The holonomic equation of the Laguerre-Sobolev-Type Orthogonal Polynomials: A nondiagonal case

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Abstract

In this paper we consider the Sobolev-Type inner product

$$\langle p,q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0),\ \alpha > -1,$$

where p and q are polynomials with real coefficients,

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix},\ \mathbb{P}(0) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix},\ \mathbb{Q}(0) = \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix},$$

and A is a positive semidefinite matrix.

First, we consider a multiplication operator that is symmetric with respect to the above inner product. As a consequence, We prove that the sequence of monic polynomials orthogonal with respect to the above inner product satisfes a five-term recurrence relation. On the other hand, we obtain raising and lowering operators associated with them. As a consequence, an holonomic equation satisfied by these polynomials is given.

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1 Introduction.

Orthogonal polynomials with respect to a Sobolev-Type inner product

$$\langle p, q \rangle = \int_{\mathbb{R}} p(x)q(x)d\mu(x) + \mathbb{P}(c)^t A \mathbb{Q}(c), \tag{1}$$

where $d\mu$ is a nontrivial positive measure supported on the real line, $A \in \mathbb{R}^{(N,N)}$ is a positive semidefinite matrix, p and q are polynomials with real coefficients, and $\mathbb{Q}(c) = \big(q(c), q'(c), \ldots, q^{(N-1)}(c)\big)$ with $c \in \mathbb{R}$, have been introduced in [1]. When A is a diagonal matrix, i.e. $A = diag(M_0, M_1, \ldots, M_{N-1})$ we get the so called diagonal Sobolev-Type case. This situation attracted the interest of many researchers mainly because of qualitative analytic properties of the corresponding sequences of monic orthogonal polynomials. From an algebraic point of view, they do not satisfy the three-term recurrence relation associated with standard orthogonal polynomials. Furthermore, their zeros are not located in the convex hull of the support of the measure of orthogonality.

R. Koekoek (see [9]) studied the second order linear differential equation satisfied by such orthogonal polynomials as well as a five term recurrence relation when $d\mu = x^\alpha e^{-x}dx$, $\alpha > -1$, $c = 0$.

Later on, when $N = 2$, and $M_0, M_1 > 0$ in [10] the authors analyzed the location of the zeros of such orthogonal polynomials. The study of the relative asymptotics of these polynomials with respect to the standard Laguerre orthogonal polynomials has been given in [3] and [12].

On the other hand, when $N > 2$, if $d\mu = x^\alpha e^{-x}dx$, $\alpha > -1$, $c = 0$, and $M_0 = M_1 = \cdots = M_{N-2} = 0$, $M_{N-1} > 0$, then some analog problems concerning the zeros have been studied in [14]. In particular, in [6] an electrostatic interpretation of them is given in terms of a logarithmic potential interaction under an external field.

The first situation of a non-diagonal Sobolev-Type inner product like (1) for a Hermite weight function was considered in [2]. There, he authors analyzed scaled asymptotics for the corresponding orthogonal polynomials (both strong outer asymptotics as well as a Mehler-Heine formula). As a consequence the asymptotic behaviour of their zeros follows.

More recently, in [7] we have studied asymptotic properties of polynomials orthogonal with respect to a Sobolev-Type inner product (1), when $N = 2$, $c = 0$, and $d\mu = x^\alpha e^{-x}dx$.

The aim of our contribution is to analyze the higher order recurrence relations that such a sequence of monic orthogonal polynomials satisfies. Furthermore, we can find a lowering and a raising operator associated with such a sequence of polynomials. As a consequence, we deduce the second order linear differential equation with polynomial coefficients that those polynomials of Sobolev-Type satisfy.

The structure of the paper is as follows. In section 2, the basic background about Laguerre polynomials orthogonal polynomials is presented.

Section 3 deals with the five term recurrence relation that these polynomials satisfy. It is a consequence of the simmetry of the multiplication operator by x^2 with respect to the Sobolev-Type inner product. The behaviour of the coefficients of such a recurrence relation is given.

Section 4 is focused on the analysis of raising and lowering operators for the sequence of orthogonal polynomials of Sobolev-Type. Thus, the second order linear differential (holonomic) equation associated with them follows in a natural way. Notice that the techniques used here are based on those used in [8] for standard orthogonal polynomials.

2 Preliminaries.

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers and let μ be the linear functional defined in the linear space $\mathbb{P}$ of the polynomials with real coefficients, such that

$$\langle \mu, x^n \rangle = \mu_n, \ n = 0, 1, 2, \dots$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Furthermore μ_n is the $n-th$ *moment* of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if

(i) The degree of P_n is n.

(ii) $\langle \mu, P_n(x)P_m(x) \rangle = 0$, $m \neq n$.

(iii) $\langle \mu, P_n^2(x) \rangle \neq 0$, $n = 0, 1, 2, ...$

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials.*

The next theorem, whose proof appears in [5] , gives necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1 *([5]) Let μ be the moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $[\mu_{i+j}]_{i,j\in\mathbb{N}}$ are non singular.*

A moment functional such that there exists the corresponding sequence of orthogonal polynomials is said to be *regular* or *quasi-definite* ([5]).

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle p, q \rangle_\alpha = \int_0^\infty pqx^\alpha e^{-x} dx, \ \alpha > -1, \ p, q \in \mathbb{P}. \tag{2}$$

In the following proposition we will summarize some properties of the Laguerre monic orthogonal polynomials. The details of the proof can be founded in [4], [5], [8], [11], [15], and [16].

Proposition 1 *Let* $\{L_n^\alpha\}_{n\geq 0}$ *be the sequence of Laguerre monic orthogonal polynomials.*

1. *For every* $n \in \mathbb{N}$,
$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha)\, L_n^\alpha(x) + n\,(n+\alpha)\, L_{n-1}^\alpha(x), \quad (3)$$
with $L_0^\alpha(x) = 1, L_1^\alpha(x) = x -\ (\alpha+1)$.

2. *For every* $n \in \mathbb{N}$,
$$\|L_n^\alpha\|_\alpha^2 = n!\Gamma(n+\alpha+1), \quad (4)$$
where $\|L_n^\alpha\|_\alpha^2 = \langle L_n^\alpha, L_n^\alpha\rangle_\alpha$.

3. *For every* $n \in \mathbb{N}$
$$L_n^\alpha(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \quad (5)$$

4. *For every* $n \in \mathbb{N}$
$$\left(L_n^\alpha\right)'(x) = nL_{n-1}^{\alpha+1}(x). \quad (6)$$

5. *For every* $n \in \mathbb{N}$,
$$x\left(L_n^\alpha(x)\right)' = nL_n^\alpha(x) + n\,(n+\alpha)\, L_{n-1}^\alpha(x). \quad (7)$$

3 The five-term recurrence relation

If p is a polynomial with real coefficients, then we will denote
$$\mathbb{P}(x) = \begin{pmatrix} p(x) \\ p'(x) \end{pmatrix}.$$

In the linear space of polynomials with real coefficients we define the following Sobolev type inner product
$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0),\ \ \alpha > -1, \quad (8)$$
where
$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix},$$
$M_0, M_1 \geq 0$, A is a positive semidefinite matrix, i.e $\det A = |A| \geq 0$, and $p, q \in \mathbb{P}$.

Let $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ be the sequence of monic polynomials orthogonal with respect to (8). In this section we will focus our attention to find a five-term recurrence relation that the sequence of monic Laguerre-Sobolev type orthogonal polynomials $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ satisfies. In order to do that, we will use the following results: The connection formula (9) was obtained in [7] and the Proposition 2 is a straightforward consequence of (8).

Theorem 2 *Let* $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ *be the sequence of monic polynomials orthogonal with respect to (8). For every* $n\in\mathbb{N}$

$$\widetilde{L}_n^\alpha(x)=L_n^{\alpha+2}(x)+A_{n,\alpha}L_{n-1}^{\alpha+2}(x)+B_{n,\alpha}L_{n-2}^{\alpha+2}(x). \tag{9}$$

where

$$\begin{aligned} A_{n,\alpha} &= 2n-\frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)}\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A\begin{pmatrix} -\frac{1}{n-1} \\ \frac{1}{(\alpha+1)} \end{pmatrix}\sim 2n-(\alpha+1)(\alpha+2), \\ B_{n,\alpha} &= n(n-1)-\frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)}\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A\begin{pmatrix} -1 \\ \frac{n}{\alpha+1} \end{pmatrix}\sim n(n-1)-(\alpha+1)(\alpha+2)(n-1). \end{aligned}$$

Proposition 2 *The multiplication operator by* x^2 *is a symmetric operator with respect to the Sobolev inner product (8). In other words, if* p *and* q *are polynomials with real coefficients, then*

$$\left\langle x^2p,q\right\rangle_S=\left\langle p,x^2q\right\rangle_S. \tag{10}$$

Let consider the Fourier expansion of $x^2\widetilde{L}_n^\alpha$ in terms of $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$

$$x^2\widetilde{L}_n^\alpha(x)=\widetilde{L}_{n+2}^\alpha(x)+\sum_{k=0}^{n+1}a_{n,k}\widetilde{L}_k^\alpha(x) \tag{11}$$

where

$$a_{n,k}=\frac{\left\langle x^2\widetilde{L}_n^\alpha(x),\widetilde{L}_k^\alpha(x)\right\rangle_S}{\left\|\widetilde{L}_k^\alpha(x)\right\|_S^2},\ \ k=0,\dots,n+1,$$

and

$$\left\|\widetilde{L}_k^\alpha(x)\right\|_S^2=\left\langle \widetilde{L}_k^\alpha(x),\widetilde{L}_k^\alpha(x)\right\rangle_S.$$

From (10)

$$a_{n,k}=\frac{\left\langle \widetilde{L}_n^\alpha(x),x^2\widetilde{L}_k^\alpha(x)\right\rangle_S}{\left\|\widetilde{L}_k^\alpha(x)\right\|_S^2}.$$

Thus, $a_{n,k}=0$ for $k=0,\dots,n-3$, and (11) becomes

$$x^2\widetilde{L}_n^\alpha(x)=\widetilde{L}_{n+2}^\alpha(x)+a_{n,n+1}\widetilde{L}_{n+1}^\alpha(x)+a_{n,n}\widetilde{L}_n^\alpha(x)+a_{n,n-1}\widetilde{L}_{n-1}^\alpha(x)+a_{n,n-2}\widetilde{L}_{n-2}^\alpha(x).$$

The next step is to find the coefficients $a_{n,k}$, $k=n-2,\dots,n+1$. Taking into account that $\left(x^2\widetilde{L}_k^\alpha\right)(0)=\left(x^2\widetilde{L}_k^\alpha\right)'(0)=0$, then

$$\left\langle \widetilde{L}_n^\alpha(x), x^2 \widetilde{L}_k^\alpha(x) \right\rangle_S = \left\langle \widetilde{L}_n^\alpha(x), \widetilde{L}_k^\alpha(x) \right\rangle_{\alpha+2},$$

and using the connection formula (9) we get

$$\begin{aligned}
a_{n,n+1} &= \frac{A_{n+1,\alpha}\left\|L_n^{\alpha+2}\right\|_{\alpha+2}^2 + A_{n,\alpha}B_{n+1,\alpha}\left\|L_{n-1}^{\alpha+2}\right\|_{\alpha+2}^2}{\left\|\widetilde{L}_{n+1}^\alpha(x)\right\|_S^2} \sim 4n \\
a_{n,n} &= \frac{\left\|L_n^{\alpha+2}\right\|_{\alpha+2}^2 + A_{n,\alpha}^2\left\|L_{n-1}^{\alpha+2}\right\|_{\alpha+2}^2 + B_{n,\alpha}^2\left\|L_{n-2}^{\alpha+2}\right\|_{\alpha+2}^2}{\left\|\widetilde{L}_n^\alpha(x)\right\|_S^2} \sim 6n^2 \\
a_{n,n-1} &= \frac{A_{n,\alpha}\left\|L_{n-1}^{\alpha+2}\right\|_{\alpha+2}^2 + B_{n,\alpha}A_{n-1,\alpha}\left\|L_{n-2}^{\alpha+2}\right\|_{\alpha+2}^2}{\left\|\widetilde{L}_{n-1}^\alpha(x)\right\|_S^2} \sim 4n^3 \\
a_{n,n-2} &= \frac{B_{n,\alpha}\left\|L_{n-2}^{\alpha+2}\right\|_{\alpha+2}^2}{\left\|\widetilde{L}_{n-2}^\alpha(x)\right\|_S^2} \sim n^4
\end{aligned}$$

As a consequence, we get

Theorem 3 *(The five term recurrence formula) For every $n \in \mathbb{N}$*

$$x^2\widetilde{L}_n^\alpha(x) = \widetilde{L}_{n+2}^\alpha(x) + a_{n,n+1}\widetilde{L}_{n+1}^\alpha(x) + a_{n,n}\widetilde{L}_n^\alpha(x) + a_{n,n-1}\widetilde{L}_{n-1}^\alpha(x) + a_{n,n-2}\widetilde{L}_{n-2}^\alpha(x). \tag{12}$$

where $a_{n,k}$, $k = n-2, \ldots, n+1$, are defined as above and $\widetilde{L}_{-1}^\alpha(x) = \widetilde{L}_{-2}^\alpha(x) =$ 0.

3.1 Example

Let consider the inner product (8) with $M_0 = M$, $M_1 = 4N$, and $\lambda = 0$. In other words, if p, q are polynomials with real coefficients, we introduce the inner product

$$\langle p, q \rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + Mp(0)q(0) + 4Np'(0)q'(0). \tag{13}$$

Let $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ be the sequence of monic Laguerre-Sobolev polynomials orthogonal with respect to (13) and $\{P_n\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the following inner product

$$\langle p, q \rangle_H = \int_{-\infty}^\infty p(x)q(x)\,|x|^{2\alpha}\,e^{-x^2}dx + Mp(0)q(0) + Np''(0)q''(0), \tag{14}$$

$M, N \in \mathbb{R}^+$. Then, according to the following proposition, we find a relation between the sequences $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$.

Proposition 3 *For every $n \in \mathbb{N}$*

$$\begin{aligned} P_{2n}(x) &= \widetilde{L}_n^{\alpha-1/2}(x^2) \\ P_{2n+1}(x) &= xL_n^{\alpha+1/2}(x^2). \end{aligned}$$

Proof. In order to prove that $P_{2n+1}(x) = xL_n^{\alpha+1/2}(x^2)$, we will see that $\left\langle xL_n^{\alpha+1/2}(x^2), x^{2k+1} \right\rangle_H = 0$, when $n < k$. Thus

$$\begin{aligned} \left\langle xL_n^{\alpha+1/2}(x^2), x^{2k+1} \right\rangle_H &= \int_{-\infty}^{\infty} L_n^{\alpha+1/2}(x^2)x^{2k+2} \left|x\right|^{2\alpha} e^{-x^2} dx \\ &= \int_0^{\infty} L_n^{\alpha+1/2}(t)t^k t^{\alpha+1/2} e^{-t} dt \\ &= \delta_{n,k} \left\| L_n^{\alpha+1/2} \right\|_{\alpha+1/2}^2, k \leq n, \end{aligned}$$

where $\delta_{n,k}$ is the delta Kronecker function.

Furthermore

$$\left\langle xL_n^{\alpha+1/2}(x^2), x^{2k} \right\rangle_H = 0.$$

i.e,

$$P_{2n+1}(x) = xL_n^{\alpha+1/2}(x^2).$$

On the other hand $P_{2n}(x) = \widetilde{L}_n^{\alpha-1/2}(x^2)$. Indeed, if $1 \leq k \leq n$,

$$\left\langle \widetilde{L}_n^{\alpha-1/2}(x^2), x^{2k} \right\rangle_H$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} x^{2k} \widetilde{L}_n^{\alpha-1/2}(x^2) \left|x\right|^{2\alpha} e^{-x^2} dx + \left[(2k)(2k-1)x^{2k-2}\right]_{x=0} 2 \left(\widetilde{L}_n^{\alpha-1/2}\right)'(0)N \\ &= \int_0^{\infty} t^k \widetilde{L}_n^{\alpha-1/2}(t) t^{\alpha-1/2} e^{-t} dt + 4N \left[k(2k-1)x^{2k-2}\right]_{x=0} \left(\widetilde{L}_n^{\alpha}\right)'(0) \\ &= \int_0^{\infty} t^k \widetilde{L}_n^{\alpha-1/2}(t) t^{\alpha-1/2} e^{-t} dt + 4N(t^k)'(0) \left(\widetilde{L}_n^{\alpha}\right)'(0) \\ &= \left\langle \widetilde{L}_n^{\alpha-1/2}(t), t^k \right\rangle_S \\ &= \delta_{nk} \left\| \widetilde{L}_n^{\alpha-1/2}(x^2) \right\|_S^2. \end{aligned}$$

Because

$$\begin{aligned} \left\langle \widetilde{L}_n^{\alpha-1/2}(x^2), 1 \right\rangle_H &= \int_{-\infty}^{\infty} \widetilde{L}_n^{\alpha-1/2}(x^2) \left|x\right|^{2\alpha} e^{-x^2} dx + M\widetilde{L}_n^{\alpha-1/2}(0) \\ &= \int_0^{\infty} \widetilde{L}_n^{\alpha-1/2}(t) t^{\alpha-1/2} e^{-t} dt + M\widetilde{L}_n^{\alpha-1/2}(0) \\ &= \left\langle \widetilde{L}_n^{\alpha-1/2}(x), 1 \right\rangle_S, \end{aligned}$$

then

$$P_{2n}(x) = \widetilde{L}_n^{\alpha-1/2}(x^2).$$

■

Notice that some results obtained in [2] can be deduced from this simple relation between sequences of orthogonal polynomials. See also [13].

4 The holonomic equation

In this section we will deduce a second order linear differential equation that the sequence of monic Laguerre-Sobolev type orthogonal polynomials $\left\{\widetilde{L}_n^{\alpha}\right\}_{n\geq 0}$ satisfy. In order to do that, we will find two first order linear differential operators $\mathcal{J}_n$ and $\mathcal{K}_n$, such that

$$\begin{aligned}
\mathcal{J}_n\left(\widetilde{L}_n^{\alpha}\right) &= H(x;n)\widetilde{L}_{n-1}^{\alpha} \\
\mathcal{K}_n\left(\widetilde{L}_{n-1}^{\alpha}\right) &= K(x;n)\widetilde{L}_n^{\alpha}.
\end{aligned}$$

for some polynomials $H(x;n)$ and $K(x;n)$. These operators are called in the literature lowering and raising operators ([8]).

Replacing (3) in (9) we get

$$\widetilde{L}_n^{\alpha}(x) = f(x;n)L_{n-1}^{\alpha+2}(x) + M_n L_{n-2}^{\alpha+2}(x) \tag{15}$$

as well as for the polynomial of degree $n-1$

$$\widetilde{L}_{n-1}^{\alpha}(x) = K_n L_{n-1}^{\alpha+2}(x) + g(x;n)L_{n-2}^{\alpha+2}(x), \tag{16}$$

where

$$\begin{aligned}
f(x;n) &= A_{n,\alpha} + (x - (2n+1+\alpha)) \\
M_n &= B_{n,\alpha} - (n-1)(n+\alpha+1) \\
K_n &= 1 - \frac{B_{n-1,\alpha}}{(n-2)(n+\alpha)} \\
g(x;n) &= A_{n-1,\alpha} + \frac{B_{n-1,\alpha}(x-(2n+\alpha-1))}{(n-2)(n+\alpha)}.
\end{aligned}$$

Derivating in both sides of (15)

$$\left(\widetilde{L}_n^{\alpha}\right)'(x) = L_{n-1}^{\alpha+2}(x) + f(x;n)\left(L_{n-1}^{\alpha+2}\right)'(x) + M_n\left(L_{n-2}^{\alpha+2}\right)'(x).$$

Multiplying in both sides of the above expression by x, and using (7), we get

$$
\begin{aligned}
x\left(\widetilde{L}_n^{\alpha}\right)'(x) \quad &= \quad xL_{n-1}^{\alpha+2}(x)+f(x;n)\left[(n-1)L_{n-1}^{\alpha+2}(x)+(n-1)(n+\alpha+1)L_{n-2}^{\alpha+2}(x)\right]+\\
&\quad\; M_n\left[(n-2)L_{n-2}^{\alpha+2}(x)+(n-2)(n+\alpha)L_{n-3}^{\alpha+2}(x)\right].
\end{aligned}
$$

From (3)

$$
L_{n-3}^{\alpha+2}(x)=-\frac{1}{(n-2)(n+\alpha)}L_{n-1}^{\alpha+2}(x)+\frac{(x-(2n+\alpha-1))}{(n-2)(n+\alpha)}L_{n-2}^{\alpha+2}(x).
$$

Thus

$$
\begin{aligned}
x\left(\widetilde{L}_n^{\alpha}\right)'(x) \quad &= \quad \left[x+(n-1)f(x;n)-M_n\right]L_{n-1}^{\alpha+2}(x)+\\
&\quad\; \left[(n-1)(n+\alpha+1)f(x;n)+M_n\left(x-(n+\alpha+1)\right)\right]L_{n-2}^{\alpha+2}(x),
\end{aligned}
$$

i.e

$$
\begin{aligned}
x\left(\widetilde{L}_n^{\alpha}\right)'(x) \quad &= \quad \left[nx+A_{n,\alpha}n-n-B_{n,\alpha}-A_{n,\alpha}-n^2\right]L_{n-1}^{\alpha+2}(x)+\\
&\quad\; \left[B_{n,\alpha}x+A_{n,\alpha}\left(-1-\alpha+n\alpha+n^2\right)-B_{n,\alpha}\left(1+n+\alpha\right)+n(\alpha-n^2-n\alpha+1)\right]L_{n-2}^{\alpha+2}(x).
\end{aligned}
$$

Then, if we define

$$
\begin{aligned}
\phi(x;n) \quad &= \quad nx+A_{n,\alpha}n-n-B_{n,\alpha}-A_{n,\alpha}-n^2\\
\sigma(x;n) \quad &= \quad B_{n,\alpha}x+A_{n,\alpha}\left(-1-\alpha+n\alpha+n^2\right)-B_{n,\alpha}\left(1+n+\alpha\right)+n(\alpha-n^2-n\alpha+1)
\end{aligned}
$$

then we obtain

$$
x\left(\widetilde{L}_n^{\alpha}\right)'(x)=\phi(x;n)L_{n-1}^{\alpha+2}(x)+\sigma(x;n)L_{n-2}^{\alpha+2}(x). \qquad (17)
$$

In a similar way, we get

$$
x\left(\widetilde{L}_{n-1}^{\alpha}\right)'(x)=\tau(x;n)L_{n-1}^{\alpha+2}(x)+\upsilon(x;n)L_{n-2}^{\alpha+2}(x),
$$

where

$$
\tau(x;n)=\frac{-xB_{n-1,\alpha}}{(n-2)(n+\alpha)}
$$

$$
+\frac{2n+2B_{n-1,\alpha}+2\alpha+nB_{n-1,\alpha}-3n\alpha+B_{n-1,\alpha}\alpha-3n^2+n^3+n^2\alpha}{(n-2)(n+\alpha)}-A_{n-1,\alpha}
$$

$$\upsilon(x;n) = K_n(n-1)(n+\alpha+1) + \frac{B_{n-1,\alpha}x}{(n-2)(n+\alpha)} + g(x;n)(x-(n+\alpha+1)).$$

From (15) and (16) we obtain

$$L_{n-1}^{\alpha+2}(x) = \frac{g(x;n)\widetilde{L}_n^{\alpha}(x) - M_n\widetilde{L}_{n-1}^{\alpha}(x)}{g(x;n)f(x;n) - K_nM_n} \tag{18}$$

$$L_{n-2}^{\alpha+2}(x) = \frac{f(x;n)\widetilde{L}_{n-1}^{\alpha}(x) - K_n\widetilde{L}_n^{\alpha}(x)}{g(x;n)f(x;n) - K_nM_n}. \tag{19}$$

As a consequence, replacing (18) and (19) in (17),

$$\begin{aligned} & x\left[g(x;n)f(x;n) - K_nM_n\right]\left(\widetilde{L}_n^{\alpha}\right)'(x) + \left[g(x;n)\phi(x;n) - K_n\phi(x;n)\right]\widetilde{L}_n^{\alpha}(x) \\ = \;& \left[f(x;n)\sigma(x;n) - M_n\phi(x;n)\right]\widetilde{L}_{n-1}^{\alpha}(x). \end{aligned}$$

In a similar way, we can prove that

$$\begin{aligned} & x\left[g(x;n)f(x;n) - K_nM_n\right]\left(\widetilde{L}_{n-1}^{\alpha}\right)'(x) + \left[M_n\tau(x;n) - \upsilon(x;n)f(x;n)\right]\widetilde{L}_{n-1}^{\alpha}(x) \\ = \;& \left[\tau(x;n)g(x;n)\sigma(x;n) - K_n\upsilon(x;n)\right]\widetilde{L}_n^{\alpha}(x). \end{aligned}$$

Therefore, we get

Proposition 4 *Let* $\left\{\widetilde{L}_n^{\alpha}\right\}_{n\geq 0}$ *be the sequence of monic polynomials orthogonal with respect to (8). Then, the differential operators* $\mathcal{J}_n$ *and* $\mathcal{K}_n$ *defined by*

$$\begin{aligned} \mathcal{J}_n &= F(x;n)D + G(x;n)I \\ \mathcal{K}_n &= F(x;n)D + J(x;n)I \end{aligned}$$

where D *is the derivate operator,* I *the identity operator, and*

$$\begin{aligned} F(x;n) &= x\left[g(x;n)f(x;n) - K_nM_n\right], \\ G(x;n) &= \phi(x;n)\left(g(x;n) - K_n\right), \\ H(x;n) &= f(x;n)\sigma(x;n) - M_n\phi(x;n), \\ J(x;n) &= M_n\tau(x;n) - \upsilon(x;n)f(x;n), \\ K(x;n) &= \tau(x;n)g(x;n)\sigma(x;n) - K_n\upsilon(x;n), \end{aligned}$$

satisfy

$$\begin{aligned} \mathcal{J}_n\left(\widetilde{L}_n^{\alpha}\right) &= H(x;n)\widetilde{L}_{n-1}^{\alpha}, \\ \mathcal{K}_n\left(\widetilde{L}_{n-1}^{\alpha}\right) &= K(x;n)\widetilde{L}_n^{\alpha}. \end{aligned}$$

In other words $\mathcal{J}_n$ is a lowering operator and $\mathcal{K}_n$ is a raising operator associated with the sequence $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$.

From the above proposition

$$\frac{1}{H(x;n)}\mathcal{J}_n\left(\widetilde{L}_n^\alpha\right)=\widetilde{L}_{n-1}^\alpha.$$

Thus, applying the differential operator $\mathcal{K}_n$ in both sides of the above identity

$$\mathcal{K}_n\left(\frac{1}{H(x;n)}\mathcal{J}_n\left(\widetilde{L}_n^\alpha\right)\right)=K(x;n)\widetilde{L}_n^\alpha,$$

i.e

$$F(x;n)D\left(\frac{1}{H(x;n)}\mathcal{J}_n\left(\widetilde{L}_n^\alpha\right)\right)+\frac{J(x;n)}{H(x;n)}\mathcal{J}_n\left(\widetilde{L}_n^\alpha\right)=K(x;n)\widetilde{L}_n^\alpha.$$

Taking into account that

$$\begin{aligned}D\left(\frac{1}{H(x;n)}\mathcal{J}_n\left(\widetilde{L}_n^\alpha\right)\right) &= D\left(\frac{1}{H(x;n)}\left(F(x;n)\left(\widetilde{L}_n^\alpha\right)'+G(x;n)\widetilde{L}_n^\alpha\right)\right)\\ &= -\frac{H'(x;n)}{H^2(x;n)}\left(F(x;n)\left(\widetilde{L}_n^\alpha\right)'+G(x;n)\widetilde{L}_n^\alpha\right)+\\ &\quad \frac{1}{H(x;n)}\left(F'(x;n)\left(\widetilde{L}_n^\alpha\right)'+F(x;n)\left(\widetilde{L}_n^\alpha\right)''\right)+\\ &\quad \frac{1}{H(x;n)}\left(G'(x;n)\widetilde{L}_n^\alpha+G(x;n)\left(\widetilde{L}_n^\alpha\right)'\right),\end{aligned}$$

we deduce

$$\begin{aligned}&\frac{F^2(x;n)}{H(x;n)}\left(\widetilde{L}_n^\alpha\right)''+\frac{F(x;n)}{H(x;n)}\left[-\frac{F(x;n)H'(x;n)}{H(x;n)}+F'(x;n)+G(x;n)+J(x;n)\right]\left(\widetilde{L}_n^\alpha\right)'\\ &+\frac{1}{H(x;n)}\left[-\frac{F(x;n)G(x;n)H'(x;n)}{H(x;n)}+F(x;n)G'(x;n)+J(x;n)G(x;n)-K(x;n)H(x;n)\right]\widetilde{L}_n^\alpha\\ =\;& 0.\end{aligned}$$

As a consequence, if we use the notation of Proposition 4, then we get

Theorem 4 *Let* $\left\{\widetilde{L}_n^a\right\}_{n\geq 0}$ *be the sequence of monic Laguerre-Sobolev polynomials orthogonal with respect to (8). Then*

$$A(x;n)\left(\widetilde{L}_n^\alpha(x)\right)''+B(x;n)\left(\widetilde{L}_n^\alpha(x)\right)'+C(x;n)\widetilde{L}_n^\alpha(x)=0,$$

where

$$\begin{aligned} A(x;n) &= F^2(x;n) \\ B(x;n) &= F(x;n)\left[F'(x;n)+G(x;n)+J(x;n)-\frac{F(x;n)H'(x;n)}{H(x;n)}\right] \\ C(x;n) &= F(x;n)G'(x;n)+J(x;n)G(x;n)-\frac{F(x;n)G(x;n)H'(x;n)}{H(x;n)}-K(x;n)H(x;n). \end{aligned}$$

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