



A unified view on compensation criteria in the real nonnegative inverse eigenvalue problem

Alberto Borobia ^{a,*}, Julio Moro ^{b,2}, Ricardo L. Soto ^{c,3}

^a *Departamento de Matemáticas, UNED, 28040 Madrid, Spain*

^b *Departamento de Matemáticas, Universidad Carlos III de Madrid, Madrid, Spain*

^c *Departamento de Matemáticas, Universidad Católica del Norte, Antofagasta, Chile*

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Abstract

A connection is established between the problem of characterizing all possible real spectra of entrywise nonnegative matrices (the so-called real nonnegative inverse eigenvalue problem) and a combinatorial process consisting in repeated application of three basic manipulations on sets of real numbers. Given realizable sets (i.e., sets which are spectra of some nonnegative matrix), each of these three elementary transformations constructs a new realizable set. This defines a special kind of realizability, called C-realizability and this is closely related to the idea of compensation. After observing that the set of all C-realizable sets is a strict subset of the set of realizable ones, we show that it strictly includes, in particular, all sets satisfying several previously known sufficient realizability conditions in the literature. Furthermore, the proofs of these conditions become much simpler when approached from this new point of view.

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* Corresponding author.

E-mail addresses: aborobia@mat.uned.es (A. Borobia), jmoro@math.uc3m.es (J. Moro), rsoato@ucn.cl (R.L. Soto).

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1. Introduction

The *nonnegative inverse eigenvalue problem* is the problem of characterizing all possible spectra $A = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of entrywise nonnegative matrices. If A is the spectrum of a nonnegative matrix, then A is said to be a *realizable* set. When all elements of A are real numbers, the problem reduces to the *real nonnegative inverse eigenvalue problem* (henceforth abbreviated as RNIEP). A complete solution of the RNIEP is known so far only for $n \leq 4$ (see §2.1 in [3]). Many different points of view have been adopted to find sufficient conditions for the RNIEP (see [3] and references therein for a comprehensive survey, or [9] for more recent approaches). Our aim in this paper is to present a new, combinatorial approach to the RNIEP, which may help to identify a subset of the set of realizable spectra, namely those described in short as *realizable by compensation*. Our main tool is the following result, obtained by Guo in [4].

Theorem 1.1 [4]. *Let $A = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{C}$ be a realizable set, let λ_1 be the Perron root and let λ_2 be real. Then for every $\epsilon > 0$ the set $\Gamma = \{\lambda_1 + \epsilon, \lambda_2 \pm \epsilon, \lambda_3, \dots, \lambda_n\}$ is also realizable.*

When $\lambda_2 \leq 0$, Theorem 1.1 amounts to a compensation: the negative eigenvalue λ_2 can be decreased as much as we want, provided the dominant eigenvalue λ_1 is increased by the same amount. In other words, the increase in negativity of λ_2 is *compensated* by the increase of positivity in λ_1 .⁴ Therefore, if $\lambda_2 \leq 0$ we will say that $\Gamma = \{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \dots, \lambda_n\}$ is obtained from $A = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ by *compensation* (see [2] for more on the relationship between the idea of compensation and the real nonnegative inverse eigenvalue problem). The goal of this paper is to show that repeated application of this compensation procedure, combined with two other, mostly trivial, manipulations on the spectrum leads to a special kind of realizability, which strictly includes several previously known realizability criteria in the literature for the RNIEP, like those by Suleimanova [10], Kellogg [5], Borobia [1] or Soto [8]. In particular, it includes every single sufficient condition for realizability mentioned in Section 2.1 of the survey paper [3].

The two additional operations we need are given by the two following, trivial results:

Theorem 1.2. *Let $A = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a realizable set, let λ_1 be the Perron root and let $\epsilon > 0$. Then $\Gamma = \{\lambda_1 + \epsilon, \lambda_2, \dots, \lambda_n\}$ is also realizable.*

Theorem 1.3. *Let A_1 and A_2 be realizable sets. Then the set $\Gamma = A_1 \cup A_2$ is realizable.*

Notice that in all three results (Theorems 1.1–1.3) we produce a *new* realizable set Γ starting from realizable sets. This suggests the definition of a new class of realizable sets, namely those which can be reached, starting from the trivially realizable zero set, by means of successively applying either of the three theorems above. This new kind of realizability, which will be defined in Section 2, is called *C-realizability* (the C standing for compensation). Its basic properties are explored in Section 2, among them its connection with majorization (see Theorem 2.1). In Section 3 we show that C-realizability is implied by all previous RNIEP realizability criteria in the literature which somehow involve compensation. Moreover, the proofs of these criteria become much simpler than the original ones via this new approach. Therefore, *C-realizability can be viewed as a unifying notion for all these sufficient conditions*. Moreover, we show, by means of

⁴ Notice that the compensation only takes place if λ_1 is the dominant eigenvalue.

an example (see (9) below), that the set of all C-realizable sets is strictly larger than the reunion of the sets satisfying these compensation realizability criteria.

2. C-realizability

Definition 2.1. A set A of n real numbers is said to be *C-realizable* if it can be reached starting from the n realizable sets

$$\{0\}, \{0\}, \dots, \{0\}$$

and successively applying, in any order and any number of times, either Theorem 1.1, Theorem 1.2 or Theorem 1.3.

Obviously, any C-realizable set is in particular realizable, since the zero sets are realizable and all three Theorems 1.1–1.3 preserve realizability. However, as will be shown below (see (4) or (5)), there are realizable sets which are not C-realizable. Furthermore, there are sets, like the one in example (9) below, which are C-realizable but do not satisfy any of the previously known realizability criteria connected with compensation. We illustrate Definition 2.1 with an example: consider the set

$$A = \{9, 6, 3, 3, -5, -5, -5, -5\}. \quad (1)$$

We will show that A is C-realizable, i.e., we will see that A can be obtained starting from the eight realizable sets $\{0\}, \{0\}, \dots, \{0\}$ and repeatedly applying one of the three results above. The successive stages in this transformation may be described as follows:

Procedure:

0. $\{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}, \{0\}$
1. $\{0, 0\}, \{0, 0\}, \{0, 0\}, \{0, 0\}$
2. $\{5, -5\}, \{3, -3\}, \{5, -5\}, \{3, -3\}$
3. $\{5, 3, -3, -5\}, \{5, 3, -3, -5\}$
4. $\{6, 3, -4, -5\}, \{7, 3, -5, -5\}$
5. $\{7, 6, 3, 3, -4, -5, -5, -5\}$
6. $\{8, 6, 3, 3, -5, -5, -5, -5\}$
7. $\{9, 6, 3, 3, -5, -5, -5, -5\}$

Stage 1 is obtained applying Theorem 1.3 four times pairwise on the eight initial sets. Stage 2 amounts to applying Theorem 1.1 to the four subsets in stage 1, twice with $\epsilon = 5$ and twice with $\epsilon = 3$. In stage 3 we just merge each two of the sets in stage 2 via Theorem 1.3, while stage 4 follows from applying Theorem 1.1 successively, first with $\epsilon = 1$ on the outcome of the merging and then with $\epsilon = 2$. Finally, after merging the two remaining sets in stage 5, we apply Theorem 1.1 in stage 6, and Theorem 1.2 in stage 7 to obtain the required set A .

An elementary, but interesting, property of C-realizability is that it is preserved under so-called *negative subdivision*:

Definition 2.2. The set $\{\rho_1, \dots, \rho_{i-1}, \gamma, \delta, \rho_{i+1}, \dots, \rho_n\}$ is a **negative subdivision** of $\{\rho_1, \dots, \rho_n\}$ if $\gamma + \delta = \rho_i$ with $\gamma, \delta, \rho_i < 0$.

Lemma 2.1. *If A is C-realizable then so is any set obtained by successively applying any number of negative subdivisions on A .*

Proof. It suffices to prove the C-realizability of the set obtained from applying one single negative subdivision on A . Let $A = \{\alpha_1, \dots, \alpha_p, \gamma_1, \dots, \gamma_q\}$ be C-realizable with $\alpha_i \geq 0 > \gamma_j$, and let \tilde{A} be obtained from A by splitting γ_k into $\delta, \eta < 0$ with $\delta + \eta = \gamma_k$.

Since A is C-realizable, during the process of arriving to A from $p + q$ sets $\{0\}, \dots, \{0\}$, one of the zeroes must be decreased via several compensations (i.e., via applying Theorem 1.1 several times) until it reaches the value γ_k . Suppose that in these compensations the corresponding element takes the successive values

$$\gamma_k^0 = 0 > \gamma_k^1 > \dots > \gamma_k^{s-1} > \gamma_k^s = \gamma_k,$$

that is, γ_k is involved in the compensations

$$\begin{aligned} \{\dots, 0, \dots\} &\rightarrow \{\dots, \gamma_k^1, \dots\} \\ \{\dots, \gamma_k^1, \dots\} &\rightarrow \{\dots, \gamma_k^2, \dots\} \\ &\vdots \quad \quad \quad \vdots \\ \{\dots, \gamma_k^{s-1}, \dots\} &\rightarrow \{\dots, \gamma_k, \dots\}, \end{aligned}$$

and suppose that $\gamma_k^t \geq \delta > \gamma_k^{t+1}$ for some $t \in \{1, \dots, s - 1\}$. Then if we start with $p + q + 1$ sets $\{0\}, \dots, \{0\}$ of zeroes and replace the compensations above with the compensations

$$\begin{aligned} \{\dots, 0, 0, \dots\} &\rightarrow \{\dots, \gamma_k^1, 0, \dots\} \\ \{\dots, \gamma_k^1, 0, \dots\} &\rightarrow \{\dots, \gamma_k^2, 0, \dots\} \\ &\vdots \quad \quad \quad \vdots \\ \{\dots, \gamma_k^{t-1}, 0, \dots\} &\rightarrow \{\dots, \gamma_k^t, 0, \dots\} \\ \{\dots, \gamma_k^t, 0, \dots\} &\rightarrow \{\dots, \delta, 0, \dots\} \\ \{\dots, \delta, 0, \dots\} &\rightarrow \{\dots, \delta, \gamma_k^{t+1} - \delta, \dots\} \\ \{\dots, \delta, \gamma_k^{t+1} - \delta, \dots\} &\rightarrow \{\dots, \delta, \gamma_k^{t+2} - \delta, \dots\} \\ &\vdots \quad \quad \quad \vdots \\ \{\dots, \delta, \gamma_k^{s-1} - \delta, \dots\} &\rightarrow \{\dots, \delta, \eta, \dots\}, \end{aligned}$$

in the end we reach the set \tilde{A} . Thus, the C-realizability of A implies the C-realizability of \tilde{A} . \square

Another, less obvious, property of C-realizability is related with *weak majorization*: for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let

$$x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$$

denote the components of x in decreasing order. Given $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we say that x *weakly majorizes* y , or that y is *weakly majorized* by x , and denote it by $x_w \succ y$ if

$$\sum_{i=1}^k x_{[i]} \geq \sum_{i=1}^k y_{[i]}, \quad k = 1, \dots, n. \tag{2}$$

We will show that the positive and the negative part of any C-realizable spectrum must satisfy a weak majorization inequality.

Theorem 2.1. *Let $\Lambda = \{\alpha_1, \dots, \alpha_p, -\beta_1, \dots, -\beta_q\} \subset \mathbb{R}$ be a C-realizable set with $\alpha_i > 0, i = 1, \dots, p$ and $\beta_j \geq 0, j = 1, \dots, q$. Set $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}^p$ and $\beta = (\beta_1, \dots, \beta_q) \in \mathbb{R}^q$ and let $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^s, s = \max\{p, q\}$, be the vectors obtained respectively from α, β by adjoining an appropriate number of zeros to one of them. Then*

$$\tilde{\alpha}_w > \tilde{\beta}. \tag{3}$$

Proof. Starting with two sets with s zero elements each one, which trivially majorize each other, $\tilde{\alpha}$ and $\tilde{\beta}$ are obtained by successively applying either of Theorems 1.1, 1.2 or 1.3. Therefore, it suffices to prove that applying each of the three theorems preserves the weak majorization between the positive and the negative part of C-realizable sets.

Obviously, Theorem 1.2 preserves weak majorization: if the positive part of a set weakly majorizes its negative part, the same will happen for the set obtained from the positive part by strictly increasing the largest positive value. A similar argument can be employed for Theorem 1.1, where two eigenvalues change: the largest positive eigenvalue is again increased by ϵ , and a second value is either increased or decreased by the same amount ϵ . The change in the largest positive eigenvalue increases by ϵ every single partial sum of the positive values, and the change in the second value can, at worst, compensate this increase. In any case, the gap between the two sums in Eq. (2) is either maintained or widened. Finally, the fact that Theorem 1.3 preserves weak majorization follows from a well known basic property of majorization (see [6, Proposition A.7(ii), p. 121]). \square

Notice that the converse of Theorem 2.1 is not true: although the set $\{5, 5; -3, -3, -3\}$ satisfies (3), it is not realizable, since it should be realized by a reducible matrix, and the set cannot be partitioned into nonnegative realizable subsets. Hence, it is not C-realizable either. Moreover, Theorem 2.1 cannot be extended to realizable sets, because, for instance,

$$\{6, 1, 1, -4, -4\} \tag{4}$$

is realizable (see [7]), but $\alpha^\downarrow = (6, 1, 1)$ does not majorize $\beta^\downarrow = (4, 4, 0)$. This example actually shows, as announced, that *the set of C-realizable sets is a strict subset of the set of realizable ones* (see the end of this section for another example).

The last result in this section gives necessary and sufficient conditions for C-realizability whenever $p = 2$, i.e., for sets Λ with only two positive elements.

Theorem 2.2. *Let $\Lambda = \{\alpha_1, \alpha_2, -\beta_1, \dots, -\beta_q\}$ with $\alpha_1 \geq \alpha_2 > 0, \beta_j \geq 0, j = 1, \dots, q$ and*

$$\alpha_1 + \alpha_2 \geq \beta_1 + \dots + \beta_q.$$

Then the following statements are equivalent:

- (i) Λ is C-realizable.
- (ii) There exists a partition $J \cup K$ of $\{1, \dots, q\}$ such that

$$\alpha_1 \geq \sum_{j \in J} \beta_j \quad \text{and} \quad \alpha_2 \geq \sum_{k \in K} \beta_k.$$

(iii) There exist $\delta_1, \dots, \delta_q \geq 0$ with $\sum_{i=1}^q \delta_i \leq \alpha_1 - \alpha_2$ such that

$$\tilde{A} = \{\alpha_2, \alpha_2, -\beta_1 + \delta_1, \dots, -\beta_q + \delta_q\}$$

is realizable.

Proof. (i) \Rightarrow (iii) Since A is C-realizable, there is a procedure to reach A starting from $\{0\}, \dots, \{0\}$ and successively applying Theorems 1.1–1.3 in some order. Note that once there are several positive values in the same C-realizable set, the only positive value that can be modified is the largest one. This means that at some point along the process we must reach two C-realizable sets

$$\{\alpha_2, -\beta'_1, \dots, -\beta'_r\} \quad \text{and} \quad \{\alpha_2, -\beta'_{r+1}, \dots, -\beta'_{r+s}\}.$$

At this point we slightly modify the original process: we apply Theorem 1.3 several times to obtain

$$A' = \{\alpha_2, \alpha_2, -\beta'_1, \dots, -\beta'_{r+s}, 0, \dots, 0\},$$

and then continue with the original process (omitting the appropriate applications of Theorem 1.3). This new process leads to A just the same, since we always increase the dominant positive value. Since A can be obtained from A' via Theorems 1.1 and 1.2, we may take $\tilde{A} = A'$.

(iii) \Rightarrow (ii) By the Perron–Frobenius Theorem applied to \tilde{A} , there exists a partition $J \cup K$ of $\{1, \dots, q\}$ such that

$$\alpha_2 \geq \sum_{j \in J} (\beta_j - \delta_j) \quad \text{and} \quad \alpha_2 \geq \sum_{k \in K} (\beta_k - \delta_k).$$

Then we have

$$\alpha_1 = \alpha_2 + (\alpha_1 - \alpha_2) \geq \sum_{j \in J} \beta_j - \sum_{j \in J} \delta_j + (\alpha_1 - \alpha_2) \geq \sum_{j \in J} \beta_j$$

and

$$\alpha_1 = \alpha_2 + (\alpha_1 - \alpha_2) \geq \sum_{k \in K} \beta_k - \sum_{k \in K} \delta_k + (\alpha_1 - \alpha_2) \geq \sum_{k \in K} \beta_k.$$

(ii) \Rightarrow (i) With no loss of generality we assume $\sum_{j \in J} \beta_j \leq \sum_{k \in K} \beta_k$.

Suppose $\alpha_2 \geq \sum_{j \in J} \beta_j$. Then

$$\{\alpha_1\} \cup \{-\beta_k : k \in K\} \quad \text{and} \quad \{\alpha_2\} \cup \{-\beta_j : j \in J\}$$

are both C-realizable sets and its union A is C-realizable.

Suppose $\alpha_2 < \sum_{j \in J} \beta_j$. Let $J = \{j_1, \dots, j_r\}$ and $K = \{k_1, \dots, k_s\}$. Then take any $\eta_1, \dots, \eta_r \geq 0$ such that $\sum_{i=1}^r \eta_i = \alpha_2$ and $\eta_i \leq \beta_{j_i}$ for $i = 1, \dots, r$. Then the sets

$$\{\alpha_2, -\eta_1, \dots, -\eta_r\} \quad \text{and} \quad \left\{ \sum_{h=1}^s \beta_{k_h}, -\beta_{k_1}, \dots, -\beta_{k_s} \right\}$$

are both C-realizable. By Theorem 1.3,

$$A^* = \left\{ \sum_{i=1}^s \beta_{k_i}, \alpha_2, -\eta_1, \dots, -\eta_r, -\beta_{k_1}, \dots, -\beta_{k_s} \right\}$$

is also C-realizable. Now, since

$$\alpha_1 + \alpha_2 = \alpha_1 + \sum_{i=1}^r \eta_i \geq \beta_1 + \dots + \beta_q,$$

we have

$$\alpha_1 - \sum_{h=1}^s \beta_{k_h} \geq \sum_{i=1}^r (\beta_{j_i} - \eta_i).$$

Thus, applying Theorem 1.1 and then Theorem 1.2 to A^* we conclude that A is C-realizable. \square

We finish this section by pointing out that a set A of real numbers with exactly two positive numbers may be realizable without satisfying condition (ii) in Theorem 2.2. One such example is, for instance,

$$\{97, 71, -44, -54, -70\} \tag{5}$$

(see [3]). This is another instance of a set which is realizable, but not C-realizable.

3. Compensation criteria and C-realizability

In this section we recall several previously known realizability criteria for the RNIEP, all of them related to compensation, and prove that each of them implies C-realizability. We do this by exhibiting in each case a procedure leading from the zero set to the desired one via Theorems 1.1–1.3. We begin with Kellogg’s realizability criterion.

Theorem 3.1 [5]. *Let $A = \{\alpha_0, \alpha_1, \dots, \alpha_s, \gamma_1, \dots, \gamma_t\} \subset \mathbb{R}$ with*

$$\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_s > 0 \geq \gamma_t \geq \dots \geq \gamma_1.$$

Define the set

$$K(A) = \{k \in \{1, \dots, \min\{s, t\}\} : \alpha_k + \gamma_k < 0\}$$

and suppose that the following conditions are satisfied:

- (i) $\alpha_0 + \sum_{i \in K(A), i < k} (\alpha_i + \gamma_i) + \gamma_k \geq 0$ for all $k \in K(A)$,
- (ii) $\alpha_0 + \sum_{i \in K(A)} (\alpha_i + \gamma_i) + \sum_{j=s+1}^t \gamma_j \geq 0$ if $t > s$.

Then A is realizable.

The following theorem shows that Kellogg’s conditions imply C-realizability.

Theorem 3.2. *Let A be a set of real numbers. If A is realizable by Theorem 3.1 then A is C-realizable.*

Proof. Let

$$\begin{aligned} \tilde{A} &= \{\alpha_0\} \cup_{i \in K(A)} \{\alpha_i, \gamma_i\} && \text{if } s \geq t, \\ \tilde{A} &= \{\alpha_0\} \cup_{i \in K(A)} \{\alpha_i, \gamma_i\} \cup \{\gamma_{s+1}, \dots, \gamma_t\} && \text{if } s < t. \end{aligned}$$

Note that the set $A - \tilde{A}$ is composed of couples $\{\alpha_i, \gamma_i\}$ with $\alpha_i + \gamma_i \geq 0$ and of the set $\{\alpha_{t+1}, \dots, \alpha_s\}$ if $s > t$. Each one of these sets is trivially C-realizable. Thus, the C-realizability of \tilde{A} implies the C-realizability of A . Notice also that conditions (i) and (ii) of Theorem 3.1 are exactly the same for A and for \tilde{A} . Therefore we may assume that

$$A = \{\alpha_0, \alpha_1, \dots, \alpha_s, \gamma_1, \dots, \gamma_{s+h}\}$$

for a certain $h \geq 0$ with $\alpha_i + \gamma_i < 0$ for each $i = 1, \dots, s$.

In order to prove that A is C-realizable we will use the following auxiliary sets: for $k = 0, 1, \dots, s$, define

$$A_k = \left\{ \alpha_0 + \sum_{i=1}^k (\alpha_i + \gamma_i), \gamma_{s+1}, \dots, \gamma_{s+h} \right\} \cup_{j=k+1}^s \{\alpha_j, \gamma_j\}$$

and for $k = 1, \dots, s$ define

$$B_k = A_k \cup \{\alpha_k, -\alpha_k\}.$$

We will also use the inequality

$$\alpha_0 + \sum_{i=1}^k (\alpha_i + \gamma_i) = \alpha_k + \left[\alpha_0 + \sum_{i=1}^{k-1} (\alpha_i + \gamma_i) + \gamma_k \right] \geq \alpha_k, \tag{6}$$

which is a consequence of condition (i) in Theorem 3.1.

We start with the C-realizable set $\{0, 0, \dots, 0\}$ containing $h + 1$ zeroes. By applying Theorem 1.1 repeatedly it can be transformed into the C-realizable set

$$\left\{ - \sum_{j=s+1}^{s+h} \gamma_j, \gamma_{s+1}, \dots, \gamma_{s+h} \right\}.$$

Condition (ii) of Theorem 3.1 allows to apply Theorem 1.2 to this set in order to obtain the C-realizable set A_s . Then, B_s is constructed via Theorem 1.3. Inequality (6) allows to apply Theorem 1.2 to B_s in order to obtain the C-realizable set A_{s-1} . Then we apply repeatedly the same argument: Theorem 1.3 allows to construct B_k from A_k , and inequality (6) allows to apply Theorem 1.2 to B_k in order to obtain the C-realizable set A_{k-1} . In the last step, the C-realizable set $A_0 = A$ is obtained. \square

Once Theorem 3.2 has been proved, the analogous result for the realizability criterion obtained by Borobia in [1] trivially follows from Lemma 2.1, since Borobia’s criterion is obtained from Kellogg’s by negative subdivision:

Theorem 3.3. *If A is realizable by Theorem 3.1 then any negative subdivision of A is C-realizable.*

Another sufficient condition for realizability is the following one, obtained by Soto in [8].

Theorem 3.4 [8]. *Let $A = \{\alpha_1, \dots, \alpha_s, \gamma_1, \dots, \gamma_t\} \subset \mathbb{R}$ with*

$$\alpha_1 \geq \dots \geq \alpha_s > 0 \geq \gamma_t \geq \dots \geq \gamma_1.$$

Define the set

$$S(A) = \{k \in \{2, \dots, \min\{s, t\}\} : \alpha_k + \gamma_k < 0\}$$

and suppose that

$$(\alpha_1 + \gamma_1) + \sum_{i \in S(A)} (\alpha_i + \gamma_i) + \sum_{j=s+1}^t \gamma_j \geq 0 \tag{7}$$

where $\sum_{j=s+1}^t \gamma_j$ is understood to be 0 if $s \geq t$. Then A is realizable.

The following result shows that this realizability criterion also implies C-realizability.

Theorem 3.5. *Let A be a set of real numbers. If A is realizable by Theorem 3.4 then A is C-realizable.*

Proof. Let

$$\begin{aligned} \tilde{A} &= \{\alpha_1, \gamma_1\} \cup \bigcup_{i \in S(A)} \{\alpha_i, \gamma_i\} && \text{if } s \geq t, \\ \tilde{A} &= \{\alpha_1, \gamma_1\} \cup \bigcup_{i \in S(A)} \{\alpha_i, \gamma_i\} \cup \{\gamma_{s+1}, \dots, \gamma_t\} && \text{if } s < t. \end{aligned}$$

Note that the set $A - \tilde{A}$ is composed of couples $\{\alpha_i, \gamma_i\}$ with $\alpha_i + \gamma_i \geq 0$ and of the set $\{\alpha_{t+1}, \dots, \alpha_s\}$ if $s > t$. Each of these sets is trivially C-realizable. Thus, the C-realizability of \tilde{A} implies the C-realizability of A . Notice also that condition (7) is exactly the same for A and for \tilde{A} . Therefore we may assume that

$$A = \{\alpha_1, \dots, \alpha_s, \gamma_1, \dots, \gamma_{s+h}\}$$

for a certain $h \geq 0$, and that $\alpha_i + \gamma_i < 0$ for each $i = 2, \dots, s$.

Consider the C-realizable set

$$\{-\gamma_1, \gamma_1\} \cup \{\alpha_2, -\alpha_2\} \cup \dots \cup \{\alpha_s, -\alpha_s\} \cup \{0, \dots, 0\}$$

with the last set containing h zeroes. Note that $-\gamma_1 \geq -\gamma_k > \alpha_k$ for each $k = 2, \dots, s$. Applying several times Theorem 1.1 we obtain the C-realizable set

$$\left\{ -\gamma_1 - \sum_{k=2}^s (\alpha_k + \gamma_k), \gamma_1 \right\} \cup \{\alpha_2, \gamma_2\} \cup \dots \cup \{\alpha_s, \gamma_s\} \cup \{0, \dots, 0\}.$$

And applying again several times Theorem 1.1 we obtain the C-realizable set

$$\left\{ -\gamma_1 - \sum_{k=2}^s (\alpha_k + \gamma_k) - \sum_{j=s+1}^{s+h} \gamma_j, \gamma_1 \right\} \cup \{\alpha_2, \gamma_2\} \cup \dots \cup \{\alpha_s, \gamma_s\} \cup \{\gamma_{s+1}, \dots, \gamma_{s+h}\}.$$

Finally, inequality (7) allows to apply Theorem 1.2 to obtain the C-realizable set

$$\{\alpha_1, \gamma_1\} \cup \{\alpha_2, \gamma_2\} \cup \dots \cup \{\alpha_s, \gamma_s\} \cup \{\gamma_{s+1}, \dots, \gamma_{s+h}\}. \quad \square$$

We conclude with the following extension of Theorem 3.4.

Theorem 3.6 [8]. *Let $A = A_0 \cup A_1 \cup \dots \cup A_q \subset \mathbb{R}$ such that, for $i = 0, 1, \dots, q$,*

$$A_i = \{\alpha_1^{(i)}, \dots, \alpha_{s_i}^{(i)}, \gamma_1^{(i)}, \dots, \gamma_{t_i}^{(i)}\}$$

with $\alpha_1^{(i)} \geq \dots \geq \alpha_{s_i}^{(i)} > 0 \geq \gamma_{t_i}^{(i)} \geq \dots \geq \gamma_1^{(i)}$. For each $i = 1, \dots, q$ let

$$\tilde{A}_i = \{\alpha_1^{(i)} + \varepsilon_i, \alpha_2^{(i)}, \dots, \alpha_{s_i}^{(i)}, \gamma_1^{(i)}, \dots, \gamma_{t_i}^{(i)}\}$$

with $\varepsilon_i > 0$ and

$$\tilde{A}_0 = \{\alpha_1^{(0)} - \eta_0, \alpha_2^{(0)}, \dots, \alpha_{s_0}^{(0)}, \gamma_1^{(0)}, \dots, \gamma_{t_0}^{(0)}\}$$

with $\eta_0 > 0$. If $\tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_q$ satisfy the conditions of Theorem 3.4 and

$$\alpha_1^{(0)} - \sum_{i=1}^q \varepsilon_i \geq \max\{\alpha_1^{(1)}, \dots, \alpha_1^{(q)}, \alpha_1^{(0)} - \eta_0\}, \tag{8}$$

then A is realizable.

Theorem 3.7. *Let A be a set of real numbers. If A is realizable by Theorem 3.6 then A is C-realizable.*

Proof. The fact that each \tilde{A}_i with $i = 0, 1, \dots, q$ satisfies the conditions of Theorem 3.4 implies, by Theorem 3.5, that each \tilde{A}_i is C-realizable.

From inequality (8) it follows that $\sum_{i=1}^q \varepsilon_i \leq \eta_0$. Hence, Theorem 1.2 applied to \tilde{A}_0 implies that

$$A_0^* = \left\{ \alpha_1^{(0)} - \sum_{i=1}^q \varepsilon_i, \alpha_2^{(0)}, \dots, \alpha_{s_0}^{(0)}, \gamma_1^{(0)}, \dots, \gamma_{t_0}^{(0)} \right\}$$

is C-realizable. By Theorem 1.3 the set $A_0^* \cup \tilde{A}_1 \cup \dots \cup \tilde{A}_q$ is C-realizable.

From inequality (8) it follows that $\alpha_1^{(0)} - \sum_{i=1}^q \varepsilon_i \geq \alpha_k$ for each $k = 1, \dots, q$. Applying Theorem 1.1 several times to $A_0^* \cup \tilde{A}_1 \cup \dots \cup \tilde{A}_q$ we obtain that A is C-realizable. \square

Before we conclude, we stress that *the concept of C-realizability is more general than the reunion of the compensation criteria analyzed in this section*: consider, for instance, the set

$$A = \{25, 21, 18, 16, -10, -10, -10, -10, -10, -10, -10\}, \tag{9}$$

which satisfies neither the conditions of Theorem 3.1 nor the ones of Theorems 3.4 or 3.6. However, one can easily check that A is C-realizable: starting from four zero sets of cardinal three, repeated application of Theorem 1.1 leads to the four sets

$$\{20, -10, -10\}, \{18, -10, -8\}, \{20, -10, -10\}, \{16, -10, -6\}. \tag{10}$$

The first two sets may be joined via Theorem 1.3, and applying Theorem 1.1 with $\epsilon = 1$ to the union leads to the set

$$A_1 = \{21, 18, -10, -10, -10, -9\}.$$

Likewise, the two last sets in (10) can be merged and transformed into

$$A_2 = \{24, 16, -10, -10, -10, -10\}.$$

Finally, merging A_1 with A_2 and applying Theorem 1.1, again with $\epsilon = 1$, leads to the set A . Therefore, A is C-realizable.

We finish by pointing out that based on Theorems 1.1–1.3, which have extremely simple statements, we have easily proved in this section results whose original proofs were quite complicated. This shows the power of the joint action of the three results in order to construct realizable sets.

It would be of the utmost interest to translate the collective effect of these three theorems into a checkable set of conditions for C -realizability. In view of example (9), such conditions would be, strictly speaking, more general than the already known compensation criteria analyzed in this section.

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