Jacobi–Sobolev orthogonal polynomials: asymptotics and a Cohen type inequality

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Abstract

Let $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$, $\alpha, \beta > -1$, be the Jacobi measure supported on the interval $[-1, 1]$. Let us introduce the Sobolev inner product

$$\langle f, g\rangle_S = \sum_{j=0}^{N} \lambda_j \int_{-1}^{1} f^{(j)}(x)g^{(j)}(x)d\mu_{\alpha,\beta}(x),$$

where $\lambda_j \geq 0$ for $0 \leq j \leq N-1$ and $\lambda_N > 0$. In this paper we obtain some asymptotic results for the sequence of orthogonal polynomials with respect to the above Sobolev inner product. Furthermore, we prove a Cohen type inequality for Fourier expansions in terms of such polynomials.

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1. Introduction

Let $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$, $\alpha, \beta > -1$, be the Jacobi measure supported on the interval $[-1, 1]$. We define the linear space $L^p(d\mu_{\alpha,\beta})$, $1 \leq p \leq$

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∞, of all measurable functions f on $[-1,1]$ such that $\|f\|_{L^p(d\mu_{\alpha,\beta})}< \infty$, where

$$\|f\|_{L^p(d\mu_{\alpha,\beta})}= \begin{cases} \left(\int_{-1}^{1}|f(x)|^p d\mu_{\alpha,\beta}(x)\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{-1<x<1} |f(x)|, & \text{if } p = \infty. \end{cases}$$

Let us now introduce the Sobolev–type spaces (see, for instance, [1, Chapter III] in a more general framework)

$$W^{N,p} = \{f \ : \ \|f\|^p_{W^{N,p}}= \sum_{j=0}^{N} \lambda_j \|f^{(j)}\|^p_{L^p(d\mu_{\alpha,\beta})}< \infty\}, \quad 1 \le p < \infty,$$

where $\lambda_j \ge 0$ for $0 \le j \le N-1$ and $\lambda_N > 0$. When $p = \infty$, we have

$$W^{N,\infty} = \{f \ : \ \|f\|_{W^{N,\infty}}= \sum_{j=0}^{N} \lambda_j \|f^{(j)}\|_{L^\infty(d\mu_{\alpha,\beta})}< \infty\}.$$

Let f and g belong to $W^{N,2}$. We can introduce the Sobolev inner product

$$\langle f, g\rangle_S = \sum_{j=0}^{N} \lambda_j \int_{-1}^{1} f^{(j)}(x) g^{(j)}(x) d\mu_{\alpha,\beta}(x). \tag{1.1}$$

Using the standard Gram–Schmidt method for the canonical basis $(x^n)_{n\ge0}$ in the linear space of polynomials, we obtain a unique sequence (up to a constant factor) of polynomials $(Q_n^{(\alpha,\beta,N)})_{n\ge0}$ orthogonal with respect to the above inner product. In the sequel they will be called Jacobi–Sobolev orthogonal polynomials and we denote them by Q_n to simplify the notation.

In [15] the authors established the outer strong asymptotics as well as the distribution of the zeros of the polynomials orthogonal with respect to the above Sobolev inner product (1.1) when $N = 1$. Outer strong asymptotics as well as outer relative asymptotics were obtained for polynomials orthogonal with respect to measures satisfying Szegő's condition and supported on Jordan curves in [13] and for $N \ge 1$ in [12]. A nice survey of these questions is presented in [14]. More recently, when $\alpha = \beta$ and $N = 1$ some asymptotic results (more precisely, inner strong asympotics, outer relative asymptotics, Mehler–Heine type formula, as well as some norm estimates) for the corresponding Sobolev orthogonal polynomials have been deduced in [6].

One of the aims of this paper is to study inner strong asympotics, Mehler–Heine type formulas, and Sobolev–norm estimates for Jacobi–Sobolev orthogonal polynomials. Thus, we extend the results of [6] and [15].

On the other hand, it is well known that the linear space $W^{N,2}$ is a Hilbert space and the linear space of polynomials is a dense subset therein. Thus, $||S_n(f)||_{W^{N,2}} \le c||f||_{W^{N,2}}$ for any function $f \in W^{N,2}$, where by $S_n(f)$ we denote the nth partial sum of the Fourier expansion of f in terms of our Sobolev orthogonal polynomials. Then, a natural question concerning the existence of $p \neq 2$ such that $||S_n(f)||_{W^{N,p}} \leq c||f||_{W^{N,p}}$ for any function $f \in W^{N,p}$ arises.

In this way, we analyze the $W^{N,p}$–norm convergence of the Fourier expansion of $f \in W^{N,p}$ in terms of Sobolev orthogonal polynomials. More precisely, we prove a Cohen type inequality for such a Fourier expansion. Notice that, for $N = 1$ and $\alpha = \beta$, the analog problem has been considered in [7]. For classical orthogonal polynomials ($N = 0$), we cite [3], [4], and [11].

The structure of the paper is as follows. In Section 2, the basic background concerning asymptotic properties and estimates of Jacobi polynomials is introduced. In Section 3, asymptotic properties of Jacobi–Sobolev polynomials are deduced. Finally, in Section 4 a Cohen type formula for Fourier expansion with respect to Jacobi–Sobolev orthogonal polynomials is obtained.

Throughout this paper positive constants are denoted by $c, c_1, ...$ and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $c_1 u_n \leq v_n \leq c_2 u_n$ for n large enough. We will denote by $\kappa(\pi_n)$ the leading coefficient of any polynomial π_n and $\hat{\pi}_n(x) = (\kappa(\pi_n))^{-1} \pi_n(x)$.

2. Preliminaries

For $\alpha, \beta > -1$, let us denote by $(P_n^{(\alpha,\beta)})_{n\geq 0}$ the sequence of Jacobi polynomials which are orthogonal on $[-1, 1]$ with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} fg d\mu_{\alpha,\beta},$$

with the normalization $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$.

The following proposition summarizes some properties of Jacobi polynomials which will be used in the sequel.

Proposition 2.1. *(a) The leading coefficient of $P_n^{(\alpha,\beta)}$ is (see [16, formula (4.21.6)])*

$$\kappa\left(P_n^{(\alpha,\beta)}\right) = \kappa_n = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

Moreover,

$$\frac{\kappa_n}{\kappa_{n-k}} \cong 2^k, \qquad k \in \mathbb{N} \cup \{0\}.$$

(b) For $\alpha, \beta > -1$ (see [16, formula (4.3.3)])

$$\int_{-1}^{1} \left(P_n^{(\alpha,\beta)}(x)\right)^2 d\mu_{\alpha,\beta}(x) = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

(c) *It holds (see [16, formula (4.5.5)])*

$$P_n^{(\alpha,\beta)\ (k)}(x) = \frac{(n+\alpha+\beta+1)_k}{2^k} P_{n-k}^{(\alpha+k,\beta+k)}(x), \quad k = 0,\ 1,\ ...,\ n,$$

where $(n)_j$ is the Pochhammer symbol given by $(n)_0 = 1$, $(n)_j = n\cdots(n+j-1)$, $j \geq 1$.

(d) *The monic Jacobi polynomials $(\hat{P}_n^{(\alpha,\beta)})_{n\geq 0}$ satisfy the relation (see [10])*

$$\hat{R}_{n,1}(x) := n\int \hat{P}_{n-1}^{(\alpha,\beta)}(x)dx = \hat{P}_n^{(\alpha,\beta)}(x) + a_{n-1}\hat{P}_{n-1}^{(\alpha,\beta)}(x) + b_{n-1}\hat{P}_{n-2}^{(\alpha,\beta)}(x), \tag{2.1}$$

where $\hat{P}_{-1} = 0$, $\hat{P}_0 = 1$, and

$$a_{n-1} = \frac{2(\alpha-\beta)n}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)} = O\left(\frac{1}{n}\right),$$

$$b_{n-1} = -\frac{4n(n-1)(n+\alpha-1)(n+\beta-1)}{(2n+\alpha+\beta-3)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-1)} \cong -\frac{1}{4}.$$

Note that the polynomial $\hat{R}_{n,1}$ is uniquely determined for each n.

(e) *We deduce from [16, formula (7.32.5)] the following estimate*

$$|P_n^{(\alpha,\beta)}(\cos\theta)| \leq \begin{cases} c_1 n^{-1/2}\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2}, & \text{if } c/n \leq \theta \leq \pi - c/n, \\ c_2 n^{\alpha}, & \text{if } 0 \leq \theta \leq c/n, \\ c_3 n^{\beta}, & \text{if } \pi - c/n \leq \theta \leq \pi, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$.

(f) *Mehler–Heine formula (see [16, Theorem 8.1.1]). We have for a fixed integer k,*

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos(x/(n+k))\right) = 2^{\alpha} x^{-\alpha} J_{\alpha}(x),$$

where α, β are real numbers, and $J_\alpha(x)$ is the Bessel function of the first kind. This formula holds locally uniformly, that is, on every compact subset of the complex plane and $k \in \mathbf{Z}$.

(g) *Inner strong asymptotics. For $\theta \in [\epsilon, \pi-\epsilon]$ and $\epsilon > 0$, (see [16, Theorem 8.21.8])*

$$P_n^{(\alpha,\beta)}(\cos\theta) = \pi^{-1/2} n^{-1/2}\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\cos(\ell\theta+\gamma) + O(n^{-3/2}),$$

where $\ell = n + (\alpha+\beta+1)/2$, $\gamma = -(\alpha+1/2)\pi/2$.

(h) Outer relative asymptotics. Uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$ (see [16, p. 196])

$$\lim_{n\to\infty}\frac{\hat{P}_{n-k}^{(\alpha,\beta)}(x)}{\hat{P}_{n}^{(\alpha,\beta)}(x)}=\left(\frac{2}{\varphi(x)}\right)^{k},\qquad k\in\mathbb{N}\cup\{0\}.$$

Here $\varphi(x)=x+(x^2-1)^{1/2}$ where $(x^2-1)^{1/2}>0$ if $x>1$.

(i) For $\alpha,\beta,\mu,\nu>-1$ and $1\le p\le\infty$ (see [16, p.391, Exercise 91], [2])

$$\left(\int_0^1(1-x)^{\mu}|P_n^{(\alpha,\beta)}(x)|^p dx\right)^{1/p}\sim\begin{cases}n^{-1/2}, & \text{if } 2\mu>\mathrm{p}\alpha-2+\mathrm{p}/2,\\ n^{-1/2}(\log\ n)^{1/p}, & \text{if } 2\mu=\mathrm{p}\alpha-2+\mathrm{p}/2,\\ n^{\alpha-\frac{2\mu+2}{p}}, & \text{if } 2\mu<\mathrm{p}\alpha-2+\mathrm{p}/2,\end{cases}$$

$$\left(\int_{-1}^0(1+x)^{\nu}|P_n^{(\alpha,\beta)}(x)|^p dx\right)^{1/p}\sim\begin{cases}n^{-1/2}, & \text{if } 2\nu>p\beta-2+p/2,\\ n^{-1/2}(\log\ n)^{1/p}, & \text{if } 2\nu=p\beta-2+p/2,\\ n^{\beta-\frac{2\nu+2}{p}}, & \text{if } 2\nu<p\beta-2+p/2.\end{cases}$$

Next, we will consider the monic polynomials $\hat{R}_{n,N}^{(\alpha,\beta)}$, $N\ge 0$, defined as

$$\hat{R}_{n,0}^{(\alpha,\beta)}(x):=\hat{P}_{n}^{(\alpha,\beta)}(x),$$
$$\left(\hat{R}_{n,N}^{(\alpha,\beta)}(x)\right)':=n\hat{R}_{n-1,N-1}^{(\alpha,\beta)}(x),\qquad N\ge 1.$$

Observe that using (2.1) in a recursive way the polynomials $\hat{R}_{n,N}^{(\alpha,\beta)}$ are uniquely determined. For example,

$$\left(\hat{R}_{n,2}^{(\alpha,\beta)}(x)\right)'=n\hat{R}_{n-1,1}^{(\alpha,\beta)}(x)=n\left(\hat{P}_{n-1}^{(\alpha,\beta)}(x)+a_{n-2}\hat{P}_{n-2}^{(\alpha,\beta)}(x)+b_{n-2}\hat{P}_{n-3}^{(\alpha,\beta)}(x)\right),$$

and integrating directly, we obtain

$$\hat{R}_{n,2}^{(\alpha,\beta)}(x)=\hat{R}_{n,1}^{(\alpha,\beta)}(x)+\frac{na_{n-2}}{n-1}\hat{R}_{n-1,1}^{(\alpha,\beta)}(x)+\frac{nb_{n-2}}{n-2}\hat{R}_{n-2,1}^{(\alpha,\beta)}(x).$$

Hence, for $j=0,1,...,N$,

$$\left(\hat{R}_{n,N}^{(\alpha,\beta)}(x)\right)^{(j)}:=[n]_j\ \hat{R}_{n-j,N-j}^{(\alpha,\beta)}(x), \tag{2.2}$$

where the symbol $[n]_j$ is used to represent the falling factorial, i.e., $[n]_0=1$, $[n]_j=n\cdots(n-j+1),\ j\ge 1$.

Proposition 2.2. *It holds*

$$\hat{R}_{n,N}^{(\alpha,\beta)}(x)=\sum_{k=0}^{2N}A_{k,n}^{[N]}\hat{P}_{n-k}^{(\alpha,\beta)}(x),\quad n>2N, \tag{2.3}$$

where $A_{0,n}^{[N]}=1,\ A_{2i-1,n}^{[N]}=O\left(\frac{1}{n}\right),\ A_{2i,n}^{[N]}\cong\frac{(-1)^i}{4^i}\binom{N}{i}$, for $i=1,\ ...,\ N$.

Proof. We will prove (2.3) by induction on N. When $N=1$ it is trivially deduced from Proposition 2.1 (d). Now, we assume (2.3) holds for $N \geq 1$. Then, taking into account Proposition 2.1 (d), we have

$$\begin{aligned}\hat{R}^{(\alpha,\beta)}_{n,N+1}(x) &= n \int \hat{R}^{(\alpha,\beta)}_{n-1,N}(x)dx \\ &= n \sum_{k=0}^{2N} A^{[N]}_{k,n-1} \int \hat{P}^{(\alpha,\beta)}_{n-k-1}(x)dx = \sum_{k=0}^{2N} \frac{n}{n-k} A^{[N]}_{k,n-1} \\ &\quad \times \left[\hat{P}^{(\alpha,\beta)}_{n-k}(x) + a_{n-k-1}\hat{P}^{(\alpha,\beta)}_{n-k-1}(x) + b_{n-k-1}\hat{P}^{(\alpha,\beta)}_{n-k-2}(x) \right].\end{aligned}$$

Since

$$\begin{aligned}\sum_{k=0}^{2N} \frac{n}{n-k} A^{[N]}_{k,n-1} \hat{P}^{(\alpha,\beta)}_{n-k}(x) &= A^{[N]}_{0,n-1}\hat{P}^{(\alpha,\beta)}_{n}(x) \\ &+ \frac{n}{n-1} A^{[N]}_{1,n-1}\hat{P}^{(\alpha,\beta)}_{n-1}(x) + \sum_{k=0}^{2N-2} \frac{n}{n-k-2} A^{[N]}_{k+2,n-1}\hat{P}^{(\alpha,\beta)}_{n-k-2}(x),\end{aligned}$$

$$\begin{aligned}&\sum_{k=0}^{2N} \frac{n a_{n-k-1}}{n-k} A^{[N]}_{k,n-1} \hat{P}^{(\alpha,\beta)}_{n-k-1}(x) = a_{n-1}A^{[N]}_{0,n-1}\hat{P}^{(\alpha,\beta)}_{n-1}(x) \\ &+ \sum_{k=0}^{2N-2} \frac{n a_{n-k-2}}{n-k-1} A^{[N]}_{k+1,n-1}\hat{P}^{(\alpha,\beta)}_{n-k-2}(x) + \frac{n a_{n-2N-1}}{n-2N} A^{[N]}_{2N,n-1}\hat{P}^{(\alpha,\beta)}_{n-2N-1}(x),\end{aligned}$$

$$\begin{aligned}&\sum_{k=0}^{2N} \frac{n b_{n-k-1}}{n-k} A^{[N]}_{k,n-1} \hat{P}^{(\alpha,\beta)}_{n-k-2}(x) = \sum_{k=0}^{2N-2} \frac{n b_{n-k-1}}{n-k} A^{[N]}_{k,n-1}\hat{P}^{(\alpha,\beta)}_{n-k-2}(x) \\ &+ \frac{n b_{n-2N}}{n-2N+1} A^{[N]}_{2N-1,n-1}\hat{P}^{(\alpha,\beta)}_{n-2N-1}(x) + \frac{n b_{n-2N-1}}{n-2N} A^{[N]}_{2N,n-1}\hat{P}^{(\alpha,\beta)}_{n-2N-2}(x).\end{aligned}$$

Thus,

$$\hat{R}^{(\alpha,\beta)}_{n,N+1}(x) = \sum_{k=0}^{2N+2} A^{[N+1]}_{k,n} \hat{P}^{(\alpha,\beta)}_{n-k}(x),$$

where

$$A^{[N+1]}_{0,n} = A^{[N]}_{0,n} = 1,$$

$$A^{[N+1]}_{1,n} = \frac{n}{n-1} A^{[N]}_{1,n-1} + a_{n-1} A^{[N]}_{0,n-1} = O\left(\frac{1}{n}\right),$$

$$A^{[N+1]}_{k,n} = \frac{n}{n-k} A^{[N]}_{k,n-1} + \frac{n a_{n-k}}{n-k+1} A^{[N]}_{k-1,n-1} + \frac{n b_{n-k+1}}{n-k+2} A^{[N]}_{k-2,n-1}, \quad 2 \leq k \leq 2N,$$

$$A^{[N+1]}_{2N+1,n} = \frac{n a_{n-2N-1}}{n-2N} A^{[N]}_{2N,n-1} + \frac{n b_{n-2N}}{n-2N+1} A^{[N]}_{2N-1,n-1} = O\left(\frac{1}{n}\right),$$

$$A_{2N+2,n}^{[N+1]} = \frac{nb_{n-2N-1}}{n-2N} A_{2N,n-1}^{[N]} \cong \frac{(-1)^{N+1}}{4^{N+1}}.$$

Finally, let us estimate the coefficients $A_{k,n}^{[N+1]}$, $2 \leq k \leq 2N$. If k is an odd number, i.e. $k = 2i+1$, $i = 1, \dots, N-1$, then

$$A_{2i+1,n}^{[N+1]} = O\left(\frac{1}{n}\right).$$

If k is an even number, i.e. $k = 2i$, $i = 1, \dots, N$, then

$$A_{2i,n}^{[N+1]} \cong \frac{(-1)^i}{4^i}\binom{N}{i} - \frac{1}{4}\frac{(-1)^{i-1}}{4^{i-1}}\binom{N}{i-1} = \frac{(-1)^i}{4^i}\binom{N+1}{i},$$

which proves the statement of this proposition. □

Let us denote by $R_{n,N}^{(\alpha,\beta)} := \kappa_n \hat{R}_{n,N}^{(\alpha,\beta)}$. From (2.2) and Proposition 2.2, we have

$$\left(R_{n,N}^{(\alpha,\beta)}(x)\right)^{(j)} = [n]_j \frac{\kappa_n}{\kappa_{n-j}} R_{n-j,N-j}^{(\alpha,\beta)}(x), \tag{2.4}$$

and

$$R_{n,N}^{(\alpha,\beta)}(x) = \sum_{k=0}^{2N} B_{k,n}^{[N]} P_{n-k}^{(\alpha,\beta)}(x), \quad n > 2N, \tag{2.5}$$

where $B_{k,n}^{[N]} = \frac{\kappa_n}{\kappa_{n-k}} A_{k,n}^{[N]} \cong 2^k A_{k,n}^{[N]}$.

Remark 2.1. *Notice that from the construction of the polynomials $R_{n,N}^{(\alpha,\beta)}(x)$ we can deduce that*

$$R_{n,N}^{(\alpha,\beta)}(x) = P_n^{(\alpha-N,\beta-N)}(x). \tag{2.6}$$

In this way, Proposition 2.2 corresponds to a connection problem between two families of Jacobi polynomials (see, for example, [9, Proposition 9.1.1]). For our purposes we are interested on the right–hand side of (2.3) (or (2.5)) and on the asymptotic behaviour of the corresponding coefficients. Thus, taking this in mind, we will use the notation $R_{n,N}^{(\alpha,\beta)}(x)$ for these polynomials because it is more comfortable for our objectives.

Now, we can state some results for the polynomials $R_{n,N}^{(\alpha,\beta)}$ which are useful for our purposes. To simplify the notation we use $R_{n,N} := R_{n,N}^{(\alpha,\beta)}$ when there is no confusion.

Proposition 2.3. *(a) For $j = 0, 1, ..., N$, we have the following estimate*

$$|R_{n,N}^{(j)}(\cos\theta)| \leq \begin{cases} c_1 n^{j-1/2} \theta^{N-j-\alpha-1/2} (\pi-\theta)^{N-j-\beta-1/2}, & \text{if } c/n \leq \theta \leq \pi - c/n, \\ c_2 n^{\alpha+2j-N}, & \text{if } 0 \leq \theta \leq c/n, \\ c_3 n^{\beta+2j-N}, & \text{if } \pi - c/n \leq \theta \leq \pi, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$.

(b) Mehler–Heine type formula. For $j = 0, 1, ..., N$, it holds

$$\lim_{n\to\infty} \frac{R_{n,N}^{(N-j)}(\cos(x/(n+k)))}{n^{\alpha+N-2j}} = 2^{N+\alpha} x^{-\alpha+j} J_{\alpha-j}(x), \tag{2.7}$$

uniformly on compact subsets of $\mathbb{C}$ and $k \in \mathbf{Z}$.

(c) Inner strong asymptotics. For $\theta \in [\epsilon, \pi-\epsilon]$ and $\epsilon > 0$,

$$R_{n,N}^{(j)}(\cos\theta) = \pi^{-1/2}\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\frac{[n]_j\ \kappa_n}{\kappa_{n-j}}$$
$$\times\left[\sum_{k=0}^{2N-2j}(n-k-j)^{-1/2}B_{k,n-j}^{[N-j]}\cos((\ell-k-j)\theta+\gamma)+O(n^{-3/2})\right],$$

where ℓ, γ, and $B_{k,n-j}^{[N-j]}$ are given in Proposition 2.1 (g) and (2.5).

(d) Outer relative asymptotics. It holds,

$$\lim_{n\to\infty}\frac{\hat{R}_{n,N}^{(j)}(x)}{n^j\,\hat{P}_{n-j}^{(\alpha,\beta)}(x)} = \left(\frac{2}{\varphi'(x)}\right)^{N-j},$$

$$\lim_{n\to\infty}\frac{\hat{R}_{n,N}^{(j)}(x)}{(\hat{P}_{n-j}^{(\alpha,\beta)})^{(j)}(x)} = \frac{2^{N-j}\left(\sqrt{x^2-1}\right)^j}{(\varphi'(x))^{N-j}},$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$, where φ is given in Proposition 2.1 (h).

Proof. All these results can be deduced from Propositions 2.1 and 2.2, (2.5), and (2.6). □

3. Jacobi–Sobolev orthogonal polynomials. Asymptotics

Let $\{Q_n^{(\alpha,\beta,N)}\}_{n\geq 0}$ denote the sequence of polynomials orthogonal with respect to (1.1) normalized by the condition that they have the same leading coefficient as $R_{n,N}^{(\alpha,\beta)}$, i.e. $\kappa(Q_n^{(\alpha,\beta,N)}) = \kappa_n$. On the other hand, it is very important to observe that we can deduce from (1.1) the following symmetry property

$$Q_n^{(\beta,\alpha,N)}(-x) = (-1)^n Q_n^{(\alpha,\beta,N)}(x). \tag{3.1}$$

As we have commented before, to simplify notation we will use Q_n and $R_{n,N}$ for denoting the polynomials $Q_n^{(\alpha,\beta,N)}$ and $R_{n,N}^{(\alpha,\beta)}$, respectively.

First, we introduce a relation between the families of polynomials Q_n and $R_{n,N}$.

Proposition 3.1. *For $\alpha, \beta > -1$, it holds*

(*i*) *If $\lambda_j = 0$, $0 \le j \le N-1$, then*

$$R_{n,N}(x) = Q_n(x).$$

(*ii*) *If there exists $0 \le j \le N-1$ such that $\lambda_j \neq 0$, then*

$$R_{n,N}(x) = Q_n(x) + \sum_{k=1}^{2N-2s} \alpha_{n-k}^{(n)} Q_{n-k}(x), \quad n > 2N-2s, \tag{3.2}$$

where $s = \min\{j: \ \lambda_j \neq 0\}$, $0 \le j \le N$, and

$$\alpha_{n-k}^{(n)} = \frac{\sum_{j=0}^{[N-k/2]} \lambda_j [n]_j \frac{\kappa_n}{\kappa_{n-j}} \sum_{i=k}^{2N-2j} B_{i,n-j}^{[N-j]} \langle P_{n-i-j}^{(\alpha,\beta)}, Q_{n-k}^{(j)} \rangle}{||Q_{n-k}||_{W^{N,2}}^2}.$$

Moreover, $|\alpha_{n-2N+2s}^{(n)}| = O\left(\frac{1}{n^{2N-2s}}\right) = O\left(\frac{1}{n^2}\right)$ and $|\alpha_{n-k}^{(n)}| = O\left(\frac{1}{n^{N-l}}\right) = O\left(\frac{1}{n}\right)$, $(l = \max\{j : \lambda_j > 0, \ 0 \le j \le [N-k/2]\})$, for $1 \le k < 2N-2s$.

Proof. Expanding $R_{n,N}$ with respect to the basis $\{Q_k\}_{k=0}^n$ of the linear space of polynomials with degree at most n, we get

$$R_{n,N}(x) = Q_n(x) + \sum_{k=1}^{n} \alpha_{n-k}^{(n)} Q_{n-k}(x),$$

where,

$$\alpha_{n-k}^{(n)} = \frac{\langle R_{n,N}, Q_{n-k} \rangle_S}{\langle Q_{n-k}, Q_{n-k} \rangle_S}.$$

Using (2.4) and (2.5) as well as the fact that deg $Q_{n-k}^{(j)} = n-k-j$, we have

$$\int_{-1}^{1} R_{n,N}^{(j)}(x) Q_{n-k}^{(j)}(x) d\mu_{\alpha,\beta}(x) = [n]_j \frac{\kappa_n}{\kappa_{n-j}} \int_{-1}^{1} R_{n-j,N-j}(x) Q_{n-k}^{(j)}(x) d\mu_{\alpha,\beta}(x)$$
$$= [n]_j \frac{\kappa_n}{\kappa_{n-j}} \sum_{i=0}^{2N-2j} B_{i,n-j}^{[N-j]} \int_{-1}^{1} P_{n-i-j}^{(\alpha,\beta)}(x) Q_{n-k}^{(j)}(x) d\mu_{\alpha,\beta}(x)$$
$$= [n]_j \frac{\kappa_n}{\kappa_{n-j}} \sum_{i=k}^{2N-2j} B_{i,n-j}^{[N-j]} \int_{-1}^{1} P_{n-i-j}^{(\alpha,\beta)}(x) Q_{n-k}^{(j)}(x) d\mu_{\alpha,\beta}(x).$$

Therefore,

$$\langle R_{n,N}, Q_{n-k} \rangle_S = 0, \quad k > 2N,$$

and we get,

$$\alpha_{n-k}^{(n)} = 0, \quad k > 2N,$$

$$\alpha_{n-k}^{(n)} = \frac{\sum_{j=0}^{[N-k/2]} \lambda_j [n]_j \frac{\kappa_n}{\kappa_{n-j}} \sum_{i=k}^{2N-2j} B_{i,n-j}^{[N-j]} \langle P_{n-i-j}^{(\alpha,\beta)}, Q_{n-k}^{(j)} \rangle}{||Q_{n-k}||_{W^{N,2}}^2}, \quad 1 \le k \le 2N. \tag{3.3}$$

Let $s=\min\{j:\ \lambda_j\neq 0\}$, $0\le j\le N$. Then for $k>2(N-s)$ we have $\alpha_{n-k}^{(n)}=0$. In particular, if $s=N$ we have $R_{n,N}(x)=Q_n(x)$. Thus, in the sequel we suppose that $s\le N-1$.

Using the extremal property for monic orthogonal polynomials with respect to the corresponding norm, i.e.

$$||\hat{P}_n^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})}=\inf\{||P||^2_{L^2(d\mu_{\alpha,\beta})}:\deg P=n,\ P\text{ monic}\},$$

we get

$$||\hat{Q}_n||^2_{W^{N,2}}\ge\lambda_N||\hat{Q}_n^{(N)}||^2_{L^2(d\mu_{\alpha,\beta})}\ge cn^{2N}||\hat{P}_{n-N}^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})}.$$

Thus, using Proposition 2.1 (a)–(b), we obtain

$$||Q_n||^2_{W^{N,2}}\ge cn^{2N}\left(\frac{\kappa_n}{\kappa_{n-N}}\right)^2||P_{n-N}^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})}\ge c_1n^{2N-1}.$$

Finally, taking into account (3.3) and fact that $Q_{n-2N+2s}^{(s)}=[n-2N+2s]_s\frac{\kappa_{n-2N+2s}}{\kappa_{n-2N+s}}P_{n-2N+s}^{(\alpha,\beta)}+$ *lower degree terms*, we find out that

$$\begin{aligned}|\alpha_{n-2N+2s}^{(n)}| &= \frac{\lambda_s[n]_s\frac{\kappa_n}{\kappa_{n-s}}|B_{2N-2s,n-s}^{[N-s]}|\,|\langle P_{n-2N+s}^{(\alpha,\beta)},Q_{n-2N+2s}^{(s)}\rangle|}{||Q_{n-2N+2s}||^2_{W^{N,2}}}\\ &\le c_2\frac{||P_{n-2N+s}^{(\alpha,\beta)}||^2_{L^2(d\mu_{\alpha,\beta})}}{n^{2N-2s-1}}=O\left(\frac{1}{n^{2N-2s}}\right)=O\left(\frac{1}{n^2}\right),\end{aligned}$$

and using the Cauchy–Schwarz inequality

$$|\langle P_{n-i-j}^{(\alpha,\beta)},Q_{n-k}^{(j)}\rangle|\le||P_{n-i-j}^{(\alpha,\beta)}||_{L^2(d\mu_{\alpha,\beta})}||Q_{n-k}^{(j)}||_{L^2(d\mu_{\alpha,\beta})}\le\frac{c_3||Q_{n-k}||_{W^{N,2}}}{n^{1/2}}.$$

Thus

$$|\alpha_{n-k}^{(n)}|\le\frac{c_4}{n^N}\sum_{j=0}^{[N-k/2]}\lambda_jn^j\le\frac{c_5}{n^{N-l}},$$

where $l=\max\{j:\lambda_j>0,\ 0\le j\le[N-k/2]\}$, $1\le k<2N-2s$. $\square$

Applying (3.2) and using a similar technique as used in [8, Proposition 3.2], for $s\le N-1$, we get

Proposition 3.2. *It holds*

$$Q_n(x)=\sum_{m=0}^{n}b_{1,n}^{(m)}R_{n-m,N}(x),$$

where $b_{1,n}^{(0)}=1$, $b_{k,n}^{(1)}=-\alpha_{n-k}^{(n)}$, *and* $b_{k,n}^{(m)}=-b_{1,n}^{(m-1)}\alpha_{n-k-m+1}^{(n-m+1)}+b_{k+1,n}^{(m-1)}$, $k=1,2,\dots,2(N-s)$, $m=2,3,\dots,n$. *Moreover,* $|b_{k,n}^{(m)}|=O\left(\frac{1}{n}\right)$ *for* $m=1,2,...,2(N-s)-1$, *and* $|b_{k,n}^{(m)}|=O\left(\frac{1}{n^2}\right)$ *for* $m=2(N-s),2(N-s)+1,...,n$.

With these previous results, now we can give some asymptotic properties of the Jacobi–Sobolev orthogonal polynomials.

Proposition 3.3. *(a) For $j = 0, 1, ..., N,$ we have the following estimate*

$$|Q_n^{(j)}(\cos\theta)| \leq \begin{cases} c_1 n^{j-1/2}\theta^{N-j-\alpha-1/2}\,(\pi-\theta)^{N-j-\beta-1/2}, & \text{if } c/n \leq \theta \leq \pi - c/n, \\ c_2 n^{\alpha+2j-N}, & \text{if } 0 \leq \theta \leq c/n, \\ c_3 n^{\beta+2j-N}, & \text{if } \pi - c/n \leq \theta \leq \pi, \end{cases}$$

where $\alpha, \beta > -1$.

(b) Mehler–Heine type formula. It holds,

$$\lim_{n\to\infty} \frac{Q_n^{(N-j)}(\cos(x/n))}{n^{\alpha+N-2j}} = 2^{N+\alpha} x^{-\alpha+j} J_{\alpha-j}(x),$$

uniformly on compact subsets of $\mathbb{C}$.

(c) Inner strong asymptotics. For $\theta \in [\epsilon, \pi-\epsilon]$ and $\epsilon > 0$,

$$Q_n^{(j)}(\cos\theta) = \pi^{-1/2}\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\frac{[n]_j\ \kappa_n}{\kappa_{n-j}}$$
$$\times\left[\sum_{k=0}^{2N-2j}(n-k-j)^{-1/2}B_{k,n-j}^{[N-j]}\cos((\ell-k-j)\theta+\gamma) + O(n^{-3/2})\right],$$

where ℓ, γ, and $B_{k,n-j}^{[N-j]}$ are given in Proposition 2.1 (g) and (2.5).

(d) Outer relative asymptotics. It holds,

$$\lim_{n\to\infty}\frac{\hat{Q}_n^{(j)}(x)}{n^j\hat{P}_{n-j}^{(\alpha,\beta)}(x)} = \left(\frac{2}{\varphi'(x)}\right)^{N-j},$$

$$\lim_{n\to\infty}\frac{\hat{Q}_n^{(j)}(x)}{\hat{P}_{n-j}^{(\alpha,\beta)(j)}(x)} = \frac{2^{N-j}\left(\sqrt{x^2-1}\right)^j}{(\varphi'(x))^{N-j}},$$

uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$, where φ is given in Proposition 2.1 (h).

Proof. If $s = N$, then the results follows from Proposition 2.3. Thus in sequel we assume $s \leq N-1$.

(a) Using Proposition 3.2, we get

$$Q_n^{(j)}(x) = \sum_{m=0}^{n} b_{1,n}^{(m)} R_{n-m,N}^{(j)}(x) = R_{n,N}^{(j)}(x) + \sum_{m=1}^{n} b_{1,n}^{(m)} R_{n-m,N}^{(j)}(x).$$

The result follows taking into account the asymptotic behaviour of the coefficients $b_{1,n}^{(m)}$ given in Proposition 3.2 and the estimate given in Proposition 2.3 (a).

(b) Using again Proposition 3.2, we get

$$Q_n^{(N-j)}(x) = R_{n,N}^{(N-j)}(x) + \sum_{m=1}^{n} b_{1,n}^{(m)} R_{n-m,N}^{(N-j)}(x),$$

where

$$\begin{aligned} |b_{1,n}^{(m)}| &= O\left(\tfrac{1}{n}\right), \quad \text{for } \ m = 1, 2, ..., 2(N-s)-1, \\ |b_{1,n}^{(m)}| &= O\left(\tfrac{1}{n^2}\right), \quad \text{for } \ m = 2(N-s), 2(N-s)+1, ..., n. \end{aligned} \tag{3.4}$$

Now, scaling properly, we get

$$\begin{aligned} \frac{Q_n^{(N-j)}(\cos(x/n))}{n^{\alpha+2N-j}} &= \frac{R_{n,N}^{(N-j)}(\cos(x/n))}{n^{\alpha+2N-j}} \\ &+ \sum_{m=1}^{n} b_{1,n}^{(m)} \frac{(n-m+1)^{\alpha+2N-j}}{n^{\alpha+2N-j}} \frac{R_{n-m,N}^{(N-j)}(\cos(x/n))}{(n-m+1)^{\alpha+2N-j}}. \end{aligned}$$

Taking into account (2.7), we have for a fixed compact $K \subset \mathbb{C}$

$$\left| \frac{R_{n-m,N}^{(N-j)}(\cos(x/n))}{(n-m+1)^{\alpha+2N-j}} \right| < C, \quad m = 1, \cdots, n, \quad x \in K,$$

and by using (3.4)

$$\frac{Q_n^{(N-j)}(\cos(x/n))}{n^{\alpha+2N-j}} = \frac{R_{n,N}^{(N-j)}(\cos(x/n))}{n^{\alpha+2N-j}} + O(n^{-1}).$$

Taking limits when $n \to \infty$ and applying again (2.7), we obtain the Mehler–Heine type formula for the Sobolev polynomials Q_n and their derivatives up to order N.

(c) Using (b) of this Proposition, we can establish that the sequence $\left(\frac{Q_n^{(j)}}{n^{j-1/2}}\right)_{n\geq 0}$ is uniformly bounded on compact subsets of $(-1,1)$, thus from Proposition 3.1, we have

$$Q_n^{(j)}(x) = R_{n,N}^{(j)}(x) + O\left(\frac{1}{n}\right).$$

Now, using Proposition 2.3 (c) the statement holds.

(d) It was obtained in a more general framework in [12]. □

Next, we give an estimate for the Sobolev norms of the Jacobi–Sobolev orthogonal polynomials. Such norm estimates are useful in Lagrange interpolation as well as in the orthogonal expansions in the corresponding norm spaces.

Proposition 3.4. *For* $\alpha, \beta \geq -1/2$, $\tau = \max\{\alpha, \beta\} > -1/2$, *and* $1 \leq p \leq \infty$

$$||Q_n||_{W^{N,p}} \sim \begin{cases} cn^{N-1/2}, & \text{if } 2\tau > p\tau - 2 + p/2, \\ cn^{N-1/2}(\log\ n)^{1/p}, & \text{if } 2\tau = p\tau - 2 + p/2, \\ cn^{N+\tau-\frac{2\tau+2}{p}}, & \text{if } 2\tau < p\tau - 2 + p/2. \end{cases}$$

Proof. From Proposition 3.3 and (3.1), we have for $\tau = \max\{\alpha, \beta\} > -1/2$,

$$|Q_n^{(j)}(\cos\theta)| \leq cn^{2j-N+\tau} \leq cn^{j+\tau} \leq cn^{N+\tau}.$$

Thus, for $p = \infty$ the upper estimate holds. For $1 \leq p < \infty$, following the same way as in [16, Theorem 7.34] by using the bounds given in Proposition 3.3 (a), we get

$$||Q_n^{(j)}||_{L^p(d\mu_{\alpha,\beta})} \leq \begin{cases} cn^{j-1/2}, & \text{if } 2\tau > p\tau - 2 + p/2, \\ cn^{j-1/2}(\log\ n)^{1/p}, & \text{if } 2\tau = p\tau - 2 + p/2, \\ cn^{j+\tau-\frac{2\tau+2}{p}}, & \text{if } 2\tau < p\tau - 2 + p/2. \end{cases} \tag{3.5}$$

On the other hand, according to (3.1) and Proposition 3.3 (b), and using a technique similar to [16, Theorem 7.34] (see also [8]), we prove that

$$||Q_n^{(N)}||_{L^p(d\mu_{\alpha,\beta})} \geq \begin{cases} cn^{N-1/2}, & \text{if } 2\tau > p\tau - 2 + p/2, \\ cn^{N-1/2}(\log\ n)^{1/p}, & \text{if } 2\tau = p\tau - 2 + p/2, \\ cn^{N+\tau-\frac{2\tau+2}{p}}, & \text{if } 2\tau < p\tau - 2 + p/2. \end{cases}$$

Thus, the proof of this result is complete. □

Remark 3.1. *Notice that the Sobolev norm of* Q_n *given in Proposition 3.4 depends on Lebesgue norm of* $Q_n^{(N)}$. *Taking into account that the zeros of* $Q_n^{(j)}$ *accumulate in* $[-1, 1]$, *this can be proved even in other way by using Turan's type inequality (see for instance, [5], [17]).*

4. A Cohen type inequality for Jacobi–Sobolev expansions

For $f \in W^{N,1}$, its Fourier expansion in terms of Jacobi–Sobolev polynomials is

$$\sum_{k=0}^{\infty} \hat{f}(k) Q_k(x), \tag{4.1}$$

where

$$\hat{f}(k) = \frac{\langle f, Q_k \rangle_S}{\|Q_k\|^2_{W^{N,2}}}, \quad k = 0, 1, ...$$

The Cesàro means of order δ of the expansion (4.1) is defined by (see [18, p. 76–77])

$$\sigma_n^\delta f(x) = \sum_{k=0}^{n} \frac{C_{n-k}^\delta}{C_n^\delta} \hat{f}(k) Q_k(x),$$

where $C_k^\delta = \binom{k+\delta}{k}$. For $\delta = 0$, the nth partial sum of the expansion (4.1) appear.

For a function $f \in W^{N,p}$ and a fixed sequence $\{c_{k,n}\}_{k=0}^{n}$, $n \in \mathbf{N} \cup \{\mathbf{0}\}$, of real numbers with $c_{k,n} = o(n^{k-n}c_{n,n})$, $k = n-2N, \ldots, n-1$, we define the operators $\mathcal{T}_n$ by

$$\mathcal{T}_n(f) = \sum_{k=0}^{n} c_{k,n} \hat{f}(k) Q_k.$$

Let us denote by $[X]$ the space of all bounded, linear operators from the space X into itself, with the usual operator norm $||\cdot||_{[X]}$. Now we can establish the main result of this section.

Theorem 4.1. *For $\alpha, \beta \geq -1/2$ and $\tau = \max\{\alpha, \beta\} > -1/2$,*

$$\|\mathcal{T}_n\|_{[W^{N,p}]} \geq c|c_{n,n}| \begin{cases} n^{\frac{2\tau+2}{p} - \frac{2\tau+3}{2}}, & \text{if } 1 \leq p < p_0, \\ (\log\ n)^{\frac{2\tau+1}{4\tau+4}}, & \text{if } p = p_0,\ p = q_0, \\ n^{\frac{2\tau+1}{2} - \frac{2\tau+2}{p}}, & \text{if } q_0 < p \leq \infty, \end{cases}$$

where $q_0 = (4\tau+4)/(2\tau+1)$ and p_0 is the conjugate of q_0.

Proof. We shall use as test functions

$$g_{n,N}(x) := g_{n,N}^{\alpha,\beta,m}(x) := (1-x^2)^{m+N} P_n^{(\alpha+m,\beta+m)}(x), \tag{4.2}$$

where $m \in \mathbb{N}$.

Applying the operator $\mathcal{T}_n$ to $g_{n,N}^{\alpha,\beta,m}$, for some $m > \tau + 1/2 - 2(\tau+1)/p$ we get

$$\mathcal{T}_n(g_{n,N}) = \sum_{k=0}^{n} c_{k,n} \widehat{g_{n,N}}(k) Q_k, \tag{4.3}$$

where, for $k = 0, 1, \ldots, n$,

$$\widehat{g_{n,N}}(k) = (\|Q_k\|^2_{W^{N,2}})^{-1} \langle g_{n,N}, Q_k \rangle_S = (\|Q_k\|^2_{W^{N,2}})^{-1} \sum_{j=0}^{N} \lambda_j \int_{-1}^{1} g_{n,N}^{(j)}(x) Q_k^{(j)}(x) d\mu_{\alpha,\beta}(x),$$

and, using Proposition 3.4 we deduce

$$\|Q_n\|^2_{W^{N,2}} \sim n^{2N-1}.$$

Taking into account Proposition 2.1 (c), (4.2) and by using the Leibniz rule we have

$$\int_{-1}^{1} g_{n,N}^{(j)}(x) Q_k^{(j)}(x) d\mu_{\alpha,\beta}(x) = \sum_{i=0}^{j} \binom{j}{i} \frac{(n+\alpha+\beta+2m+1)_{j-i}}{2^{j-i}}$$
$$\times \int_{-1}^{1} (1-x^2)^{m+N-i} R_i(x) P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}(x) Q_k^{(j)}(x) d\mu_{\alpha,\beta}(x)$$
$$= \sum_{i=0}^{j} \binom{j}{i} \frac{(n+\alpha+\beta+2m+1)_{j-i}}{2^{j-i}} \int_{-1}^{1} P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}(x)$$
$$\times S_{k+2N-3j+i}(x) d\mu_{\alpha+m+j-i,\beta+m+j-i}(x),$$

where

$$
\begin{aligned}
R_i(x) &= (-1)^i \sum_{l=0}^{i} \binom{i}{l} [m+N]_l [m+N]_{i-l} (x-1)^{i-l}(x+1)^l \\
&= r_i x^i + \cdots + r_0, \\
S_{k+2N-3j+i}(x) &= (1-x^2)^{N-j} R_i(x) Q_k^{(j)}(x).
\end{aligned}
$$

Since $\deg S_{k+2N-3j+i} = k+2N-3j+i$, then for $k < n-2N+2j$,

$$
\int_{-1}^{1} g_{n,N}^{(j)}(x) Q_k^{(j)}(x) d\mu_{\alpha,\beta}(x) = 0.
$$

Now, for $k=n$ and $j=N$, using (2.4) and Proposition 3.2 we have

$$
\begin{aligned}
S_{n-N+i}(x) &= R_i(x) \sum_{m=0}^{n} [n-m]_N \frac{\kappa_{n-m}}{\kappa_{n-m-N}} b_{1,n}^{(m)} P_{n-m-N}^{(\alpha,\beta)}(x) \\
&= [n]_N \frac{\kappa_n}{\kappa_{n-N}} R_i(x) P_{n-N}^{(\alpha,\beta)}(x) + T_{n-N+i-1}(x) \\
&= \gamma_{n,i,N}^{\alpha,\beta} P_{n-N+i}^{(\alpha+m+n-i,\beta+m+n-i)}(x) + V_{n-N+i-1}(x),
\end{aligned}
$$

where $\deg T_{n-N+i-1} \le n-N+i-1$, $\deg V_{n-N+i-1} \le n-N+i-1$, and taking into account Proposition 2.1 (a),

$$
\gamma_{n,i,N}^{\alpha,\beta} = [n]_N \frac{r_i \kappa_n}{\kappa\left(P_{n-N+i}^{(\alpha+m+N-i,\beta+m+N-i)}\right)} \sim n^N.
$$

Finally, we consider the case $k \ge n-2N+2j$ and $j = 0,1,\ldots,[\frac{k-n+2N}{2}]$. Then, using the Cauchy–Schwarz inequality, Proposition 2.1 (b), (3.5), and taking into account that $\max_{-1\le x\le 1}|R_i(x)| \le c$, $0 \le i \le j$, we obtain

$$
\begin{aligned}
&\left| \int_{-1}^{1} g_{n,N}^{(j)}(x) Q_k^{(j)}(x) d\mu_{\alpha,\beta}(x) \right| \le c_1 \sum_{i=0}^{j} n^{j-i} \\
&\times ||(1-x^2)^{m+N-i} P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}||_{L^2(d\mu_{\alpha,\beta})} ||Q_k^{(j)}||_{L^2(d\mu_{\alpha,\beta})} \\
&\le c_2 n^{2j-1/2} ||P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}||_{L^2(d\mu_{\alpha+m+j-i,\beta+m+j-i})} \le c_3 n^{k-n+2N-1}.
\end{aligned}
$$

Moreover, using again Proposition 2.1 (b), we get

$$\int_{-1}^{1} g_{n,N}^{(N)}(x) Q_n^{(N)}(x) d\mu_{\alpha,\beta}(x) = \sum_{i=0}^{N} \binom{N}{i} \frac{(n+\alpha+\beta+2m+1)_{N-i}}{2^{N-i}} \gamma_{n,i,N}^{\alpha,\beta}$$
$$\times \int_{-1}^{1} \left(P_{n-N+i}^{(\alpha+m+N-i,\beta+m+N-i)}(x) \right)^2 d\mu_{\alpha+m+N-i,\beta+m+N-i}(x)$$
$$= \frac{(n+\alpha+\beta+2m+1)_N}{2^N} \gamma_{n,0,N}^{\alpha,\beta}$$
$$\times \int_{-1}^{1} \left(P_{n-N}^{(\alpha+m+N,\beta+m+N)}(x) \right)^2 d\mu_{\alpha+m+N,\beta+m+N}(x) + O(n^{2N-2})$$
$$\sim n^{2N-1}.$$

Gathering all the cases we obtain,

$$\begin{cases} \widehat{g_{n,N}}(k) = 0, & \text{if } 0 \le k < n-2N, \\ \widehat{g_{n,N}}(k) = O(n^{k-n}), & \text{if } n-2N \le k \le n-1, \\ \widehat{g_{n,N}}(n) \sim c. & \end{cases} \tag{4.4}$$

Now, we are going to estimate

$$\|g_{n,N}\|_{W^{N,p}}^p = \sum_{j=0}^{N} \lambda_j \|g_{n,N}^{(j)}\|_{L^p(d\mu_{\alpha,\beta})}^p.$$

For $0 \le j \le N$,

$$\begin{aligned} \left| g_{n,N}^{(j)}(x) \right| &\le \sum_{i=0}^{j} \binom{j}{i} \frac{(n+\alpha+\beta+2m+1)_{j-i}}{2^{j-i}} \\ &\times (1-x^2)^{m+N-i} |R_i(x)| \left| P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}(x) \right| \\ &\le cn^j \sum_{i=0}^{j} (1-x^2)^{m+j-i} \left| P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}(x) \right|. \end{aligned}$$

Therefore, using the Minkowski inequality and taking into account Proposition 2.1 (i), we obtain for $m > \tau + 1/2 - 2(\tau+1)/p - j + i$,

$$\|g_{n,N}^{(j)}\|_{L^p(d\mu_{\alpha,\beta})} \le c_1 n^j \sum_{i=0}^{j} \left[\int_{-1}^{1} (1-x^2)^{p(m+j-i)} |P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}(x)|^p d\mu_{\alpha,\beta} \right]^{1/p}$$
$$\le c_2 n^j \sum_{i=0}^{j} \left[\int_{0}^{1} (1-x)^{p(m+j-i)+\alpha} |P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}(x)|^p dx \right.$$
$$\left. + \int_{-1}^{0} (1+x)^{p(m+j-i)+\beta} |P_{n-j+i}^{(\alpha+m+j-i,\beta+m+j-i)}(x)|^p dx \right]^{1/p} \le c_3 n^{N-1/2}.$$

Hence, we have for $m > \tau + 1/2 - 2(\tau+1)/p$,

$$\|g_{n,N}\|_{W^{N,p}} \leq cn^{N-1/2}. \tag{4.5}$$

Finally, by duality, it is enough to assume that $q_0 \leq p \leq \infty$. Using (4.3), (4.4), and (4.5) we deduce

$$\begin{aligned} \|\mathcal{T}_n\|_{[W^{N,p}]} &\geq [\|g_{n,N}\|_{W^{N,p}}]^{-1} \|\mathcal{T}_n(g_{n,N})\|_{W^{N,p}} \\ &\geq c_5 n^{1/2-N} \Big[|c_{n,n}\widehat{g_{n,N}}(n)| \, ||Q_n||_{W^{N,p}} \\ &\quad -c \sum_{k=n-2N}^{n-1} |c_{k,n}\widehat{g_{n,N}}(k)| \, ||Q_k||_{W^{N,p}} \Big] \\ &\sim cn^{1/2-N}|c_{n,n}|||Q_n||_{W^{N,p}} \left(1 - c\left|\sum_{k=n-2N}^{n-1} \frac{c_{k,n}}{c_{n,n}} n^{k-n}\right|\right). \end{aligned}$$

Now, from Proposition 3.4 the statement of the theorem follows. □

Corollary 4.1. *Let* $\alpha, \beta \geq -1/2$, $\tau = \max\{\alpha, \beta\} > -1/2$, $1 \leq p \leq \infty$ *and* $\delta \geq 0$. *Let*

$$p_0(\delta) = \frac{4\tau+4}{2\tau+2\delta+3} \quad \text{and} \quad q_0(\delta) = \frac{4\tau+4}{2\tau-2\delta+1}.$$

Then, for $p \notin (p_0(0), q_0(0))$ *or if* $\delta > 0$, $p \notin [p_0(\delta), q_0(\delta)]$, *we get*

$$||\sigma_n^\delta||_{[W^{N,p}]} \to \infty, \qquad n \to \infty.$$

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