

A COHEN TYPE INEQUALITY FOR GEGENBAUER-SOBOLEV EXPANSIONS

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ABSTRACT. Let introduce the Sobolev-type inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)d\mu(x) + M[f(1)g(1) + f(-1)g(-1)] \\ + N[f'(1)g'(1) + f'(-1)g'(-1)],$$

where

$$d\mu(x) = \frac{\Gamma(2\alpha + 2)}{2^{2\alpha+1}\Gamma^2(\alpha + 1)}(1 - x^2)^\alpha dx, \quad M, N \geq 0, \quad \alpha > -1.$$

In this paper we prove a Cohen type inequality for the Fourier expansion in terms of the orthonormal polynomials associated with the above Sobolev inner product.

1. INTRODUCTION AND MAIN RESULT

The aim of this contribution is to derive a lower bound for the norm of the Fourier expansions in terms of orthonormal polynomials with respect to a Sobolev inner product, the well known Cohen type inequality of the approximation theory literature. For Fourier expansions in terms of classical orthogonal polynomials (Hermite, Laguerre, and Jacobi) such inequalities were proved by Dreseler and Soardi ([4],[5]) and Markett ([8]).

Let us first introduce some notation. Given the Gegenbauer measure $d\mu(x) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma^2(\alpha+1)}(1-x^2)^\alpha dx$, $\alpha > -1$, supported in the interval $[-1, 1]$ we say that $f \in L^p(d\mu)$ if f is measurable on $[-1, 1]$ and $\|f\|_{L^p(d\mu)} < \infty$, where

$$\|f\|_{L^p(d\mu)} = \begin{cases} \left(\int_{-1}^1 |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \text{ess sup}_{-1 < x < 1} |f(x)| & \text{if } p = \infty. \end{cases}$$

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Now let us introduce the Sobolev-type spaces

$$W_p = \{f : \|f\|_{W_p}^p = \|f\|_{L^p(d\mu)}^p + M[|f(1)|^p + |f(-1)|^p] \\ + N[|f'(1)|^p + |f'(-1)|^p] < \infty\}, \quad 1 \leq p < \infty,$$

$$W_\infty = \{f : \|f\|_{W_\infty} = \|f\|_{L^\infty(d\mu)} < \infty\}, \quad p = \infty.$$

If $f, g \in W_2$ then we can introduce the discrete Sobolev-type inner product

$$(1) \quad \langle f, g \rangle = \int_{-1}^1 f(x)g(x)d\mu(x) + M[f(1)g(1) + f(-1)g(-1)] \\ + N[f'(1)g'(1) + f'(-1)g'(-1)],$$

where $M, N \geq 0$. Notice that the linear space of polynomials with real coefficients is a linear subspace of W_2 . Thus, applying the Gram-Schmidt orthogonalization process to the canonical basis $\{x^n\}_{n \geq 0}$ we get a sequence of orthonormal polynomials with respect to the above inner product. We denote it by $\{\hat{B}_n^{(\alpha)}\}_{n \geq 0}$ (see [2], [3], [9]). They are called Gegenbauer-Sobolev type polynomials. For $M = N = 0$, we get the sequence of classical Gegenbauer (ultraspherical) orthonormal polynomials that we will denote $\{p_n^{(\alpha)}\}_{n \geq 0}$. In the sequel, we will use the notation $P_n^{(\alpha)}(x)$ for the n -th Gegenbauer polynomial with the normalization $P_n^{(\alpha)}(1) = \frac{(\alpha+1)_n}{n!}$, where $(a)_0 = 1$ and $(a)_n = a(a+1) \dots (a+n-1)$, for $n \geq 1$, is the standard Pochhammer symbol.

An interesting and natural question is to analyze the convergence of Fourier expansions in terms of Gegenbauer-Sobolev type orthonormal polynomials as well as to compare them with the convergence of Fourier expansions given in terms of standard Gegenbauer orthonormal polynomials. More precisely, we will study some estimates for the Sobolev norm of a projection operator associated with some balanced summation for the n th partial sum of the Fourier expansion of a function $f \in W_p$, in terms of Gegenbauer-Sobolev type orthonormal polynomials. The divergence of the Fourier expansion is related to the unboundedness of the sequence of such projection operators.

Indeed if $f \in W_1$, then the Fourier expansion in terms of Gegenbauer-Sobolev type polynomials is

$$(2) \quad \sum_{k=0}^{\infty} \hat{f}(k) \hat{B}_k^{(\alpha)}(x),$$

where

$$\hat{f}(k) = \langle f, \hat{B}_k^{(\alpha)} \rangle, \quad k = 0, 1, \dots$$

The n th Cesàro means of order δ of the Fourier expansion (2) is defined by (see [10, p. 20-21] and [13, p. 76-77])

$$\sigma_n^\delta f(x) = \sum_{k=0}^n \frac{A_{n-k}^\delta}{A_n^\delta} \hat{f}(k) \hat{B}_k^{(\alpha)}(x),$$

where $A_k^\delta = \binom{k+\delta}{k}$.

For a function $f \in W_p$ and a given sequence $\{c_{k,n}\}_{k=0}^n$ of complex numbers with $c_{n,n} \neq 0$, we define the operator $T_n^{\alpha, M, N}$ by

$$T_n^{\alpha, M, N}(f) = \sum_{k=0}^n c_{k,n} \hat{f}(k) \hat{B}_k^{(\alpha)}.$$

Let denote $p_0 = (4\alpha + 4)/(2\alpha + 3)$ and its conjugate $q_0 = (4\alpha + 4)/(2\alpha + 1)$. Now we formulate the main result of the manuscript.

Theorem 1. *Let $\alpha > -1/2$ and $1 \leq p \leq \infty$. There exists a positive constant c , independent of n , such that*

$$\|T_n^{\alpha, M, N}\|_{[W_p]} \geq c |c_{n,n}| \begin{cases} n^{\frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}} & \text{if } 1 \leq p < p_0 \\ (\log n)^{\frac{2\alpha+1}{4\alpha+4}} & \text{if } p = p_0, p = q_0 \\ n^{\frac{2\alpha+1}{2} - \frac{2\alpha+2}{p}} & \text{if } q_0 < p \leq \infty, \end{cases}$$

where $[W_p]$ denotes the space of all bounded, linear operators from the space W_p into itself, with the usual operator norm $\|\cdot\|_{[W_p]}$.

Corollary 1. *Let $\alpha > -1/2$ and $1 \leq p \leq \infty$. For $c_{k,n} = 1$, $k = 0, \dots, n$, and for p outside the Pollard interval (p_0, q_0)*

$$\|S_n\|_{[W_p]} \rightarrow \infty, \quad n \rightarrow \infty,$$

where S_n denotes the Fourier projector associated with the n th partial sum of the expansion (2).

For $c_{k,n} = \frac{A_{n-k}^p}{A_n^p}$, $0 \leq k \leq n$, as a consequence of Theorem 1 we get

Corollary 2. *Let α , p , and δ be real numbers such that $\alpha > -1/2$, $1 \leq p \leq \infty$, and*

$$\begin{cases} 0 \leq \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2} & \text{if } 1 \leq p < p_0 \\ 0 \leq \delta < \frac{2\alpha+1}{2} - \frac{2\alpha+2}{p} & \text{if } q_0 < p \leq \infty. \end{cases}$$

Then, for $p \notin [p_0, q_0]$

$$\|\sigma_n^\delta\|_{[W_p]} \rightarrow \infty, \quad n \rightarrow \infty.$$

The structure of the manuscript is as follows. In Section 2 we will present the basic background concerning some analytic properties of Gegenbauer-Sobolev type orthonormal polynomials. In Section 3, the proof of our main result (Theorem 1) is given.

2. GEGENBAUER-SOBOLEV TYPE ORTHOGONAL POLYNOMIALS

We summarize some properties of Gegenbauer-Sobolev type polynomials that we will need in the sequel (see [2], [3], [9]). Throughout the article positive constants are denoted by c, c_1, \dots ; unless specified, their values may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $c_1 u_n \leq v_n \leq c_2 u_n$ for n large enough.

The representation of $\hat{B}_n^{(\alpha)}$ in terms of Gegenbauer orthonormal polynomials is (see [2], [3], [9])

$$(3) \quad \hat{B}_n^{(\alpha)}(x) = A_n(1-x^2)^2 p_{n-4}^{(\alpha+4)}(x) + B_n(1-x^2) p_{n-2}^{(\alpha+2)}(x) + C_n p_n^{(\alpha)}(x)$$

where

i) If $M = 0, N > 0$, then

$$A_n \cong \frac{2^{\alpha+1}\Gamma(\alpha+1)}{\alpha+2} \sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}}, \quad B_n \cong -2^{\alpha+1}\Gamma(\alpha+1) \sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}}, \quad C_n \cong -A_n,$$

ii) If $M > 0, N > 0$, then

$$A_n \cong 2^{\alpha+1}\Gamma(\alpha+1) \sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}}, \quad B_n \sim -n^{-2\alpha-2}, \quad C_n \sim -n^{-2\alpha-2},$$

iii) If $M > 0, N = 0$, then

$$A_n = 0, \quad B_n \cong -2^{\alpha+1}\Gamma(\alpha+1) \sqrt{\frac{\alpha+1}{\Gamma(2\alpha+3)}}, \quad C_n \sim n^{-2\alpha-2}.$$

For the polynomials $\hat{B}_n^{(\alpha)}$ we get the following estimate

$$(4) \quad |\hat{B}_n^{(\alpha)}(\cos \theta)| = \begin{cases} O(\theta^{-\alpha-1/2}) & \text{if } c/n < \theta \leq \pi/2 \\ O(n^{\alpha+1/2}) & \text{if } 0 \leq \theta \leq c/n. \end{cases}$$

The inner strong asymptotic behaviour of $\hat{B}_n^{(\alpha)}$, for $\theta \in [\epsilon, \pi - \epsilon]$ and $\epsilon > 0$, is given by

$$(5) \quad \hat{B}_n^{(\alpha)}(\cos \theta) = c(\sin \theta)^{-\alpha-1/2} \cos(k\theta + \gamma) + O(n^{-1}),$$

where $k = n + \alpha + 1/2$, $\gamma = -(\alpha + 1/2)\pi/2$, and c is positive constant independent of n and θ .

The formula of Mehler-Heine type for Gegenbauer orthonormal polynomials is (see [12, Theorem 8.1.1] and [12, (4.3.4)])

$$(6) \quad \lim_{n \rightarrow \infty} n^{-\alpha-1/2} p_n^{(\alpha)} \left(\cos \frac{z}{n+k} \right) = z^{-\alpha} J_\alpha(z),$$

uniformly on compact subsets of \mathbf{C} , α is a real number, $k \in \mathbf{N} \cup 0$, and $J_\alpha(z)$ is the Bessel function of the first kind of order α . Indeed, the above formula in [12, Theorem 8.1.1] is for $k = 0$ but it can be shown that this formula is also true for any fixed $k \in \mathbf{N}$.

Lemma 1. *Uniformly on compact subsets of \mathbf{C}*

$$(7) \quad \lim_{n \rightarrow \infty} n^{-\alpha-1/2} \hat{B}_n^{(\alpha)} \left(\cos \frac{z}{n} \right) = z^{-\alpha} (c_1 J_{\alpha+4}(z) - c_2 J_{\alpha+2}(z) - c_3 J_\alpha(z))$$

where

- i) If $M = 0$, $N > 0$, then $c_i > 0$, $i = 1, 2, 3$, and $c_1 = c_3$,
- ii) If $M > 0$, $N > 0$, then $c_1 > 0$ and $c_2 = c_3 = 0$,
- iii) If $M > 0$, $N = 0$, then $c_2 > 0$ and $c_1 = c_3 = 0$.

Proof. From (3) we have

$$\begin{aligned} n^{-\alpha-1/2} \hat{B}_n^{(\alpha)} \left(\cos \frac{z}{n} \right) &= A_n \left(\sin \frac{z}{n} \right)^4 n^{-\alpha-1/2} p_{n-4}^{(\alpha+4)} \left(\cos \frac{z}{n} \right) \\ &\quad + B_n \left(\sin \frac{z}{n} \right)^2 n^{-\alpha-1/2} p_{n-2}^{(\alpha+2)} \left(\cos \frac{z}{n} \right) \\ &\quad + C_n n^{-\alpha-1/2} p_n^{(\alpha)} \left(\cos \frac{z}{n} \right). \end{aligned}$$

Finally, taking the limit $n \rightarrow \infty$ and using the fact that $\sin \frac{z}{n} \cong \frac{z}{n}$ from (6) we get (7). \square

Now we will estimate the W_p norms for Gegenbauer-Sobolev type orthogonal polynomials

$$(8) \quad \|\hat{B}_n^{(\alpha)}\|_{W_p}^p = \int_{-1}^1 |\hat{B}_n^{(\alpha)}(x)|^p d\mu(x) + M \left[|(\hat{B}_n^{(\alpha)})(1)|^p + |(\hat{B}_n^{(\alpha)})(-1)|^p \right] \\ + N \left[|(\hat{B}_n^{(\alpha)})'(1)|^p + |(\hat{B}_n^{(\alpha)})'(-1)|^p \right]$$

where $1 \leq p < \infty$. Hence, it is sufficient to estimate the $L^p(d\mu)$ norms for $\hat{B}_n^{(\alpha)}$. For $M = N = 0$ the calculation of this norm appears in [12, p.391. Exercise 91] (see also [8, (2.2)]).

Lemma 2. Let $M, N \geq 0$ and $\gamma > -1/p$. For $\alpha \geq -1/2$

$$\int_0^1 (1-x)^\gamma |\hat{B}_n^{(\alpha)}(x)|^p dx \sim \begin{cases} c & \text{if } 2\gamma > p\alpha - 2 + p/2, \\ \log n & \text{if } 2\gamma = p\alpha - 2 + p/2, \\ n^{p\alpha + p/2 - 2\gamma - 2} & \text{if } 2\gamma < p\alpha - 2 + p/2. \end{cases}$$

Proof. In order to prove the lemma, we will follow the same steps as in [12, Theorem 7.34]. From (4), for $p\alpha + p/2 - 2\gamma - 2 \neq 0$, we have

$$\begin{aligned} \int_0^1 (1-x)^\gamma |\hat{B}_n^{(\alpha)}(x)|^p dx &\sim \int_0^{\pi/2} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p d\theta \\ &= O(1) \int_0^{n^{-1}} \theta^{2\gamma+1} n^{p\alpha + p/2} d\theta + O(1) \int_{n^{-1}}^{\pi/2} \theta^{2\gamma+1} \theta^{-p\alpha - p/2} d\theta \\ &= O(n^{p\alpha + p/2 - 2\gamma - 2}) + O(1), \end{aligned}$$

and for $p\alpha + p/2 - 2\gamma - 2 = 0$ we have

$$\int_0^{\pi/2} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p d\theta = O(\log n).$$

Now we will prove the lower estimates for integrals involving Gegenbauer-Sobolev type orthogonal polynomials when $M = 0$ and $N > 0$. The proof of the other cases can be done in a similar way. According to Lemma 1 and [11, Lemma 2.1] we have

$$\begin{aligned} \int_0^{\pi/2} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p d\theta &> \int_0^{n^{-1}} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p d\theta \\ &\cong c \int_0^1 (z/n)^{2\gamma+1} n^{p\alpha + p/2} n^{-1} |z^{-\alpha} (c_1 J_{\alpha+4}(z) - c_2 J_{\alpha+2}(z) - c_1 J_\alpha(z))|^p dz \\ &\sim n^{p\alpha + p/2 - 2\gamma - 2}. \end{aligned}$$

Using a similar argument as above, for $2\gamma = p\alpha + p/2 - 2$, we have

$$\begin{aligned} \int_0^{\pi/2} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p dx &> c \int_0^{n^{-1/2}} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p dx \\ &\cong c \int_0^{n^{1/2}} z^{2\gamma+1} |z^{-\alpha} (c_1 J_{\alpha+4}(z) - c_2 J_{\alpha+2}(z) - c_1 J_\alpha(z))|^p dz \geq c \log n. \end{aligned}$$

Finally, from (5) we obtain

$$\int_0^{\pi/2} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p d\theta > \int_{\pi/4}^{\pi/2} \theta^{2\gamma+1} |\hat{B}_n^{(\alpha)}(\cos\theta)|^p d\theta \sim c.$$

□

Note that, some of the above results appear in another framework in [7].

3. PROOF OF THEOREM 1

For the proof of Theorem 1 we will use the test functions

$$g_n^{(\alpha,j)}(x) = (1-x^2)^j p_n^{(\alpha+j)}(x),$$

where $\alpha > -1/2$ and $j \in \mathbb{N} \setminus \{1\}$. By applying the operators $T_n^{\alpha,M,N}$ to the test functions $g_n^{(\alpha,j)}$, for some $j > \alpha + 1/2 - (2\alpha + 2)/p$, we get

$$(9) \quad T_n^{\alpha,M,N}(g_n^{(\alpha,j)}) = \sum_{k=0}^n c_{k,n} (g_n^{(\alpha,j)})^\wedge(k) \hat{B}_k^{(\alpha)},$$

where

$$(g_n^{(\alpha,j)})^\wedge(k) = \langle g_n^{(\alpha,j)}, \hat{B}_k^{(\alpha)} \rangle, \quad k = 0, 1, \dots, n.$$

From (3) the Fourier coefficients of the functions $g_n^{(\alpha,j)}(x)$ in terms of the Gegenbauer-Sobolev type orthogonal polynomials are

$$\begin{aligned} (g_n^{(\alpha,j)})^\wedge(k) &= \int_{-1}^1 (1-x^2)^j p_n^{(\alpha+j)}(x) \hat{B}_k^{(\alpha)}(x) d\mu(x) \\ &= A_k \int_{-1}^1 (1-x^2)^j p_n^{(\alpha+j)}(x) (1-x^2)^2 p_{k-4}^{(\alpha+4)}(x) d\mu(x) \\ &\quad + B_k \int_{-1}^1 (1-x^2)^j p_n^{(\alpha+j)}(x) (1-x^2) p_{k-2}^{(\alpha+2)}(x) d\mu(x) \\ &\quad + C_k \int_{-1}^1 (1-x^2)^j p_n^{(\alpha+j)}(x) p_k^{(\alpha)}(x) d\mu(x) \\ &= I_1(k, n) + I_2(k, n) + I_3(k, n), \end{aligned}$$

where $0 \leq k \leq n$, and, as a convention, $p_i^{(\alpha)}(x) = 0$, for $i = -1, -2, -3, -4$.

For $k \geq 4$, according to [12, formula (4.3.4)] we obtain

$$\begin{aligned} I_1(k, n) &= A_k \{h_n^{(\alpha+j)}\}^{-1/2} \{h_{k-4}^{(\alpha+4)}\}^{-1/2} \\ &\quad \times \int_{-1}^1 (1-x^2)^j P_n^{(\alpha+j)}(x) (1-x^2)^2 P_{k-4}^{(\alpha+4)}(x) d\mu(x), \end{aligned}$$

where

$$h_n^{(\alpha)} = \frac{2^{2\alpha+1}}{2n+2\alpha+1} \frac{(\Gamma(n+\alpha+1))^2}{\Gamma(n+1)\Gamma(n+2\alpha+1)} \cong \frac{4^\alpha}{n}.$$

On the other hand, from [8, formula (2.8)]

$$\begin{aligned} (10) \quad (1-x^2)^j P_n^{(\alpha+j)}(x) &= \sum_{m=0}^{2j} b_{m,j}(\alpha, \alpha, n) P_{n+m}^{(\alpha)}(x) \\ &= b_{0,j}(\alpha, \alpha, n) P_n^{(\alpha)}(x) + \dots + b_{2j,j}(\alpha, \alpha, n) P_{n+2j}^{(\alpha)}(x) \end{aligned}$$

and

$$\begin{aligned} (1-x^2)^2 P_{k-4}^{(\alpha+4)}(x) &= \sum_{l=0}^4 b_{l,2}(\alpha+2, \alpha+2, k-4) P_{k+l-4}^{(\alpha+2)}(x) \\ &= b_{4,2}(\alpha+2, \alpha+2, k-4) P_k^{(\alpha+2)}(x) + Q_{k-1}(x), \end{aligned}$$

where $\deg Q_{k-1} \leq n-1$, and

$$\begin{aligned} b_{0,j}(\alpha, \alpha, n) &= 4^j \frac{(\Gamma(n+\alpha+j+1))^2}{(\Gamma(n+\alpha+1))^2} \frac{\Gamma(2n+2\alpha+2)}{\Gamma(2n+2\alpha+2j+2)} \cong 4^j, \\ b_{2j,j}(\alpha, \alpha, n) &= (-4)^j \frac{\Gamma(n+2j+1)}{\Gamma(n+1)} \frac{\Gamma(2n+2\alpha+2j+1)}{\Gamma(2n+2\alpha+4j+1)} \cong (-4)^j. \end{aligned}$$

Thus

$$I_1(k, n) = 0, \quad 0 \leq k \leq n-1.$$

Let $n \geq 4$. Then

$$\begin{aligned} I_1(n, n) &= A_n \{h_n^{(\alpha+j)}\}^{-1/2} \{h_{n-4}^{(\alpha+4)}\}^{-1/2} \\ &\quad \times b_{0,j}(\alpha, \alpha, n) b_{4,2}(\alpha+2, \alpha+2, n-4) \int_{-1}^1 P_n^{(\alpha)}(x) P_n^{(\alpha+2)}(x) d\mu(x). \end{aligned}$$

Since (see [1, p. 359, Theorem 7.1.4])

$$P_n^{(\alpha+2)}(x) = \frac{(2\alpha+1)_4 (n+\alpha+1/2)_2 (n+\alpha+1)_2}{(\alpha+1/2)_2 (\alpha+1)_2 (n+2\alpha+1)_4} P_n^{(\alpha)}(x) + Q_{n-1}(x),$$

we get

$$\begin{aligned} (11) \quad &\frac{2^{2\alpha+1} \Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)} I_1(n, n) \\ &= \frac{A_n (2\alpha+1)_4 (n+\alpha+1/2)_2 (n+\alpha+1)_2}{(\alpha+1/2)_2 (\alpha+1)_2 (n+2\alpha+1)_4} \{h_n^{(\alpha+j)}\}^{-1/2} \\ &\quad \times \{h_{n-4}^{(\alpha+4)}\}^{-1/2} h_n^{(\alpha)} b_{0,j}(\alpha, \alpha, n) b_{4,2}(\alpha+2, \alpha+2, n-4) \cong 16 \cdot 2^j A_n. \end{aligned}$$

In a similar way, for $k \geq 2$, using [12, (4.3.4)] and (10)

$$\begin{aligned} I_2(k, n) &= B_k \{h_n^{(\alpha+j)}\}^{-1/2} \{h_{k-2}^{(\alpha+2)}\}^{-1/2} \sum_{m=0}^{2j} b_{m,j}(\alpha, \alpha, n) \\ &\quad \times \int_{-1}^1 P_{n+m}^{(\alpha)}(x) (1-x^2) P_{k-2}^{(\alpha+2)}(x) d\mu(x). \end{aligned}$$

Again, from [8, formula (2.8)] and [1, p. 359, Theorem 7.1.4] we get

$$\begin{aligned} (1-x^2) P_{k-2}^{(\alpha+2)}(x) &= b_{2,1}(\alpha+1, \alpha+1, k-2) P_k^{(\alpha+1)}(x) \\ &\quad + b_{1,1}(\alpha+1, \alpha+1, k-2) P_{k-1}^{(\alpha+1)}(x) + b_{0,1}(\alpha+1, \alpha+1, k-2) P_{k-2}^{(\alpha+1)}(x) \end{aligned}$$

and

$$P_n^{(\alpha+1)}(x) = \frac{(2\alpha+1)_2(n+\alpha+1/2)(n+\alpha+1)}{(\alpha+1/2)(\alpha+1)(n+2\alpha+1)_2} P_n^{(\alpha)} + Q_{n-1}(x).$$

Thus

$$I_2(k, n) = 0, \quad 0 \leq k \leq n-1.$$

For $n \geq 2$ we have

$$(12) \quad \frac{2^{2\alpha+1}\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)} I_2(n, n) \\ = \frac{(2\alpha+1)_2(n+\alpha+1/2)(n+\alpha+1)B_n}{(\alpha+1/2)(\alpha+1)(n+2\alpha+1)_2} \{h_n^{(\alpha+j)}\}^{-1/2} \\ \times \{h_{n-2}^{(\alpha+2)}\}^{-1/2} h_n^{(\alpha)} b_{0,j}(\alpha, \alpha, n) b_{2,1}(\alpha+1, \alpha+1, n-2) \cong -4 \cdot 2^j B_n.$$

Finally, for $k \geq 0$

$$I_3(k, n) = C_k \{h_n^{(\alpha+j)}\}^{-1/2} \{h_k^{(\alpha)}\}^{-1/2} \\ \times \sum_{m=0}^{2j} b_{m,j}(\alpha, \alpha, n) \int_{-1}^1 P_{n+m}^{(\alpha)}(x) P_k^{(\alpha)}(x) d\mu(x).$$

Thus

$$(13) \quad \begin{cases} I_3(k, n) = 0, & 0 \leq k \leq n-1 \\ \frac{2^{2\alpha+1}\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)} I_3(n, n) = C_n \{h_n^{(\alpha+j)}\}^{-1/2} \{h_n^{(\alpha)}\}^{1/2} b_{0,j}(\alpha, \alpha, n) \cong 2^j C_n. \end{cases}$$

From (11), (12), and (13) we can conclude that

$$(14) \quad \begin{cases} (g_n^{(\alpha,j)})^\wedge(k) = 0, & 0 \leq k \leq n-1, \\ (g_n^{(\alpha,j)})^\wedge(n) \cong c \neq 0. \end{cases}$$

On the other hand, for $1 \leq p < \infty$

$$\|g_n^{(\alpha,j)}\|_{W_p}^p = \|g_n^{(\alpha,j)}\|_{L^p(d\mu)}^p = \int_{-1}^1 (1-x^2)^{jp+\alpha} |p_n^{(\alpha+j)}(x)|^p dx \\ \leq c_1 \int_0^1 (1-x)^{jp+\alpha} |p_n^{(\alpha+j)}(x)|^p dx \\ + c_2 \int_{-1}^0 (1+x)^{jp+\alpha} |p_n^{(\alpha+j)}(x)|^p dx.$$

Taking $M = N = 0$ and $\gamma = jp + \alpha$ in Lemma 2, we have

$$(15) \quad \|g_n^{(\alpha,j)}\|_{W_p}^p \leq c$$

for $j > \alpha + 1/2 - (2\alpha + 2)/p$ and $q_0 \leq p < \infty$.

It is well known (see [6, Theorem 1]) that

$$|p_n^{(\alpha+j)}(x)| \leq c(1-x^2)^{-\alpha/2-j/2-1/4}$$

for $x \in (-1, 1)$. Therefore

$$(16) \quad \|g_n^{(\alpha, j)}\|_{W_\infty} = \|g_n^{(\alpha, j)}\|_{L^\infty} \leq c(1-x^2)^{j/2-\alpha/2-1/4} \leq c,$$

for $j > \alpha + 1/2$.

Now we are in position to prove our main result:

Proof of Theorem 1. By duality, it is enough to assume that $q_0 \leq p \leq \infty$. From (9), (14), (15), and (16) we have

$$(17) \quad \|T_n^{\alpha, M, N}\|_{W_p} \geq [\|g_n^{(\alpha, j)}\|_{W_p}]^{-1} \|T_n^{\alpha, M, N} g_n^{(\alpha, j)}\|_{W_p} \geq c|c_{n, n}| \|\hat{B}_n^{(\alpha)}\|_{W_p}.$$

From (8) and Lemma 2 we obtain

$$(18) \quad \|\hat{B}_n^{(\alpha)}\|_{W_p} \geq c \begin{cases} (\log n)^{1/p} & \text{if } p = q_0, \\ n^{(2\alpha+1)/2-(2\alpha+2)/p} & \text{if } q_0 < p < \infty. \end{cases}$$

Now, using the estimate (18) in (17) the statement of Theorem 1 follows. \square

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