FIRST ORDER ASYMPTOTIC EXPANSIONS FOR EIGENVALUES OF MULTIPLICATIVELY PERTURBED MATRICES.

FREDY SOSA* AND JULIO MORO*

Abstract. Given an arbitrary square matrix A, we obtain the leading terms of the asymptotic expansions in the small, real parameter ε of multiplicative perturbations $\widehat{A}(\varepsilon) = (I + \varepsilon B)A(I + \varepsilon C)$ of A for arbitrary matrices B and C. The analysis is separated in two rather different cases, depending on whether the unperturbed eigenvalue is zero or not. It is shown that in either case the leading exponents are obtained from the partial multiplicities of the eigenvalue of interest, and the leading coefficients generically involve only appropriately normalized left and right eigenvectors of A associated with that eigenvalue, with no need of generalized eigenvectors. Similar results are obtained for multiplicative perturbation of singular values as well.

Key words. multiplicative perturbation of eigenvalues, perturbation of singular values, perturbation theory for linear operators, asymptotic expansions, Newton polygon

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1. Introduction.

First order eigenvalue perturbation theory deals with identifying the leading term in the asymptotic expansions of the eigenvalues of a perturbed operator $\widehat{A} = \widehat{A}(\varepsilon)$, depending on a small, usually positive real, perturbation parameter ε. The operator $\widehat{A}(\varepsilon)$ is assumed to be a slight deviation from some close, simpler matrix or operator A, for which the spectral problem is completely (and, in most cases, easily) solved.

The most usual approach to this question is to model the perturbation *additively*, by assuming that

$$\widetilde{A} = \widetilde{A}(\varepsilon) = A + \varepsilon E, \tag{1.1}$$

for some appropriate perturbation matrix E, where ε is a small, positive parameter. In this paper we will instead analyze *multiplicative* perturbations of the form

$$\widehat{A} = \widehat{A}(\varepsilon) = (I + \varepsilon C)A(I + \varepsilon B), \tag{1.2}$$

where A, B and C are square complex matrices of the same dimensions.

There are several contexts in which multiplicative perturbations are more natural than additive ones (see [7, §5], where some of these situations are discussed at length): sometimes, for instance, it is easier to express an entry-wise perturbation of a sparse matrix as a multiplicative, rather than as an additive, perturbation. This often opens the possibility of performing a finer perturbation analysis, which is able to quantify the influence of changes in specific entries of the matrix. Consider, for instance, the following example, taken from [1]:

$$A = \begin{pmatrix} 0 & \alpha_1 & & & \\ \alpha_1 & 0 & \alpha_2 & & \\ & \alpha_2 & 0 & \alpha_3 & \\ & & \alpha_3 & 0 & \alpha_4 \\ & & & \alpha_4 & 0 \end{pmatrix}.$$

Such real, symmetric tridiagonal matrices appear, for instance, whenever the singular value problem of a bidiagonal matrix is transformed into an eigenvalue problem. An entry-wise

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round-off perturbation to a single off-diagonal pair, say α_3, produces a matrix

$$\widehat{A} = \begin{pmatrix} 0 & \alpha_1 & & & \\ \alpha_1 & 0 & \alpha_2 & & \\ & \alpha_2 & 0 & \beta\alpha_3 & \\ & & \beta\alpha_3 & 0 & \alpha_4 \\ & & & \alpha_4 & 0 \end{pmatrix},$$

where $\beta = 1 + \varepsilon$ and $|\varepsilon|$ does not exceed the size of the roundoff unit (roughly 10^{-16} for the IEEE double precision standard). Then, the perturbed matrix $\widehat{A}$ can be written as a multiplicative perturbation $\widehat{A} = DAD$, where D is the diagonal matrix

$$D = \mathrm{diag}(\beta^{1/2}, \beta^{-1/2}, \beta^{1/2}, \beta^{1/2}, \beta^{-1/2}) = I + \varepsilon B, \qquad \|B\| = O(1).$$

Notice that one might of course rewrite the perturbation as an additive one of the form (1.1), but this would lead to a perturbation matrix E depending on α_3, while D does not depend on the entries of A.

This specific example illustrates some relevant features of multiplicative vs. additive perturbations: quite often, the interest in multiplicative perturbations is motivated by a quest for relative perturbation bounds, i.e., bounds on the relative error

$$\frac{|\widehat{\lambda} - \lambda|}{|\lambda|}$$

between the perturbed eigenvalue $\widehat{\lambda}$ and the unperturbed one λ, instead of bounding just the absolute error $|\widehat{\lambda} - \lambda|$. That is the case in [1], as well as in several other references since the 1990s ([4, 10, 11, 19], see also [9]). Such bounds are needed for the error analysis of high relative accuracy eigenvalue algorithms, to show that, for certain specific classes of matrices, small relative changes in the entries of the matrix cause only small relative changes in the eigenvalues. This allows such algorithms, tailored to the specific class of matrices under study, to compute all eigenvalues, even tiny ones, to high relative accuracy. From this point of view, multiplicative perturbations are often better suited to this kind of error analysis, since additive perturbations lead typically to absolute error bounds, while phrasing the perturbation multiplicatively leads naturally to error bounds in the relative sense.

The example above also shows that, although a multiplicative perturbation (1.2) can always be disguised, up to higher order terms, as an additive one by taking $E = CA + AB$, this has the awkward consequence of the additive perturbation matrix E depending on the unperturbed one A. It should be noted, however, that multiplicative perturbations are in a way less powerful than additive ones, since, unlike the latter, *multiplicative perturbations preserve the rank* of A for small ε. This will make a big difference when the unperturbed eigenvalue under examination is zero, since in that case the unperturbed eigenvalue zero persists as an eigenvalue of the perturbed matrix (see § 3.2 below).

Another context where writing perturbations multiplicatively may be advantageous is that of *structured* perturbations: recent interest in spectral algorithms specifically designed to preserve some relevant, physically meaningful structure has brought with it the development of corresponding *structured* eigenvalue perturbation theories, which constrain both perturbed and unperturbed matrices $\widehat{A}$ and A to belong to the same class of structured matrices of interest. Whenever that class of matrices has an underlying multiplicative structure, it may be more natural to write the perturbation in multiplicative form. Take, for instance, symplectic matrices: the sum of two symplectic matrices is in general not symplectic, but their product always is. It has recently been shown in [13, Lemma 7.1], for instance, that any J-symplectic rank-one perturbation $\widehat{S}$ of a J-symplectic matrix S can be written as

$$\widehat{S} = (I + uu^T J)S$$

for an appropriate vector u. In other words, any rank-one symplectic perturbation of a symplectic matrix is necessarily multiplicative.

Results on multiplicative perturbations go back at least to Ostrowski [17], who bounded the ratio between the eigenvalues of a multiplicative perturbation $\widehat{A} = DAD^*$ and those of the unperturbed Hermitian A in terms of the largest and smallest eigenvalues of DD^* for nonsingular D. As has already been mentioned, the 1990s witnessed the publication of several papers containing multiplicative perturbation results for eigenvalues, motivated by the error analysis of high relative accuracy algorithms (see [1, 4, 10, 11, 9] or the comprehensive survey [7]). Most of them also contain results on the multiplicative perturbation of singular values, which, as we will see in § 4 below, can be essentially reduced to the structured perturbation of eigenvalues of Hermitian matrices. In any case, all these results present finite perturbation bounds, i.e., no small perturbation parameter is present as in (1.2), and the perturbation matrices are allowed to be arbitrarily large.

Our focus in this paper, however, lies on determining the local behavior of (possibly multiple) eigenvalues under asymptotically small multiplicative perturbations. It is well known (see [3, §9.3.1] or [8, §II.1.2]) that each eigenvalue and eigenvector of (1.1) admits an expansion in fractional powers of ε whose zero-th order term is an eigenvalue or eigenvector of the unperturbed matrix A. The same kind of argument leads to the same conclusion for multiplicative perturbations (1.2) as well, and our goal here is to find both the leading term and the leading coefficient of such asymptotic expansions in ε: the leading exponent will measure how fast the perturbed eigenvalues move away from unperturbed ones, while the leading coefficient will indicate the directions along which they escape. In this sense, our results are the multiplicative analogue of classical results, going back to either Vishik and Lyusternik, or Lidskii, for additive parameter-dependent perturbations: Vishik and Lyusternik's results [22], motivated originally by infinite-dimensional differential operators, were later specialized by Lidskii [12] to the finite-dimensional case. Lidskii obtained simpler explicit formulas for the leading coefficients and provided, at the same time, a much more elementary proof. This classical perturbation theory was revisited, and extended, in [14], by means of the so-called Newton diagram, an elementary geometrical construction first devised by Sir Isaac Newton (see e.g. [2, §8.3]). This will be our main tool here as well.

Our goal in this paper is, therefore, to find multiplicative analogues for Lidskii's additive first order perturbation expansions. The expansions we shall find are similar to Lidskii's, with a sharp distinction between the case of a nonzero (Theorem 3.1) and that of a zero unperturbed eigenvalue (Theorem 3.2), as expected given the preservation of rank. What is not so expected and, therefore, worth remarking, is that, although the case of a zero eigenvalue is highly nongeneric from the additive point of view, we still recover expansions which are very similar in structure to those for the additive case and, furthermore, only eigenvectors are involved, with no influence at all of generalized eigenvectors. Also surprising is, in our opinion, the fact that the leading term in the asymptotic expansion for two-sided multiplicative perturbations of the form (1.2) is the same as that for what we might call the 'equivalent' one-sided perturbations $A(I + \varepsilon(B + C))$, or $(I + \varepsilon(B + C))A$.

The paper is organized as follows: in section 2 we briefly recall some preliminaries which will be needed later on: § 2.1 describes the basics of the Newton diagram technique, while §2.2 is devoted to briefly recalling Lidskii's first order expansions for additive perturbations [12]. Section 3 contains the main results of this paper: Theorem 3.1 for nonzero unperturbed eigenvalues, and Theorem 3.2 for zero unperturbed eigenvalue. Since the proof of the latter is quite more involved than that of the former, we have chosen to postpone to a final Appendix the proofs of certain intermediate results of a quite technical nature.

2. Preliminaries.

2.1. The Newton Diagram. As stated in the introduction, our goal in this work is to find the leading terms of asymptotic expansions in the parameter ε of the eigenvalues of multiplicative perturbations (1.2) of a matrix A. Obviously, such eigenvalues are the roots of the corresponding characteristic polynomial

$$p(\lambda, \varepsilon) = \det(\lambda I - (I + \varepsilon C)A(I + \varepsilon B)),$$

which is a polynomial in λ with ε-dependent coefficients. The classical tool to find the leading term in the asymptotic expansions of roots of such polynomials is the so-called *Puiseux-Newton Diagram* (in short, Newton Diagram, or also Newton Polygon), an elementary geometrical construction going back to Sir Isaac Newton (but only rigorously founded by Puiseux [18]), which provides us with both leading powers and leading coefficients of the expansions (see [3, Appendix A7], [2, §8.3] or [14, 16] for more details). The Newton Diagram technique applies to any complex polynomial[1]

$$P(\lambda\,,\,\varepsilon) = \lambda^n + \alpha_1(\varepsilon)\lambda^{n-1} + \ldots + \alpha_{n-1}(\varepsilon)\lambda + \alpha_n(\varepsilon). \tag{2.1}$$

in a variable λ with coefficients depending analytically on a parameter ε. In order to simplify the exposition, we assume there is only one zero root of multiplicity n for $\varepsilon = 0$, i.e., the coefficients $\alpha_k(\varepsilon)$ satisfy

$$\alpha_k(\varepsilon) = \widehat{\alpha}_k \varepsilon^{a_k} + o(\varepsilon^{a_k})\,, \qquad k = 1,\, 2,\, \ldots,\, n,$$

with $\widehat{\alpha}_k \neq 0$, and no term of order lower than a_k appears in the expansion of $\alpha_k(\varepsilon)$. Otherwise, we just shift $\lambda \mapsto \lambda - \lambda_0$ for any nonzero root λ_0 of $P(\lambda, 0)$.

It is well known [3, 8] that in this situation the roots of equation (2.1) can be written as a series in fractional powers of ε, and we are interested in finding the leading term (i.e., both the leading exponent and leading coefficient) of these series.

The Newton Diagram associated with equation (2.1) is obtained as follows: let $I_d = \{k \in \{0,\, \ldots,\, n\} \,:\, \alpha_k(\varepsilon) \neq 0\}$ and $k_{max} = \max I_d$; notice that $\alpha_0(\varepsilon) = 1$ and, as a consequence, $a_0 = 0$. Now we plot the set of points $\{(k\,,\, a_k) \,:\, k \in I_d\} \subset \mathbb{Z}^2$ on a Cartesian grid, and draw the segments on the lower boundary of the convex hull of the plotted points. These segments constitute the so-called *Newton Diagram* associated with the polynomial $P(\lambda\,,\,\varepsilon)$ in (2.1). For instance, the diagram associated with the polynomial

$$P(\lambda, \varepsilon) = \lambda^5 + (2\varepsilon^2 - \varepsilon^3)\lambda^4 - \varepsilon\lambda^3 + (-6\varepsilon^2 + 3\varepsilon^5)\lambda + \varepsilon^3 - \varepsilon^4$$

is as illustrated in Fig. 2.1.

It turns out that *the leading exponents of the asymptotic expansions of the different roots of P are just the slopes of the different segments in the Newton Diagram.* More specifically, let S be an arbitrary segment in the diagram, and $I_S = \{k \in I_d \,:\, (k\,,\, a_k) \in S\}$. If we denote by η the slope of S, $k_{\min} = \min I_S$ and $k_{\max} = \max I_S$, then there are $k_{\max} - k_{\min}$ nonzero roots of $P(\lambda\,,\,\varepsilon)$ with asymptotic expansions.

$$\lambda_j(\varepsilon) = \mu_j \varepsilon^{\eta} + \sum_{s=2}^{\infty} a_{js}\, \varepsilon^{s\eta}\,, \qquad j = 1,\, \ldots,\, k_{\max} - k_{\min}. \tag{2.2}$$

[1]The Newton Diagram technique applies, in fact, to more general analytic functions, but we restrict ourselves to the special case of polynomials in λ with coefficients analytic in a parameter ε.

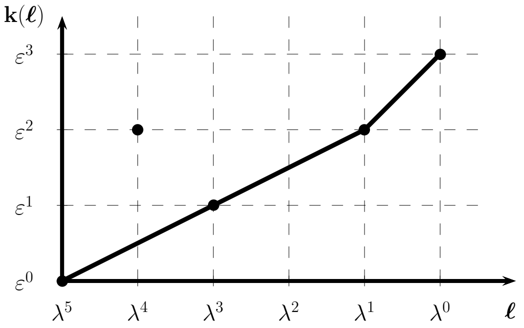

FIG. 2.1. Newton diagram for $P(\lambda,\varepsilon) = \lambda^5 + (2\varepsilon^2 - \varepsilon^3)\lambda^4 - \varepsilon\lambda^3 + (-6\varepsilon^2 + 3\varepsilon^5)\lambda + \varepsilon^3 - \varepsilon^4$.

Futhermore, *the leading coefficients* $\{\mu_j\}_{j=1}^{k_{max}-k_{min}}$ *are just the roots of the polynomial*

$$\sum_{k\in I_S} \widehat{\alpha}_k\, \mu^{k_{\max}-k}, \tag{2.3}$$

which is, in general, of a much lower order than $P(\lambda\,,\,\varepsilon)$.

Summarizing, to obtain both the leading exponent η and the leading coefficient μ_j in the asymptotic expansions (2.2), all we have to do is

1. Draw the associated Newton Diagram;
2. Compute the different slopes η of the segments on the Newton Diagram. These are the leading exponents of the different roots of (2.1);
3. For each slope η, find the length of the projection on the horizontal axis of the segment with slope η. This is the number of roots of the order of ε^η;
4. The leading coefficient μ_j for each root of order ε^η is each of the roots of equation (2.3), where S is the segment of the Newton diagram with slope η.

2.2. First order expansions for additive perturbation of eigenvalues. In this section we briefly describe some classical first order additive perturbation results which will be needed later in the multiplicative case. They go back to Vishik and Lyusternik [22], who first obtained them in the context of differential operators, but we will be using mostly the expansions obtained by Lidskii [12], who specialized them to the finite-dimensional case.

In order to state Lidskii's result, we first need to introduce some notation: let A be an arbitrary complex $n\times n$ matrix, and consider an additive perturbation

$$\widetilde{A}(\varepsilon) = A + \varepsilon B \tag{2.4}$$

for arbitrary $B \in \mathbb{C}^{n\times n}$ and small, real $\varepsilon > 0$. Suppose that the unperturbed matrix A in (2.4) has Jordan structure

$$\left[\begin{array}{c|c} J & \\ \hline & \widehat{J} \end{array}\right] = \left[\begin{array}{c} Q \\ \hline \widehat{Q} \end{array}\right] A \left[\begin{array}{c|c} P & \widehat{P} \end{array}\right] \tag{2.5}$$

with

$$\left[\begin{array}{c} Q \\ \hline \widehat{Q} \end{array}\right] \left[\begin{array}{c|c} P & \widehat{P} \end{array}\right] = I, \tag{2.6}$$

where J corresponds to a (possibly multiple) eigenvalue λ_0, and $\widehat{J}$ is the part of the Jordan form containing the other eigenvalues of A. Moreover, we take J to be partitioned in the form

$$J = \mathrm{Diag}(\Gamma_1^1, \ldots, \Gamma_1^{r_1}, \ldots, \Gamma_q^1, \ldots, \Gamma_q^{r_q}), \tag{2.7}$$

where, for $j = 1, \ldots, q$, each Γ_j^k, $k = 1, \ldots, r_j$ is a Jordan block of dimension n_j, in decreasingly order according to their dimensions, i.e., $n_1 > n_2 > \ldots > n_q$.

Notice that the columns of P (resp. the rows of Q) form right (resp. left) Jordan chains of A associated with λ_0. If we partition P in blocks of columns (resp. Q in blocks of rows) conformally with (2.7), then the first column (resp. the last row) of each block is a right (resp. left) eigenvector of A associated with λ_0. We collect all those eigenvectors cumulatively as follows: for each $s \in \{1, \ldots, q\}$ we define

$$f_s = \sum_{j=1}^{s} r_j, \tag{2.8}$$

and we denote by Z_s (resp. W_s), $s \in \{1, \ldots, q\}$, the $n \times f_s$ matrix (resp. the $f_s \times n$ matrix) whose columns (resp. rows) are all right (resp. left) eigenvectors taken from the Jordan chains in P (resp. in Q) of length at least n_s.

Finally, given any arbitrary matrix $K \in \mathbb{C}^{n \times n}$, we define associated square $f_s \times f_s$ matrices $\Phi_s(K)$ and E_s by

$$\begin{array}{lll} \Phi_s(K) = W_s K Z_s, & s = 1, \ldots, q, & \\ E_1 = I, & E_s = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} & \text{for } s = 2, \ldots, q, \end{array} \tag{2.9}$$

where the identity block in E_s has dimension r_s. Note that, due to the cumulative definitions of W_s and Z_s, every $\Phi_{s-1}(K)$, is the upper left block of $\Phi_s(K)$ for $s = 2, \ldots, q$.

We are now in the position to state Lidskii's Theorem [12]:

THEOREM 2.1. *(Lidskii [12]) Let A be a complex $n \times n$ matrix with an eigenvalue λ_0 and Jordan structure (2.5). Let B be any complex $n \times n$ matrix, and let $j \in \{1, \ldots, q\}$ be such that, if $j > 1$, $\Phi_{j-1}(B)$ is nonsingular. Then there are $r_j n_j$ eigenvalues of the perturbed matrix $A + \varepsilon B$ admitting first-order expansions*

$$\widehat{\lambda}_{j,k,l} = \lambda_0 + (\xi_{j,k})^{1/n_j}\, \varepsilon^{1/n_j} + o\left(\varepsilon^{1/n_j}\right) \tag{2.10}$$

for $k = 1, \ldots, r_j$, $l = 1, \ldots, n_j$, where

(i) *the $\xi_{j,k}$, $k = 1, \ldots, r_j$, are the roots of equation*

$$\det\left(\Phi_j(B) - \xi\, E_j\right) = 0. \tag{2.11}$$

where Φ_j and E_j are as in (2.9). Equivalently, the $\xi_{j,k}$ are the eigenvalues of the Schur complement† *of $\Phi_{j-1}(B)$ in $\Phi_j(B)$ (if $j = 1$, the $\xi_{1,k}$ are just the r_1 eigenvalues of*

†Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A invertible. Then the Schur complement of A in M is $D - CA^{-1}B$.

$\Phi_1(B)$);

(ii) *the different values $\widehat{\lambda}_{j,k,l}(\varepsilon)$ for $l = 1, \ldots, n_j$ are obtained by taking the n_j distinct n_j-th roots of $\xi_{j,k}$.*

Notice that Theorem 2.1 applies to any perturbation matrix B except those for which some $\Phi_s(B)$ is singular. The singularity of any such matrix amounts to a polynomial condition on the entries of B, so the set of matrices B for which some $\Phi_s(B)$ is singular has zero measure within the set $\mathbb{C}^{n\times n}$ of complex $n \times n$ matrices. In other words, Theorem 2.1 describes the *generic behavior* of λ_0 under additive matrix perturbations.

Theorem 2.1 can be proved in different ways, but the one most relevant to our purposes is the one making use of the Newton Diagram (see [14, 16]).

3. First order expansions for eigenvalues of multiplicative perturbations. The main goal of this section is to obtain a result similar to Theorem 2.1, but for multiplicative, instead of additive perturbations. Let $A \in \mathbb{C}^{n\times n}$ be a matrix with arbitrary Jordan form (2.5), and consider a multiplicative perturbation

$$\widehat{A}(\varepsilon) = (I + \varepsilon C)\, A\, (I + \varepsilon B). \tag{3.1}$$

for arbitrary matrices B, $C \in \mathbb{C}^{n\times n}$. In the additive case, the asymptotic expansions do not depend on whether the eigenvalue λ_0 under examination is zero or nonzero. For multiplicative perturbations, however, the two cases are very different, since, as we already observed, *multiplicative perturbations preserve rank*. This means that $\lambda_0 = 0$ will persist as an eigenvalue of $\widehat{A}(\varepsilon)$ for any ε. Therefore, we will need to analyze both cases separately. We begin with the simpler case of nonzero eigenvalues, which will turn out to be a straightforward consequence of the additive case.

3.1. The case $\lambda_0 \neq 0$. Suppose the unperturbed eigenvalue λ_0 of A is different from zero. We may rewrite the perturbation (3.1) additively as

$$\widehat{A}(\varepsilon) = A + \varepsilon\,(CA + AB) + O\left(\varepsilon^2\right),$$

and apply Lisdkii's Theorem 2.1 for additive perturbations to conclude that the asymptotic behavior of λ_0 under perturbation depends on the matrices $\Phi_s(CA + AB)$, as defined in (2.9). Now, recall that the rows of W_s (respectively, columns of Z_s) are left eigenvectors (respectively., right eigenvectors) of A associated with λ_0, i.e.

$$\begin{aligned} W_s A &= \lambda_0 W_s \\ A Z_s &= \lambda_0 Z_s\,, \qquad s = 1,\, \ldots,\, q \end{aligned}$$

and, therefore,

$$\Phi_s(CA + AB) = W_s\,(CA + AB)\,Z_s = \lambda_0 W_s\,(C + B)\,Z_s = \lambda_0 \Phi_s\,(B + C)$$

Hence, although the additive perturbation matrix $CA + AB$ does depend on A, the corresponding $\Phi_s(\cdot)$ does not. This leads directly to the following result:

THEOREM 3.1. *Let $\lambda_0 \neq 0$ be an eigenvalue of a complex $n \times n$ matrix A with Jordan structure (2.5), and let B, C be arbitrary $n \times n$ complex matrices. Let $j \in \{1,\, \ldots,\, q\}$ be given and assume that if $j > 1$, $\Phi_{j-1}(B + C)$ is nonsingular, where $\Phi_{j-1}(\cdot)$ is defined as in (2.9). Then there are $r_j n_j$ eigenvalues of the perturbed matrix $\widehat{A}(\varepsilon) = (I + \varepsilon C)\, A\, (I + \varepsilon B)$ admitting first order expansions*

$$\widehat{\lambda}_{j,k,l} = \lambda_0 + (\lambda_0 \xi_{j,k})^{1/n_j} \varepsilon^{1/n_j} + o\left(\varepsilon^{1/n_j}\right) \tag{3.2}$$

where

(i) *the* $\xi_{j,k},\ k = 1, \ldots, r_j,$ *are the roots of the equation*

$$\det\left(\Phi_j(B + C) - \xi\, E_j\right) = 0 \tag{3.3}$$

where Φ_j *and* E_j *are as in (2.9). Equivalently, the* $\xi_{j,k},\ k = 1, \ldots, r_j,$ *are the eigenvalues of the Schur complement of* $\Phi_{j-1}(B+C)$ *in* $\Phi_j(B+C)$ *(if* $j = 1,$ *the* $\xi_{1,k}$ *are just the* r_1 *eigenvalues of* $\Phi_1(B + C)$*);*

(ii) *the different values* $\widehat{\lambda}_{j,k,l}(\varepsilon)$ *for* $l = 1, \ldots, n_j$ *are defined by taking the* n_j *distinct* n_j*-th roots of* $\xi_{j,k}$.

Notice that the form of the expansion (3.2) leads to an asymptotic *relative* perturbation bound

$$\frac{|\widehat{\lambda}_{j,k,l} - \lambda_0|}{|\lambda_0|} = O(\varepsilon^{1/n_j}),$$

only if either $\lambda_0 = O(1)$ or $n_j = 1$.

3.2. The case $\lambda_0 = 0$**.** Let us now consider the case when the eigenvalue under examination is zero. Notice that the argument in section 3.1 above gives no information whatsoever, since now both AZ_s and W_sA are zero, so $\Phi_s(CA + AB) = 0$. Furthermore, the fact that A and $\widehat{A}(\varepsilon)$ have the same rank forces $\lambda_0 = 0$ to be an eigenvalue of both matrices, and with the same geometric multiplicity. Hence, both matrices A and $\widehat{A}$ have the same number of Jordan blocks associated with $\lambda_0 = 0$. The algebraic multiplicity, however, will generically decrease: it is well known (see, for instance, [5, 15, 20]) that the larger the dimension of a Jordan block, the more unstable it is under perturbation. Hence, the most likely behavior of the zero eigenvalue under multiplicative perturbations is that any of its 1×1 Jordan blocks in A is preserved in $\widehat{A}$, while any Jordan block of A of dimension larger than one becomes a 1×1 Jordan block of $\widehat{A}$. In other words, the algebraic multiplicity of $\lambda_0 = 0$ is expected to go from its initial value of $n_1 r_1 + \ldots + n_q r_q$ down to $f_q = r_1 + \ldots + r_q$, creating $r_1(n_1 - 1) + \ldots + r_q(n_q - 1)$ nonzero eigenvalues in the process.

In terms of the Newton Diagram, this amounts to the diagram being formed by q segments with slopes $1/(n_j - 1),\ j = 1, \ldots, q$, instead of q segments of slope $1/n_j$ as in the additive case. The length of their horizontal projections is therefore smaller, $r_j(n_j - 1)$ instead of $r_j n_j$. Loosely speaking, the Newton Diagram for the multiplicative case should be obtained by moving the first point $(n_1, 1)$ on the additive Newton Diagram one unit to the left, the second point on the additive diagram two units to the left, an so on. Consider, for instance the following 8×8 example,

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.4}$$

In this case, for instance, the two Newton Diagrams, additive and multiplicative, will be the ones in Figure 3.1 below. The one for additive perturbations is the dashed one on the bottom,

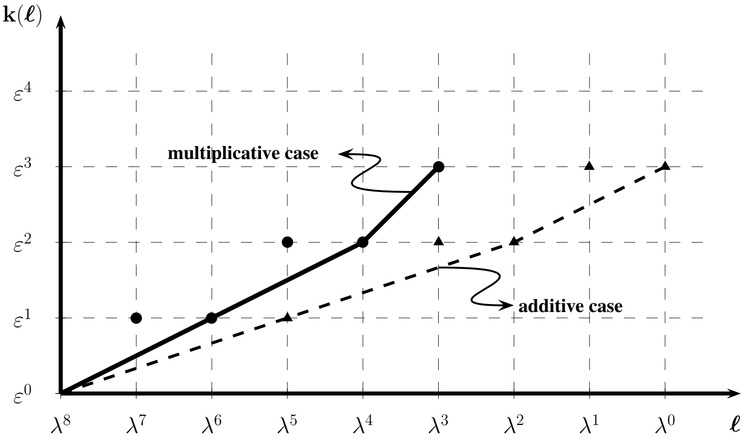

FIG. 3.1. Newton diagrams for example (3.4). The Newton diagram corresponding to the multiplicative case is depicted with a solid line, the one with the dashed line is associated with additive perturbations.

while the diagram for the multiplicative ones is the solid line on top.

More specifically, the expected behavior under multiplicative perturbations of a zero eigenvalue is described by our main Theorem in this section:

THEOREM 3.2. *Let A be any complex $n \times n$ matrix with Jordan structure (2.5) and $\lambda_0 = 0$. Let B, C be arbitrary $n \times n$ complex matrices, let $j \in \{1, \ldots, q\}$ be given and assume that if $j > 1$, $\Phi_{j-1}(B + C)$ is nonsingular. Then, there are $r_j\,(n_j - 1)$ eigenvalues of the perturbed matrix $\widehat{A}(\varepsilon) = (I + \varepsilon C)\,A\,(I + \varepsilon B)$ admitting first order expansions*

$$\widehat{\lambda}_{j,k,l} = (\xi_{j,k})^{\frac{1}{n_j - 1}}\,\varepsilon^{\frac{1}{n_j-1}} + o\left(\varepsilon^{\frac{1}{n_j-1}}\right). \tag{3.5}$$

Moreover, if $n_j \geq 2$, then

(i) *the $\xi_{j,k}$, $k = 1, \ldots, r_j$, are the roots of equation*

$$\det\left(\Phi_j(B + C) - \xi\,E_j\right) = 0 \tag{3.6}$$

where Φ_j and E_j are as in (2.9) or, equivalently, the eigenvalues of the Schur complement of $\Phi_{j-1}(B+C)$ in $\Phi_j(B+C)$ (if $j = 1$, the $\xi_{1,k}$ are just the r_1 eigenvalues of $\Phi_1(B+C)$);

(ii) *the different values $\widehat{\lambda}_{j,k,l}(\varepsilon)$ for $l = 1, \ldots, n_j - 1$ are defined by taking the $(n_j - 1)$ distinct $(n_j - 1)$-th roots of ξ_j^k.*

(iii) *the remaining r_j eigenvalues are zero, i.e.,*

$$\widehat{\lambda}_{j,k,n_j} = 0,\;\; k = 1, \ldots, r_j \tag{3.7}$$

Notice that one consequence of Theorem 3.2 (and of Theorem 3.1 as well, for that matter) is that the leading coefficients depend on the perturbation matrices B an C only through their sum, so the perturbed eigenvalues of (1.2) have the same leading term as those of the matrix

$(I+\varepsilon(B+C))A$, where A is perturbed only from the left, or those of $A(I+\varepsilon(B+C))$, since all three perturbations give rise to the same matrices $\Phi_j(B+C)$. Of course the two latter matrices have the same eigenvalues, since they are products of the same two matrices in reverse order, but this is not true for (1.2). One can easily check that the eigenvalues of (1.2) are close to those of the one-sided perturbations only for small values of ε.

Now, in order to prove Theorem 3.2 we will separately prove the validity of the expansions (3.5) and of the formula (3.6) for the leading coefficients.

Mirroring the idea of the proof of Lidskii's theorem in [14], we first prove the validity of (3.5) by finding, for every height $l \in \{1,\ldots,f_q\}$, the rightmost possible point on the Newton Diagram at that specific height. This amounts to identifying the lowest possible Newton Diagram in this situation (the *Newton envelope* in the terminology of [14]). The corresponding result is as follows:

THEOREM 3.3. *Let $A \in \mathbb{C}^{m\times m}$ be a complex matrix with Jordan form (2.5) and $\lambda_0 = 0$. For every $l \in \{1,\ldots,f_q\}$, let $k(l)$ be the largest possible integer such that there exist perturbation matrices B, C for which $a_{k(l)} = l$. If $l = f_{j-1}+\rho$ for some $j \in \{1,\ldots,q\}$ with $n_j \geq 2$ and $1 \leq \rho \leq r_j$, then*

$$k(l) = r_1\,(n_1-1) + \ldots + r_{j-1}\,(n_{j-1}-1) + \rho\,(n_j-1)\,. \tag{3.8}$$

(the case $n_j = 1$ is left out of both Theorems 3.2 and 3.3, since we know that 1 by 1 Jordan blocks corresponding to a zero eigenvalue are preserved by multiplicative perturbations). In order to prove Theorem 3.3, we will need to carefully analyze all possible ways of constructing principal minors of order ε^l of an appropriately chosen ε-dependent matrix $\widehat{H}(\varepsilon)$, to be defined in the next subsection. We first identify the matrix $\widehat{H}(\varepsilon)$ in section 3.2.1, and describe basic properties of its principal minors in section 3.2.2 below. Section 3.2.3 contains the proof of Theorem 3.3, while the proof of Theorem 3.2 is included in § 3.2.4.

3.2.1. The matrix $\widehat{H}(\varepsilon)$. We begin by transforming the characteristic polynomial of the perturbed matrix $\widehat{A}(\varepsilon)$ in order to apply the Newton Diagram technique:

$$\begin{aligned} P(\lambda\,,\,\varepsilon) &= \det\left(\lambda I - \widehat{A}(\varepsilon)\right) = \det\left(\lambda I - (I+\varepsilon C)\,A\,(I+\varepsilon B)\right) = \\ &= \det\left(\lambda I - \left(I+\varepsilon\widetilde{C}\right)\left[\begin{array}{c|c} J_0 & \\ \hline & \widehat{J}\end{array}\right]\left(I+\varepsilon\widetilde{B}\right)\right), \end{aligned}$$

where J_0 contains the Jordan blocks associated with $\lambda_0 = 0$, $\widehat{J}$ contains the Jordan blocks corresponding to nonzero eigenvalues, and

$$\widetilde{B} = \left[\begin{array}{c} Q \\ \hline \widehat{Q}\end{array}\right] B \left[\begin{array}{c|c} P & \widehat{P}\end{array}\right] = \left[\begin{array}{c|c} QBP & QB\widehat{P} \\ \hline \widehat{Q}BP & \widehat{Q}B\widehat{P}\end{array}\right] = \left[\begin{array}{c|c} \widetilde{B}_{11} & \widetilde{B}_{12} \\ \hline \widetilde{B}_{21} & \widetilde{B}_{22}\end{array}\right]$$

$$\widetilde{C} = \left[\begin{array}{c} Q \\ \hline \widehat{Q}\end{array}\right] C \left[\begin{array}{c|c} P & \widehat{P}\end{array}\right] = \left[\begin{array}{c|c} QCP & QC\widehat{P} \\ \hline \widehat{Q}CP & \widehat{Q}C\widehat{P}\end{array}\right] = \left[\begin{array}{c|c} \widetilde{C}_{11} & \widetilde{C}_{12} \\ \hline \widetilde{C}_{21} & \widetilde{C}_{22}\end{array}\right]$$

Partitioning the matrix λI, and using the properties of the Schur complement, we may factorize $P(\lambda\,,\,\varepsilon)$ as

$$P(\lambda\,,\,\varepsilon)=\widehat{\pi}(\lambda\,,\,\varepsilon)\pi(\lambda\,,\,\varepsilon),$$

for

$$\begin{aligned}&\widehat{\pi}(\lambda\,,\,\varepsilon)=\det\left(M\right),\\&\pi(\lambda\,,\,\varepsilon)=\det\left(\lambda I-\left(I+\varepsilon\widetilde{C}_{11}\right)J_0\left(I+\varepsilon\widetilde{B}_{11}\right)-\varepsilon^2\left(\widetilde{C}_{12}\widehat{J}\widetilde{B}_{21}+\widehat{S}(\lambda\,,\,\varepsilon)\right)\right),\end{aligned}$$

where

$$\begin{aligned}&M=\lambda I-\left(I+\varepsilon\widetilde{C}_{22}\right)\widehat{J}\left(I+\varepsilon\widetilde{B}_{22}\right)-\varepsilon^2\widetilde{C}_{21}J_0\widetilde{B}_{12}\\&\widehat{S}(\lambda\,,\,\varepsilon)=\\&\Big(\Big(I+\varepsilon\widetilde{C}_{11}\Big)J_0\widetilde{B}_{12}+\widetilde{C}_{12}\widehat{J}\left(I+\varepsilon\widetilde{B}_{22}\right)\Big)M^{-1}\Big(\widetilde{C}_{21}J_0\Big(I+\varepsilon\widetilde{B}_{11}\Big)+\Big(I+\varepsilon\widetilde{C}_{22}\Big)\widehat{J}\widetilde{B}_{21}\Big)\,.\end{aligned}$$

For small ε, the matrices $I+\varepsilon B$, $I+\varepsilon C$ and $\widehat{J}$ are nonsingular. Hence, if λ is an eigenvalue of the perturbed matrix $\widehat{A}(\varepsilon)$ close to zero, it cannot be a root of $\widehat{\pi}(\lambda\,,\,\varepsilon)$, so it must be a root of the polynomial $\pi(\lambda\,,\,\varepsilon)$. Hence, we write $\pi(\lambda\,,\,\varepsilon)=\det\left(\lambda I-\widehat{H}_1(\varepsilon)\right)$ for

$$\widehat{H}_1(\varepsilon)=J_0+\varepsilon\left(\widetilde{C}_{11}J_0+J_0\widetilde{B}_{11}\right)+\varepsilon^2\left(\widetilde{C}_{12}\widehat{J}\widetilde{B}_{21}+\widetilde{C}_{11}J_0\widetilde{B}_{11}+\widehat{S}(\lambda\,,\,\varepsilon)\right). \qquad (3.9)$$

One can easily check that the sum $\widetilde{C}_{11}J_0+J_0\widetilde{B}_{11}$ has zero entries on the lower left corner of every submatrix resulting from the partition conformal with J_0. Consequently, the entries of $\widehat{H}_1(\varepsilon)$ in those positions are of order $O(\varepsilon^2)$. The remaining entries $\widehat{H}_1(\varepsilon)$ are either of order $O(1)$ (the ones coming from the superdiagonal 1s in J_0) or of order $O(\varepsilon)$ (all the rest). To illustrate this, consider again the 8×8 example $J_0=J_3(0)\oplus J_3(0)\oplus J_2(0)$ in (3.4). In that case, the sum $\widetilde{C}_{11}J_0+J_0\widetilde{B}_{11}$ has a zero on the lower left corner of each block in the 3×3 block partition conformal with J_0, and the corresponding entries of $\widehat{H}(\varepsilon)$ will be of order $O(\varepsilon^2)$. We may schematically write

$$\widehat{H}_1(\varepsilon)=\left[\begin{array}{ccc|ccc|cc}*&+1&*&*&*&*&*&*\\ *&*&+1&*&*&*&*&*\\ \bullet&*&*&\bullet&*&*&\bullet&*\\ \hline *&*&*&*&+1&*&*&*\\ *&*&*&*&*&+1&*&*\\ \bullet&*&*&\bullet&*&*&\bullet&*\\ \hline *&*&*&*&*&*&*&+1\\ \bullet&*&*&\bullet&*&*&\bullet&*\end{array}\right]$$

where the bullets denote $O(\varepsilon^2)$ entries, the asterisks denote $O(\varepsilon)$ entries, and the '$+1$' denote entries of type $1+O(\varepsilon)$. This is roughly the general form of $\widehat{H}_1(\varepsilon)$ for any dimension.

We now concentrate on the matrix $\widetilde{C}_{12}\widehat{J}\widetilde{B}_{21}+\widehat{S}(\lambda\,,\,\varepsilon)$ in the second order term of $\widehat{H}_1(\varepsilon)$. We shall prove that the entries of this matrix lying in the same positions as the bullet entries in $\widehat{H}_1(\varepsilon)$ are $O\left(\varepsilon^{\eta}\right)$ for some $\eta>0$. As a consequence, we will write

$$\widehat{H}_1(\varepsilon)=J_0+\varepsilon\left(J_0\widetilde{B}_{11}+\widetilde{C}_{11}J_0\right)+\varepsilon^2\left(\widetilde{C}_{11}J_0\widetilde{B}_{11}+R\right)+O\left(\varepsilon^{2+\eta}\right) \qquad (3.10)$$

for some matrix R whose entries in the bullet positions of $\widehat{H}_1(\varepsilon)$ are all zero.

In order to prove this, we begin by noting that any principal minor of $\widehat{H}_1(\varepsilon)$ is of order at least ε. Hence, the Newton diagram applied to the polynomial $\pi(\lambda\,,\,\varepsilon)$ implies that any of its nonzero roots is $O\left(\varepsilon^{\eta}\right)$ for some $\eta,\ 0<\eta\leq 1$. Taking $\lambda = O\left(\varepsilon^{\eta}\right)$ in the formula for M leads to $M=-\widehat{J}+\phi(\varepsilon)$ for some matrix $\phi(\varepsilon)$ of order $O\left(\varepsilon^{\eta}\right)$ and, consequently, $-\widehat{J}^{-1}M = I-\widehat{J}^{-1}\phi(\varepsilon)$. Since $\widehat{J}^{-1}\phi(\varepsilon)$ can be made arbitrarily small by taking ε small enough, we obtain

$$M^{-1} = -\left(I-\widehat{J}^{-1}\phi(\varepsilon)\right)^{-1}\widehat{J}^{-1} = -\widehat{J}^{-1}+O\left(\varepsilon^{\eta}\right).$$

Replacing this in the expression for $\widehat{S}(\lambda\,,\,\varepsilon)$, we obtain

$$\begin{aligned}\widehat{S}(\lambda\,,\,\varepsilon) &= -\left(J_0\widetilde{B}_{12}+\widetilde{C}_{12}\widehat{J}\right)\widehat{J}^{-1}\left(\widetilde{C}_{21}J_0+\widehat{J}\widetilde{B}_{21}\right)+O\left(\varepsilon^{\eta}\right)=\\ &= -\widetilde{C}_{12}\widehat{J}\widetilde{B}_{21}+R+O\left(\varepsilon^{\eta}\right)\end{aligned}$$

where

$$R = -J_0\widetilde{B}_{12}\left(\widehat{J}^{-1}\widetilde{C}_{21}J_0+\widetilde{B}_{21}\right)-\widetilde{C}_{12}\widetilde{C}_{21}J_0$$

is the matrix we announced in (3.10). This implies

$$\widehat{H}_1(\varepsilon) = J_0+\varepsilon\left(J_0\widetilde{B}_{11}+\widetilde{C}_{11}J_0\right)+\varepsilon^2\left(\widetilde{C}_{11}J_0\widetilde{B}_{11}+R\right)+O\left(\varepsilon^{2+\eta}\right),$$

so the leading matrix for $\widehat{H}_1(\varepsilon)$ is

$$\widehat{H}(\varepsilon) = \left(I+\varepsilon\widetilde{C}_{11}\right)J_0\left(I+\varepsilon\widetilde{B}_{11}\right)$$

and, therefore, the leading terms in the coefficients of the polynomial π will be sums of principal minors of $\widehat{H}(\varepsilon)$. Thus, the crucial question from now on is to identify which principal minors of $\widehat{H}(\varepsilon)$ give rise to terms of a given order in ε. Notice that the $O(\varepsilon^2)$ entries of $\widehat{H}(\varepsilon)$ are placed precisely in those positions which were most important in the expansions for the additive case. This will somewhat complicate the analysis.

If we write

$$\pi(\lambda,\varepsilon) = \det(\lambda I-\widehat{H}_1(\varepsilon)) = \lambda^m+\alpha_1(\varepsilon)\lambda^{m-1}+\ldots+\alpha_{m-1}(\varepsilon)\lambda+\alpha_m(\varepsilon) \qquad (3.11)$$

with coefficients

$$\alpha_k(\varepsilon) = \widehat{\alpha}_k\varepsilon^{a_k}+o\left(\varepsilon^{a_k}\right), \qquad k=1,\,\ldots,\,m,$$

we now know that under the conditions of Theorem 3.2 the polynomial π has a root $\lambda_0=0$ with multiplicity

$$f_q = r_1+\ldots+r_q.$$

If we denote $\widetilde{m}=m-f_q$, then $\alpha_k(\varepsilon)=0$ for $k=\widetilde{m}+1,\,\ldots,\,m$. On the other hand, it is well known [6, §1.2] that each coefficient of a characteristic polynomial, except for a sign, is a sum of principal minors of the matrix. Hence, every coefficient α_k can be written as

$$\alpha_k(\varepsilon) = (-1)^k\mathbb{E}_k\left[\widehat{H}(\varepsilon)\right], \qquad k=1,\,\ldots,\,\widetilde{m}, \qquad (3.12)$$

where $\mathbb{E}_k\left[\widehat{H}(\varepsilon)\right]$ denotes the sum of all k-by-k principal minors of $\widehat{H}(\varepsilon)$.

3.2.2. The principal minors of $\widehat{H}(\varepsilon)$. In this subsection we will identify which principal minors of $\widehat{H}(\varepsilon)$ give rise to terms of a given order in ε. We begin by fixing the notation for principal minors: let $M \in \mathbb{C}^{m\times m}$ be an arbitrary matrix, and, for each $k \in \{1,\ldots,m\}$, let Ξ_k be the family of all increasingly ordered lists γ of length k with entries taken from $\{1,\ldots,m\}$. For each $\gamma \in \Xi_k$ we denote by $M[\gamma]$ the k-by-k principal sub-matrix of M whose entries are those lying on the rows and columns of M with indices in γ. In a more general context, and for $\gamma,\,\theta \in \Xi_k$, we denote by $M[\gamma\,|\,\theta]$ the sub-matrix of M whose entries are those lying on the rows with indices in γ and on the columns with indices in θ. With this definition,

$$\mathbb{E}_k\left[\widehat{H}(\varepsilon)\right] = \sum_{\beta\in\Xi_k} \det\, \widehat{H}(\varepsilon)[\beta]$$

For simplicity, set $M_1 = I + \varepsilon C_{11}$ and $M_2 = I + \varepsilon B_{11}$, so

$$\widehat{H}(\varepsilon) = M_1 J_0 M_2.$$

Then, the Cauchy-Binet formula [6] applied to $\widehat{H}(\varepsilon)[\beta]$ leads to

$$\mathbb{E}_k\left[\widehat{H}(\varepsilon)\right] = \sum_{\beta\in\Xi_k}\sum_{\gamma\in\Xi_k}\sum_{\theta\in\Xi_k} \det\left(M_1[\beta\,|\,\gamma]\right)\det\left(J_0[\gamma\,|\,\theta]\right)\det\left(M_2[\theta\,|\,\beta]\right).$$

One can easily check that $\det\left(J_0[\gamma\,|\,\theta]\right) \in \{0\,,\,1\}\quad \forall\gamma,\,\theta\in\Xi_k$.
More precisely, $\det(J_0[\gamma\,|\,\theta]) = 1$ if and only if

$$\gamma(i) \in \{1,\ldots,m\}\setminus\Omega \qquad \wedge \qquad \theta(i) = \gamma(i)+1, \qquad i = 1,\ldots,k,$$

where

$$\Omega = \left\{ \sum_{i=1}^{j-1} n_i r_i + \rho n_j \;\middle|\; \begin{array}{ccc} j & = & 1,\ldots,q \\ \rho & = & 1,\ldots,r_j \end{array} \right\} \tag{3.13}$$

is the set of indices corresponding to the last row of each Jordan block in J_0, and we denote by $\xi(j)$ the index placed in the j-th position in a list ξ. Therefore, $\mathbb{E}_k\left[\widehat{H}(\varepsilon)\right]$ can be written as

$$\mathbb{E}_k\left[\widehat{H}(\varepsilon)\right] = \sum_{\gamma\in\widehat{\Xi}_k}\sum_{\beta\in\Xi_k} \det\left(M_1[\beta\,|\,\gamma]\right)\det\left(M_2[\theta\,|\,\beta]\right)\quad, \qquad \theta(i)=\gamma(i)+1,\; i=1,\ldots,l, \tag{3.14}$$

where $\widehat{\Xi}_k$ denotes the family of all increasingly ordered lists of length k and entries in $\{1,\ldots,m\}\setminus\Omega$.

We now highlight some entries of M_1 and M_2 which will play a crucial role in our analysis: consider the lower left entry of each block in the Jordan partition (2.7), and denote with a club ♣ (respectively, a spade ♠) the entry of M_1 (resp. of M_2) in that specific position. In our 8 by 8 previous example, the highlighted positions are as follows:

$$M_1 = \left[\begin{array}{ccc|ccc|cc}
+1 & * & * & * & * & * & * & * \\
* & +1 & * & * & * & * & * & * \\
\clubsuit_{11}\,\varepsilon & * & +1 & \clubsuit_{12}\,\varepsilon & * & * & \clubsuit_{13}\,\varepsilon & * \\ \hline
* & * & * & +1 & * & * & * & * \\
* & * & * & * & +1 & * & * & * \\
\clubsuit_{21}\,\varepsilon & * & * & \clubsuit_{22}\,\varepsilon & * & +1 & \clubsuit_{23}\,\varepsilon & * \\ \hline
* & * & * & * & * & * & +1 & * \\
\clubsuit_{31}\,\varepsilon & * & * & \clubsuit_{32}\,\varepsilon & * & * & \clubsuit_{33}\,\varepsilon & +1
\end{array}\right],\quad M_2 = \left[\begin{array}{ccc|ccc|cc}
+1 & * & * & * & * & * & * & * \\
* & +1 & * & * & * & * & * & * \\
\spadesuit_{11}\,\varepsilon & * & +1 & \spadesuit_{12}\,\varepsilon & * & * & \spadesuit_{13}\,\varepsilon & * \\ \hline
* & * & * & +1 & * & * & * & * \\
* & * & * & * & +1 & * & * & * \\
\spadesuit_{21}\,\varepsilon & * & * & \spadesuit_{22}\,\varepsilon & * & +1 & \spadesuit_{23}\,\varepsilon & * \\ \hline
* & * & * & * & * & * & +1 & * \\
\spadesuit_{31}\,\varepsilon & * & * & \spadesuit_{32}\,\varepsilon & * & * & \spadesuit_{33}\,\varepsilon & +1
\end{array}\right].$$

The reason these entries are singled out is that the club (resp., the spade) entries are precisely the entries of the nested matrices $\Phi_j(C)$ (resp., $\Phi_j(B)$) defined in (2.9).

We are now in the position of introducing the following auxiliary result, which will be the basis of the proof of Theorem 3.3.

LEMMA 3.4. *Let $k \in \{1, \ldots, m\}$ and $\gamma \in \widehat{\Xi}_k$, where $\widehat{\Xi}_k$ is as defined in (3.14). Let $\theta(i) = \gamma(i) + 1, i = 1, \ldots, k$, and let η be the cardinal of the set*[3] *$\gamma \bigcap \theta$. Then the lowest possible order in ε of*

$$\det\left(M_1[\beta \,|\, \gamma]\right) \det\left(M_2[\theta \,|\, \beta]\right) \tag{3.15}$$

for any $\beta \in \Xi_k$ is $O\left(\varepsilon^{k-\eta}\right)$, and the order $k - \eta$ is actually attained if and only if β satisfies the two following properties:

$$\begin{aligned} &\beta \supset \gamma \bigcap \theta, \\ &\beta \subset \gamma \bigcup \theta. \end{aligned} \tag{3.16}$$

Proof: our goal is to make the exponent of ε as small as possible by choosing an appropriate β. Let β be any set of indices in Ξ_k and let $\eta_1 = \mathrm{card}\,(\beta \bigcap \gamma)$, $\eta_2 = \mathrm{card}\,(\beta \bigcap \theta)$. Then it is easy to check that

$$\begin{aligned} &\det\left(M_1[\beta \,|\, \gamma]\right) = O\left(\varepsilon^{k-\eta_1}\right) \text{ and} \\ &\det\left(M_2[\theta \,|\, \beta]\right) = O\left(\varepsilon^{k-\eta_2}\right), \end{aligned}$$

since all entries in both M_1 and M_2 are $O(\varepsilon)$, except diagonal ones, which are $O(1)$. Hence, the number of $O(1)$ entries in $M_1[\beta \,|\, \gamma]$ (resp. in $M_2[\theta \,|\, \beta]$) is precisely the cardinal of the intersection between β and γ (resp. β and θ). Therefore,

$$\det\left(M_1[\beta \,|\, \gamma]\right) \det\left(M_2[\theta \,|\, \beta]\right) = O\left(\varepsilon^{2k-\eta_1-\eta_2}\right)$$

Now, to make $\eta_1 + \eta_2$ as large as possible , we need the index set β to have as much overlapping as possible with both γ and θ, in order to include as many 1 entries as possible in the minors $M_1[\beta \,|\, \gamma]$ and $M_2[\theta, \beta]$. One can easily check that this is equivalent to both conditions in (3.16).

□

The importance of Lemma 3.4 lies in identifying the situations which produce the lowest possible order in ε as those when η takes its maximum possible value. Obviously, the more consecutive elements γ has, the more common elements the sets γ and $\theta = \{i + 1 \,|\, i \in \gamma\}$ will have. These are the ideas we will be using in the following subsection to characterize the principal minors of $\widehat{H}(\varepsilon)$ with lower order in ε for a given size k.

3.2.3. Proof of Theorem 3.3. Let $l = f_{j-1} + \rho$ for some index $j \in \{1, \ldots, q\}$ with $n_j \geq 2$ and $1 \leq \rho \leq r_j$. We must prove two things:

(i) there exist k-by-k principal minors of $\widehat{H}(\varepsilon)$ of order $O(\varepsilon^l)$, where

$$k = r_1(n_1 - 1) + \ldots + r_{j-1}(n_{j-1} - 1) + \rho(n_j - 1);$$

[3] Strictly speaking, γ and θ are not sets, but lists. However, we have chosen not to make this distinction explicit, since that would complicate the notation even more. In other words, from now on we identify, when needed, each increasingly ordered list with the corresponding set of indices.

(ii) any $(k+1)$-by-$(k+1)$ principal minor of $\widehat{H}(\varepsilon)$ is $O(\varepsilon^{l+1})$.

The first statement is easy to prove: take γ as the set of indices of all rows, except the last one, in each of the first (i.e., largest) l Jordan blocks in J_0. Then, the cardinal of γ is k and there are exactly l indices in $\{i+1 \mid i \in \gamma\}$ which are not in γ, so

$$\operatorname{card}\left(\gamma \bigcap \theta\right) = k - l.$$

Hence, Lemma 3.4 shows the existence of index sets β with

$$\det\left(M_1[\beta \mid \gamma]\right) \det\left(M_2[\theta \mid \beta]\right) = O\left(\varepsilon^l\right).$$

As to statement (ii), notice that, since the choice of γ excludes the last row of each Jordan block (recall that $\gamma \subset \{1, \ldots, m\} \setminus \Omega$ for Ω given by (3.13)), any choice of $k+1$ rows must be taken from at least $l+1$ Jordan blocks. As a consequence, the cardinal of $\gamma \cap \theta$ becomes $k-(l+1)$, and the corresponding principal minor of $\widehat{H}(\varepsilon)$ is $O\left(\varepsilon^{l+1}\right)$ as claimed.

□

It should be noted that, in fact, the restriction $\gamma \subset \{1, \ldots, m\} \setminus \Omega$ implies that the only possible choices of $k(l)$ rows for γ in order to produce principal minors of order $O(\varepsilon^l)$ are those described when proving statement (i) above, i.e., to choose all rows, except the last one, taken from l among the largest Jordan blocks (notice that if $\rho < f_j$, the choice of l largest Jordan blocks is not unique).

We are finally in the position of proving the main result in this paper.

3.2.4. Proof of Theorem 3.2 . Up to this point we have already established that under the conditions in the statement of Theorem 3.2 the Newton Diagram associated with the polynomial

$$\pi(\lambda\,,\,\varepsilon) = \det\left(\lambda I - \widehat{H}_1(\varepsilon)\right) \tag{3.17}$$

is generically the concatenation of the q segments of slopes $1/(n_j - 1),\ j = 1, \ldots, q$, joining the $q+1$ points $(k(f_j), f_j),\ j = 0, 1, \ldots, q$, where $k(\cdot)$ is given by (3.8) and we make the convention $f_0 = k(f_0) = 0$. Hence, the leading exponents in the asymptotic expansion (3.5) are correct and it only remains to show that the leading coefficients ξ_j^k are given by formula (3.6).

In order to do that, we will find formulas for the coefficients $\widehat{\alpha}_{k(l)}$ associated with each point $(k(l), l)$ in the Newton Diagram (recall that $(k(l), l)$ lies on the diagram if and only if $\widehat{\alpha}_{k(l)} \neq 0$). Such $\widehat{\alpha}_{k(l)}$ are (up to a sign) just the leading coefficients in the sum (3.12) of $k(l) \times k(l)$ principal minors of $\widehat{H}(\varepsilon)$.

Let $l = f_{j-1} + \rho$ for some index $j \in \{1, \ldots, q\}$ with $n_j \geq 2$ and $1 \leq \rho \leq r_j$. Our first goal is to show that the coefficient of $\varepsilon^l \lambda^{m-k(l)}$ in (3.17) is

$$\widehat{\alpha}_{k(l)} = (-1)^l \sum_{\varpi} \det\left(\Phi_j(B+C)[\varpi]\right) \tag{3.18}$$

where $k(l)$ is given by (3.8) and the sum on ϖ is over all index sets with cardinality l whose first f_{j-1} entries are $1, \ldots, f_{j-1}$ and whose remaining ρ entries are taken from the set $\{f_{j-1}+1, \ldots, f_j\}$.

We have seen in §3.2.2 and §3.2.3 that a term of order $O(\varepsilon^l)$ can only be obtained in the sum (3.14) if the corresponding principal minor is associated with sets γ, θ and β, each with cardinal $k(l)$, satisfying the following conditions:

1. γ contains the indices of all rows, except the last one, taken among l of the largest blocks in J_0 (if $\rho < f_j$, the choice of l largest Jordan blocks is not unique) ;
2. $\theta = \{i+1 \mid i \in \gamma\}$;
3. $\beta \subset \gamma \bigcup \theta$; (3.19)
4. $\beta \supset \gamma \bigcap \theta$.

Since we are only interested in the leading terms of products of the form (3.15), we may replace minors of the ε-dependent matrices $M_1 = I + \varepsilon C_{11}$ and $M_2 = I + \varepsilon B_{11}$ with 'simplified minors' of constant matrices as follows:

- Replace every entry of type $1 + O(\varepsilon)$ in the appropriate submatrix of M_1 or M_2 by 1, and set all entries in the same row or column of the submatrix to zero;
- For each of the remaining entries in the submatrix, if any, replace the entry by its coefficient in ε (recall that if any entry remains, it must be $O(\varepsilon)$).

If we denote the simplified minors with $\widetilde{\ }$, it is obvious that if γ, θ and β satisfy conditions 1.– 4. in (3.19) above, then

$$\det\left(M_1[\beta \mid \gamma]\right) \det\left(M_2[\theta \mid \beta]\right) = \det\left(\widetilde{M}_1[\beta \mid \gamma]\right) \det\left(\widetilde{M}_2[\theta \mid \beta]\right) \varepsilon^l + O\left(\varepsilon^{l+1}\right) \quad (3.20)$$

In order not to complicate the proof unnecessarily, we will only analyze in full the case when $\gamma = \{1, \ldots, k(l)+l\} \setminus \Omega$. Any other choice for γ can be analyzed analogously.

The way we will identify the coefficients of all $O(\varepsilon^l)$ terms of the form (3.20) is by

(i) first locating all the $O(1)$ entries in both $\widetilde{M}_1[\beta \mid \gamma]$ and $\widetilde{M}_2[\theta \mid \beta]$, and then

(ii) expanding the corresponding minors along the rows where those $O(1)$ entries lie.

Of course, such locations will very much depend on the set β. Table 3.2.4 below summarizes the relevant information on all possible positions of the $O(1)$ entries, as well as the size of the diagonal blocks (according to the Jordan partition induced by J_0) including them. We distinguish four cases, depending on whether the index of the first row, say r_F, and the index of the last row, say r_L, of the diagonal block is included in β or not.
In order to illustrate this and simplify subsequent proofs let us introduce some examples based on the 8×8 example introduced in (3.4) with $l = 3$. Here $k(3) = 5$ and we have only one possibility for γ:

$$\Omega = \{3, 6, 8\},\ \gamma = \{1, 2, 4, 5, 7\},\ \theta = \{2, 3, 5, 6, 8\}$$

Now, depending on the choice of the index set β satisfying conditions (3.16), there are different possibilities:

EXAMPLE 3.5. $4, 7, 8 \in \beta \quad \wedge \quad 1, 3, 6 \notin \beta$
Here we have chosen the indices in β in such a way that the first diagonal block is in Case 4, as described in Table 3.2.4, the second block is in Case 2, and the third block in Case 1. Hence, $\beta = \{1, 2, 3, 5, 7\}$ and

$$\widetilde{M}_1[\beta \mid \gamma] = \left[\begin{array}{c|c|c|c|c} 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \clubsuit_{31} & 0 & 0 & 0 & 0 \end{array}\right], \qquad \widetilde{M}_2[\theta \mid \beta] = \left[\begin{array}{c|cc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & \spadesuit_{12} & 0 & \spadesuit_{13} & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & \spadesuit_{22} & 0 & \spadesuit_{23} & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Cases	Diagonal block of $\widetilde{M}_1[\beta\,|\,\gamma]$	Diagonal block of $\widetilde{M}_2[\theta\,|\,\beta]$
$1) \to \left\{ \begin{array}{l} r_F \in \beta \\ r_L \in \beta \end{array} \right.$	$n_i \times (n_i - 1)$ block with the 1s on the main diagonal. The last row does not contain a 1.	$(n_i - 1) \times n_i$ block with the 1s on the superdiagonal. The first column does not contain a 1.
$2) \to \left\{ \begin{array}{l} r_F \in \beta \\ r_L \notin \beta \end{array} \right.$	The block is the identity matrix of order $(n_i - 1)$.	An $(n_i - 1) \times (n_i - 1)$ block with the 1s on the superdiagonal. Neither the first column nor the last row contain a 1.
$3) \to \left\{ \begin{array}{l} r_F \notin \beta \\ r_L \in \beta \end{array} \right.$	$(n_i - 1) \times (n_i - 1)$ block with the 1s on the superdiagonal. Neither the first column nor the last row contain a 1.	The block is the identity matrix of order $(n_i - 1)$.
$4) \to \left\{ \begin{array}{l} r_F \notin \beta \\ r_L \notin \beta \end{array} \right.$	$(n_i - 2) \times (n_i - 1)$ block with the 1s on the superdiagonal. The first column does not contain a 1.	$(n_i - 1) \times (n_i - 2)$ block with the 1s on the superdiagonal. The last row does not contain a 1.

Table 3.2.4

EXAMPLE 3.6. $1, 4, 7 \in \beta \quad \wedge \quad 3, 6, 8 \notin \beta$
In this case we have chosen the indices in β so that all three diagonal blocks are in Case 2, so $\beta = \{1\,,\,2\,,\,4\,,\,5\,,\,7\}$ and

$$\widetilde{M}_1[\beta\,|\,\gamma] = \left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right], \qquad \widetilde{M}_2[\theta\,|\,\beta] = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ \spadesuit_{11} & 0 & \spadesuit_{12} & 0 & \spadesuit_{13} \\ \hline 0 & 0 & 0 & 1 & 0 \\ \spadesuit_{21} & 0 & \spadesuit_{22} & 0 & \spadesuit_{23} \\ \hline \spadesuit_{31} & 0 & \spadesuit_{32} & 0 & \spadesuit_{33} \end{array} \right]$$

EXAMPLE 3.7. $3, 4, 8 \in \beta \quad \wedge \quad 1, 6, 7 \notin \beta$
Finally, we consider β such that the first and third diagonal blocks are in Case 3 and the second one is in Case 2. Now, $\beta = \{2\,,\,3\,,\,4\,,\,5\,,\,8\}$ and

$$\widetilde{M}_1[\beta\,|\,\gamma] = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ \clubsuit_{11} & 0 & 0 & 0 & \clubsuit_{13} \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline \clubsuit_{31} & 0 & 0 & 0 & \clubsuit_{33} \end{array} \right], \qquad \widetilde{M}_2[\theta\,|\,\beta] = \left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \spadesuit_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

One can check that in this configuration there are twenty different possible choices for β. If we compute $\mathbb{E}_5\left[\widehat{H}(\varepsilon)\right]$ via (3.14) and (3.20) we obtain

$$\mathbb{E}_5\left[\widehat{H}(\varepsilon)\right] = \left((\clubsuit_{31}) \det \begin{bmatrix} \spadesuit_{12} & \spadesuit_{13} \\ \spadesuit_{22} & \spadesuit_{23} \end{bmatrix} + \det \begin{bmatrix} \spadesuit_{11} & \spadesuit_{12} & \spadesuit_{13} \\ \spadesuit_{21} & \spadesuit_{22} & \spadesuit_{23} \\ \spadesuit_{31} & \spadesuit_{32} & \spadesuit_{33} \end{bmatrix} + \right.$$
$$\left. - \det \begin{bmatrix} \clubsuit_{11} & \clubsuit_{13} \\ \clubsuit_{31} & \clubsuit_{33} \end{bmatrix} (-\spadesuit_{22}) + \cdots \right) \varepsilon^3 + O(\varepsilon^4)$$
$$= \det\left(\Phi_2(B) + \Phi_2(C)\right) \varepsilon^3 + O(\varepsilon^4)$$

In general by construction, the only entries in the modified submatrix $\widetilde{M_1}[\beta \,|\, \gamma]$ (resp., $\widetilde{M_2}[\theta \,|\, \beta]$) which are not zero or 1 are placed in those rows and columns not containing a 1, and are just the coefficients in ε of entries of $\Phi_j(C)$ (resp. $\Phi_j(B)$). Thus, if we expand the determinants in (3.15) along the rows containing 1s, then for each choice of β the leading coefficient in the product (3.15) will be, up to a sign, just a product of two appropriately chosen minors of $\Phi_j(C)$ and $\Phi_j(B)$.

The sign, of course, will depend on the positions of the 1s in both $\widetilde{M_1}[\beta \,|\, \gamma]$ and $\widetilde{M_2}[\theta \,|\, \beta]$. In order to identify both the sign and the minors we need to further specify, for each diagonal block, to which of the four cases in Table 3.2.4 it belongs: for each $i \in \{1, \ldots, l\}$ we include the index i in either of four increasingly ordered lists $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ depending on which, among the four cases in Table 3.2.4, is the one corresponding to the i-th diagonal block.

Since we will need to concatenate some of these lists, for each pair $\xi,\ \chi$ of disjoint lists of indices with lengths a and b, we denote by (ξ, χ) the list of length $a + b$ obtained from concatenating ξ and χ in that order, i.e.,

$$(\xi, \chi) = \xi(1),\ \ldots,\ \xi(a),\ \chi(1),\ \ldots,\ \chi(b).$$

Furthermore, we denote by $\xi + \chi$ the increasingly ordered list obtained from reordering the concatenation (ξ, χ) and by $\operatorname{sgn}(\xi, \chi)$ the sign of the permutation $\begin{pmatrix} \xi + \chi \\ \xi, \chi \end{pmatrix}$ which transforms the reordered list $\xi + \chi$ into the concatenation (ξ, χ). Finally, for each increasingly ordered list υ with a indices taken from $\{1, \ldots, l\}$, we denote by υ^c its complementary list, i.e., the list of length $l - a$ containing those indices in $\{1, \ldots, l\}$ which are not in υ, increasingly ordered.

With this notation, one can prove that, if we denote

$$\vartheta = \upsilon_1 + \upsilon_3, \qquad \zeta = \upsilon_3 + \upsilon_4,$$

then

$$\det\left(\widetilde{M_1}[\beta \,|\, \gamma]\right) \det\left(\widetilde{M_2}[\theta \,|\, \beta]\right) = S\ \det\left(\Phi_j(C)[\vartheta \,|\, \zeta]\right) \det\left(\Phi_j(B)[\vartheta^c \,|\, \zeta^c]\right) \tag{3.21}$$

where S is the sign

$$S = (-1)^{k(l)-l} \operatorname{sgn}(\vartheta, \vartheta^c) \operatorname{sgn}(\zeta, \zeta^c) \tag{3.22}$$

for $k(l)$ given by (3.8). Since the proof of (3.21) is not central to our argument, we defer it to the Appendix.

Obviously, the four lists $\upsilon_1,\ \upsilon_2,\ \upsilon_3,\ \upsilon_4$ amount to a partition of $\{1, \ldots, m\}$. Furthermore, notice that, according to Table 3.2.4,

1. each block whose index is in υ_1 contributes n_i indices to the set β;
2. each block whose index is either in υ_2 or in υ_3 contributes $n_i - 1$ indices to the set β; and
3. each block whose index is in υ_4 contributes $n_i - 2$ indices to the set β.

Since we know that the cardinal of β is

$$k(l) = r_1(n_1 - 1) + \ldots + r_{j-1}(n_{j-1} - 1) + \rho(n_j - 1),$$

we conclude that *the lengths of* υ_1 *and* υ_4 *coincide*.

Now that we have a formula for each product (3.15), we focus on the inner sum in (3.14), which runs over all index sets β for a fixed γ. Recall that we have fixed γ by taking rows from only the first l Jordan blocks, and that each choice of β induces a choice of $\upsilon_1,\, \upsilon_2,\, \upsilon_3,\, \upsilon_4$ and, therefore, of $\vartheta,\, \zeta$, in such a way that υ_1 and υ_4 have the same length. Let us now show that taking all possible choices for β is equivalent to making all possible choices of pairs (ϑ, ζ) in the cartesian product $\Lambda_i \times \Lambda_i$, where Λ_i denotes the family of all index sets with entries in $\{1, \ldots, l\}$ and cardinality

$$i = \mathrm{card}(\vartheta) = \mathrm{card}(\upsilon_3) + \mathrm{card}(\upsilon_1) = \mathrm{card}(\upsilon_3) + \mathrm{card}(\upsilon_4) = \mathrm{card}(\zeta),$$

with i varying from 0 to l: on one hand, we have already seen that every choice of β produces two sets ϑ, ζ of the same cardinal i for some appropriate i, satisfying the constraint on υ_1 and υ_4. We only need to show that, given $i \in \{0, \ldots, l\}$, and given any two sets $\vartheta, \zeta \in \Lambda_i$, we can uniquely define four lists $\upsilon_1,\, \upsilon_2,\, \upsilon_3,\, \upsilon_4$ covering all indices in $\{1, \ldots, l\}$ and such that the lengths of υ_1 and υ_4 coincide. One can easily check that

$$\begin{aligned}
\upsilon_1 &= \vartheta \setminus \zeta \\
\upsilon_2 &= \{1, \ldots, l\} \setminus \left(\vartheta \bigcap \zeta\right) \\
\upsilon_3 &= \vartheta \bigcap \zeta \\
\upsilon_4 &= \zeta \setminus \vartheta
\end{aligned}$$

is such a choice, and it is the only one satisfying the required conditions. Hence, we may use (3.21) to rewrite the coefficient in ε^l of

$$\sum_{\beta} \det\left(M_1[\beta \,|\, \gamma]\right) \det\left(M_2[\theta \,|\, \beta]\right)$$

in formula (3.14) as

$$\sum_{i=0}^{l} \sum_{\vartheta, \in \Lambda_i} \sum_{\zeta \in \Lambda_i} S \, \det\left(\Phi_j(C)[\vartheta \,|\, \zeta]\right) \det\left(\Phi_j(B)[\widetilde{\vartheta} \,|\, \widetilde{\zeta}]\right) \tag{3.23}$$

with S given by (3.22).

We now make use of the following technical lemma, whose proof is also deferred to the Appendix.

LEMMA 3.8. *Let* $M,\, N \in \mathbb{C}^{l\times l}$ *and, for* $i = 0, \ldots, l$, *let* Λ_i *be the family of increasingly ordered lists of indices, taken from* $\{1, \ldots, l\}$, *and with length* i. *Then*

$$\det(M+N) = \sum_{i=0}^{l} \sum_{\vartheta \in \Lambda_i} \sum_{\zeta \in \Lambda_i} \mathrm{sgn}\left(\vartheta, \vartheta^c\right) \mathrm{sgn}\left(\zeta, \zeta^c\right) \det\left(M\left(\vartheta \,|\, \zeta\right)\right) \det\left(N\left(\vartheta^c \,|\, \zeta^c\right)\right), \tag{3.24}$$

where ^{c} *denotes the complementary list in* $\{1,\ldots,l\}$.

This Lemma implies that, since $\Phi_j(B+C) = \Phi_j(B)+\Phi_j(C)$, if we take $M = \Phi_j(C)[1:l]$ and $N=\Phi_j(B)[1:l]$, then (3.14) is equal to

$$(-1)^{k(l)-l}\det\left(\Phi_j(B+C)[1:l]\right).$$

This, of course, corresponds to our initial simplifying choice $\gamma=\{1,\,\ldots,\,k(l)+l\}\setminus\Omega$. If we repeat the same argument for any other admissible choice of γ in the outer sum of (3.14), we obtain

$$\mathbb{E}_k\left[\widehat{H}(\varepsilon)\right] = (-1)^{k(l)-l}\varepsilon^l\sum_{\varpi}\det\left(\Phi_j(B+C)[\varpi]\right)+O\left(\varepsilon^{l+1}\right) \tag{3.25}$$

where the sum on ϖ is over all index sets with cardinality l whose first f_{j-1} entries are $1,\ldots,f_{j-1}$ and whose remaining ρ entries are taken from the set $\{f_{j-1}+1,\,\ldots,\,f_j\}$. Thus, using (3.12) we obtain

$$\widehat{\alpha}_k = (-1)^l\sum_{\varpi}\det\left(\Phi_j(B+C)[\varpi]\right).$$

In other words, except for the sign, $\widehat{\alpha}_k$ is the sum of all $k\times k$ principal minors of $\Phi_j(B+C)$ which include the first f_{j-1} rows, together with ρ rows chosen among the last r_j ones.

Now, we know from (2.3) that the leading coefficients $\xi_{j,k}$ in Theorem 3.2 are solutions of a polynomial equation

$$\sum_{k\in I_{S_j}}\widehat{\alpha}_k\,\mu^{k_{max}-k}=0, \tag{3.26}$$

where $k_{max}=k(f_j)$ and $I_{S_j}=\{k(f_{j-1})+\rho(n_j-1)\,:\,\rho=0,\,\ldots,\,r_j\}$ is the set of possible abscissae for points on the segment S_j of the Newton diagram joining $(k(f_{j-1}),f_{j-1})$ and $(k(f_j),f_j)$. Of course, only those terms with nonzero $\widehat{\alpha}_k$ appear on the equation. If we multiply (3.26) by $(-1)^{f_{j-1}}(\det\left(\Phi_{j-1}(B+C)\right))^{-1}$ and make the change of variables $\xi=\mu^{n_j-1}$, we get the equivalent equation

$$\xi^{r_j}-\frac{\mathbb{E}_1^*[\Phi_j(B+C)]}{\det\left(\Phi_{j-1}(B+C)\right)}\xi^{r_j-1}+\ldots+(-1)^{r_j}\frac{\mathbb{E}_{r_j}^*[\Phi_j(B+C)]}{\det\left(\Phi_{j-1}(B+C)\right)}=0 \tag{3.27}$$

where $E_\rho^*[\Phi_j(B+C)]$, $\rho=1,\ldots,r_j$, stand for the sum of all principal minors of $\Phi_j(B+C)$ including the first f_{j-1} rows together with ρ rows chosen from the last r_j ones.

Finally, consider a principal minor of $\Phi_j(B+C)$ including its first f_{j-1} rows and ρ other rows among the r_j last ones. Then such a minor is just the determinant of a matrix

$$M=\left[\begin{array}{c|c}\Phi_{j-1}(B+C) & M_{12}\\ \hline M_{21} & M_{22}\end{array}\right],$$

and the basic properties of Schur complements imply that

$$\det\left(M_{22}-M_{21}\Phi_{j-1}^{-1}(B+C)M_{12}\right)=\frac{\det(M)}{\det\left(\Phi_{j-1}(B+C)\right)}.$$

Hence, if we denote by Ξ_j the Schur complement of $\Phi_{j-1}(B+C)$ in $\Phi_j(B+C)$, one can easily prove that, for $\rho = 1, \ldots, r_j$,

$$\frac{\mathbb{E}_\rho^*[\Phi_j(B+C)]}{\det\left(\Phi_{j-1}(B+C)\right)} = \mathbb{E}_\rho[\Xi_j],$$

so equation (3.27) may be rewritten as

$$\xi^{r_j} - \mathbb{E}_1\left[\Xi_j\right]\xi^{r_j-1} + \ldots + (-1)^{r_j}\mathbb{E}_{r_j}\left[\Xi_j\right] = 0.$$

But this is just the characteristic equation of Ξ_j, so the solutions of equation (3.26) or, equivalently, of (3.27), are just the eigenvalues of the Schur complement Ξ_j of $\Phi_{j-1}(B)$ in $\Phi_j(B)$. This completes the proof of Theorem 3.2.

□

4. Asymptotic singular value expansions for multiplicative perturbations. All the ideas above can be easily translated into the context of multiplicative perturbation of singular values: let

$$A = U\,\Sigma\,V^*, \quad \Sigma = \begin{bmatrix} \Sigma_n \\ 0 \end{bmatrix}, \quad \Sigma_n = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{n\times n}$$

be a singular value decomposition of $A \in \mathbb{C}^{m\times n}$ with $m \geq n$. It is well known [21] that the Hermitian $(m+n)\times(m+n)$ matrix

$$M = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \tag{4.1}$$

has $2n$ eigenvalues $\pm\sigma_i,\ i = 1, \ldots, n$, plus $m-n$ zero eigenvalues. Furthermore, if we partition $U = [\,U_1 \,|\, U_2]$, with $U_1 \in \mathbb{C}^{m\times n}$, then M can be unitarily diagonalized as

$$\begin{bmatrix} \Sigma_n & & \\ & -\Sigma_n & \\ & & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} U_1^* & V^* \\ U_1^* & -V^* \\ \sqrt{2}U_2^* & 0 \end{bmatrix}\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\begin{bmatrix} U_1 & U_1 & \sqrt{2}U_2 \\ V & -V & 0 \end{bmatrix}$$

In this setting, it is straightforward to prove the following result on multiplicative perturbation of nonzero singular values (recall that zero singular values of A correspond to zero eigenvalues of M, which are unchanged by multiplicative perturbations):

COROLLARY 4.1. *Let $A \in \mathbb{C}^{m\times n}$, $m \geq n$, and let σ_0 be a nonzero singular value of A with multiplicity k. Let $U_0 \in \mathbb{C}^{m\times k}$ and $V_0 \in \mathbb{C}^{n\times k}$ be matrices whose columns span simultaneous bases of the respective left and right singular subspaces of A associated with σ_0. Then, for any $C \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$, the matrix $\widehat{A}(\varepsilon) = (I+\varepsilon C)A(I+\varepsilon\, B)$ has k singular values analytic in ε which can be expanded as*

$$\sigma_j(\varepsilon) = \sigma_0 + \xi_j\ \varepsilon + O(\varepsilon^2), \tag{4.2}$$

where the $\xi_j,\ j = 1, \ldots, k$ are the eigenvalues of the $k\times k$ matrix

$$\Phi = \frac{1}{2}\left(U_0^*\,(C+C^*)\,U_0 + V_0^*\,(B+B^*)\,V_0\right).$$

Proof. We view the nonzero singular values of $\widehat{A}(\varepsilon) = (I+\varepsilon C)A(I+\varepsilon\, B)$ as the positive eigenvalues of $\widehat{M} = (I+\varepsilon\widetilde{C})M(I+\varepsilon\widetilde{B})$ for

$$\widetilde{C} = \left[\begin{array}{cc} C & 0 \\ 0 & B^* \end{array}\right], \quad \widetilde{B} = \left[\begin{array}{cc} C^* & 0 \\ 0 & B \end{array}\right],$$

and σ_0 as the unperturbed eigenvalue of the matrix M in (4.1) with (algebraic and geometric) multiplicity k. Hence, we are in the simplest case

$$q = n_1 = 1, \qquad r_1 = k$$

of Theorem 3.2. Since the columns of

$$\frac{1}{\sqrt{2}}\left[\begin{array}{c} U_0 \\ V_0 \end{array}\right] \in \mathbb{C}^{(m+n)\times k}$$

form an orthonormal basis of the eigenspace associated with the semisimple eigenvalue σ_0 of $\widehat{M}$, straightforward application of Theorem 3.2 leads to the expansions (4.2).

□

5. Conclusions. The discussion above shows that additive and multiplicative perturbations produce expansions of the same asymptotic order for nonzero eigenvalues, only replacing the additive perturbation matrix by the sum of the two one-sided multiplicative perturbations matrices B and C in (1.1). If the eigenvalue under examination is zero, however, the fact that multiplicative perturbations preserve rank makes both situations quite different. In that case we still identify the asymptotic expansions of perturbed eigenvalues: the leading exponent is $1/(n_j-1)$ instead of $1/n_j$, where n_j is the dimension of the corresponding Jordan block, and only eigenvectors, no generalized vectors, are generically involved in the leading coefficients. Furthermore, perturbation expansions for singular values can be easily derived from expansions for perturbed eigenvalues via the so-called Jordan-Wielandt form.

Appendix.

A.1. Proof of (3.21). Given $l = f_{j-1}+\rho$ for some index $j \in \{1,\ldots,q\}$ with $n_j \geq 2$ and $1 \leq \rho \leq r_j$, our goal is to find the leading coefficient $\det(\widetilde{M_1}[\beta\,|\,\gamma])\det(\widetilde{M_2}[\theta\,|\,\beta])$ of the product $\det(M_1[\beta\,|\,\gamma])\det(M_2[\theta\,|\,\beta])$, where $\gamma = \{1,\ldots,k(l)+l\}\setminus\Omega$ with $k(l)$ given by (3.8), and β and θ are index sets satisfying conditions 2 – 4 in (3.19).

The choice of β gives rise to the lists $\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4$ of indices labelling which diagonal blocks of $\widetilde{M_1}[\beta\,|\,\gamma]$, $\widetilde{M_2}[\theta\,|\,\beta]$ are in each of the four cases described in Table 3.2.4. Also, recall that the only entries in these matrices which are not 0 or 1 are those placed on the rows and columns not containing a $O(1)$ entry, and they are just entries of $\Phi_j(B)$ and $\Phi_j(C)$, respectively.

Now, we need to expand both determinants along the rows and columns where the 1 entries lie in order to simplify the formula. This will lead to a product of a sign and two minors of matrices $\Phi_j(B)$ and $\Phi_j(C)$, and the sign will be given by the position of those 1s in the matrices $\widetilde{M_1}[\beta\,|\,\gamma]$ and $\widetilde{M_2}[\theta\,|\,\beta]$ (see Table 3.2.4). Such positions, however, can widely vary, so a direct analysis becomes impractical. To avoid this, we rearrange both matrices in such a way that all the 1 entries lie either on the main diagonal or on the first superdiagonal. This will largely simplify the analysis of the sign: first consider $\widetilde{M_1}[\beta\,|\,\gamma]$ with the block partition induced by (2.7), and permute its rows and columns to obtain a new matrix

$$\widehat{M_1} = P_1\widetilde{M_1}[\beta\,|\,\gamma]P_2$$

for appropriate permutation matrices $P_1,\ P_2$ in such a way that the diagonal blocks of $\widehat{M}_1$ are those of $\widetilde{M}_1[\beta\,|\,\gamma]$ in the following order

- first, all the diagonal blocks in Case 2 (see Table 3.2.4),
- next, all the diagonal blocks in Case 3,
- then, alternate pairs of one block in Case 1 and one block in Case 4,

all this without changing the relative order among the original blocks in the same Case.

In Example 3.5 above, for instance, where $4,7,8 \in \beta$, we would have

$$\widetilde{M}_1[\beta\,|\,\gamma] = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \clubsuit_{31} & 0 & 0 & 0 & 0 \end{array}\right], \qquad \widehat{M}_1 = \left[\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \clubsuit_{31} & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

for permutation matrices

$$P_1 = \left[\begin{array}{c|c|c} 0 & I_2 & 0 \\ \hline 0 & 0 & I_2 \\ \hline I_1 & 0 & 0 \end{array}\right], \qquad P_2 = \left[\begin{array}{c|c|c} 0 & 0 & I_2 \\ \hline I_2 & 0 & 0 \\ \hline 0 & I_1 & 0 \end{array}\right]. \tag{A.1}$$

In general, according to Table 3.2.4, the diagonal blocks in Case 2 are all identity matrices, and those in Case 3 are also square with the 1s placed on the superdiagonal. Although the blocks in Cases 1 and 4 are not square, one can easily check that by pairing them we obtain larger square matrices, of dimension $(n_i+n_j-2)\times(n_i+n_j-2)$, with all its 1 entries on the main diagonal. Hence, the total amount of 1s on the superdiagonal is $\widehat{M}_1$ is $\sum_{i\in\upsilon_3}(n_i-2)$, and all the remaining 1 entries of $\widehat{M}_1$ lie on the main diagonal

To rearrange $\widetilde{M}_2[\theta\,|\,\beta]$ we use the same permutations, but transposed, i.e.,

$$\widehat{M}_2 = P_2^T\,\widetilde{M}_2[\theta\,|\,\beta]\,P_1^T.$$

This produces the exact same order of Cases in the diagonal blocks as above, since the Case for each diagonal block is fixed by the choice of β, and while β selects rows in $\widetilde{M}_1[\beta\,|\,\gamma]$, it selects columns in $\widetilde{M}_2[\theta\,|\,\beta]$, thus the transposes. In Example 5, we get

$$\widetilde{M}_2[\theta\,|\,\beta] = \left[\begin{array}{c|cc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & \spadesuit_{12} & 0 & \spadesuit_{13} & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & \spadesuit_{22} & 0 & \spadesuit_{23} & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right], \qquad \widehat{M}_2 = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ \spadesuit_{22} & 0 & \spadesuit_{23} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \spadesuit_{12} & 0 & \spadesuit_{13} & 0 & 0 \end{array}\right]$$

for the same permutations $P_1,\ P_2$ in (A.1). Now, according to the last column in Table 3.2.4, the upper left blocks in Case 2 are square with the 1s on the superdiagonal, the next diagonal blocks in Case 3 are identity matrices, and each pair of diagonal blocks in Cases 1 and 4 forms a square $(n_i+n_j-2)\times(n_i+n_j-2)$ block with $\sum_{i\in\upsilon_1}(n_i-1)+\sum_{j\in\upsilon_4}(n_j-2)$ entries equal to one on the superdiagonal of $\widehat{M}_2$.

Hence, after reordering, both $\widehat{M}_1$ and $\widehat{M}_2$ have all their 1 entries either on the main diagonal or on the superdiagonal, and the number of superdigonal 1 entries in both matrices together is

$$\sum_{i\in\upsilon_2}(n_i-2)+\sum_{i\in\upsilon_3}(n_i-2)+\sum_{i\in\upsilon_1}(n_i-1)+\sum_{i\in\upsilon_4}(n_i-2) = k(l)-l+\mathrm{card}(\upsilon_1).$$

As a consequence of this,

$$\begin{aligned}\det(\widetilde{M}_1[\beta\,|\,\gamma])\det(\widetilde{M}_2[\theta\,|\,\beta]) &= \det(\widehat{M}_1)\det(\widehat{M}_2)\det(P_1)^2\det(P_2)^2 = \\ = (-1)^{k(l)-l+\mathrm{card}(\upsilon_1)}\det\left(\Phi_j(C)[(\upsilon_3,\upsilon_1)\,|\,(\upsilon_3,\upsilon_4)]\right) &\det\left(\Phi_j(B)[(\upsilon_2,\upsilon_4)\,|\,(\upsilon_2,\upsilon_1)]\right)\end{aligned} \tag{A.2}$$

Now, since the entries of $\Phi_j(C)[(\upsilon_3,\upsilon_1)\,|\,(\upsilon_3,\upsilon_4)]$, $\Phi_j(B)[(\upsilon_2,\upsilon_4)\,|\,(\upsilon_2,\upsilon_1)]$ are not in their natural order on $\Phi_j(B)$ and $\Phi_j(C)$, we need to restore the order changed by reordering the diagonal blocks in $\widetilde{M}_1[\beta\,|\,\gamma]$ and $\widetilde{M}_2[\theta\,|\,\beta]$: let $\vartheta = \upsilon_3 + \upsilon_1$ be the increasingly ordered tuple containing the indices in both υ_3 and υ_1, and, similarly, $\zeta = \upsilon_3 + \upsilon_4$. Then, using the notation introduced in §3.2.4 for signs of permutations,

$$\begin{aligned}\det\left(\Phi_j(C)[(\upsilon_3,\upsilon_1)\,|\,(\upsilon_3,\upsilon_4)]\right) &= \mathrm{sgn}(\upsilon_3,\upsilon_1)\,\mathrm{sgn}(\upsilon_3,\upsilon_4)\det\left(\Phi_j(C)[\vartheta\,|\,\zeta]\right) \\ \det\left(\Phi_j(B)[(\upsilon_2,\upsilon_4)\,|\,(\upsilon_2,\upsilon_1)]\right) &= \mathrm{sgn}(\upsilon_2,\upsilon_4)\,\mathrm{sgn}(\upsilon_2,\upsilon_1)\det\left(\Phi_j(B)[\vartheta^c\,|\,\zeta^c]\right)\end{aligned} \tag{A.3}$$

where, as before, υ^c and ζ^c denote the complementary in $\{1,\ldots,l\}$.

Let us now prove that

$$\mathrm{sgn}\,(\upsilon_3,\upsilon_1)\,\mathrm{sgn}\,(\upsilon_2,\upsilon_4)\,\mathrm{sgn}\,(\upsilon_3,\upsilon_4)\,\mathrm{sgn}\,(\upsilon_2,\upsilon_1) = (-1)^{\mathrm{card}(\upsilon_1)}\mathrm{sgn}\,(\vartheta\,,\,\vartheta^c)\,\mathrm{sgn}\,(\zeta\,,\,\zeta^c) \tag{A.4}$$

In order to do that, consider the auxiliary l-tuples

$$\begin{aligned}\upsilon_R &= (\upsilon_3,\upsilon_1,\upsilon_2,\upsilon_4)\,, \\ \upsilon_C &= (\upsilon_3,\upsilon_4,\upsilon_2,\upsilon_1)\,.\end{aligned}$$

First we check how many transpositions are needed to transform υ_R into υ_C. Below we detail, step by step, the required transformations:

$$\begin{aligned}(\upsilon_3,\upsilon_1,\upsilon_2,\upsilon_4) &\longmapsto (\upsilon_3,\upsilon_2,\upsilon_1,\upsilon_4) \quad &&: \ \mathrm{card}(\upsilon_1)\,\mathrm{card}(\upsilon_2) \text{ transpositions} \\ (\upsilon_3,\upsilon_2,\upsilon_1,\upsilon_4) &\longmapsto (\upsilon_3,\upsilon_2,\upsilon_4,\upsilon_1) \quad &&: \ \mathrm{card}(\upsilon_1)\,\mathrm{card}(\upsilon_4) \text{ transpositions} \\ (\upsilon_3,\upsilon_2,\upsilon_4,\upsilon_1) &\longmapsto (\upsilon_3,\upsilon_4,\upsilon_2,\upsilon_1) \quad &&: \ \mathrm{card}(\upsilon_2)\,\mathrm{card}(\upsilon_4) \text{ transpositions}\end{aligned}$$

Hence, the total number of transpositions needed to transform υ_R into υ_C is

$$\mathrm{card}(\upsilon_1)\,\mathrm{card}(\upsilon_2) + \mathrm{card}(\upsilon_1)\,\mathrm{card}(\upsilon_4) + \mathrm{card}(\upsilon_2)\,\mathrm{card}(\upsilon_4)$$

But we know that $\mathrm{card}(\upsilon_4) = \mathrm{card}(\upsilon_1)$, so

$$\begin{aligned}\mathrm{sgn}\,(\upsilon_R) &= (-1)^{\mathrm{card}(\upsilon_1)^2 + 2\mathrm{card}(\upsilon_1)\mathrm{card}(\upsilon_2)}\mathrm{sgn}\,(\upsilon_C) \\ &= (-1)^{\mathrm{card}(\upsilon_1)}\mathrm{sgn}\,(\upsilon_C)\end{aligned} \tag{A.5}$$

Finally, take for instance υ_R: if we rearrange increasingly both its first half (υ_3,υ_1) and its second half (υ_2,υ_4), we obtain (ϑ,ϑ^c) . The same goes for υ_C and (ζ,ζ^c). Therefore,

$$\begin{aligned}\mathrm{sgn}(\upsilon_R) &= \mathrm{sgn}\,(\upsilon_3,\upsilon_1,\upsilon_2,\upsilon_4) = \mathrm{sgn}\,(\upsilon_3,\upsilon_1)\,\mathrm{sgn}\,(\upsilon_2,\upsilon_4)\,\mathrm{sgn}\,(\vartheta\,,\,\vartheta^c)\,, \\ \mathrm{sgn}(\upsilon_C) &= \mathrm{sgn}\,(\upsilon_3,\upsilon_4,\upsilon_2,\upsilon_1) = \mathrm{sgn}\,(\upsilon_3,\upsilon_4)\,\mathrm{sgn}\,(\upsilon_2,\upsilon_1)\,\mathrm{sgn}\,(\zeta\,,\,\zeta^c)\,.\end{aligned}$$

Combining this with (A.5) proves (A.4). Finally, if we substitute (A.3) into (A.2) and make use of (A.4), we obtain (3.21), as claimed.

A.2. Proof of Lemma 3.8. Let $D = \det(M+N)$. Then

$$D = \sum_{\sigma\in S_l}\left(\operatorname{sgn}(\sigma)\prod_{i=1}^{l}\left(m_{i,\sigma(i)}+n_{i,\sigma(i)}\right)\right),$$

where S_l denotes the group of permutations of l elements. The product inside can be expanded as

$$\prod_{i=1}^{l}\left(m_{i,\sigma(i)}+n_{i,\sigma(i)}\right) = \sum_{i=0}^{l}\left(\sum_{\vartheta\in\Lambda_i}\left(\prod_{j=1}^{i} m_{\vartheta(j),\sigma(\vartheta(j))}\prod_{j=1}^{l-i} n_{\vartheta^c(j),\sigma(\vartheta^c(j))}\right)\right),$$

where Λ_i is the family of all increasingly ordered lists of indices, taken from $\{1,\ \ldots,\ l\}$ with length i, and $\vartheta^c\in\Lambda_{l-i}$ denotes, as before, the complement of ϑ in $\{1,\ldots,l\}$. Substituting this expression in the determinant formula above we obtain

$$D = \sum_{\sigma\in S_l}\operatorname{sgn}(\sigma)\left(\sum_{i=0}^{l}\left(\sum_{\vartheta\in\Lambda_i}\left(\prod_{j=1}^{i} m_{\vartheta(j),\sigma(\vartheta(j))}\prod_{j=1}^{l-i} n_{\vartheta^c(j),\sigma(\vartheta^c(j))}\right)\right)\right).$$

Now, since the sum over i and the sum over ϑ are finite and independent of σ, we may swap the three of them,

$$D = \sum_{i=0}^{l}\left(\sum_{\vartheta\in\Lambda_i}\left(\sum_{\sigma\in S_l}\left(\operatorname{sgn}(\sigma)\prod_{j=1}^{i} m_{\vartheta(j),\sigma(\vartheta(j))}\prod_{j=1}^{l-i} n_{\vartheta^c(j),\sigma(\vartheta^c(j))}\right)\right)\right).$$

Our next step is to partition S_l in order to split the sum over σ. Given $\vartheta\in S_l$, let ζ be any set in Λ_i and let $S_{\vartheta,\zeta}\subset S_l$ be the family of all permutations transforming ϑ into ζ, i.e.,

$$S_{\vartheta,\zeta} = \{\sigma\in S_l \mid\ \sigma(a)\in\{\zeta(j)\mid 1\le j\le i\}\,,\forall a\in\{\vartheta(j)\mid 1\le j\le i\}\}\,.$$

Clearly, the symmetric group S_l is a disjoint union of all $S_{\vartheta,\zeta}$,

$$S_l = \bigcup_{i=0}^{l}\ \bigcup_{\vartheta,\zeta\in\Lambda_i} S_{\vartheta,\zeta},$$

so we can split the sum over S_l as

$$\sum_{\sigma\in S_l}(\qquad) = \sum_{i=0}^{l}\ \sum_{\vartheta,\zeta\in\Lambda_i}\ \sum_{\sigma\in S_{\vartheta,\zeta}}(\qquad)$$

We also split the sign of each $\sigma\in S_l$ into the product of signs of other permutations in order to further simplify the formula: let $\vartheta,\zeta\in\Lambda_i$, and suppose $\sigma\in S_{\vartheta,\zeta}$. We consider the following auxiliary permutations:

$$\widehat{\sigma} = \begin{pmatrix} 1 & \dots & i & i+1 & \dots & l \\ \sigma(\vartheta(1)) & \dots & \sigma(\vartheta(i)) & \sigma(\vartheta^c(1)) & \dots & \sigma(\vartheta^c(l-i)) \end{pmatrix} \in S_l$$
$$\sigma_\vartheta = \begin{pmatrix} 1 & \dots & i & i+1 & \dots & l \\ \vartheta(1) & \dots & \vartheta(i) & \vartheta^c(1) & \dots & \vartheta^c(l-i) \end{pmatrix} \in S_l$$
$$\sigma_\zeta = \begin{pmatrix} 1 & \dots & i & i+1 & \dots & l \\ \zeta(1) & \dots & \zeta(i) & \zeta^c(1) & \dots & \zeta^c(l-i) \end{pmatrix} \in S_l$$
$$\sigma_1 = \begin{pmatrix} \zeta(1) & \zeta(2) & \dots & \zeta(i) \\ \sigma(\vartheta(1)) & \sigma(\vartheta(2)) & \dots & \sigma(\vartheta(i)) \end{pmatrix} \in S_i$$
$$\sigma_2 = \begin{pmatrix} \widetilde{\zeta}(1) & \widetilde{\zeta}(2) & \dots & \widetilde{\zeta}(l-i) \\ \sigma\left(\widetilde{\vartheta}(1)\right) & \sigma(\vartheta^c(2)) & \dots & \sigma(\vartheta^c(l-i)) \end{pmatrix} \in S_{l-i}$$

Notice, on one hand, that $\widehat{\sigma}(i) = \sigma(\sigma_\vartheta(i))$, so $\widehat{\sigma}$ is the composition of σ and σ_ϑ, and, consequently,

$$\operatorname{sgn}(\widehat{\sigma}) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma_\vartheta) \tag{A.6}$$

On the other hand, the concatenation $(\sigma_1, \sigma_2) \in S_l$ satisfies $\widehat{\sigma}(i) = (\sigma_1, \sigma_2)(\sigma_\zeta(i))$, so $\widehat{\sigma}$ is the composition of the concatenation (σ_1, σ_2) with σ_ζ. Hence,

$$\operatorname{sgn}(\widehat{\sigma}) = \operatorname{sgn}(\sigma_\zeta)\operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) \tag{A.7}$$

Combining equations (A.6) and (A.7) we get

$$\begin{aligned} \operatorname{sgn}(\sigma) &= \operatorname{sgn}(\sigma_\zeta)\operatorname{sgn}(\sigma_\vartheta)\operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) \\ &= \operatorname{sgn}(\zeta, \zeta^c)\operatorname{sgn}(\vartheta, \vartheta^c)\operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) \end{aligned} \tag{A.8}$$

Finally, the sum over $\sigma \in S_{\vartheta,\zeta}$, can be rewritten as a double sum as

$$\sum_{\sigma \in S_l} (\dots) = \sum_{i=0}^{l} \sum_{\vartheta,\zeta \in \Lambda_i} \sum_{\sigma_1 \in S_i} \sum_{\sigma_2 \in S_{l-i}} (\dots)$$

Taking into account all of the above, we rewrite the formula for D as

$$D = \sum_{i=0}^{l} \sum_{\vartheta,\,\zeta \in \Lambda_i} \sum_{\sigma_1 \in S_i} \sum_{\sigma_2 \in S_{l-i}} (\operatorname{sgn}(\zeta, \zeta^c)\operatorname{sgn}(\vartheta, \vartheta^c)\operatorname{sgn}(\sigma_1)\ \operatorname{sgn}(\sigma_2)\, P(\vartheta\,,\,\zeta))$$

where

$$P(\vartheta\,,\,\zeta) = \prod_{j=1}^{i} m_{\vartheta(j),\sigma_1(\zeta(i))} \prod_{j=1}^{l-i} n_{\vartheta^c(j),\sigma_2(\zeta^c(j))}.$$

But

$$\det(M[\vartheta \,|\, \zeta]) = \sum_{\sigma_1} \operatorname{sgn}(\sigma_1) \prod_{j=1}^{i} m_{\vartheta(j),\sigma_1(\zeta(i))} \qquad \text{and}$$
$$\det(N[\vartheta^c \,|\, \zeta^c]) = \sum_{\sigma_2} \operatorname{sgn}(\sigma_2) \prod_{j=1}^{l-i} n_{\vartheta^c(j),\sigma_2(\zeta^c(j))},$$

so finally

$$D = \sum_{i=0}^{l} \left(\sum_{\vartheta,\, \zeta \in \Lambda_i} \operatorname{sgn}(\zeta, \zeta^c) \operatorname{sgn}(\vartheta, \vartheta^c) \det\left(M\left[\vartheta \mid \zeta\right]\right) \det\left(N\left[\vartheta^c \mid \zeta^c\right]\right) \right),$$

which completes the proof .

□

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