{n\geq 0}$ is $\widehat{\Omega}_{1,\widetilde{c}}$ $-$classical. ∎

In the sequel, we write $Q_n(x) := Q_n(x;c)$, $n \geq 0$, if there is no ambiguity. Our next goal is to describe all the $\widehat{\Omega}_{1,c}-$classical polynomial sequences. Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ \gamma_{n+1} \neq 0,\ n \geq 0, \end{cases} \tag{19}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \xi_0, \\ Q_{n+2}(x) = (x-\xi_{n+1})Q_{n+1}(x) - \lambda_{n+1}Q_n(x),\ \lambda_{n+1} \neq 0,\ n \geq 0. \end{cases} \tag{20}$$

The dual sequences of $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ will be denoted by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, respectively. By Proposition 2.1 (i), we get

$$u_n = \frac{P_n}{\langle u_0, P_n^2\rangle}u_0,\ n \geq 0 \quad ; \quad v_n = \frac{Q_n}{\langle v_0, Q_n^2\rangle}v_0,\ n \geq 0. \tag{21}$$

Let us start with some auxiliary results.

Lemma 3.2. *The MOPS $\{Q_n\}_{n\geq 0}$ is classical and we have*

(i) (SSR) $\quad Q_n(x) = Q_n^{[1]}(x) + c_nQ_{n-1}^{[1]}(x) + d_nQ_{n-2}^{[1]}(x),\ n \geq 0,$ *where*

$$c_n = \frac{n}{2}\big(\beta_{n+1} - \xi_n\big),\ n \geq 0,\ d_n = \frac{n-1}{2}\big(\frac{n}{n+2}\gamma_{n+1} - \lambda_n\big),\ n \geq 1,\ d_0 = 0.$$

(ii) (PE) $(\Phi v_0)' + \Psi v_0 = 0$, *where* $\kappa\Phi = d_2(\lambda_1\lambda_2)^{-1}Q_2 + c_1\lambda_1^{-1}Q_1 + 1$, $\Psi = (\kappa\lambda_1)^{-1}Q_1$ *and* κ *is a normalization factor.*

Proof. Let us introduce the MPS $\{Z_n\}_{n\geq 0}$ given by

$$(n+1)Z_n(x) := (x-c)P_n'(x) + P_n(x),\ n \geq 0. \tag{22}$$

Taking derivatives in both hand sides of (22), where n is replaced by $n+1$, and using (16), we get

$$Z_n^{[1]}(x) = Q_n(x),\ n \geq 0. \tag{23}$$

Notice that $Z_n(x)$ is a monic primitive of $Q_n(x)$.
From (19) and (22), we get $(n+3)Z_{n+2} = (n+2)(x-\beta_{n+1})Z_{n+1} - (n+$

$1)\gamma_{n+1}Z_n + (x-c)P_{n+1},\ n \geq 0$. Differentiating in both hand sides of the previous identity and inserting (23), $(n+3)(n+2)Q_{n+1} = (n+1)(n+2)(x-\beta_{n+1})Q_n - n(n+1)\gamma_{n+1}Q_{n-1} + 2(n+2)Z_{n+1}$. Then, by (20), it follows that

$$Z_n(x) = Q_n(x) + a_nQ_{n-1}(x) + b_nQ_{n-2}(x),\ n \geq 0, \tag{24}$$

where $a_n = \frac{n}{2}(\beta_n - \xi_{n-1}),\ n \geq 1,\ a_0 = 0,\ b_n = \frac{n}{2}(\frac{n-1}{n+1}\gamma_n - \lambda_{n-1}),\ n \geq 2$, $b_0 = b_1 = 0$.
By differentiating both hand sides of (24) and using (23), (i) holds.
Let $\{v_n^{[1]}\}_{n\geq 0}$ be the dual sequence of $\{Q_n^{[1]}\}_{n\geq 0}$. By (i), $\langle v_0^{[1]}, Q_n\rangle = 0$, $n \geq 3$, $\langle v_0^{[1]}, Q_2\rangle = d_2$, $\langle v_0^{[1]}, Q_1\rangle = c_1$, and $\langle v_0^{[1]}, Q_0\rangle = 1$. So, $v_0^{[1]} = d_2v_2 + c_1v_1 + v_0$, and by (21), we get $v_0^{[1]} = \kappa\Phi v_0$, where $\kappa\Phi = d_2\lambda_1^{-1}\lambda_2^{-1}Q_2 + c_1\lambda_1^{-1}Q_1 + 1$ and κ is a normalization factor. Since $(v_0^{[1]})' = -v_1 = -\lambda_1^{-1}Q_1v_0$, then $(\Phi v_0)' + \Psi v_0 = 0$, where $\Psi = (\kappa\lambda_1)^{-1}Q_1$. Hence, (ii) holds. ■

Lemma 3.3. *There exist two non-zero polynomials F and G, with* $\deg F \leq 3$ *and* $\deg G \leq 2$ *such that*

(i) $(x-c)v_0 = Fu_0$.
(ii) $FQ_n'' + GQ_n' + \rho_0P_1Q_n = \rho_nP_{n+1},\ n \geq 0,$ *where*

$$\begin{aligned} F &= (1/2)\big(\rho_2P_3 - GQ_2' - \rho_0P_1Q_2\big), \quad G = \rho_1P_2 - \rho_0P_1Q_1, \text{ and} \\ \rho_n &= (n+1)(n+2)\frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+1}^2\rangle} = \frac{F^{(3)}(0)}{6}n(n-1) + \frac{G^{(2)}(0)}{2}n + \rho_0. \end{aligned}$$

(iii) *The following relations hold*
 (a) $AF = \rho_0P_1\Phi^2$,
 (b) $-2\big(\Phi' + \Psi\big)F = \Phi G$,
 (c) $\gamma_1\Phi^2 = \Big[\big((\rho_0^{-1} - \rho_1^{-1})x - \beta_1\rho_0^{-1} + \xi_0\rho_1^{-1}\big)A - \rho_1^{-1}B\Big]F$, *where*
 $A = (2\Phi' + \Psi)(\Phi' + \Psi) - (\Phi'' + \Psi')\Phi$, *and* $B = -2\Phi(\Phi' + \Psi)$.

Proof. From (15), (17) and (21), we obtain

$$(x-c)Q_nv_0'' + 2(x-c)Q_n'v_0' + (x-c)Q_n''v_0 = \rho_nP_{n+1}u_0,\ n \geq 0, \tag{25}$$

where $\rho_n = (n+1)(n+2)\langle v_0, Q_n^2\rangle\langle u_0, P_{n+1}^2\rangle^{-1},\ n \geq 0$.
From (25) with $n = 0$, we obtain

$$(x-c)v_0'' = \rho_0P_1u_0. \tag{26}$$

Using (25) and (26), it follows that

$$(x-c)Q_n''v_0 + 2(x-c)Q_n'v_0' = (\rho_n P_{n+1} - \rho_0 P_1 Q_n)u_0. \tag{27}$$

For $n = 1$, (27) becomes

$$2(x-c)v_0' = Gu_0, \tag{28}$$

where $G = \rho_1 P_2 - \rho_0 P_1 Q_1$.
By inserting (28) in (27), we obtain

$$(x-c)Q_n''v_0 = (\rho_n P_{n+1} - \rho_0 P_1 Q_n - GQ_n')u_0. \tag{29}$$

Hence, taking $n = 2$ in (29), (i) holds.

So, by substituting $(x-c)v_0 = Fu_0$ in (29) and taking into account the quasi-definiteness of u_0, we deduce (ii).

By using Lemma 3.2 (ii), we can write

$$\Phi v_0' = -(\Phi' + \Psi)v_0, \quad \Phi^2 v_0'' = Av_0, \tag{30}$$

where $A = (2\Phi' + \Psi)(\Phi' + \Psi) - (\Phi'' + \Psi')\Phi$.
So, if we multiply (26) by Φ^2 and we take into account (30), (i) as well as the quasi-definiteness of u_0, we get (a). If we multiply (28) by Φ and we use (30), (i) and the quasi-definiteness of u_0 then (b) follows. Finally, multiplying (25) by Φ^2 and using (30), (i) as well as the quasi-definiteness of u_0, we get

$$(\Phi^2 Q_n'' + BQ_n' + AQ_n)F = \rho_n \Phi^2 P_{n+1}, \ n \geq 0, \tag{31}$$

where $B = -2\Phi(\Phi' + \Psi)$.
For $n = 1$ in (31), $n = 0$ in (19), $n = 0$ in (31), and (20), we obtain (c). ∎

Theorem 3.1. *The $\widehat{\Omega}_{1,c}-$classical polynomial sequences are, up to a suitable shifting, one of the following $D-$classical polynomial sequences:*

(i) $P_n = Q_n = L_n^{(1)}, \ n \geq 0,$ *with* $c = 0$.

(ii) $P_n = P_n^{(\alpha-2,1)}, \ n \geq 0,$ *with* $\alpha \neq -n+2, \ n \geq 1, \ Q_n = P_n^{(\alpha,1)}, \ n \geq 0,$ *and* $c = 1$.

Proof. From Lemma 3.2, $\{Q_n\}_{n\geq 0}$ is $D-$classical. By Lemma 3.1, we will analyze the four canonical situations given in Table 1.

($\mathbf{C}_1$). $\{Q_n\}_{n\geq 0}$ is the Hermite MOPS. From Table 1 **($\mathbf{C}_1$)**, $A = 2(2x^2-1)$ and $B = -4x$. Since Lemma 3.3 (iii) (a), $2(2x^2-1)F = \rho_0 P_1$. This yields a contradiction, since $\deg P_1 = 1$ and $\rho_0 \neq 0$.

($\mathbf{C}_2$). $\{Q_n\}_{n\geq 0}$ is the Laguerre MOPS. From Table 1 **($\mathbf{C}_2$)**, $A = x^2 - 2\alpha x + \alpha(\alpha-1)$ and $B = -2x(x-\alpha)$. By Lemma 3.3, (iii), we get

$$\big(x^2 - 2\alpha x + \alpha(\alpha-1)\big)F = \rho_0 x^2 P_1, \tag{32}$$

$$-2(x-\alpha)F = xG, \tag{33}$$

$$\gamma_1 x^2 = \Big(\big((\rho_0^{-1} - \rho_1^{-1})x - \beta_1\rho_0^{-1} + \xi_0\rho_1^{-1}\big)A - \rho_1^{-1}B\Big)F, \tag{34}$$

From (32), $\deg F = 1$. So, $F = \rho_0 x$, by (32) and (34). Hence, (32) becomes $(x^2 - 2\alpha x + \alpha(\alpha-1)) = xP_1$. Thus, $\alpha(\alpha-1) = 0$. Necessarily, $\alpha = 1$. Otherwise, if $\alpha = 0$ then $G = -2\rho_0 x$ and $P_1 = x$. But, by evaluating at $x = 0$ the equation given by Lemma 3.3 (ii), we get $P_{n+1}(0) = 0,\ n \geq 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.
By (32) with $\alpha = 1$ and $F = \rho_0 x$, we get $P_1 = x - 2$ and then $\beta_0 = 2$. So,

$$A = x(x-2), \quad B = -2x(x-1), \quad G = -2\rho_0(x-1). \tag{35}$$

Since $\xi_0 = 2$ (see Table 1 **($\mathbf{C}_2$)** with $\alpha = 1$) and by (35), (34) yields $\rho_1 = \rho_0$, $\beta_1 = 4$, and $\gamma_1 = 2$. Then, $\rho_0 = 2\langle v_0, Q_0^2\rangle\langle u_0, P_1^2\rangle^{-1} = 2\gamma_1^{-1} = 1$. By Table 1 **($\mathbf{C}_2$)**, with $\alpha = 1$, and by Lemma 3.2 (i), $\beta_{n+1} = 2n+4,\ n \geq 1$, and $\gamma_{n+1} = (n+1)(n+2),\ n \geq 2$. By Lemma 3.3 (ii), $xQ_n'' - 2(x-1)Q_n' + (x-2)Q_n = \rho_n P_{n+1},\ n \geq 0$. Thus, $\rho_n = 1,\ n \geq 0$. So, $xQ_n'' - 2(x-1)Q_n' + (x-2)Q_n = P_{n+1}$, $n \geq 0$. Since $1 = \rho_1 = 6\langle v_0, Q_1^2\rangle\langle u_0, P_2^2\rangle^{-1} = 6\lambda_1(\gamma_1\gamma_2)^{-1}$, then $\gamma_2 = 6$, $\beta_n = 2(n+1)$ and $\gamma_{n+1} = (n+1)(n+2),\ n \geq 0$. Hence, $P_n = Q_n = L_n^{(1)}$, $n \geq 0$, and $u_0 = v_0$. Also, by Lemma 3.3 (i), we get $(x-c)u_0 = xu_0$. This requires that $c = 0$, because of the quasi-definiteness of u_0. Thus, we have

$$L_n^{(1)} = (n+1)^{-1}(n+2)^{-1}\widehat{\Omega}_1\ L_{n+1}^{(1)},\ n \geq 0, \tag{36}$$

$$xL_n^{(1)''}(x) - 2(x-1)L_n^{(1)'}(x) + (x-2)L_n^{(1)}(x) = L_{n+1}^{(1)}(x),\ n \geq 0. \tag{37}$$

($\mathbf{C}_3$). $\{Q_n\}_{n\geq 0}$ is the Bessel MOPS with parameter $\alpha \neq -n/2,\ n \geq 0$. By Table 1 **($\mathbf{C}_3$)**, $A = 2(1-\alpha)(3-2\alpha)x^2 + 4(2\alpha-3)x + 4$. By Lemma 3.3 (iii) (a), $AF = \rho_0 x^4 P_1$. This requires that $\deg A = 2$ and $\deg F = 3$, since $\deg A \leq 2$, $\deg F \leq 3$, and $\deg A + \deg F = 5$. But, by the previous equation, we must have $A(0) = 0$, that contradicts the fact that $A(0) = 4$.

($\mathbf{C}_4$). $\{Q_n\}_{n\geq 0}$ is the Jacobi MOPS with parameters α and β satisfying $\alpha,\ \beta \neq -n,\ \alpha+\beta \neq -n-1,\ n\geq 1$. From Table 1 ($\mathbf{C}_4$), we get

$$\begin{aligned} A &= (\alpha+\beta-1)\big((\alpha+\beta)x^2+2(\beta-\alpha)x\big)+(\alpha-\beta)^2-(\alpha+\beta), && (38)\\ B &= 2(x^2-1)\big((\alpha+\beta)x+\beta-\alpha\big). && (39) \end{aligned}$$

By Lemma 3.3 (iii) (a), $AF = \rho_0(x^2-1)^2P_1$. Since $\deg A+\deg F = 5$, $\deg F \leq 3$ and $\deg A \leq 2$, then $\deg A = 2$ and $\deg F = 3$. By Lemma 3.3 (iii) (c), F divides $\Phi^2 = (x^2-1)^2$ and hence there are two situations to be treat. Either $F = \mu(x-1)(x+1)^2$ or $F = \mu(x+1)(x-1)^2$, where $\mu \neq 0$.

($\mathbf{C}_{4,1}$). $F = \mu(x-1)(x+1)^2,\ \mu\neq 0$. By Lemma 3.3 (iii),

$$\begin{aligned} \mu A &= \rho_0(x-1)P_1, && (40)\\ G &= 2\mu\big((\alpha+\beta)x+\beta-\alpha\big)(x+1), && (41)\\ \mu^{-1}\gamma_1(x-1) &= \big((\rho_0^{-1}-\rho_1^{-1})x-\beta_1\rho_0^{-1}+\xi_0\rho_1^{-1}\big)A-\rho_1^{-1}B, && (42) \end{aligned}$$

where $\xi_0 = (\alpha-\beta)(\alpha+\beta+2)^{-1}$.
From (38) and (40), $\rho_0 = \mu(\alpha+\beta-1)(\alpha+\beta)$, $\beta_0 = (\alpha-3\beta)(\alpha+\beta)^{-1}$, and $\beta(\beta-1) = 0$. We must have $\beta = 1$. Otherwise, if $\beta = 0$, then $G(1) = P_1(1) = F(1) = 0$. But, by evaluating at $x = 1$ the equation given by Lemma 3.3 (ii) we get $P_{n+1}(1) = 0,\ n\geq 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$. For $\beta = 1$, we get $\rho_0 = \mu\alpha(\alpha+1)$, $\beta_0 = (\alpha-3)(\alpha+1)^{-1}$ and $G = 2\mu\big((\alpha+1)x+1-\alpha\big)(x+1)$. This expression of G together with that given by Lemma 3.3 (ii), yields by identification of the coefficients $\beta_1 = \frac{(\alpha-1)(\alpha-3)}{(\alpha+1)(\alpha+3)}$ and $\gamma_1 = \frac{8(\alpha-1)}{(\alpha+1)^2(\alpha+2)}$. Moreover, since $\rho_0 = 2\gamma_1^{-1}$, we get $\mu = \frac{(\alpha+1)(\alpha+2)}{4\alpha(\alpha-1)}$.

From Table 1 ($\mathbf{C}_4$) with $\beta = 1$ and by Lemma 3.2 (i), $\beta_n = \frac{(\alpha-2)^2-1}{(2n+\alpha-1)(2n+\alpha+1)}$, $\gamma_{n+1} = \frac{4(n+1)(n+2)(n+\alpha)(n+\alpha-1)}{(2n+\alpha)(2n+\alpha+1)^2(2n+\alpha+2)},\ n\geq 2$. By Lemma 3.3 (ii), $(x-1)(x+1)^2Q_n''+2\big((\alpha+1)x+1-\alpha\big)(x+1)Q_n'+\alpha(\alpha+1)(x-\beta_0)Q_n = \mu^{-1}\rho_nP_{n+1}$, $n\geq 0$. This allows us to deduce that $\rho_n = \mu\big(n^2+(2\alpha+1)n+\alpha(\alpha+1)\big)$. Consequently,

$$\begin{aligned} &(x-1)(x+1)^2Q_n''+2\big((\alpha+1)x+1-\alpha\big)(x+1)Q_n'+\\ &\qquad \alpha\big((\alpha+1)x+3-\alpha\big)Q_n = (n+\alpha)(n+\alpha+1)P_{n+1},\ n\geq 0. \end{aligned}$$

Since $\frac{(\alpha+1)^2(\alpha+2)^2}{4\alpha(\alpha-1)} = \rho_1 = 6\lambda_1(\gamma_1\gamma_2)^{-1}$, then $\gamma_2 = \frac{24\alpha(\alpha+1)}{(\alpha+2)(\alpha+3)^2(\alpha+4)}$. So, $\beta_n = \frac{(\alpha-2)^2-1}{(2n+\alpha-1)(2n+\alpha+1)}$, $\gamma_{n+1} = \frac{4(n+1)(n+2)(n+\alpha)(n+\alpha-1)}{(2n+\alpha)(2n+\alpha+1)^2(2n+\alpha+2)}$, $n\geq 0$. Thus, $P_n = P_n^{(\alpha-2,1)}$, $Q_n = P_n^{(\alpha,1)},\ n\geq 0$ with $\alpha\neq -n+2,\ n\geq 1$. Besides, by (18) with $n = 1$,

(19) and (20), $c = -\beta_0 - \beta_1 + 3\xi_0 = 1$. By Lemma 3.3 (i), $(x-1)v_0 = \mu(x-1)(x+1)^2 u_0$, where $\mu = \frac{(\alpha+1)(\alpha+2)}{4\alpha(\alpha-1)}$. Thus, $v_0 = \mu(x+1)^2 u_0$. Then,

$$P_n^{(\alpha,1)}(x) = (n+1)^{-1}(n+2)^{-1}\widehat{\Omega}_{1,1}\ P_{n+1}^{(\alpha-2,1)}(x), \tag{43}$$

$$(x-1)(x+1)^2 P_n^{(\alpha,1)''}(x) + 2\big((\alpha+1)x+1-\alpha\big)(x+1)P_n^{(\alpha,1)'}(x) + \alpha\big((\alpha+1)x+3-\alpha\big)P_n^{(\alpha,1)}(x) = \varrho_n P_{n+1}^{(\alpha-2,1)}(x),\ n\geq 0, \tag{44}$$

where $\varrho_n = (n+\alpha)(n+\alpha+1),\ n\geq 0$.

($\mathbf{C}_{4,2}$). $F = \mu(x+1)(x-1)^2$, $\mu\neq 0$. By a similar computation as in **($\mathbf{C}_{4,1}$)**, $P_n = P_n^{(1,\beta-2)}$, $(\beta\neq -n+2,\ n\geq 1)$, as well as $Q_n = P_n^{(1,\beta)}$ and $c=-1$. By shifting, we get **($\mathbf{C}_{4,1}$)**. Indeed, by (11) and Lemma 3.1, the MOPS $\{\widetilde{P}_n\}_{n\geq 0}$, $\widetilde{P}_n(x) = (-1)^n P_n(-x) = P_n^{(\beta-2,1)}(x)$, is $\widehat{\Omega}_{1,1}-$classical. ■

As we have pointed in the introduction, you can compare the proof of this theorem with [11]. The techniques are different as well as the length of this one is shorter.

In the linear space $\mathbb{P}$ we can introduce two operators $\mathscr{L}$ and $\mathscr{P}_\alpha$, where $\alpha\neq -n+2$, $n\geq 1$, defined as follows: For any $p\in\mathbb{P}$,

$$\begin{aligned}\mathscr{L}(p) &= xp'' - 2(x-1)p' + (x-2)p,\\ \mathscr{P}_\alpha(p) &= (x-1)(x+1)^2p'' + 2\big((\alpha+1)x+1-\alpha\big)(x+1)p' + \\ &\quad \alpha\big((\alpha+1)x+3-\alpha\big)p.\end{aligned}$$

According to (37) and (44)

$$\mathscr{L}(L_n^{(1)}) = L_{n+1}^{(1)},\ n\geq 0, \tag{45}$$

$$\mathscr{P}_\alpha(P_n^{(\alpha,1)}) = (n+\alpha)(n+\alpha+1)P_{n+1}^{(\alpha,1)},\ n\geq 0. \tag{46}$$

Notice that $\mathscr{L}$ and $\mathscr{P}_\alpha$ are *creation/raising operators* ([4]). Applying $\widehat{\Omega}_1$, (resp. $\widehat{\Omega}_{1,1}$) to both hand sides of (45) (resp. (46)) and using (36) (resp. (43)), we get

$$\widehat{\Omega}_1\circ\mathscr{L}(L_n^{(1)}) = (n+1)(n+2)L_n^{(1)},\ n\geq 0, \tag{47}$$

$$\widehat{\Omega}_{1,1}\circ\mathscr{P}_\alpha(P_n^{(\alpha,1)}) = (n+1)(n+2)(n+\alpha)(n+\alpha+1)P_n^{(\alpha,1)},\ n\geq 0. \tag{48}$$

The operator $\widehat{\Omega}_1\circ\mathscr{L}$, (resp. $\widehat{\Omega}_{1,1}\circ\mathscr{P}_\alpha$) preserves the degree of polynomials and also the orthogonality of the MPS $\{L_n^{(1)}\}_{n\geq 0}$, (resp. $\{P_n^{(\alpha,1)}\}_{n\geq 0}$). Such

operators are called *transfer operators* (see [4, 10]).
Secondly, by (14) and (1), the intertwining operator $\mathfrak{S}_0$ satisfies

$$\mathfrak{S}_0(L_n^{(1)}) = (n+1)!(x-1)^n,\ n \geq 0. \tag{49}$$

From (9), (10), and (14), the intertwining operator $\mathfrak{S}_1$ satisfies

$$\mathfrak{S}_1(P_n^{(\alpha-2,1)})(x) = (n+1)!\ B_n^{(\frac{\alpha+1}{2})}(x-1),\ n \geq 0, \quad where \tag{50}$$

$\{B_n^{(\frac{\alpha+1}{2})}\}_{n\geq 0}$ is the Bessel MOPS with parameter $\frac{\alpha+1}{2}$, (see Table 1 $(\mathbf{C}_3)$).
Then, the following new integral relation holds

$$B_n^{(\frac{\alpha+1}{2})}(x) = \frac{1}{(n+1)!}\int_0^{+\infty} te^{-t} P_n^{(\alpha-2,1)}\big(tx+1\big)dt,\ n \geq 0,\ (\alpha \neq -n+2,\ n \geq 1).$$

4. Some expansions in series of Laguerre polynomials

Our purpose here is to give some new expansion in series of Laguerre polynomials. First, let us establish the following result:

Theorem 4.1. *Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a real function defined in the real line. The following statements are equivalent.*

(i) *f is 1−periodic.*

(ii) $\sum_{n\geq 0} \frac{a_n}{(n+1)!}\ x^n = \sum_{n\geq 0} \frac{a_n}{(n+1)!}\ L_n^{(1)}(x)$.

(iii) $(a_n)_{n\geq 0}$ *satisfies:* $a_\nu = \sum_{n\geq 0}(-1)^n\binom{n+\nu}{\nu}a_{n+\nu},\ \nu \geq 0$.

Proof. (i)⇒(ii). Let $g(x) = \sum_{n\geq 0} \frac{a_n}{(n+1)!}\ x^n - \sum_{n\geq 0} \frac{a_n}{(n+1)!}\ L_n^{(1)}(x)$. By applying $\mathfrak{S}_0$ to g and using (14) and (49), we get $\mathfrak{S}_0(g)(x) = f(x) - f(x-1)$. Since f is 1−periodic, then $\mathfrak{S}_0(g) = 0$. So, $g = 0$, since $\mathfrak{S}_0$ is one-to-one.

(ii)⇒(iii). From (1) and (ii), we get $\sum_{n\geq 0} \frac{a_n}{(n+1)!}x^n =$ $\sum_{n\geq 0} a_n \sum_{\nu=0}^{n} \frac{(-1)^{n-\nu}}{(\nu+1)!}\binom{n}{\nu}x^\nu = \sum_{\nu\geq 0}\frac{1}{(\nu+1)!}\sum_{n\geq 0}(-1)^n a_{n+\nu}\binom{n+\nu}{\nu}x^\nu$. By identification, we find (iii).

(iii)⇒(i). From (iii), we get $f(x) = \sum_{n\geq 0}\sum_{m\geq 0}(-1)^m\binom{m+n}{n}a_{m+n}\ x^n =$ $\sum_{n\geq 0}\sum_{m\geq 0}(-1)^{m-n}\binom{m}{n}a_m\ x^n = \sum_{m\geq 0} a_m(x-1)^m = f(x-1)$. ■

Examples. Let us consider the following 1−periodic functions.

• For all $x \in \mathbb{R}$, $\cos(2\pi x) = \sum_{n\geq 0} c_n x^n$, with $\begin{cases} c_{2n+1} = 0, n \geq 0, \\ c_{2n} = \frac{(-1)^n(2\pi)^{2n}}{(2n)!},\ n \geq 0. \end{cases}$

• For all $x \in \mathbb{R}$, $\sin(2\pi x) = \sum_{n\geq 0} s_n x^n$, with $\begin{cases} s_{2n} = 0, \ n \geq 0, \\ s_{2n+1} = \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}, \ n \geq 0. \end{cases}$

• For all $x \in \mathbb{R}$ such that $x \neq \frac{\pi}{2} + l\pi, \ l \in \mathbb{Z}$,

$$\tan(\pi x) = \sum_{n\geq 0} t_n x^n, \text{ with } \begin{cases} t_{2n} = 0, \ n \geq 0, \\ t_{2n+1} = \frac{(-1)^n \pi^{2n+1} 2^{2n+2} (2^{2n+2}-1)}{(2n+2)!} \ B_{2n+2}, \ n \geq 0, \end{cases}$$

and $B_k, \ k \geq 0$, the Bernoulli numbers satisfying $\sum_{k=0}^{n-1} \binom{n}{k} B_k = \delta_{n,1}, \ n \geq 1$. According to Theorem 4.1 (ii),

$$\Re e\big(\, {}_0F_1(2\pi i x, 2) \, \big) = \sum_{n\geq 0} \frac{c_{2n}}{(2n+1)!} \ x^{2n} = \sum_{n\geq 0} \frac{c_{2n}}{(2n+1)!} \ L_{2n}^{(1)}(x),$$

$$\Im m\big(\, {}_0F_1(2\pi i x, 2) \, \big) = \sum_{n\geq 0} \frac{s_{2n+1}}{(2n+2)!} \ x^{2n+1} = \sum_{n\geq 0} \frac{s_{2n+1}}{(2n+2)!} \ L_{2n+1}^{(1)}(x),$$

where ${}_0F_1(z, 2) = \sum_{n\geq 0} \frac{z^n}{(n+1)!n!}$ is the confluent hypergeometric function.

$$\sum_{n\geq 0} \frac{t_{2n+1}}{(2n+2)!} \ B_{2n+2} \ x^{2n+1} = \sum_{n\geq 0} \frac{t_{2n+1}}{(2n+2)!} \ B_{2n+2} \ L_{2n+1}^{(1)}(x).$$

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