The Laguerre-Sobolev-Type Orthogonal Polynomials

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Abstract

In this paper we study the asymptotic behaviour of polynomials orthogonal with respect to a Sobolev-Type inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + Np^{(j)}(0)q^{(j)}(0).$$

We will focus our attention on the outer relative asymptotics with respect to the standard Laguerre polynomials as well as on an analog of the Mehler-Heine formula for the rescaled polynomials.

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1 Introduction.

Let $(\mu_0, \mu_1, \cdots, \mu_j)$ be a vector of positive measures supported on the real line such that

$$\int_\Gamma |x|^n \, d\mu_k < \infty,$$

for $k = 0, 1, 2, \cdots, j$ and for every $n \in \mathbb{N}$.

In the linear space $\mathbb{P}$ of polynomials with real coefficients we can define an inner product

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$$\langle p, q\rangle = \sum_{k=0}^{j} \int_{\mathbb{R}} p^{(k)}(x)q^{(k)}(x)d\mu_k \tag{1}$$

$p, q \in \mathbb{P}$.

This inner product is known in the literature as a Sobolev inner product. The study of the sequences of monic polynomials orthogonal with respect to (1) has attracted the interest of many researchers during the last twenty years. One of the reasons is the fact that many properties of the standard polynomials ($j = 0$) are lost when (1) is considered. In particular, the existence of recurrence relations of low order is only true when the measures $\{\mu_k\}_{k=1}^{j}$ are discrete. Furthermore, the zeros can be complex or, if real, they can be located outside the convex hull of the union of the support of the measures $\{\mu_k\}_{k=0}^{j}$.

Most of the contributions (see [15] and [13] for a survey of the main results concerning the analytic properties of such polynomials) deal with measures of bounded support.

If μ_0 has an unbounded support and $\{\mu_k\}_{k=1}^{j}$ are discrete measures very few results are known. From an algebraic point of view a first approach was done in [12] where, as an example, the case of $d\mu_0 = e^{-x^2}dx$ is considered. When $j = 1$, for $d\mu_0 = x^{\alpha}e^{-x}dx + M_0\delta(x)$ and $d\mu_1 = M_1\delta(x)$ in [8] a representation of the corresponding sequence of monic orthogonal polynomials in terms of Laguerre polynomials $\{\mathcal{L}_n^{\alpha}\}_{n\geq 0}$ as well as its representation as hypergeometric series is done. Later on, in [2] the authors analyzed the outer relative asymptotics in terms of Laguerre polynomials $\{L_n^{\alpha}\}_{n\geq 0}$. Furthermore, a Mehler-Heine type formula is obtained for such polynomials. As an application, some results concerning the behaviour of their zeros are obtained. For some more extra information see the survey [11].

For higher derivatives, i.e $j > 1$, when $d\mu_0 = x^{\alpha}e^{-x}dx + M_0\delta(x)$, $d\mu_k = M_k\delta(x)$, $k = 1, 2, \cdots, j$ in [7] an explicit expression of the Sobolev orthogonal polynomials in terms of classical Laguerre polynomials is given. Furthermore, their representation as an hypergeometric function as well as the holonomic second order linear differential equation that such polynomials satisfy, are obtained.

Unfortunately, no asymptotic results for such polynomials are know. More recently, in [1] the authors analyzed asymptotic properties when $d\mu_0 = e^{-x^2}dx + M_0\delta(x)$, $d\mu_k = M_k\delta(x)$.

The structure of the manuscript is as follows. In section 2 we present the basic background concerning classical Laguerre orthogonal polynomials. Section 3 deals with the connection formula between the polynomials $\{\mathcal{L}_n^{\alpha}\}_{n\geq 0}$ orthogonal with respect the Sobolev-Type inner product

$$\langle p, q\rangle_S = \int_0^{\infty} pqx^{\alpha}e^{-x}dx + Np^{(j)}(0)q^{(j)}(0),$$

where $N \in \mathbb{R}^+$ and $j \in \mathbb{N}$, and the Laguerre polynomials $\left\{L_n^{\alpha+j+1}\right\}_{n\geq 0}$. In section 4 we obtain the outer relative asymptotic of the polynomials $\{\mathcal{L}_n^{\alpha}\}_{n\geq 0}$ in terms of $\{L_n^{\alpha}\}_{n\geq 0}$ as well a Mehler-Heine type formula i.e the behavior of $\frac{\widehat{\mathcal{L}}_n^{\alpha}(x/n)}{n^{\alpha}}$ on compact subsets of the complex plane, where $\widehat{\mathcal{L}}_n^{\alpha}(x) = \frac{(-1)}{n!}\mathcal{L}_n^{\alpha}(x)$.

This last result constitutes a partial answer to a question posed in [2] about the outer relative asymptotics and the existence of a Mehler-Heine type formula for the corresponding sequence of Sobolev orthogonal polynomials.

2 Preliminaries.

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers and let μ be a linear functional defined in the linear space $\mathbb{P}$ of the polynomials with real coefficients, such that

$$\langle \mu, x^n \rangle = \mu_n, \; n = 0, 1, 2...$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Moreover μ_n is the $n-th$ *moment* of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if

(i) The degree of P_n is n.

(ii) $\langle \mu, P_n(x)P_m(x) \rangle = 0, m \neq n.$

(iii) $\left\langle \mu, P_n^2(x) \right\rangle \neq 0, n = 0, 1, 2, ...$

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials.* It is clear that for every sequence of orthogonal polynomials there exists the corresponding family of monic orthogonal polynomials. In the sequel we will work with monic polynomials .

The next theorem, whose proof appears in [5] , gives necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1 *[5] Let μ be a moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $\left[\mu_{i+j}\right]_{i,j\in\mathbb{N}}$ are non singular.*

A moment functional such that there exists the correspondient sequence of orthogonal polynomials is said to be *regular* or *quasi-definite* ([5]). If $\phi(x)$ is a complex polynomial, we define the moment functional $\phi\mu$, the left multiplication by a polynomial ϕ, and $D\mu$, the usual distributional derivative of μ, as follows

$$\langle \phi\mu, p(x) \rangle = \langle \mu, \phi\,(x)\, p(x) \rangle\,,\;\; \langle D\mu, p(x) \rangle = -\langle \mu, p'(x) \rangle\,.$$

A sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ is said to be *classical* if there exist polynomials ϕ and ψ, with $\deg \phi \leq 2$ and $\deg \psi = 1$, such that μ satisfies the Pearson differential equation

$$D\,(\phi\mu) = \psi\mu.$$

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) are extensively used in the literature taking into account their applications in Mathematical Physics. Indeed, one of their most popular applications is the study of problems involving hypergeometric differential equations (see [3], [6], [9],[16], and [17]).

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_0^\infty pqx^\alpha e^{-x}dx,\ \alpha > -1,\ p, q \in \mathbb{P}. \tag{2}$$

We will need to summarize some properties of the Laguerre monic orthogonal polynomials that we will use in the sequel. The details of the proof can be founded in [3], [5], [6], [10], and [17].

Proposition 1 *Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials.*

1. *For every $n \in \mathbb{N}$,*

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha)\, L_n^\alpha(x) + n\,(n+\alpha)\, L_{n-1}^\alpha(x) \tag{3}$$

   with $L_0^\alpha(x) = 1, L_1^\alpha(x) = x-(\alpha+1)$.

2. *For every $n \in \mathbb{N}$,*

$$x\left(L_n^\alpha(x)\right)' = nL_n^\alpha(x) + n\,(n+\alpha)\, L_{n-1}^\alpha(x). \tag{4}$$

3. *For every $n \in \mathbb{N}$, $L_n^\alpha(x)$ satisfies the differential equation*

$$xy'' + (\alpha + 1 - x)\, y' = \lambda_n y \tag{5}$$

   with $\lambda_n = -n$.

4. *For every $n \in \mathbb{N}$,*

$$L_n^\alpha(x) = (-1)^n(\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}. \tag{6}$$

   where $(a)_n = a(a+1)\cdots(a+n-1)$, $n \geq 1$, and $(a)_0 = 1$, is the Pochhammer symbol.

5. *For every $n \in \mathbb{N}$,*

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x). \tag{7}$$

6. *For every $n \in \mathbb{N}$,*

$$\left\|L_n^\alpha\right\|_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{8}$$

7. *For every $n \in \mathbb{N}$*

$$L_n^\alpha(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{9}$$

8. *If*

$$K_n(x,y)=\sum_{j=0}^{n}\frac{L_j^{\alpha}(y)L_j^{\alpha}(x)}{\left\|L_j^{\alpha}\right\|^2}$$

denotes the n-th kernel polynomial then, for every $n\in\mathbb{N}$,

$$K_n(x,y)=\frac{L_{n+1}^{\alpha}(x)L_n^{\alpha}(y)-L_{n+1}^{\alpha}(y)L_n^{\alpha}(x)}{x-y}\frac{1}{\|L_n^{\alpha}\|^2} \tag{10}$$

and

$$K_n(x,0)=\frac{L_n^{\alpha}(0)}{n!\Gamma(n+\alpha+1)}L_n^{\alpha+1}(x). \tag{11}$$

9. *(The Mehler-Heine type formula) Let J_α be the Bessel function defined by*

$$J_\alpha(x)=\sum_{j=0}^{\infty}\frac{(-1)^j(x/2)^{2j+\alpha}}{j!\Gamma(j+\alpha+1)}.$$

Then

$$\lim_{n\to\infty}\frac{\widehat{L}_n^{\alpha}(x/(n+j))}{n^{\alpha}}=x^{-\alpha/2}J_\alpha(2\sqrt{x}) \tag{12}$$

uniformly on compact subsets $\mathbb{C}$ and, uniformly in $j\in\mathbb{N}\cup\{0\}$. Here $\widehat{L}_n^{\alpha}(x)=(-1)^n/n!L_n^{\alpha}(x)$.

We will use the following notation for the partial derivatives of $K_n(x,y)$

$$\frac{\partial^{j+k}\left(K_n(x,y)\right)}{\partial^j x\partial^k y}=K_n^{(j,k)}(x,y),$$

If $p(x)$ is a polynomial with $\deg p\le n$, then we can write it as a linear combination of the Laguerre polynomials as follows

$$p(x)=\sum_{k=0}^{n}\frac{\left\langle L_k^{\alpha}(x),p(x)\right\rangle}{\left\|L_k^{\alpha}\right\|^2}L_k^{\alpha}(x).$$

As a consequence,

$$p^{(j)}(y)=\sum_{k=0}^{n}\frac{\left\langle L_k^{\alpha}(x),p(x)\right\rangle}{\left\|L_k^{\alpha}\right\|^2}\left(L_k^{\alpha}\right)^{(j)}(y),$$

and, taking into account that

$$\begin{aligned}\left\langle K_n^{(0,j)}(x,y), p(x)\right\rangle_\alpha &= \left\langle \sum_{k=0}^{n} \frac{L_j^\alpha(x)\left(L_j^\alpha\right)^{(j)}(y)}{\left\|L_j^\alpha\right\|^2}, p(x)\right\rangle \\ &= \sum_{k=0}^{n} \frac{\left\langle L_j^\alpha(x), p(x)\right\rangle}{\left\|L_j^\alpha\right\|^2}\left(L_j^\alpha\right)^{(j)}(y),\end{aligned}$$

we get

$$\left\langle K_n^{(0,j)}(x,y), p(x)\right\rangle = p^{(j)}(y). \tag{13}$$

Let $\{P_n\}_{n\geq 0}$ a sequence of monic orthogonal polynomials. From the Christoffel Darboux Formula, (see [3], [5], [6], and [17]) , we have

$$K_{n-1}(x,y) = \frac{1}{\|P_{n-1}\|^2}\frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x-y}.$$

Calculating the $j-th$ partial derivative with respect to y, we get

$$K_{n-1}^{(0,j)}(x,y) = \frac{1}{\|P_{n-1}\|^2}\left(P_n(x)\frac{\partial^j}{\partial y^j}\left(\frac{P_{n-1}(y)}{x-y}\right) - P_{n-1}(x)\frac{\partial^j}{\partial y^j}\left(\frac{P_n(y)}{x-y}\right)\right). \tag{14}$$

Using the Leibnitz rule

$$\frac{\partial^j}{\partial y^j}\left(\frac{P_n(y)}{x-y}\right) = \sum_{k=0}^{j}\frac{j!}{k!}\frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}},$$

and replacing the last expression in (14), we obtain

$$\begin{aligned}K_{n-1}^{(0,j)}(x,y) &= \frac{1}{\|P_{n-1}\|^2}\left(P_n(x)\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_{n-1}^{(k)}(y)}{(x-y)^{j-k+1}} - P_{n-1}(x)\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}}\right) \\ &= \frac{j!}{\|P_{n-1}\|^2 (x-y)^{j+1}} \times \\ &\quad \left(P_n(x)\sum_{k=0}^{j}\frac{1}{k!}P_{n-1}^{(k)}(y)(x-y)^k - P_{n-1}(x)\sum_{k=0}^{j}\frac{1}{k!}P_n^{(k)}(y)(x-y)^k\right).\end{aligned}$$

As a consequence

$$K_{n-1}^{(0,j)}(x,0) = \frac{j!}{\|P_{n-1}\|^2 x^{j+1}}\left(P_n(x)Q_j(x,0,P_{n-1}) - P_{n-1}(x)Q_j(x,0,P_n)\right) \tag{15}$$

where $Q_j(x,0,P_{n-1})$ and $Q_j(x,0,P_n)$ denote the Taylor Polynomials of degree j of the polynomials P_{n-1} and P_n around $x=0$, respectively.

Next, we will compute $K_{n-1}^{(0,j)}(x,0)$. Indeed

$$K_{n-1}^{(0,j)}(x,0) = \tag{16}$$

$$\begin{aligned} = \quad & \frac{j!}{\|P_{n-1}\|^2 x^{j+1}} \left[\left(P_n(0) + P_n'(0)x + \frac{P_n''(0)}{2!}x^2 + ... + \frac{P_n^{(n)}(0)}{n!}x^n \right) \times \right. \\ & \left(P_{n-1}(0) + P_{n-1}'(0)x + \frac{P_{n-1}''(0)}{2!}x^2 + \cdots + \frac{P_{n-1}^{(j)}(0)}{j!}x^j \right) - \\ & \left(P_{n-1}(0) + P_{n-1}'(0)x + \frac{P_{n-1}''(0)}{2!}x^2 + \cdots + \frac{P_{n-1}^{(n-1)}(0)}{(n-1)!}x^{n-1} \right) \times \\ & \left. \left(P_n(0) + P_n'(0)x + \frac{P_n''(0)}{2!}x^2 + \cdots + \frac{P_n^{(j)}(0)}{j!}x^j \right) \right]. \end{aligned}$$

If we make some computations in the last expression, the coefficients of the monomials of degree less than or equal to j inside the bracket are cancelled. Thus, when $x = 0$, we have

$$K_{n-1}^{(0,j)}(0,0) = \frac{j!}{\|P_{n-1}\|^2} \left(P_{n-1}(0) \frac{P_n^{(j+1)}(0)}{(j+1)!} - P_n(0) \frac{P_{n-1}^{(j+1)}(0)}{(j+1)!} \right),$$

and

$$K_{n-1}^{(0,j)}(0,0) = \frac{1}{\|P_{n-1}\|^2 (j+1)} \left(P_{n-1}(0) P_n^{(j+1)}(0) - P_n(0) P_{n-1}^{(j+1)}(0) \right). \tag{17}$$

In order to find $K_{n-1}^{(j,j)}(0,0)$, we just need to deduce in (16) the coefficient of x^{2j+1} inside the bracket which is

$$\begin{aligned} & \left[\frac{P_{n-1}(0)}{0!} \frac{P_n^{(2j+1)}(0)}{(2j+1)!} + \frac{P_{n-1}'(0)}{1!} \frac{P_n^{(2j)}(0)}{(2j)!} + \cdots + \frac{P_{n-1}^{(j)}(0)}{j!} \frac{P_n^{(j+1)}(0)}{(j+1)!} \right] - \\ & \left[\frac{P_n(0)}{0!} \frac{P_{n-1}^{(2j+1)}(0)}{(2j+1)!} + \frac{P_n'(0)}{1!} \frac{P_{n-1}^{(2j)}(0)}{(2j)!} + \cdots + \frac{P_n^{(j)}(0)}{j!} \frac{P_{n-1}^{(j+1)}(0)}{(j+1)!} \right] \\ = \quad & \frac{1}{(2j+1)!} \left[\left(P_{n-1}(0) P_n^{(2j+1)}(0) + P_{n-1}'(0) P_n^{(2j)}(0) \binom{2j+1}{1} + \cdots \right. \right. \\ & \left. + \binom{2j+1}{j} P_{n-1}^{(j)}(0) P_n^{(j+1)}(0) \right) - \\ & \left(P_n(0) P_{n-1}^{(2j+1)}(0) + P_n'(0) P_{n-1}^{(2j)}(0) \binom{2j+1}{1} + \cdots \right. \\ & \left. \left. + \binom{2j+1}{j} P_n^{(j)}(0) P_{n-1}^{(j+1)}(0) \right) \right]. \end{aligned}$$

Furthermore, in the case of the Laguerre monic polynomials, we get

$$K_{n-1}^{(j,j)}(0,0) =$$

$$\frac{j!j!}{(2j+1)!\left\|L_{n-1}^{\alpha}\right\|^{2}} \times \Bigg[\Bigg(L_{n-1}^{\alpha}(0)(L_{n}^{\alpha})^{(2j+1)}(0) + \binom{2j+1}{1}(L_{n-1}^{\alpha})'(0)(L_{n}^{\alpha})^{(2j)}(0) + \cdots + \binom{2j+1}{j}(L_{n-1}^{\alpha})^{(j)}(0)(L_{n}^{\alpha})^{(j+1)}(0)\Bigg) - \Bigg(L_{n}^{\alpha}(0)(L_{n-1}^{\alpha})^{(2j+1)}(0) + \binom{2j+1}{1}(L_{n}^{\alpha})'(0)(L_{n-1}^{\alpha})^{(2j)}(0) + \cdots + \binom{2j+1}{j}(L_{n}^{\alpha})^{(j)}(0)(L_{n-1}^{\alpha})^{(j+1)}(0)\Bigg)\Bigg].$$

Thus

$$K_{n-1}^{(j,j)}(0,0) = \tag{18}$$

$$\begin{aligned}
& \frac{j!j!}{(2j+1)!\left\|L_{n-1}^{\alpha}\right\|^{2}} \sum_{k=0}^{j}\binom{2j+1}{k} \times \\
& \left[\left(L_{n-1}^{\alpha}\right)^{(k)}(0)(L_{n}^{\alpha})^{(2j+1-k)}(0) - (L_{n}^{\alpha})^{(k)}(0)(L_{n-1}^{\alpha})^{(2j+1-k)}(0)\right] \\
= \; & \frac{j!j!}{(2j+1)!(n-1)!\Gamma(n+\alpha)} \sum_{k=0}^{j}\binom{2j+1}{k} \times \\
& \Big[(n-1)\cdots(n-k)L_{n-k-1}^{\alpha+k}(0)n(n-1)\cdots(n-2j+k)L_{n-2j-1+k}^{\alpha+2j+1-k}(0) \\
& - n\cdots(n-k+1)L_{n-k}^{\alpha+k}(0)(n-1)\cdots(n-2j-1+k)L_{n-2j-2+k}^{\alpha+2j-k+1}(0)\Big] \\
= \; & \sum_{k=0}^{j} \frac{j!j!n(n-1)\cdots(n-k+1)(n-1)\cdots(n-2j+k)}{(2j+1)!(n-1)!\Gamma(n+\alpha)}\binom{2j+1}{k} \times \\
& \Bigg[(n-k)\,(-1)^{n-k-1}\,\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+k+1)}\,(-1)^{n-2j-1+k}\,\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+2j-k+2)} \\
& - (n-2j+k-1)\,(-1)^{n-k}\,\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1)}\,(-1)^{n-2j-2+k}\,\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+2j-k+2)}\Bigg] \\
= \; & \sum_{k=0}^{j}\binom{2j+1}{k} \times \\
& \frac{j!j!(2j-2k+1)n(n-1)\cdots(n-k+1)(n-1)\cdots(n-2j+k)\Gamma(n+\alpha+1)}{(2j+1)!(n-1)!\Gamma(\alpha+k+1)\Gamma(\alpha+2j-k+2)},
\end{aligned}$$

as a consequence,

$$K_{n-1}^{(j,j)}(0,0) \sim C_{\alpha,j}\frac{\Gamma(n+\alpha+1)n^{2j+1}}{n!} \tag{19}$$

where $C_{\alpha,j}$ is a real constant number that depends of α and j. More precisely,

$$C_{\alpha,j} = \frac{j!j!}{(2j+1)!}\sum_{k=0}^{j}\begin{pmatrix} 2j+1 \\ k \end{pmatrix}\frac{(2j-2k+1)}{\Gamma(\alpha+k+1)\Gamma(\alpha+2j+2-k)}. \tag{20}$$

In particular if $j = 1$

$$C_{\alpha,1} = \frac{1}{(\alpha+3)\left[\Gamma(\alpha+2)\right]^2}. \tag{21}$$

3 Connection Formula

Let introduce the following Sobolev-Type inner product

$$\langle p,q\rangle_S = \int_0^\infty pqx^\alpha e^{-x}dx + Np^{(j)}(0)q^{(j)}(0), \tag{22}$$

where $N \in \mathbb{R}^+$ and $j \in \mathbb{N}$. Let $\{\mathcal{L}_n^\alpha\}_{n\geq 0}$ be the monic Laguerre-Sobolev-Type orthogonal polynomials with respect to the above inner product.

Our aim is to obtain an explicit expression of these polynomials in terms of classical Laguerre polynomials. In order to do it, we will consider the Fourier expansion of $\mathcal{L}_n^\alpha$ in terms of $\{L_n^\alpha\}_{n\geq 0}$. Indeed

$$\mathcal{L}_n^\alpha(x) = L_n^\alpha(x) + \sum_{k=0}^{n-1} a_k^{(n)} L_k^\alpha(x),$$

where

$$a_k^{(n)} = \frac{\left\langle \mathcal{L}_n^\alpha(x), L_k^\alpha(x)\right\rangle_\alpha}{\left\|L_k^\alpha(x)\right\|_\alpha^2}.$$

But, from (22)

$$a_k^{(n)} = \frac{\left\langle \mathcal{L}_n^\alpha(x), L_k^\alpha(x)\right\rangle_S - N(\mathcal{L}_n^\alpha)^{(j)}(0)\left(L_k^\alpha\right)^{(j)}(0)}{\left\|L_k^\alpha(x)\right\|_\alpha^2},$$

and taking into account that $\left\langle \mathcal{L}_n^\alpha(x), L_k^\alpha(x)\right\rangle_S = 0$ for $k = 0, \cdots, n-1$, we get

$$a_k^{(n)} = -\frac{N(\mathcal{L}_n^\alpha)^{(j)}(0)\left(L_k^\alpha\right)^{(j)}(0)}{\left\|L_k^\alpha(x)\right\|_\alpha^2}.$$

As a consequence,

$$\mathcal{L}_n^{\alpha}(x) = L_n^{\alpha}(x) - N(\mathcal{L}_n^{\alpha})^{(j)}(0) \sum_{k=0}^{n-1} \frac{\left(L_k^{\alpha}\right)^{(j)}(0) L_k^{\alpha}(x)}{\left\|L_k^{\alpha}(x)\right\|_{\alpha}^2}$$

$$= L_n^{\alpha}(x) - N(\mathcal{L}_n^{\alpha})^{(j)}(0) K_{n-1}^{(0,j)}(x,0). \tag{23}$$

Next, we will express $K_{n-1}^{(0,j)}(x,0)$ as a linear combination of some Laguerre polynomials. Using the orthogonality of the Laguerre polynomials, we have

$$\frac{K_{n-1}^{(0,j)}(x,0)\left\|L_{n-1}^{\alpha}(x)\right\|_{\alpha}^2}{\left(L_{n-1}^{\alpha}\right)^{(j)}(0)} = L_{n-1}^{\alpha+j+1}(x) + \sum_{k=0}^{n-2} b_k^{(n)} L_k^{\alpha+j+1}(x),$$

where

$$b_k^{(n)} = \frac{\left\|L_{n-1}^{\alpha}(x)\right\|_{\alpha}^2 \left\langle K_{n-1}^{(0,j)}(x,0), L_k^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1}}{\left(L_{n-1}^{\alpha}\right)^{(j)}(0)\left\|L_k^{\alpha+j+1}(x)\right\|_{\alpha+j+1}^2}.$$

Using the fact that

$$\left\langle K_{n-1}^{(0,j)}(x,0), L_k^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1} = \left\langle K_{n-1}^{(0,j)}(x,0), x^{j+1} L_k^{\alpha+j+1}(x)\right\rangle_{\alpha} = 0,$$

for $0 \le k \le n-j-2$, then, we get

$$K_{n-1}^{(0,j)}(x,0) = \frac{\left(L_{n-1}^{\alpha}\right)^{(j)}(0)}{\left\|L_{n-1}^{\alpha}(x)\right\|_{\alpha}^2} L_{n-1}^{\alpha+j+1}(x) + \frac{\left\langle K_{n-1}^{(0,j)}(x,0), L_{n-2}^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1}}{\left\|L_{n-2}^{\alpha+j+1}(x)\right\|_{\alpha+j+1}^2} L_{n-2}^{\alpha+j+1}(x)$$

$$+ \cdots + \frac{\left\langle K_{n-1}^{(0,j)}(x,0), L_{n-j-1}^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1}}{\left\|L_{n-j-1}^{\alpha+j+1}(x)\right\|_{\alpha+j+1}^2} L_{n-j-1}^{\alpha+j+1}(x). \tag{24}$$

Now we will compute each one of the coefficients of the previous expression. From (4),

$$x L_{n-1}^{\alpha+1}(x) = L_n^{\alpha}(x) + (n+\alpha) L_{n-1}^{\alpha}(x),$$

and thus

$$\begin{aligned}
x^j x L_{n-k-1}^{\alpha+j+1}(x) &= x^j \left(L_{n-k}^{\alpha+j}(x) + (n-k+\alpha+j) L_{n-k-1}^{\alpha+j}(x)\right) \\
&= x^{j-1}\Big[L_{n-k+1}^{\alpha+j-1}(x) + (n-k+j+\alpha) L_{n-k}^{\alpha+j-1}(x) + \\
&\quad + (n-k+\alpha+j)\,(L_{n-k}^{\alpha+j-1}(x) + (n-k+\alpha+j-1) L_{n-k-1}^{\alpha+j-1}(x))\Big] \\
&= x^{j-1}\Big[L_{n-k+1}^{\alpha+j-1}(x) + 2(n-k+j) L_{n-k}^{\alpha+j-1}(x) + \\
&\quad + (n-k+\alpha+j)\,(n-k+\alpha+j-1) L_{n-k-1}^{\alpha+j-1}(x)\Big].
\end{aligned}$$

Iterating the procedure, we will prove by induction that

$$x^{j+1}L_{n-k-1}^{\alpha+j+1}(x) = \sum_{r=0}^{j+1}\binom{j+1}{r}(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r}^{\alpha}(x). \tag{25}$$

Indeed, assuming that the above expression holds for $j-1$, i.e

$$x^{j}L_{n-k-1}^{\alpha+j}(x) = \sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r)_r\, L_{n-k+j-r-1}^{\alpha}(x)$$

then

$$x^{j+1}L_{n-k-1}^{\alpha+j+1}(x) = x\left(x^{j}L_{n-k-1}^{(\alpha+1)+j}(x)\right)$$

$$\begin{aligned}
&= x\sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r-1}^{\alpha+1}(x)\\
&= \sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r+1)_r\left(L_{n-k+j-r}^{\alpha}(x)+(n-k+j-r+\alpha)L_{n-k+j-r-1}^{\alpha}(x)\right)\\
&= \sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r}^{\alpha}(x)+\\
&\quad+\sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r+1)_r\,(n-k+j-r+\alpha)L_{n-k+j-r-1}^{\alpha}(x)\\
&= \sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r}^{\alpha}(x)+\\
&\quad+\sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r)_{r+1}\, L_{n-k+j-r-1}^{\alpha}(x)\\
&= \sum_{r=0}^{j}\binom{j}{r}(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r}^{\alpha}(x)+\\
&\quad+\sum_{r=1}^{j+1}\binom{j}{r-1}(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r}^{\alpha}(x)\\
&= L_{n-k+j}^{\alpha}(x)+\sum_{r=1}^{j}\left(\binom{j}{r}+\binom{j}{r-1}\right)(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r}^{\alpha}(x)+\\
&\quad+(n-k+\alpha)_{j+1}\, L_{n-k-1}^{\alpha}(x)\\
&= \sum_{r=0}^{j+1}\binom{j+1}{r}(n-k+\alpha+j-r+1)_r\, L_{n-k+j-r}^{\alpha}(x),
\end{aligned}$$

and our results follows.

Now, we will use this to compute the coefficients in (24). Using (25) , for $0 \leq k \leq j$

$$\left\langle K_{n-1}^{(0,j)}(x,0), x^{j+1} L_{n-k-1}^{\alpha+j+1}(x) \right\rangle_{\alpha} =$$

$$\begin{aligned} &= \left\langle K_{n-1}^{(0,j)}(x,0), \sum_{r=0}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \, L_{n-k+j-r}^{\alpha}(x) \right\rangle_{\alpha} \\ &= \sum_{r=0}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \left\langle K_{n-1}^{(0,j)}(x,0), L_{n-k+j-r}^{\alpha}(x) \right\rangle_{\alpha}. \end{aligned}$$

Therefore, everything comes down calculating $\left\langle K_{n-1}^{(0,j)}(x,0), L_{n-k+j-r}^{\alpha}(x) \right\rangle_{\alpha}$ for $0 \leq r \leq j+1$. But, using (13) and (22), we get that

$$\left\langle K_{n-1}^{(0,j)}(x,0), L_{n-k+j-r}^{\alpha}(x) \right\rangle_{\alpha} = 0,$$

for $0 \leq r \leq j-k$, and

$$\left\langle K_{n-1}^{(0,j)}(x,0), L_{n-k+j-r}^{\alpha}(x) \right\rangle_{\alpha} = \left(L_{n-k+j-r}^{\alpha}\right)^{(j)}(0).$$

$j-k+1 \leq r \leq j+1$.

As a conclusion, for $0 \leq k \leq j$

$$\left\langle K_{n-1}^{(0,j)}(x,0), x^{j+1} L_{n-k-1}^{\alpha+j+1}(x) \right\rangle_{\alpha} = \sum_{r=j-k+1}^{j+1} \binom{j+1}{r} (n-k+\alpha+j-r+1)_r \left(L_{n-k+j-r}^{\alpha}\right)^{(j)}(0).$$

Therefore

$$K_{n-1}^{(0,j)}(x,0) = A_{n,1}^{(j)} L_{n-1}^{\alpha+j+1}(x) + A_{n,2}^{(j)} L_{n-2}^{\alpha+j+1}(x) + \cdots + A_{n,j+1}^{(j)} L_{n-j-1}^{\alpha+j+1}(x). \qquad (26)$$

where

$$A_{n,1}^{(j)} = \frac{\left(L_{n-1}^{\alpha}\right)^{(j)}(0)}{\left\| L_{n-1}^{\alpha}(x) \right\|_{\alpha}^{2}}$$

and

$$A_{n,r}^{(j)} = \frac{1}{\left\| L_{n-r}^{\alpha+j+1}(x) \right\|_{\alpha+j+1}^{2}} \sum_{s=j-r+2}^{j+1} \binom{j+1}{s} (n-r+\alpha+j-s+2)_s \left(L_{n-r+j-s+1}^{\alpha}\right)^{(j)}(0),$$

for $2 \leq r \leq j+1$.

In order to compute $A_{n,r}^{(j)}$, we use the next expression which is a consequence of (7)

$$\left(L_n^{\alpha}\right)^{(j)}(x) = \frac{n!}{(n-j)!} L_{n-j}^{\alpha+j}(x).$$

Indeed,

$$\begin{aligned} A_{n,1}^{(j)} &= \frac{\frac{(n-1)!}{(n-j-1)!}L_{n-j-1}^{\alpha+j}(0)}{(n-1)!\Gamma(n+\alpha)} = \frac{(-1)^{n-j-1}\ \frac{\Gamma(n+\alpha)}{\Gamma(\alpha+j+1)}}{(n-j-1)!\Gamma(n+\alpha)} \\ &= \frac{(-1)^{n-j-1}}{(n-j-1)!\Gamma(\alpha+j+1)}, \end{aligned}$$

and

$$\begin{aligned} A_{n,r}^{(j)} &= \frac{1}{(n-r)!\Gamma(n+\alpha+j-r+2)} \\ & \sum_{s=j-r+2}^{j+1} \binom{j+1}{s} (n-r+\alpha+j-s+2)_s \frac{(n-r+j-s+1)!}{(n-r-s+1)!} \left(L_{n-r-s+1}^{\alpha+j}\right)(0) \\ &= \frac{1}{(n-r)!} \sum_{s=j-r+2}^{j+1} \binom{j+1}{s} \frac{(n-r+j-s+1)!}{(n-r-s+1)!} \frac{(-1)^{n-r-s+1}}{\Gamma(\alpha+j+1)} \end{aligned}$$

$2 \leq r \leq j+1$.
Thus (23) becomes

$$\mathcal{L}_n^{\alpha}(x) = L_n^{\alpha}(x) - N(\mathcal{L}_n^{\alpha})^{(j)}(0) \sum_{r=1}^{j+1} A_{n,r}^{(j)} L_{n-r}^{\alpha+j+1}(x), \tag{27}$$

Using this equality, we can compute $(\mathcal{L}_n^{\alpha})^{(j)}(0)$. Taking the $j-th$ derivative in both hand sides,

$$\left(\mathcal{L}_n^{\alpha}\right)^{(j)}(x) = (n-j+1)_j L_{n-j}^{\alpha+j}(x) - N(\mathcal{L}_n^{\alpha})^{(j)}(0) \sum_{r=1}^{j+1} A_{n,r}^{(j)} (n-r-j+1)_j L_{n-r-j}^{\alpha+2j+1}(x),$$

and thus

$$\begin{aligned} \left(\mathcal{L}_n^{\alpha}\right)^{(j)}(0) &= \frac{(n-j+1)_j L_{n-j}^{\alpha+j}(0)}{1 + N\sum_{r=1}^{j+1} A_{n,r}^{(j)} (n-r-j+1)_j D(n,\alpha,j,r)} \\ &= \frac{(n-j+1)_j (-1)^{n-j} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1 + N\sum_{r=1}^{j+1} A_{n,r}^{(j)} (n-r-j+1)_j D(n,\alpha,j,r)}, \end{aligned}$$

with

$$D(n,\alpha,j,r) = \begin{cases} 0 & if \quad r > n-j \\ (-1)^{n-r-j} \frac{\Gamma(n+\alpha+j-r+2)}{\Gamma(\alpha+2j+2)} & if \quad r \leq n-j \end{cases}.$$

As a consequence, we obtain

Theorem 2 *Let $\{L_n^\alpha\}_{n\geq 0}$ the monic Laguerre orthogonal polynomials and $\{\mathcal{L}_n^\alpha(x)\}_{n\geq 0}$ the monic Laguerre-Sobolev-Type orthogonal polynomials corresponding to the inner product defined in (22).Then, for every $n \in \mathbb{N}$,*

$$\mathcal{L}_n^\alpha(x) = L_n^\alpha(x) - N(\mathcal{L}_n^\alpha)^{(j)}(0) \sum_{r=1}^{j+1} A_{n,r}^{(j)} L_{n-r}^{\alpha+j+1}(x), \tag{28}$$

where

$$\begin{aligned} \left(\mathcal{L}_n^\alpha\right)^{(j)}(0) &= \frac{(n-j+1)_j(-1)^{n-j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1+N\sum_{r=1}^{j+1} A_{n,r}^{(j)}(n-r-j+1)_j D(n,\alpha,j,r)}, \\ D(n,\alpha,j,r) &= \begin{cases} 0 & if \quad n-r-j<0 \\ (-1)^{n-r-j}\frac{\Gamma(n+\alpha+j-r+1)}{\Gamma(\alpha+2j+2)} & if \quad n-r-j\geq 0 \end{cases}, \end{aligned}$$

$$A_{n,1}^{(j)} = \frac{(-1)^{n-j-1}}{(n-j-1)!\Gamma(\alpha+j+1)}$$

and, for $2 \leq r \leq j+1$

$$A_{n,r}^{(j)} = \frac{1}{(n-r)!} \sum_{s=j-r+2}^{j+1} \binom{j+1}{s} \frac{(n-r+j-s+1)!}{(n-r-s+1)!} \frac{(-1)^{n-r-s+1}}{\Gamma(\alpha+j+1)}.$$

From (7) , we get

$$L_n^\alpha(x) = \sum_{r=0}^{j+1} (n-r+1)_r \binom{j+1}{r} L_{n-r}^{\alpha+j+1}(x),$$

and using the notation of the above theorem, we obtain another equivalent expression

Corollary 1 *For every $n \in \mathbb{N}$,*

$$\mathcal{L}_n^\alpha(x) = L_n^{\alpha+j+1}(x) + \sum_{r=1}^{j+1} \left[(n-r+1)_r \binom{j+1}{r} - N(\mathcal{L}_n^\alpha)^{(j)}(0) A_{n,r}^{(j)} \right] L_{n-r}^{\alpha+j+1}(x). \tag{29}$$

If we denote

$$C_{n,r}^{(j)} = (n-r+1)_r \binom{j+1}{r} - N(\mathcal{L}_n^\alpha)^{(j)}(0) A_{n,r}^{(j)} \tag{30}$$

for $r = 1, ..., j+1$, we can write the Laguerre Sobolev orthogonal polynomials as follows

$$\mathcal{L}_n^\alpha(x) = L_n^{\alpha+j+1}(x) + \sum_{r=1}^{j+1} C_{n,r}^{(j)} L_{n-r}^{\alpha+j+1}(x). \tag{31}$$

This means that $\{\mathcal{L}_n^\alpha(x)\}_{n\geq 0}$ is a quasi-orthogonal sequence of order $j+1$ with respect to the linear functional associated with the weight function $w_{\alpha+j+1}(x) = x^{\alpha+j+1}e^{-x}$ (see [4])

4 The zeros

In this section, we are going to prove that the zeros of the monic Laguerre-Sobolev-Type orthogonal polynomials, are real, simple and are interlaced. The ideas of the proofs are the same that used H. G. Meijer in [14].

Theorem 3 *The monic Laguerre Sobolev-Type orthogonal polynomial $\mathcal{L}_n^\alpha(x)$ has n real simple zeros and at most one of them is outside of* $(0,\infty)$.

Proof. Let $\xi_1, \xi_2, \ldots, \xi_k$ be the positive zeros of $\mathcal{L}_n^\alpha(x)$ of odd multiplicity. Let

$$\varphi(x) = (x-\xi_1)(x-\xi_2)\cdots(x-\xi_k),$$

thus $\varphi(x)\mathcal{L}_n^\alpha(x)$ does not change sign on $(0,\infty)$. Suppose that $\deg\varphi \leq n-2$, then, using the fact that $(x\varphi(x))^{(j)}(0) = j\varphi^{(j-1)}(0)$ we get

$$\begin{aligned}\langle \varphi(x), \mathcal{L}_n^\alpha(x)\rangle_S &= \int_0^\infty \varphi(x)\mathcal{L}_n^\alpha(x)x^\alpha e^{-x}dx + N\varphi^{(j)}(0)\left(\mathcal{L}_n^\alpha\right)^{(j)}(0) = 0,\\ \langle x\varphi(x), \mathcal{L}_n^\alpha(x)\rangle_S &= \int_0^\infty x\varphi(x)\mathcal{L}_n^\alpha(x)x^\alpha e^{-x}dx + jN\varphi^{(j-1)}(0)\left(\mathcal{L}_n^\alpha\right)^{(j)}(0) = 0.\end{aligned}$$

Taking into account that the integrals in the last expressions are positive, then $\varphi^{(j)}(0)\left(\mathcal{L}_n^\alpha\right)^{(j)}(0) < 0$ and $\varphi^{(j-1)}(0)\left(\mathcal{L}_n^\alpha\right)^{(j)}(0) < 0$. It means that $\varphi^{(j)}(0)$ and $\varphi^{(j-1)}(0)$ have the same sign. But this is a contradiction with the well known fact that if $p(x)$ is a polynomial with simple zeros in $(0,\infty)$, then $p'(0)$ and $p(0)$ have different signs.

In other words, $\deg\varphi = n-1$ or $\deg\varphi = n$, which proves our statement. ■

Theorem 4 *Let $\xi_1 < \xi_2 <, \ldots, < \xi_n$ the zeros of the monic Laguerre-Sobolev-Type orthogonal polynomials $\mathcal{L}_n^\alpha(x)$ and let $x_1 < x_2 <, \ldots, < x_n$ the zeros of the monic Laguerre orthogonal polynomials $L_n^\alpha(x)$. Then $\xi_1 < x_1$ and $x_i < \xi_{i+1} < x_{i+1}$ for $i = 1, 2, \ldots, n-1$.*

Proof. From the Gauss quadrature formula there exist positive constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\sum_{i=0}^{n} \lambda_i \mathcal{L}_n^\alpha(x_i)x_i^r = \int_0^\infty x^r \mathcal{L}_n^\alpha(x)x^\alpha e^{-x}dx, \; 0 \leq r \leq n-1$$

i.e

$$\sum_{i=0}^{n} \lambda_i \mathcal{L}_n^\alpha(x_i)x_i^r = -j!N\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)\delta_{j,r}.$$

We consider the system of n linear equations in the n unknown incognits $\mathcal{L}_n^\alpha(x_1), \mathcal{L}_n^\alpha(x_2), \ldots, \mathcal{L}_n^\alpha(x_n)$. The determinant of this system is

$$D = \lambda_1\lambda_2\cdots\lambda_n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \ddots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

which is equal to $\lambda_1\lambda_2\cdots\lambda_n$ multiplied by the Vandermonde determinant $V(x_1, x_2, \ldots, x_n)$ of $x_1, x_2, \ldots, x_n$. As a result, D is positive. Therefore, $\mathcal{L}_n^\alpha(x_i) = \frac{D_i}{D}$, where

$$D_i = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_{i-1}^2 & x_{i+1}^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{j-1} & x_2^{j-1} & \cdots & x_{i-1}^{j-1} & x_{i+1}^{j-1} & \cdots & x_n^{j-1} \\ x_1^{j+1} & x_2^{j+1} & \cdots & x_{i-1}^{j+1} & x_{i+1}^{j+1} & \cdots & x_n^{j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_{i-1}^{n-1} & x_{i+1}^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

But

$$D_i = -(-1)^{i+j} j! N \left(\mathcal{L}_n^\alpha\right)^{(j)}(0) V(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\phi(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

with $\phi(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ a symmetric function. Then D_i and D_{i+1} have different signs, which implies that $\mathcal{L}_n^\alpha(x_i)$ and $\mathcal{L}_n^\alpha(x_{i+1})$ have different signs. In other words, $\mathcal{L}_n^\alpha(x)$ has at least one zero in every interval (x_i, x_{i+1}) for $i = 1, 2, \ldots, n-1$. Finally, taking into account that the sign of $\mathcal{L}_n^\alpha(x_1)$ is $(-1)^{n+1}$, we conclude that $\xi_1 < x_1$. ■

5 Asymptotic behaviour

We will analyze the outer relative asymptotic behaviour of $\mathcal{L}_n^\alpha(x)$. From (23) we have

$$\frac{\mathcal{L}_n^\alpha(x)}{L_n^\alpha(x)} = 1 - \frac{N(\mathcal{L}_n^\alpha)^{(j)}(0) K_{n-1}^{(0,j)}(x,0)}{L_n^\alpha(x)}.$$

But from (15), we get

$$\frac{\mathcal{L}_n^\alpha(x)}{L_n^\alpha(x)} = 1 - \frac{j! N(\mathcal{L}_n^\alpha)^{(j)}(0)\left(L_n^\alpha(x) Q_j(x,0,L_{n-1}^\alpha) - L_{n-1}^\alpha(x) Q_j(x,0,L_n^\alpha)\right)}{\left\|L_{n-1}^\alpha\right\|^2 x^{j+1} L_n^\alpha(x)},$$

where $Q_j(x,0,L_{n-1}^\alpha)$ and $Q_j(x,0,L_n^\alpha)$ are the Taylor polynomials of degree j of $L_{n-1}^\alpha(x)$ and $L_n^\alpha(x)$ around $x = 0$, respectively. Therefore

$$\frac{\mathcal{L}_n^\alpha(x)}{L_n^\alpha(x)} = 1 - \frac{j! N(\mathcal{L}_n^\alpha)^{(j)}(0) Q_j(x,0,L_{n-1}^\alpha)}{\left\|L_{n-1}^\alpha\right\|^2 x^{j+1}} \left(1 - \frac{Q_j(x,0,L_n^\alpha)}{Q_j(x,0,L_{n-1}^\alpha)} \frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\right). \tag{32}$$

First, we will study the behaviour of the following three expressions

$(i)\ \dfrac{j!Q_j(x,0,L^{\alpha}_{n-1})}{\left\|L^{\alpha}_{n-1}\right\|^2},$

$(ii)\ 1-\dfrac{Q_j(x,0,L^{\alpha}_{n})}{Q_j(x,0,L^{\alpha}_{n-1})}\dfrac{L^{\alpha}_{n-1}(x)}{L^{\alpha}_{n}(x)},$

$(iii)\ (\mathcal{L}^{\alpha}_{n})^{(j)}(0).$

For the first one

$$\begin{aligned}
\frac{j!Q_j(x,0,L^{\alpha}_{n-1})}{\left\|L^{\alpha}_{n-1}\right\|^2} &\sim \frac{j!\left(L^{\alpha}_{n-1}\right)^{(j)}(0)x^j}{\left\|L^{\alpha}_{n-1}\right\|^2 j!} \\
&= \frac{(n-1)(n-2)\cdots(n-j)}{(n-1)!\Gamma(n+\alpha)}L^{\alpha+j}_{n-1-j}(0)x^j \\
&= \frac{(n-1)(n-2)\cdots(n-j)}{(n-1)!\Gamma(n+\alpha)}(-1)^{n-1-j}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha+j+1)}x^j.
\end{aligned}$$

As a consequence

$$\frac{j!Q_j(x,0,L^{\alpha}_{n-1})}{\left\|L^{\alpha}_{n-1}\right\|^2 x^{j+1}} \sim \frac{(-1)^{n-1-j}n^j}{x(n-1)!\Gamma(\alpha+j+1)}. \tag{33}$$

On the other hand

$$1-\frac{Q_j(x,0,L^{\alpha}_{n})}{Q_j(x,0,L^{\alpha}_{n-1})}\frac{L^{\alpha}_{n-1}(x)}{L^{\alpha}_{n}(x)} \sim \tag{34}$$

$$\begin{aligned}
& 1+\frac{n(n-1)\cdots(n-j+1)\Gamma(n+\alpha+1)}{(n-1)(n-2)\cdots(n-j)\Gamma(n+\alpha)}\frac{L^{\alpha}_{n-1}(x)}{L^{\alpha}_{n}(x)} \\
= \quad & 1-\frac{n+\alpha}{n-j}\frac{\widehat{L}^{\alpha}_{n-1}(x)}{\widehat{L}^{\alpha}_{n}(x)},
\end{aligned}$$

where $\widehat{L}^{\alpha}_{n}(x)$ denotes the $n-th$ Laguerre polynomial with leading coefficient $\frac{(-1)^n}{n!}$.

Finally, from (19) and (23) we get

$$\begin{aligned}
(\mathcal{L}^{\alpha}_{n})^{(j)}(0) &= \frac{(L^{\alpha}_{n})^{(j)}(0)}{1+NK^{(j,j)}_{n-1}(0,0)} \\
&\sim \frac{n(n-1)\cdots(n-j+1)(-1)^{n-j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1+NC_{\alpha,j}\frac{\Gamma(n+\alpha+1)n^{2j+1}}{n!}}.
\end{aligned}$$

Thus

$$(\mathcal{L}_n^{\alpha})^{(j)}(0) \sim \frac{1}{\Gamma(\alpha+j+1)} \frac{(-1)^{n-j} n^j \Gamma(n+\alpha+1)(n-1)!}{(n-1)! + NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}}. \tag{35}$$

Denoting by $\widehat{\mathcal{L}}_n^{\alpha}(x)$ the Laguerre-Sobolev-Type orthogonal polynomial of degree n with leading coefficient $\frac{(-1)^n}{n!}$, if $h, k \in \mathbb{Z}$, and $t \in \mathbb{R}$, then it is very well known that (see [2], and [17])

$$\lim_{n\to\infty} \frac{\widehat{L}_{n+k}^{\alpha+t}(x)}{n^{t/2}\widehat{L}_{n+h}^{\alpha}(x)} = (-x)^{-t/2}, \tag{36}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$ (see [2] and [17]). Furthermore, using (36) in (34) we get

$$\lim_{n\to\infty}\left(1 - \frac{Q_j(x,0,L_n^{\alpha})}{Q_j(x,0,L_{n-1}^{\alpha})} \frac{\widehat{L}_{n-1}^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)}\right) = 0 \tag{37}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

As a conclusion, from (32), we obtain

$$\begin{aligned}
&\frac{\mathcal{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} - 1\\
\sim\quad & -\frac{N j!(\mathcal{L}_n^{\alpha})^{(j)}(0) Q_j(x,0,L_{n-1}^{\alpha})}{\left\|L_{n-1}^{\alpha}\right\|^2 x^{j+1}} \left(1 - \frac{Q_j(x,0,L_n^{\alpha})}{Q_j(x,0,L_{n-1}^{\alpha})} \frac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)}\right)\\
=\quad & -\frac{N(-1)^{n-j} n^j \Gamma(n+\alpha+1)}{(n-1)! + NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}} \frac{(-1)^{n-1-j} n^j}{x(\Gamma(\alpha+j+1))^2} \left(1 - \frac{Q_j(x,0,L_n^{\alpha})}{Q_j(x,0,L_{n-1}^{\alpha})} \frac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)}\right)\\
=\quad & \frac{N}{x(\Gamma(\alpha+j+1))^2} \frac{n^{2j}\Gamma(n+\alpha+1)}{(n-1)! + NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}} \left(1 - \frac{Q_j(x,0,L_n^{\alpha})}{Q_j(x,0,L_{n-1}^{\alpha})} \frac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)}\right)
\end{aligned}$$

therefore, using (37) we get

Proposition 2 *For* $n \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\mathcal{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1. \tag{38}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

Taking into account the Mehler-Heine formula (12), we will deduce an analog Mehler-Heine formula for the Laguerre-Sobolev-Type orthogonal polynomials.

First, we will find an expression of the $j-th$ Taylor polynomial of $L_n^{\alpha}(x)$ replacing the variable x by x/n.

$$\begin{aligned}
Q_j(x/n,0,L_n^{\alpha}) \quad &= \quad \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} + \frac{(-1)^{n-1}n\Gamma(n+\alpha+1)}{\Gamma(\alpha+2)}\frac{x}{n} + \cdots\\
&\quad + \frac{(-1)^{n-j}n(n-1)\cdots(n-j+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}\frac{x^j}{n^j}\\
&= \quad \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\left[S_j^{\alpha}(x) - \frac{1}{n}R_j^{\alpha}(x) + O(n^{-2})\right],
\end{aligned}$$

where

$$S_j^{\alpha}(x) = 1 - \frac{x}{1!(\alpha+1)_1} + \cdots + \frac{(-1)^j x^j}{j!(\alpha+1)_j},$$
$$R_j^{\alpha}(x) = \frac{x^2}{2!(\alpha+1)_2} - \frac{3x^3}{3!(\alpha+1)_3} + \cdots + \frac{(-1)^j x^j (j-1) j}{2j!(\alpha+1)_j}.$$

Therefore

$$Q_j(x/n, 0, L_n^{\alpha}) = \frac{(-1)^n \Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \left[S_j^{\alpha}(x) - \frac{1}{n} R_j^{\alpha}(x) + O(n^{-2}) \right]. \tag{39}$$

In an analog way, we conclude that

$$Q_j(x/n, 0, L_{n-1}^{\alpha}) = \frac{(-1)^{n-1} \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \left[S_j^{\alpha}(x) + \frac{1}{n} T_j^{\alpha}(x) + O(n^{-2}) \right], \tag{40}$$

where

$$T_j^{\alpha}(x) = \frac{x}{1!(\alpha+1)} - \frac{3x^2}{2!(\alpha+1)_2} + \cdots + \frac{(-1)^{j-1} x^j j(j+1)}{2(\alpha+1)_j j!}.$$

From (8), (27), and (35), we get

$$\mathcal{L}_n^{\alpha}(x) = L_n^{\alpha}(x) - \frac{N j! (\mathcal{L}_n^{\alpha})^{(j)}(0)}{\left\| L_{n-1}^{\alpha} \right\|^2 x^{j+1}} \left[L_n^{\alpha}(x) Q_j(x, 0, L_{n-1}^{\alpha}) - L_{n-1}^{\alpha}(x) Q_j(x, 0, L_n^{\alpha}) \right],$$

and, from (7) we get that $L_n^{\alpha-1}(x) = L_n^{\alpha}(x) + n L_{n-1}^{\alpha}(x)$.

As a consequence,

$$\begin{aligned} \mathcal{L}_n^{\alpha}(x) &= L_n^{\alpha}(x) - \frac{N j! (\mathcal{L}_n^{\alpha})^{(j)}(0)}{\left\| L_{n-1}^{\alpha} \right\|^2 x^{j+1}} \left[L_n^{\alpha}(x) Q_j(x, 0, L_{n-1}^{\alpha}) - \frac{L_n^{\alpha-1}(x) - L_n^{\alpha}(x)}{n} Q_j(x, 0, L_n^{\alpha}) \right] \\ &= L_n^{\alpha}(x) - \frac{N j! (\mathcal{L}_n^{\alpha})^{(j)}(0)}{\left\| L_{n-1}^{\alpha} \right\|^2 x^{j+1}} \left[\left(Q_j(x, 0, L_{n-1}^{\alpha}) + \frac{Q_j(x, 0, L_n^{\alpha})}{n} \right) L_n^{\alpha}(x) - \frac{L_n^{\alpha-1}(x)}{n} Q_j(x, 0, L_n^{\alpha}) \right]. \end{aligned}$$

Multiplying in both sides of the above expression by $\frac{(-1)^n}{n! n^{\alpha}}$ and using the change of variable $x \to x/n$, we get

$$\frac{\widehat{\mathcal{L}}_n^{\alpha}(x/n)}{n^{\alpha}} =$$

$$\frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}} - \frac{N j! (\mathcal{L}_n^{\alpha})^{(j)}(0) n^{j+1}}{\left\| L_{n-1}^{\alpha} \right\|^2 x^{j+1}} \times$$
$$\left[\left(Q_j(x/n, 0, L_{n-1}^{\alpha}) + \frac{Q_j(x/n, 0, L_n^{\alpha})}{n} \right) \frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}} - \frac{1}{n^2} \frac{\widehat{L}_n^{\alpha-1}(x/n)}{n^{\alpha-1}} Q_j(x/n, 0, L_n^{\alpha}) \right],$$

and, from (8), (35), (39), and (40)

$$\frac{\widehat{\mathcal{L}}_n^{\alpha}(x/n)}{n^{\alpha}} =$$

$$\frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}} - \frac{\frac{Nj!}{\Gamma(\alpha+j+1)} \frac{(-1)^{n-j} n^j \Gamma(n+\alpha+1)(n-1)!}{(n-1)!+NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j}} n^{j+1}}{(n-1)!\Gamma(n+\alpha)x^{j+1}} \frac{(-1)^n \Gamma(n+\alpha)}{\Gamma(\alpha+1)} \times$$
$$\left[\left(\frac{\alpha}{n} S_j^{\alpha}(x) - \frac{1}{n} T_j^{\alpha}(x) - \frac{1}{n} R_j^{\alpha}(x) + O(n^{-2})\right) \frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}}\right.$$
$$\left. - \frac{\widehat{L}_{n-1}^{\alpha-1}(x/n)(n+\alpha)}{n^2 n^{\alpha-1}} \left(S_j^{\alpha}(x) - \frac{1}{n} R_j^{\alpha}(x) + O(n^{-2})\right)\right].$$

As a consequence

$$\lim_{n\to\infty} \frac{\widehat{\mathcal{L}}_n^{\alpha}(x/n)}{n^{\alpha}} =$$

$$x^{-\alpha/2} J_{\alpha}(2\sqrt{x}) + \frac{(-1)^{j+1} j!}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+1)x^{j+1}} \times$$
$$\left[x^{-\alpha/2} J_{\alpha}(2\sqrt{x})\left(\alpha S_j^{\alpha}(x) - T_j^{\alpha}(x) - R_j^{\alpha}(x)\right) - x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}) S_j^{\alpha}(x)\right],$$

uniformly on compact subsets of $\mathbb{C}$. Using the fact that

$$\begin{aligned} J_{\alpha-1}(2\sqrt{x}) + J_{\alpha+1}(2\sqrt{x}) &= \frac{\alpha}{\sqrt{x}} J_{\alpha}(2\sqrt{x}), \\ T_j^{\alpha}(x) + R_j^{\alpha}(x) &= \frac{x}{\alpha+1} S_{j-1}^{\alpha+1}(x), \end{aligned}$$

and doing some computations we get

Proposition 3 *For $n \in \mathbb{N}$*

$$\lim_{n\to\infty} \frac{\widehat{\mathcal{L}}_n^{\alpha}(x/n)}{n^{\alpha}} \tag{41}$$
$$= x^{-\alpha/2} J_{\alpha}(2\sqrt{x}) \left[1 + \frac{(-1)^j j! S_{j-1}^{\alpha+1}(x)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+2)x^j}\right] - x^{-\alpha/2} J_{\alpha+1}(2\sqrt{x}) \frac{(-1)^j j! S_j^{\alpha}(x)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+1)x^{j+1/2}}.$$

uniformly on compact subsets of $\mathbb{C}$.

On the other hand, using the fact that

$$J_{\alpha}(2\sqrt{x}) + J_{\alpha+2}(2\sqrt{x}) = \frac{\alpha+1}{\sqrt{x}} J_{\alpha+1}(2\sqrt{x}),$$

(41) becomes

$$\lim_{n\to\infty}\frac{\widehat{\mathcal{L}}_n^\alpha(x/n)}{n^\alpha}$$

$$= x^{-\alpha/2}J_\alpha(2\sqrt{x})\left[1+\frac{(-1)^j j!\left(S_{j-1}^{\alpha+1}(x)-S_j^{\alpha}(x)\right)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+2)x^j}\right]-x^{-\alpha/2}J_{\alpha+2}(2\sqrt{x})\frac{(-1)^j j! S_j^{\alpha}(x)}{C_{\alpha,j}\Gamma(\alpha+j+1)\Gamma(\alpha+2)x^j}.$$

If $j=1$, then we deduce after some straightforward computations a result obtained in [2].

Now, we are going to make another important asymptotic behaviour of $\mathcal{L}_n^\alpha(x)$. Taking into account

$$\begin{aligned}\left\|\mathcal{L}_n^\alpha\right\|_S^2 &= \langle \mathcal{L}_n^\alpha, L_n^\alpha\rangle_S \\ &= \langle \mathcal{L}_n^\alpha, L_n^\alpha\rangle_\alpha + N\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)\left(L_n^\alpha\right)^{(j)}(0) \\ &= \left\|L_n^\alpha\right\|_\alpha^2 + N\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)\left(L_n^\alpha\right)^{(j)}(0),\end{aligned}$$

and, using the asymptotic behaviour of $(L_n^\alpha)^{(j)}(0)$ and (35) which show us the behaviour of $(\mathcal{L}_n^\alpha)^{(j)}(0)$, then we get

$$\begin{aligned}\frac{\left\|\mathcal{L}_n^\alpha\right\|_S^2}{\|L_n^\alpha\|_\alpha^2} &= 1+\frac{N\left(\mathcal{L}_n^\alpha\right)^{(j)}(0)\left(L_n^\alpha\right)^{(j)}(0)}{\|L_n^\alpha\|_\alpha^2} \\ &\sim 1+N\frac{(-1)^{n-j}n^j\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{1+NC_{\alpha,j}\frac{\Gamma(n+\alpha+1)}{n!}n^{2j+1}}\,\frac{n^j(-1)^{n-j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)}}{n!\Gamma(n+\alpha+1)} \\ &= 1+\frac{n^{2j}\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+j+1)^2}}{n!+NC_{\alpha,j}\Gamma(n+\alpha+1)n^{2j+1}} \\ &= 1+O\left(\frac{1}{n}\right).\end{aligned}$$

Thus, we have proved

Proposition 4 *For every $n\in\mathbb{N}$,*

$$\lim_{n\to\infty}\frac{\left\|\mathcal{L}_n^\alpha\right\|_S^2}{\|L_n^\alpha\|_\alpha^2}=1.$$

Finally, we present a result about the behaviour of the norm of the Laguerre Sobolev-Type orthogonal polynomials that is a consequence of the above proposition an the fact that $n^{-1}\left\|L_n^\alpha\right\|_S^{1/n}\to e^{-1}$ when $n\to\infty$.

Corollary 2 *For $n\in\mathbb{N}$,*

$$\lim_{n\to\infty} n^{-1}\left\|\mathcal{L}_n^\alpha\right\|_S^{1/n}=e^{-1}. \tag{42}$$

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