Quadratic decomposition of a Laguerre-Hahn polynomial sequence II

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Abstract. In [4] the authors proved that a quasi-symmetric orthogonal polynomial sequence $\{R_n\}_{n\geq 0}$ is a Laguerre-Hahn sequence if and only if the component $\{P_n\}_{n\geq 0}$ in its quadratic decomposition is also a Laguerre-Hahn sequence. In this paper and under these conditions, we deduce the class s of the Laguerre-Hahn sequence $\{R_n\}_{n\geq 0}$. More precisely, if s' is the class of $\{P_n\}_{n\geq 0}$ then $2s' \leq s \leq 2s' + 3$. On the other hand the polynomial coefficients of the Riccati equation satisfied by the Stieltjes function corresponding to $\{R_n\}_{n\geq 0}$ are given in terms of those of $\{P_n\}_{n\geq 0}$. As an application, we determine all non-symmetric quasi-symmetric Laguerre-Hahn sequences of class one.

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1. Introduction

Let $\{R_n\}_{n\geq 0}$ be a monic orthogonal polynomial sequence (MOPS in short) with respect to a linear functional u in the linear space of polynomials with complex coefficients such that the following three-term recurrence relation holds

$$R_0(x) = 1,\ R_1(x) = x - \beta_0,$$
$$R_{n+2}(x) = (x - (-1)^{n+1}\beta_0)R_{n+1}(x) - \gamma_{n+1}R_n(x),\ n \geq 0, \tag{1.1}$$

where $\beta_0 \in \mathbb{C}$, and $\gamma_{n+1} \in \mathbb{C}^*,\ n \geq 0$.
Such a MOPS is characterized by the following quadratic decomposition (see

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[8] and [11])

$$R_{2n}(x) = P_n(x^2), \ \ n \geq 0,$$
$$R_{2n+1}(x) = (x - \beta_0)P_n^*(x^2), \ \ n \geq 0, \tag{1.2}$$

where $\{P_n\}_{n\geq 0}$ is a MOPS and $\{P_n^*\}_{n\geq 0}$ is the sequence of monic kernel polynomials of K-parameter β_0^2 associated with $\{P_n\}_{n\geq 0}$ defined by (see [5] and [8])

$$P_n^*(x) = \frac{1}{x - \beta_0^2}[P_{n+1}(x) - \frac{P_{n+1}(\beta_0^2)}{P_n(\beta_0^2)}P_n(x)], \ n \geq 0. \tag{1.3}$$

The family of sequences $\{R_n\}_{n\geq 0}$ satisfying (1.1) is wide enough to accommodate all symmetric sequences of orthogonal polynomials ($\beta_0 = 0$) as well as non-symmetric sequences ($\beta_0 \neq 0$).

From a structural and constructive point of view, the Laguerre-Hahn polynomials constitute a very remarkable family of orthogonal polynomials taking into account most of the MOPS considered in the literature belong to this family. In particular, semiclassical orthogonal polynomials are Laguerre-Hahn MOPS (see [12]). The Laguerre-Hahn set of linear functional is invariant under the standard perturbations of linear functionals (see [1], [6], [7], [9], and [10] among others). In a previous paper (see [4]) we dealt with a perturbation of linear functionals such that the corresponding MOPS satisfies (1.1). We have shown that $\{R_n\}_{n\geq 0}$ is a Laguerre-Hahn sequence of polynomials if and only if $\{P_n\}_{n\geq 0}$ is also a Laguerre-Hahn sequence.

In this contribution, our main goal is to complete the results of [4], focussing our attention in the analysis of the class of the MOPS $\{R_n\}_{n\geq 0}$ in terms of the class of the MOPS $\{P_n\}_{n\geq 0}$. Thus the description of all the possible situations is given. As a consequence, a constructive approach to new families of Laguerre-Hahn linear functionals in the non-classical sense is presented.

The structure of the manuscript is as follows.

In Section 2, we introduce the basic background about the algebra of linear functionals, orthogonal polynomials, and Stieltjes functions to be used in the sequel. In particular, we will remind the basic results of [4]. In Section 3, a complete analysis of the class s of the Laguerre-Hahn polynomial sequence $\{R_n\}_{n\geq 0}$ is done. More precisely, we show that $2s' \leq s \leq 2s' + 3$, where s' is the class of $\{P_n\}_{n\geq 0}$. In Section 4, we express the polynomial coefficients A, B, C, and D of the Riccati equation satisfied by the Stieltjes formal series associated with $\{R_n\}_{n\geq 0}$, in terms of the polynomial coefficients of the Riccati equation satisfied by the Stieltjes formal series corresponding to $\{P_n\}_{n\geq 0}$. Finally, in Section 5 we give explicitly the coefficients of the three-term recurrence relation coefficients as well as the polynomial coefficients of the structure relation for the Laguerre-Hahn polynomial sequences $\{R_n\}_{n\geq 0}$ of class one satisfying (1.1). This is done by analyzing the case when $\{P_n\}_{n\geq 0}$, the component of $\{R_n\}_{n\geq 0}$ in the quadratic decomposition, is a Laguerre-Hahn sequence of class zero.

2. Notations and preliminary results

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual space. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ and $S(u)(z) = -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}$ the formal Stieltjes function of u where $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, are the moments of u.
Let introduce the following operations on $\mathcal{P}'$. For more details, see [12].

(1) The left multiplication of a linear functional by a polynomial is defined by

$$\langle gu, f\rangle = \langle u, gf\rangle, f, g \in \mathcal{P}, \ \mathrm{u} \in \mathcal{P}'. \tag{2.1}$$

(2) The right multiplication of a linear functional by a polynomial is given by

$$(uf)(x) = \left\langle u, \frac{xf(x) - \xi f(\xi)}{x - \xi}\right\rangle, \ f \in \mathcal{P}, \ \mathrm{u} \in \mathcal{P}', \tag{2.2}$$

where u is acting over ξ.

(3) The product of two linear functionals reads as

$$\langle vu, f\rangle = \langle u, vf\rangle, f \in \mathcal{P}, \mathrm{u{,}v} \in \mathcal{P}'. \tag{2.3}$$

(4) The dilation of a linear functional is defined as

$$\langle h_a u, f\rangle = \langle u, h_a f\rangle, a \in \mathbb{C} - \{0\}, f \in \mathcal{P}, \mathrm{u} \in \mathcal{P}', \tag{2.4}$$

where

$$(h_a f)(x) = f(ax). \tag{2.5}$$

(5) The shift of a linear functional is

$$\langle \tau_{-b} u, f\rangle = \langle u, \tau_b f\rangle, b \in \mathbb{C}, f \in \mathcal{P}, \mathrm{u} \in \mathcal{P}', \tag{2.6}$$

where

$$(\tau_b f)(x) = f(x - b). \tag{2.7}$$

(6) The even part of a linear functional is given by

$$\langle \sigma u, f\rangle = \langle u, \sigma f\rangle, \tag{2.8}$$

where

$$\sigma f(x) = f(x^2). \tag{2.9}$$

(7) The division of a linear functional by a polynomial of first degree is the linear functional such that

$$\left\langle (x - c)^{-1} u, f\right\rangle = \langle u, \theta_c f\rangle, c \in \mathbb{C}, f \in \mathcal{P}, \mathrm{u} \in \mathcal{P}', \tag{2.10}$$

where

$$(\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}. \tag{2.11}$$

The following result can be easily proved

Lemma 2.1.

$$(\theta_c\sigma f)(x) = (x+c)(\sigma\theta_{c^2} f)(x), c \in \mathbb{C}, f \in \mathcal{P}, \tag{2.12}$$

and

$$\langle u, \theta_a f - \theta_0 f\rangle = a\,\langle u, \theta_0\theta_a f\rangle\,, a \in \mathbb{C}, f \in \mathcal{P}, u \in \mathcal{P}'. \tag{2.13}$$

Definition 2.2. (See [5]) A sequence of polynomials $\{R_n\}_{n\geq 0}$ is said to be a monic orthogonal polynomial sequence (MOPS) with respect to a linear functional w if
i) $\deg R_n = n$ and the leading coefficient of $B_n(x)$ is equal to 1, $n \geq 0$.
ii) $\langle w, R_n R_m\rangle = r_n\delta_{n,m}, n, m \geq 0, r_n \neq 0, n \geq 0$.

It is well known that a sequence of monic orthogonal polynomials satisfies a three-term recurrence relation (see [5])

$$\begin{gathered} R_0(x) = 1, R_1(x) = x - \beta_0, \\ R_{n+2}(x) = (x - \beta_{n+1})R_{n+1}(x) - \gamma_{n+1}R_n(x), n \geq 0, \end{gathered} \tag{2.14}$$

with

$$(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C}^*, n \geq 0.$$

In such conditions w is said to be regular or quasi-definite (see [5] and [12]). In the sequel we will consider regular linear functionals w with the normalization $(w)_0 = 1$.
The sequence $\{R_n^{(1)}\}_{n\geq 0}$ of associated polynomials of the first kind for the MOPS $\{R_n\}_{n\geq 0}$ is defined by

$$R_n^{(1)}(x) = \left\langle w, \frac{R_{n+1}(x) - R_{n+1}(\xi)}{x - \xi}\right\rangle, n \geq 0. \tag{2.15}$$

It satisfies the shifted recurrence relation (see [5] and [12])

$$\begin{gathered} R_0^{(1)}(x) = 1, R_1^{(1)}(x) = x - \beta_1, \\ R_{n+2}^{(1)}(x) = (x - \beta_{n+2})R_{n+1}^{(1)}(x) - \gamma_{n+2}R_n^{(1)}(x), n \geq 0. \end{gathered} \tag{2.16}$$

The orthogonality is preserved by a shifting in the variable. Indeed, for the shifted sequence $\{\widehat{R}_n\}_{n\geq 0}$ defined by $\widehat{R}_n(x) = a^{-n}R_n(ax+b)$, $n \geq 0$, the following recurrence relation holds (see [12])

$$\begin{gathered} \widehat{R}_0(x) = 1, \widehat{R}_1(x) = x - \tfrac{\beta_0 - b}{a}, \\ \widehat{R}_{n+2}(x) = (x - \tfrac{\beta_n - b}{a})\widehat{R}_{n+1}(x) - \tfrac{\gamma_{n+1}}{a^2}\widehat{R}_n(x), n \geq 0. \end{gathered} \tag{2.17}$$

Such a sequence of monic polynomials is the MOPS with respect to the linear functional $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$. A linear functional w is said symmetric if and only if $(w)_{2n+1} = 0$, $n \geq 0$, or, equivalently, in (2.16) $\beta_n = 0$, $n \geq 0$.
If the sequence $\{R_n\}_{n\geq 0}$ satisfies (1.1) then there exists a MOPS $\{P_n\}_{n\geq 0}$

such that (1.2) holds (see [11]). Furthermore the sequence $\{P_n\}_{n\geq 0}$ satisfies the following recurrence relation

$$(2.18)\qquad \begin{gathered} P_0(x)=1,\ P_1(x)=x-\beta_0^P, \\ P_{n+2}(x)=(x-\beta_{n+1}^P)\ P_{n+1}(x)-\gamma_{n+1}^P P_n(x), n\geq 0, \end{gathered}$$

where

$$(2.19)\qquad \begin{cases} \beta_0^P=\gamma_1+\beta_0^2, \\ \beta_{n+1}^P=\gamma_{2n+2}+\gamma_{2n+3}+\beta_0^2, n\geq 0, \\ \gamma_{n+1}^P=\gamma_{2n+1}\gamma_{2n+2},\ n\geq 0. \end{cases}$$

Denoting by w, u, and v the linear functionals associated with $\{R_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$, and $\{P_n^*\}_{n\geq 0}$ respectively, we get

$$(2.20),\qquad u=\sigma w$$

$$(2.21),\qquad \beta_0 u=\sigma(xw)$$

$$(2.22)\qquad v=\gamma_1^{-1}(x-\beta_0^2)u.$$

The regularity of v and (2.22) means that (see [5], [6], [7], [12])

$$(2.23)\qquad P_n(\beta_0^2)\neq 0, n\geq 0.$$

Conversely, if $\beta_0\in\mathbb{C}$ and u is a linear functional such that the corresponding MOPS $\{P_n\}_{n\geq 0}$ satisfies (2.18) and (2.23) then the linear functional v defined in (2.22) where $\gamma_1=\beta_0^P-\beta_0^2$, is regular (see [5], [6], [7], [12]). In such a case, the MOPS corresponding to v is the sequence $\{P_n^*\}_{n\geq 0}$ given in (1.3). Let us define the sequence of monic polynomials $\{R_n\}_{n\geq 0}$ by (1.2). Then it satisfies (1.1) with (2.19) and its corresponding linear functional w satisfies (2.20) and (2.21).

Definition 2.3. ([9], [10]). Let $\{R_n\}_{n\geq 0}$ be a MOPS with respect to the linear functional w. $\{R_n\}_{n\geq 0}$ is said to be a Laguerre-Hahn orthogonal polynomial sequence (respectively, w is called Laguerre-Hahn linear functional) of class s if the following conditions hold.
There exist three polynomials Φ of degree t, Ψ of degree p, and B of degree r such that

$$(2.24)\qquad (\Phi w)'+\Psi w+B(x^{-1}w^2)=0,$$

as well as

$$(2.25)\qquad \prod_{c\in Z_\Phi}\{|\Psi(c)+\Phi'(c)|+|B(c)|+\langle w,\theta_c\Psi+\theta_c^2\Phi+w\theta_0\theta_c B\rangle\}\neq 0,$$

where Z_Φ denotes the set of zeros of Φ. The class s of $\{R_n\}_{n\geq 0}$ is given by $s=\max\{t-2,p-1,r-2\}$.

Remark 2.4. 1.If c_1 is a zero of Φ such that $\Psi(c_1)+\Phi'(c_1)=0,\ B(c_1)=0$, and
$\left\langle w, \theta_{c_1}\Psi + \theta_{c_1}^2\Phi + w\theta_0\theta_{c_1}B\right\rangle = 0$, then the functional equation (2.24) can be simplified dividing by $x-c_1$. Thus, it becomes

$$(\Phi_1 w)' + \Psi_1 w + B_1(x^{-1}w^2) = 0, \tag{2.26}$$

where

$$\Phi_1 = \theta_{c_1}\Phi,\ \Psi_1 = \theta_{c_1}^2\Phi + \theta_{c_1}\Psi,\ B_1 = \theta_{c_1}B. \tag{2.27}$$

To see if (2.26) can also be simplified by $x-c$, where c is a zero of Φ_1, we must find $B_1(c),\ \Psi_1(c)+\Phi_1'(c)$, and $\left\langle w, \theta_c\Psi_1 + \theta_c^2\Phi_1 + w\theta_0\theta_c B_1\right\rangle$. This yields a tedious computational work in particular when we must simplify many times the functional equation (2.24). In the following lemma, whose proof is straightforward, we show that the two last expressions can be obtained by dividing $\Psi(c)+\Phi'(c)$ and $\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\right\rangle$, respectively, by $c-c_1$.

Lemma 2.5. *Let $c_1 \in \mathbb{C}$ be such that*

$$\Phi(c_1) = 0, \tag{2.28}$$

$$\Psi(c_1) + \Phi'(c_1) = 0, \tag{2.29}$$

$$B(c_1) = 0, \tag{2.30}$$

and

$$\left\langle w, \theta_{c_1}\Psi + \theta_{c_1}^2\Phi + w\theta_0\theta_{c_1}B\right\rangle = 0. \tag{2.31}$$

For all $c \in \mathbb{C}$ we have

$$\left\langle w, \theta_c\Psi_1 + \theta_c^2\Phi_1 + w\theta_0\theta_c B_1\right\rangle = \frac{\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\right\rangle}{c-c_1}, \tag{2.32}$$

$$(\Phi_1)'(c) + \Psi_1(c) = \frac{\Phi'(c)+\Psi(c)}{c-c_1}. \tag{2.33}$$

If w is a Laguerre-Hahn linear functional of class s fulfilling (2.24) then, $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$ is also a Laguerre-Hahn linear functional of the same class and satisfies (see [2] and [3])

$$(\widehat{\Phi}\widehat{w})' + \widehat{\Psi}\widehat{w} + \widehat{B}(x^{-1}\widehat{w}^2) = 0, \tag{2.34}$$

where

$$\widehat{\Phi}(z) = a^{-t}\Phi(az+b),\ \widehat{B}(z) = a^{-t}B(az+b),\ \widehat{\Psi}(z) = a^{1-t}\Psi(az+b). \tag{2.35}$$

A Laguerre-Hahn orthogonal polynomial sequence $\{R_n\}_{n\geq 0}$ satisfies the following structure relation (see [1], [2], [3], [6], and [7])

$$\begin{aligned}\Phi(x)R_{n+1}^{'}(x) - B(x)R_n^{(1)}(x) &= \frac{C_{n+1}(x)-C_0(x)}{2}R_{n+1}(x)\\ &\quad -\gamma_{n+1}D_{n+1}(x)R_n(x),\ n\geq 0,\end{aligned}$$

where

$$C_0(x) = -\Phi'(x) - \Psi(x), \tag{2.36}$$

(2.37) $$\chi_0(x) = B(x),$$

(2.38) $$D_0(x) = -(w\theta_0\Phi)'(z) - (w\theta_0\Psi)(z) - (w^2\theta_0^2 B)(z),$$

(2.39) $$\chi_{n+1}(x) = \gamma_{n+1} D_n(x), n \geq 0,$$

(2.40) $$C_{n+1}(x) = -C_n(x) + 2(x-\beta_n)D_n(x),\ n \geq 0,$$

(2.41)
$$\gamma_{n+1}D_{n+1}(x) = -\Phi(x) + \chi_n(x) + (x-\beta_n)^2 D_n(x) - (x-\beta_n)C_n(x),\ n \geq 0.$$

Remark 2.6. If c is a common zero of Φ, B, C_0, and D_0 then it is also a common zero of χ_n, C_n, and D_n, $n \geq 0$. Therefore, in both sides of (2.42) we can divide by $x - c$.

The sequence $\{\widehat{R}_n\}_{n\geq 0}$ given in (2.17) satisfies

$$\widehat{\Phi}(x)\widehat{R}'_{n+1}(x) - \widehat{B}(x)\widehat{R}^{(1)}_{n+1}(x) = \frac{\widehat{C}_{n+1}(x) - \widehat{C}_0(x)}{2}\widehat{R}_{n+1}(x) - \frac{\gamma_{n+1}}{a^2}\widehat{D}_{n+1}(x)\widehat{R}_n(x), n \geq 0,$$

where

(2.42) $$\widehat{C}_n(x) = a^{1-t}C_n(ax+b),\ n \geq 0,$$

(2.43) $$\widehat{\chi}_n(x) = a^{-t}\chi_n(ax+b),\ n \geq 0,$$

(2.44) $$\widehat{D}_n(x) = a^{2-t}D_n(ax+b),\ n \geq 0.$$

In the sequel we suppose that $\{R_n\}_{n\geq 0}, \{P_n\}_{n\geq 0}, \{P_n^*\}_{n\geq 0}, u, w$, and v satisfy (1.1)-(1.3), (2.16), (2.18) – (2.23).

The structure relation plays an important role in order to find the fourth order linear differential equation with polynomial coefficients that every Laguerre-Hahn MOPS satisfies (see [6]).

To conclude this section, we recall the following well known results.

Theorem 2.7. ([4]) *The sequence $\{P_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS if and only if $\{R_n\}_{n\geq 0}$ is a Laguerre-Hahn MOPS. Furthermore, if the class of u is s' and satisfies*

(2.45) $$(\Phi^P u)' + \Psi^P u + B^P(x^{-1}u^2) = 0,$$

then w is of class $s \leq 2s' + 3$ and satisfies (2.24) *with*

(2.46) $$\Phi(x) = (x+\beta_0)\Phi^P(x^2),$$

(2.47) $$B(x) = 2xB^P(x^2),$$

and

(2.48) $$\Psi(x) = 2x(x+\beta_0)\Psi^P(x^2) - 2\Phi^P(x^2).$$

Theorem 2.8. ([4]) *If the sequence* $\{P_n\}_{n\geq 0}$ *satisfies the structure relation*

$$\Phi^P(x)P'_{n+1}(x) - B^P(x)P_n^{(1)}(x) = \frac{C_{n+1}^P(x) - C_0^P(x)}{2}P_{n+1}(x) - \gamma_{n+1}^P D_{n+1}^P(x)P_n(x),\ n \geq 0,$$

then, the sequence $\{R_n\}_{n\geq 0}$, *for* $n \geq s'+2$, *satisfies* (2.36) *with*

$$C_{2n}(x) = \Phi^P(x^2) + 2x(x+\beta_0)(C_n^P(x^2) + 2\frac{\gamma_n^P D_n^P(x^2)}{\gamma_{2n-1}}), \tag{2.49}$$

$$C_{2n+1}(x) = -\Phi^P(x^2) - 2x(x+\beta_0)C_n^P(x^2) + 4x(x+\beta_0)(x^2-\beta_0^2-\frac{\gamma_n^P}{\gamma_{2n-1}})D_n^P(x^2), \tag{2.50}$$

$$D_{2n}(x) = 2x(x+\beta_0)^2 D_n^P(x^2), \tag{2.51}$$

$$D_{2n+1}(x) = x(C_{n+1}^P(x^2) - C_n^P(x^2) + 2\tfrac{\gamma_{n+1}^P}{\gamma_{2n+1}}D_{n+1}^P(x^2) + 2(x^2-\beta_0^2-\tfrac{\gamma_n^P}{\gamma_{2n-1}})D_n^P(x^2)). \tag{2.52}$$

3. The class of $\{R_n\}_{n\geq 0}$

As we have shown, Theorem 2.5 does not precise the class s of the sequence $\{R_n\}_{n\geq 0}$. The aim of this section is to study in detail this class. We will assume that the equation (2.47) satisfied by the linear functional w can not be simplified. In other words, $s' = \max\{\deg \Phi^P - 2, \deg \Psi^P - 1, \deg B^P - 2\}$. To find s we use condition (2.25).

Lemma 3.1. *For all* f *in* $\mathcal{P}$ *we have*

$$\left\langle w, xf(x^2)\right\rangle = \beta_0 \left\langle u, f(x)\right\rangle, \tag{3.1}$$

$$\left\langle w, f(x^2)\right\rangle = \left\langle u, f(x)\right\rangle, \tag{3.2}$$

$$(w\sigma f)(x) = (uf)(x^2) + \beta_0 x(u\theta_0 f)(x^2), \tag{3.3}$$

$$(w(x\sigma f))(x) = (x+\beta_0)(uf)(x^2). \tag{3.4}$$

Proof. (3.1) follows from (2.8), (2.9), and (2.21). From (2.20), we get (3.2). (3.3) and (3.4) are deduced in a straightforward way using (2.2). □

Lemma 3.2. *For all* $c \in \mathbb{C}$ *we have*

$$\Psi(c) + \Phi'(c) = \Phi^P(c^2) + 2c(c+\beta_0)(\Psi^P(c^2) + (\Phi^P)'(c^2)), \tag{3.5}$$

$$\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\right\rangle = 2c(c+\beta_0)^2 \left\langle u, \theta_{c^2}\Psi^P + \theta_{c^2}^2\Phi^P + u\theta_0\theta_{c^2}B^P\right\rangle. \tag{3.6}$$

Proof. Formula (3.5) follows in a straightforward way from (2.48) and (2.50). On the other hand, from (2.48) – (2.56) we get

(3.7)
$$\begin{aligned}\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\right\rangle &= \left\langle w, \theta_c(2x(x+\beta_0)\Psi^P(x^2)\right\rangle \\ &+ \left\langle w, \theta_c^2((x+\beta_0)\Phi^P(x^2) + \theta_c(-2\Phi^P(x^2))\right\rangle \\ &+ \left\langle w, w\theta_0\theta_c(2xB^P(x^2))\right\rangle .\end{aligned}$$

Using the definition of the operator θ_c, it is easy to see that, for two polynomials f and g, we have

$$\theta_c(fg) = g(x)\theta_c(f)(x) + f(c)(\theta_c g)(x). \tag{3.8}$$

Taking $g(x) = \Psi^P(x^2)$ and $f(x) = 2x(x+\beta_0)$ in (3.8), and using (2.12), from (3.1) and (3.2) we get

$$\begin{aligned}\left\langle w, \theta_c(2x(x+\beta_0)\Psi^P(x^2)\right\rangle &= 2(2\beta_0+c)\left\langle u, \Psi^P(x)\right\rangle \\ &+2c(c+\beta_0)^2\left\langle u, (\theta_{c^2}\Psi^P)(x)\right\rangle .\end{aligned} \tag{3.9}$$

Replacing $g(x) = \Phi^P(x^2)$ and $f(x) = x+\beta_0$ in (3.8), and using (2.12) we deduce

(3.10)
$$\theta_c^2((x+c)\Phi^P(x^2)) = (x+c)(\theta_{c^2}\Phi^P)(x^2) + (c+\beta_0)\theta_c((x+c)(\theta_{c^2}\Phi^P)(x^2)).$$

Next, let consider (3.8) with $g(x) = (\theta_{c^2}\Phi^P)(x^2)$ and $f(x) = x+c$. From (2.12) we get $\theta_c((x+c)(\theta_{c^2}\Phi^P)(x^2)) = (\theta_{c^2}\Phi^P)(x^2) + 2c(x+c)(\theta_{c^2}^2\Phi^P)(x^2))$.

From (3.1), (3.2), and (3.10) we get

(3.11)
$$\left\langle w, \theta_c^2((x+\beta_0)\Phi^P(x^2)) + \theta_c(-2\Phi^P(x^2))\right\rangle = 2c(c+\beta_0)^2\left\langle u, (\theta_{c^2}^2\Phi^P)(x)\right\rangle .$$

If we consider in (3.8) $g(x) = B^P(x^2)$ and $f(x) = 2x$, then,

$$\theta_c(2xB^P(x^2)) = 2B^P(x^2) + 2c(x+c)(\theta_{c^2}B^P)(x^2).$$

But, from (2.12,)

$$\theta_0\theta_c(2xB^P(x^2)) = 2x(\theta_0 B^P)(x^2) + 2c\theta_0((x+c)(\theta_{c^2}B^P)(x^2)).$$

Hence, applying (3.8) to $g(x) = (\theta_{c^2}B^P)(x^2)$ and $f(x) = x+c$, we get

$$\theta_0\theta_c(2xB^P(x^2)) = 2x(\theta_0 B^P)(x^2) + 2c(\theta_{c^2}B^P)(x^2) + 2c^2x(\theta_0\theta_c B^P)(x^2).$$

Therefore, by (3.3) and (3.4), and taking into account (3.1) and (3.2) we get

(3.12)
$$\begin{aligned}\left\langle w, w\theta_0\theta_c(2xB^P(x^2))\right\rangle &= 4\beta_0\left\langle u^2,(\theta_0B^P)(x)\right\rangle + 2c\left\langle u^2,(\theta_{c^2}B^P)(x)\right\rangle\\ &+2\beta_0^2c\left\langle u^2,(\theta_0\theta_{c^2}B^P)(x)\right\rangle\\ &+4\beta_0c^2\left\langle u^2,(\theta_0\theta_{c^2}B^P)(x)\right\rangle.\end{aligned}$$

Using (2.13) we deduce

$$\left\langle u^2,(\theta_{c^2}B^P)(x)\right\rangle = c^2\left\langle u^2,(\theta_0\theta_{c^2}B^P)(x)\right\rangle + \left\langle u^2,(\theta_0B^P)(x)\right\rangle.$$

As a consequence, (3.12) becomes

$$\begin{aligned}\left\langle w^2,\theta_0\theta_c(2xB^P(x^2))\right\rangle &= 2(2\beta_0+c)\left\langle u^2,(\theta_0B^P)(x)\right\rangle\\ &+2c(c+\beta_0)^2\left\langle u^2,(\theta_0\theta_{c^2}B^P)(x)\right\rangle.\end{aligned} \tag{3.13}$$

Finally, replacing (3.9), (3.11), and (3.13) in (3.7)

$$\begin{aligned}\left\langle w,\theta_c\Psi+\theta_c^2\Phi+w\theta_0\theta_cB\right\rangle &= 2c(c+\beta_0)^2\\ &\left\langle u,\theta_{c^2}\Psi^P+\theta_{c^2}^2\Phi^P+u\theta_0\theta_{c^2}B^P\right\rangle\\ &+2(2\beta_o+c)(\left\langle u^2,(\theta_0B^P)(x)\right\rangle\\ &+\left\langle u,\Psi^P(x)\right\rangle).\end{aligned}$$

This yields (3.6) since $\left\langle u^2,(\theta_0B^P)(x)\right\rangle + \left\langle u,\Psi^P(x)\right\rangle = 0$. □

Proposition 3.3. *The class of w depends only on the zeros $x=0$ and $x=-\beta_0$.*

Proof. Let c be a zero of Φ such that $c\neq 0$ and $c\neq -\beta_0$. If $\Phi'(c)+\Psi(c)=0$ and $B(c)=0$, then from (2.54), (2.55) and (3.5), c^2 is a common zero of Φ^P, B^P, and $(\Phi^P)'+\Psi^P$. But, since (2.48) can not be simplified, then $\left\langle u_0,\theta_{c^2}\Psi^P+\theta_{c^2}^2\Phi^P+u_0\theta_0\theta_{c^2}B^P\right\rangle \neq 0$. Therefore, from (3.6) we have $\left\langle w,\theta_c\Psi+\theta_c^2\Phi+w\theta_0\theta_cB\right\rangle\neq 0$. So we cannot divide in (2.24) by $x-c$. □

Notice that if $c=0$ or $c=-\beta_0$, then we can divide by $x-c$ in the functional equation. Thus, we will analyze the cases $\beta_0=0$ and $\beta_0\neq 0$, separately.

CASE 1.- $\beta_0=0$.

According to Theorem 2.5, the linear functional w satisfies (2.24) with

$$\Phi(x)=x\Phi^P(x^2),\ \Psi(x)=2x^2\Psi^P(x^2)-2\Phi^P(x^2),\ B(x)=2xB^P(x^2).$$

The class of w is at most $2s'+3$. From (3.5) we have

$$\Phi(0)=0,\ \Phi'(0)+\Psi(0)=-\Phi^P(0),\ B(0)=0.$$

But from (3.6)

$$\left\langle w,\theta_c\Psi+\theta_c^2\Phi+w\theta_0\theta_cB\right\rangle = 2c^3\left\langle u,\theta_{c^2}\Psi^P+\theta_{c^2}^2\Phi^P+u\theta_0\theta_{c^2}B^P\right\rangle. \tag{3.14}$$

Thus,

$$\left\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B \right\rangle = 0.$$

1.A. If $\Phi^P(0) \neq 0$ then (2.24) can not be simplified and we have $s = 2s' + 3$.

1.B. If $\Phi^P(0) = 0$ then (2.24) can be simplified and, according to (2.27) and (2.12), w satisfies (2.24) with

$$\Phi(x) = \Phi^P(x^2),\ \Psi(x) = 2x\Psi^P(x^2) - x(\theta_0\Phi^P)(x^2),\ B(x) = 2B^P(x^2).$$

Application of Lemma 2.4 yields

$$\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B \right\rangle = 2c^2 \left\langle u, \theta_{c^2}\Psi^P + \theta_{c^2}^2\Phi^P + u\theta_0\theta_{c^2} B^P \right\rangle.$$

Because $c = 0$, we get

$$\Phi(0) = 0,\ \Phi'(0) + \Psi(0) = 0,\ B(0) = 2B^P(0), \text{ and}$$

$$\left\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B \right\rangle = 0.$$

1.B.-1. If $B^P(0) \neq 0$, then (2.24) can not be simplified and we have $s = 2s' + 2$.

1.B.-2. If $B^P(0) = 0$, then the functional equation can be simplified and w satisfies (2.24) with

$$\Phi(x) = x(\theta_0\Phi^P)(x^2),\ \Psi(x) = 2\Psi^P(x^2),\ B(x) = 2x(\theta_0 B^P)(x^2).$$

Using Lemma 2.4 we get

$$\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B \right\rangle = 2c \left\langle u, \theta_{c^2}\Psi^P + \theta_{c^2}^2\Phi^P + u\theta_0\theta_{c^2} B^P \right\rangle.$$

Hence

$$\Phi(0) = 0,\ \Phi'(0) + \Psi(0) = 2\Psi^P(0) + (\Phi^P)'(0),\ B(0) = 0,$$

and $\left\langle w, \theta_0\Psi + \theta_0^2\Phi + w\theta_0^2 B \right\rangle = 0$

1.B.-2.-1. If $2\Psi^P(0) + (\Phi^P)'(0) \neq 0$, then the functional equation can not be simplified and thus $s = 2s' + 1$.

1.B.-2.-2. If $2\Psi^P(0) + (\Phi^P)'(0) = 0$, then w satisfies (2.24) with

$$\Phi(x) = (\theta_0\Phi^P)(x^2),\ \Psi(x) = 2x(\theta_0\Psi^P)(x^2) + x(\theta_0^2\Phi^P)(x^2),$$

$$B(x) = 2(\theta_0 B^P)(x^2), \text{ and}$$

$$\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B \right\rangle = 2 \left\langle u, \theta_{c^2}\Psi^P + \theta_{c^2}^2\Phi^P + u\theta_0\theta_{c^2} B^P \right\rangle.$$

In this case, the functional equation can not be simplified. Indeed, in the contrary, we have $\left\langle u, \theta_0\Psi^P + \theta_0^2\Phi^P + u\theta_0^2 B^P \right\rangle = 0$ and $(\theta_0\Phi^P)(0) = 0$.

The last equality is equivalent to $(\Phi^P)'(0) = 0$. Taking into account that $2\Psi^P(0)+(\Phi^P)'(0) = 0$, we obtain $\Psi^P(0) = 0$. Therefore $\Psi^P(0)+(\Phi^P)'(0) = 0$. But we also have $\Phi^P(0) = 0$ and $B^P(0) = 0$. Then we can divide in (2.47) by x and this yields a contradiction. Hence $s = 2s'$.

CASE 2.- $\beta_0 \neq 0$.

2.A. If $\Phi^P(0) \neq 0$, then in (2.24) we can not divide by x. Thus we can analyze the possible division by $x + \beta_0$.

2.A.-1. If $\Phi^P(\beta_0^2) \neq 0$ or $B^P(\beta_0^2) \neq 0$, then $s = 2s' + 3$.

2.A.-2. If $\Phi^P(\beta_0^2) = B^P(\beta_0^2) = 0$, then w satisfies (2.24), with

$$\Phi(x) = \Phi^P(x^2),\ \Psi(x) = 2x\Psi^P(x^2) - (x - \beta_0)(\theta_{\beta_0^2}\Phi^P)(x^2),$$

$$B(x) = 2x(x - \beta_0)(\theta_{\beta_0^2}B)^P(x^2).$$

From Lemma 2.4 we get

$$\left\langle w, \theta_c\Psi + \theta_c^2\Phi + w\theta_0\theta_c B\right\rangle = 2c(c + \beta_0)\left\langle u, \theta_{c^2}\Psi^P + \theta_{c^2}^2\Phi^P + u\theta_0\theta_{c^2}B^P\right\rangle.$$

2.A.-2.-1. If $(B^P)'(\beta_0^2) \neq 0$ or $\Psi^P(\beta_0^2) \neq 0$, then $s = 2s' + 2$.

2.A.-2.-2. If $(B^P)'(\beta_0^2) = \Psi^P(\beta_0^2) = 0$, then w satisfies (2.24) with

$$\Phi(x) = (x - \beta_0)(\theta_{\beta_0^2}\Phi^P)(x^2),\ \Psi(x) = 2x(x - \beta_0)(\theta_{\beta_0^2}\Psi^P)(x^2),$$

$$B(x) = 2(x - 2\beta_0)(\theta_{\beta_0^2}B)^P(x^2) + 4\beta_0^2(x - \beta_0)(\theta_{\beta_0^2}^2B)^P(x^2).$$

Thus, the functional equation can not be simplified and $s = 2s' + 1$.

2.B. If $\Phi^P(0) = 0$, then w satisfies (2.24) with

$$\Phi(x) = x(x + \beta_0)(\theta_0\Phi^P)(x^2), B(x) = 2B^P(x^2),$$

$$\Psi(x) = 2(x + \beta_0)\Psi^P(x^2) - (x - \beta_0)(\theta_0\Phi^P)(x^2).$$

2.B.-1. If $\Phi^P(\beta_0^2) \neq 0$ or $B^P(\beta_0^2) \neq 0$, then $s = 2s' + 2$.

2.B.-2. If $\Phi^P(\beta_0^2) = B^P(\beta_0^2) = 0$, then w satisfies (2.24), with

$$\Phi(x) = x(\theta_0\Phi^P)(x^2),\ B(x) = 2(x - \beta_0)(\theta_{\beta_0^2}B)^P(x^2),$$

$$\Psi(x) = 2\Psi^P(x^2) + \beta_0(x-\beta_0)(\theta_{\beta_0^2}\theta_0\Phi^P)(x^2).$$

2.B.-2.-1. If $(B^P)'(\beta_0^2) \neq 0$ or $\Psi^P(\beta_0^2) \neq 0$, then $s = 2s' + 1$.

2.B.-2.-2. If $(B^P)'(\beta_0^2) = \Psi^P(\beta_0^2) =, 0$ then w satisfies (2.26), with

$$\Phi(x) = x(x-\beta_0)(\theta_{\beta_0^2}\theta_0\Phi^P)(x^2),\ B(x) = 2(x-\beta_0)^2(\theta_{\beta_0^2}^2 B)^P(x^2),$$

$$\Psi(x) = 2(x-\beta_0)(\theta_{\beta_0^2}\Psi^P)(x^2) + (x-\beta_0)(\theta_{\beta_0^2}\theta_0\Phi^P)(x^2).$$

Thus, the functional equation can not be simplified and we have $s = 2s'$.

The previous discussion can be summarized as follows.

Proposition 3.4. *We distinguish the following cases.*

(A) $\beta_0 = 0$.

(A$_1$) $\Phi^P(0) \neq 0$. *Then* $s = 2s' + 3$.

(A$_2$) $\Phi^P(0) = 0$ *and* $B^P(0) \neq 0$. *Then* $s = 2s' + 2$.

(A$_3$) $\Phi^P(0) = 0$, $B^P(0) = 0$ *and* $2\Psi^P(0) + (\Phi^P)'(0) \neq 0$. *Then* $s = 2s' + 1$.

(A$_4$) $\Phi^P(0) = 0$, $B^P(0) = 0$ *and* $2\Psi^P(0) + (\Phi^P)'(0) = 0$. *Then* $s = 2s'$.

(B) $\beta_0 \neq 0$.

(B$_1$) $\Phi^P(0) \neq 0$ *and* $|\Phi^P(\beta_0^2)| + |B^P(\beta_0^2)| \neq 0$. *Then* $s = 2s' + 3$.

(B$_2$) *Either* $\Phi^P(0) \neq 0$, $\Phi^P(\beta_0^2) = 0$, $B^P(\beta_0^2) = 0$, $|(B^P)'(\beta_0^2)| + |\Psi^P(\beta_0^2)| \neq 0$, *or* $\Phi^P(0) = 0$, $|\Phi^P(\beta_0^2)| + |B^P(\beta_0^2)| \neq 0$. *Then* $s = 2s' + 2$.

(B$_3$) *Either* $\Phi^P(0) \neq 0$, $\Phi^P(\beta_0^2) = 0$, $B^P(\beta_0^2) = 0$, $(B^P)'(\beta_0^2) = 0$, $\Psi^P(\beta_0^2) = 0$, *or* $\Phi^P(0) = 0$, $\Phi^P(\beta_0^2) = 0$, $B^P(\beta_0^2) = 0$, $|(B^P)'(\beta_0^2)| + |\Psi^P(\beta_0^2)| \neq 0$. *Then* $s = 2s' + 1$.

(B$_4$) $\Phi^P(0) = 0$, $\Phi^P(\beta_0^2) = 0$, $B^P(\beta_0^2) = 0$, $(B^P)'(\beta_0^2) = 0$, $\Psi^P(\beta_0^2) = 0$. *Then* $s = 2s'$.

4. Riccati equation

It is well known that the linear functional w satisfies (2.24) if and only if its formal Stieltjes function $S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}$ satisfies the following

Riccati equation (see [2], [3], [6], [9],[10])

$$A(z)S'(w)(z) = B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) \tag{4.1}$$

with

$$\begin{aligned} A(z) &= \Phi(z), \\ C(z) &= -\Phi'(z) - \Psi(z), \\ D(z) &= -(w\theta_0\Phi)'(z) - (w\theta_0\Psi)(z) - (w^2\theta_0^2 B)(z). \end{aligned} \tag{4.2}$$

Condition (2.25) is equivalent to the fact that polynomials A, B, C, and D are coprime.

Let assume that $\{P_n\}_{n\geq 0}$ is a Laguerre-Hahn sequence. According to Theorem 2.5, $\{R_n\}_{n\geq 0}$ is also a Laguerre-Hahn sequence and, as a consequence, its corresponding Stieltjes function satisfies a Riccati equation. In this section our aim is to express the polynomial coefficients A, B, C, and D of the Riccati equation (4.1) corresponding to w in terms of those of u which we will denote A^P, B^P, C^P, and D^P. We need the following Lemma whose proof is a straightforward exercise.

Lemma 4.1.

$$S(w)(z) = (z + \beta_0)S(u)(z^2). \tag{4.3}$$

As a consequence, we get

Proposition 4.2. *If* $S(u)(z)$ *satisfies*

$$A^P(z)S'(u)(z) = B^P(z)S^2(u)(z) + C^P(z)S(u)(z) + D^P(z) \tag{4.4}$$

then, $S(w)(z)$ *satisfies* (4.1) *with*

$$\begin{aligned} A(z) &= (z + \beta_0)A^P(z^2), \\ B(z) &= 2zB^P(z^2), \\ C(z) &= A^P(z^2) + 2z(z + \beta_0)C^P(z^2), \\ D(z) &= 2z(z + \beta_0)^2 D^P(z^2). \end{aligned}$$

Proof. Taking formal derivatives in (4.3) we get

$$2\ zS^{'}(u)(z^2) = \frac{(z + \beta_0)S^{'}(w)(z) - S(w)(z)}{(z + \beta_0)^2}.$$

Therefore

$$S^{'}(u)(z^2) = \frac{(z + \beta_0)S^{'}(w)(z) - S(w)(z)}{2z(z + \beta_0)^2}. \tag{4.5}$$

In (4.4) the change of variable $z \to z^2$ yields

$$A^P(z^2)S'(u)(z^2) = B^P(z^2)S^2(u)(z^2) + C^P(z^2)S(u)(z^2) + D^P(z^2). \tag{4.6}$$

Replacing (4.3) and (4.5) in (4.6) we get

$$\begin{aligned} A^P(z^2)\left(\frac{(z + \beta_0)S^{'}(w)(z) - S(w)(z)}{2z(z + \beta_0)^2}\right) &= B^P(z^2)\left(\frac{S(w)(z)}{(z + \beta_0)}\right)^2 \\ &\quad + C^P(z^2)\frac{S(w)(z)}{(z + \beta_0)} + D^P(z^2). \end{aligned}$$

Hence,

$$\begin{aligned}(z+\beta_0)\ A^P(z^2)\ S'(w)(z) \quad = \quad & 2zB^P(z^2)S^2(w)(z) \\ & +(A^P(z^2)+2\ z\ (z+\beta_0)\ C^P(z^2))S(w)(z) \\ & +2\ z\ (z+\beta_0)^2\ D^P(z^2).\end{aligned}$$

□

5. Example

In this section we will give the coefficients of the three term recurrence relation and those of the structure relation coefficients for all non-symmetric Laguerre-Hahn orthogonal polynomial sequences $\{R_n\}_{n\geq 0}$ of class $s=1$ satisfying (1.1). This method can be used to determine all symmetric Laguerre-Hahn sequences of class $s=1$. But we don't do it here because the coefficients of such sequences are obtained in an alternative way in [1] by solving a non linear system.

Let assume $\beta_0 \neq 0$. Using a suitable dilation in the variable that, for a sake of simplicity, yields $\beta_0 = 1$, from Theorem 2.5 and Proposition 3.4, $\{R_n\}_{n\geq 0}$ is a Laguerre-Hahn sequence of class $s=1$ if and only if $\{P_n\}_{n\geq 0}$ is a Laguerre-Hahn sequence of class $s'=0$ and one of the following conditions hold

(1) $\quad \Phi^P(0)=0, \Phi^P(1)=0,\ B^P(1)=0,\ \mid (B^P)'(1)\mid + \mid \Psi^P(1) \mid \neq 0,$

(2) $\quad \Phi^P(0)\neq 0, \Phi^P(1)=0,\ B^P(1)=0,\ (B^P)'(1)=0,\ \Psi^P(1)=0.$

First we will analyze the conditions in (1).
Since $s'=0$ implies $\deg \Phi^P \leq 2$, then from the conditions $\Phi^P(0)=0,\ \Phi^P(1)=0$ we have

$$\Phi^P(x)=x^2-x. \tag{5.1}$$

Therefore, from (2.34) and (2.35), u can be obtained by shifting the linear functional satisfying (2.47) with $\Phi^P(x)=x^2-1$. Indeed, we use the shift $h_{-\frac{1}{2}}\circ \tau_{-1}$.

The sequence $\{P_n\}_{n\geq 0}$ corresponding to the above linear functional appears in [2], [3], and [13]. These polynomials are related to some perturbations of Jacobi polynomials and their associated polynomials of a real order τ (a shift by τin the parameters of the recurrence relation). Thus, using the previous results, we can give the characteristic elements of the sequence $\{R_n\}_{n\geq 0}$. The main difficulty is that given γ_n^P and β_n^P, $n\geq 0$, the recurrence coefficients of the sequence $\{P_n\}_{n\geq 0}$, it is not easy, by solving directly the non linear system (2.19), to give explicitly $\gamma_n,\ n\geq 0$, the coefficients of the recurrence relation of $\{R_n\}_{n\geq 0}$. To solve this problem, we state the following

Proposition 5.1. *Let denote by $C_n^P(x)$ and $D_n^P(x)$, $n\geq 0$, the polynomial coefficients of the structure relation of $\{P_n\}_{n\geq 0}$. Then*

$$\gamma_{2n+1}=\frac{-2\gamma_{n+1}^P D_{n+1}^P(1)}{C_{n+1}^P(1)-C_0^P(1)},\ n\geq 0, \tag{5.2}$$

(5.3) $$\gamma_{2n+2} = -(\frac{C^P_{n+1}(1) - C^P_0(1)}{2D^P_{n+1}(1)}),\ n \geq 0.$$

Proof. From (1.1) we have

$$R_{2n+2}(x) = (x+1)R_{2n+1}(x) - \gamma_{2n+1}R_{2n}(x).$$

Setting $x = -1$, we obtain

$$R_{2n+2}(-1) = -\ \gamma_{2n+1}R_{2n}(-1).$$

Taking into account (1.2), we have

$$P_{n+1}(1) = -\gamma_{2n+1}P_n(1).$$

Hence

(5.4) $$\gamma_{2n+1} = -\frac{P_{n+1}(1)}{P_n(1)},\ n \geq 0,$$

and from (2.19) we have

(5.5) $$\gamma_{2n+2} = -\gamma^P_{n+1}\frac{P_n(1)}{P_{n+1}(1)},\ n \geq 0.$$

Substituting $x = 1$ in the structure relation (2.51) and taking into account (5.1) as well as $B^P(1) = 0$, we obtain

$$\frac{C^P_{n+1}(1) - C^P_0(1)}{2}P_{n+1}(1) = \gamma^P_{n+1}D^P_{n+1}(1)P_n(1), n \geq 0.$$

Therefore

(5.6) $$\frac{P_{n+1}(1)}{P_n(1)} = \frac{2\gamma^P_{n+1}D^P_{n+1}(1)}{C^P_{n+1}(1) - C^P_0(1)},\ n \geq 0.$$

Using (5.4) – (5.6), we get (5.2) and (5.3). □

According to [2] (see also [3]) as well as (2.34) and (2.35), we distinguish two cases

(1.A)

$$B^P(x) = \tfrac{1}{4}\Big((1-\alpha)(-2x+1)^2 + (\alpha\lambda - \tfrac{2\mu(\alpha-1)}{\alpha-2})(-2x+1) - \tfrac{\mu^2}{\alpha-2} + \lambda\mu - 1 + \rho(1+\alpha)\Big).$$

In this case,

(5.7) $$\psi^P(x) = -\frac{1}{2}((\alpha-2)(-2x+1) + \mu),$$

(5.8) $$C^P_0(x) = \frac{1}{2}(\alpha(-2x+1) + \mu),$$

(5.9) $$C^P_{n+1}(x) - C^P_0(x) = -\tfrac{1}{2}\Big(2(n+\alpha)(-2x+1) + 2\alpha\ (\lambda + \tfrac{\mu}{2-\alpha}) - 2(\alpha\lambda - \tfrac{2\mu(\alpha-1)}{\alpha-2}) - \alpha\mu(\tfrac{1}{\alpha} - \tfrac{1}{2n+\alpha})\Big), n \geq 0,$$

(5.10) $$D^P_0(x) = 0,\ D^P_{n+1}(x) = 2n + \alpha + 1, n \geq 0,$$

$$\gamma_1^P = \frac{\rho}{4}, \tag{5.11}$$

$$\gamma_{n+1}^P = \frac{n(n+\alpha)(2n+\alpha-\mu)(2n+\alpha+\mu)}{4(2n+\alpha+1)(2n+\alpha-1)(2n+\alpha)^2},\ n \geq 1, \tag{5.12}$$

where $\alpha \neq -n, n \geq 0$ and $\alpha \neq \pm\mu - 2n, n \geq 1$.

Condition $B^P(1) = 0$ is equivalent to $\rho = \frac{(\mu-\alpha)(\mu-\alpha+2-\alpha\lambda+2\lambda)}{(\alpha+1)(\alpha-2)}$. Hence, from (2.48), (2.49), and (2.50) we obtain, after division by $(x+1)x$,

$$\Phi(x) = x^3 - x, \tag{5.13}$$

$$\Psi(x) = -x^3 + (-1+2\alpha)\,x^2 + 2x - \alpha - \mu, \tag{5.14}$$

$$\begin{aligned} B(x) \quad &= \quad (-2\,\alpha+2)\,x^3 + (2\,\alpha-2)\,x^2 + \frac{\left(-\alpha^2\lambda + 2\,\alpha\,\lambda + 2\,\mu\,\alpha - 2\,\mu\right)x}{\alpha-2} \\ &\quad - \frac{-\alpha^2\lambda + 2\,\alpha\,\lambda + 2\,\mu\,\alpha - 2\,\mu}{\alpha-2}. \end{aligned}$$

Using (5.2), (5.3), and (5.9) – (5.12) we get

$$\gamma_1 = -\frac{(\mu-\alpha+2-\alpha\lambda+2\lambda)}{2(\alpha-2)}, \tag{5.15}$$

$$\gamma_{2n+1} = -\frac{1}{2}\frac{n(2n+\alpha+\mu)}{(2n+\alpha-1)(2n+\alpha)}, n \geq 1, \tag{5.16}$$

$$\gamma_{2n+2} = -\frac{1}{2}\frac{(n+\alpha)(2n+\alpha-\mu)}{(2n+\alpha)(2n+\alpha+1)},\ n \geq 0. \tag{5.17}$$

From (2.37), (2.39), (5.13), and (5.14), dividing by $x(x+1)$ we get

$$C_0(x) = (1-2\alpha)x^2 - x + \alpha + \mu, \tag{5.18}$$

$$D_0(x) = 0. \tag{5.19}$$

From (2.52) – (2.55), (5.8) – (5.12), and (5.1) we obtain after a division by $x(x+1)$

$$C_{2n+1}(x) = (4n-1+2\alpha)x^2 + x - 4n - \alpha - \mu,\ n \geq 0, \tag{5.20}$$

$$C_{2n+2}(x) = (1+4n+2\alpha)x^2 - x - 4n - 3\alpha + \mu,\ n \geq 0, \tag{5.21}$$

$$D_{2n+1}(x) = 2(2n+\alpha)(x-1),\ n \geq 0, \tag{5.22}$$

$$D_{2n+2}(x) = 2(2n+\alpha+1)(x+1), n \geq 0. \tag{5.23}$$

We can summarize our results about the basic information for the linear functional in the following Table 1.

Table 1
$\Phi(x) = x^3 - x,$
$\Psi(x) = -x^3 + (-1+2\alpha)\,x^2 + 2x - \alpha - \mu,$
$B(x) = (-2\,\alpha+2)\,x^3 + (2\,\alpha-2)\,x^2 + \frac{\left(-\alpha^2\lambda+2\,\alpha\,\lambda+2\,\mu\,\alpha-2\,\mu\right)x}{\alpha-2}$ $-\frac{-\alpha^2\lambda+2\,\alpha\,\lambda+2\,\mu\,\alpha-2\,\mu}{\alpha-2},$
$\gamma_1 = -\frac{(\mu-\alpha+2-\alpha\lambda+2\lambda)}{2(\alpha-2)}$ $\gamma_{n+1} = \frac{-1}{4}\frac{\left(n+(\alpha-\frac{1}{2})(1-(-1)^n)\right)\left(n+\alpha+\mu+(1+2\mu)(\frac{(-1)^n-1}{2})\right)}{(n+\alpha)(n+\alpha-1)},\ \ n \geq 1,$
$C_0(x) = (1-2\alpha)x^2 - x + \alpha + \mu,$
$C_{n+1}(x) = (-1+2\alpha+2n)x^2 + (-1)^n x - 2n - \alpha - \mu$ $+(1-(-1)^n)(\mu-\alpha+1),\ n \geq 0,$
$D_0(x) = 0,$ $D_{n+1}(x) = 2(n+\alpha)(x-(-1)^n),\ n \geq 0.$

(1.B)

$$B^P(x) = \frac{1}{4}[b_2(-2x+1)^2 + b_1(-2x+1) + b_0],$$

with

$$\begin{aligned}
b_2 &= (\frac{1}{\rho}-1)(2\tau+\alpha+\beta+1),\\
b_1 &= 2(1-\frac{1}{\rho})(1+\frac{1}{2\tau+\alpha+\beta})(\frac{\alpha^{2-\beta^2}}{2\tau+\alpha+\beta+2})\\
&\quad +\lambda[(1-\frac{2}{\rho})(2\tau+\alpha+\beta) - 2(\frac{1}{\rho}-1)],\\
b_0 &= \{1-b_2\}((\beta_0^P)^2) - b_1\beta_0^P - 1\\
&\quad +4\frac{\rho(1+\tau)(1+\tau+\alpha)(1+\tau+\beta)(1+\tau+\alpha+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+2)^2},\\
\beta_0^P &= \lambda + \frac{\alpha^2-\beta^2}{(2\tau+\alpha+\beta+2)(2\tau+\alpha+\beta)}.
\end{aligned}$$

In this case,

$$\psi^P(x) = [(1-\frac{2}{\rho})(2\tau+\alpha+\beta) - \frac{2}{\rho}]x + 2\frac{\alpha+\beta+2\tau+1}{\rho}\beta_0 + \mu, \tag{5.24}$$

(5.25)
$$\begin{aligned} C_0^P(x) &= \frac{-1}{2}\Big(\{(\frac{2}{\rho}-1)(2\tau+\alpha+\beta)+\frac{2}{\rho}-2\}(-2x+1)+\frac{\beta^2-\alpha^2}{2\tau+\alpha+\beta} \\ &\quad -2\lambda(2\tau+\alpha+\beta+1)+2(1-\tfrac{1}{\rho})(1+\tfrac{1}{2\tau+\alpha+\beta})(\tfrac{\alpha^2-\beta^2}{2\tau+\alpha+\beta+2})\Big), \end{aligned}$$

(5.26)
$$\begin{aligned} C_{n+1}^P(x) &= C_0^P(x) - \frac{1}{2}\Big(2(n+1-b_2)(-2x+1)+2(1-b_2)\beta_0-2b_1 \\ &\quad +(\alpha^2-\beta^2)(\tfrac{1}{2\tau+\alpha+\beta+2}-\tfrac{1}{2n+2\tau+\alpha+\beta+2})\Big), \end{aligned}$$

$$D_0^P(x) = \frac{1}{\rho}(2\tau+\alpha+\beta+1),\ D_{n+1}^P(x) = (2n+2\tau+\alpha+\beta+3), n \geq 0, \tag{5.27}$$

$$\gamma_1^P = \frac{\rho(1+\tau)(1+\tau+\alpha)(1+\tau+\beta)(1+\tau+\alpha+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+3)(2\tau+\alpha+\beta+2)^2}, \tag{5.28}$$

(5.29)
$$\gamma_{n+1}^P = \frac{(n+1+\tau)(n+1+\tau+\alpha)(n+1+\tau+\beta)(n+1+\tau+\alpha+\beta)}{(2n+2\tau+\alpha+\beta+1)(2n+2\tau+\alpha+\beta+2)^2(2n+2\tau+\alpha+\beta+3)},$$

$n \geq 1$, where $\tau \neq -n-1, \tau+\alpha \neq -n-1, \tau+\beta \neq -n-1$, $n \geq 0$ and $2\tau+\alpha+\beta \neq -n-2, n \geq -2$.

Taking into account that $B^P(1) = 0$ becomes a quadratic equation in λ, we can consider two values for λ which we denote λ_1 and λ_2.

For $\lambda = \lambda_1$, we get

$$\Phi(x) = x^3 - x, \tag{5.30}$$

(5.31)
$$\begin{aligned} \Psi(x) &= -x^3 + 2x + \frac{(-4+3\rho+4\rho\tau+2\rho\alpha+2\rho\beta-8\tau-4\alpha-4\beta)\,x^2}{\rho} \\ &\quad -2\frac{2\rho\tau-4\tau+\rho\beta-2-2\beta+2\rho-2\alpha}{\rho}, \end{aligned}$$

(5.32)
$$\begin{aligned} &B(x) = \\ &-2\frac{(-1+\rho)(\alpha+1+\beta+2\tau)}{\rho}x^3 + 2\frac{(-1+\rho)(\alpha+1+\beta+2\tau)}{\rho}x^2 \\ &+2\frac{(-\alpha-1-\beta+\rho+\rho\beta+\rho\tau-2\tau)(\alpha+1+\beta-\rho+2\tau-\rho\tau)}{(\alpha+1+\beta+2\tau)\rho}x \\ &-2\frac{(-\alpha-1-\beta+\rho+\rho\beta+\rho\tau-2\tau)(\alpha+1+\beta-\rho+2\tau-\rho\tau)}{(\alpha+1+\beta+2\tau)\rho}. \end{aligned}$$

Using (5.2) and (5.3) we get

(5.33) $$\gamma_1 = -\frac{\rho(1+\tau)(1+\tau+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+2)},$$

(5.34) $$\gamma_{2n+1} = -\frac{(n+1+\tau)(n+1+\tau+\beta)}{(2n+2\tau+\alpha+\beta+1)(2n+2\tau+\alpha+\beta+2)},\ n \geq 1,$$

(5.35) $$\gamma_{2n+2} = -\frac{(n+1+\tau+\alpha)(n+1+\tau+\alpha+\beta)}{(2n+2\tau+\alpha+\beta+2)(2n+2\tau+\alpha+\beta+3)},\ n \geq 0.$$

From (2.37), (2.39), (5.30) – (5.32) if we divide by $x(x+1)$ we get

(5.36)
$$C_0(x) = -\left(\frac{-4+3\rho+4\rho\tau+2\rho\alpha+2\rho\beta-8\tau-4\alpha-4\beta}{\rho}\right)x^2 - x$$
$$2\frac{2\rho\tau-4\tau-2+2\rho-2\beta+\rho\beta-2\alpha}{\rho},$$

(5.37) $$D_0(x) = \frac{2(2\tau+\alpha+\beta+1)}{\rho}(x+1).$$

Using (2.58) – (2.61) the division by $x(x+1)$ yields

(5.38) $$C_{2n+1}(x) = (3+4n+2\alpha+4\tau+2\beta)x^2 + x - 4n - 4 - 4\tau - 2\beta,$$

$n \geq 0$

(5.39) $$C_{2n+2}(x) = (5+4n+4\tau+2\alpha+2\beta)x^2 - x - 4n - 2\beta - 4\alpha - 4\tau - 4,\ n \geq 0,$$

(5.40) $$D_{2n+1}(x) = 2(2n+2\tau+\alpha+\beta+2)(x-1),\ n \geq 0,$$

(5.41) $$D_{2n+2}(x) = 2(2n+2\tau+\alpha+\beta+3)(x+1), n \geq 0.$$

In the following table we summarize our results concerning this case.

Tabla 2
$\Phi(x) = x^3 - x,$
$\Psi(x) = -x^3 + \frac{(-4+3\rho+4\rho\tau+2\rho\alpha+2\rho\beta-8\tau-4\alpha-4\beta)}{\rho}x^2 + 2x - 2\frac{2\rho\tau-4\tau+\rho\beta-2-2\beta+2\rho-2\alpha}{\rho},$
$B(x) = -2\frac{(-1+\rho)(\alpha+1+\beta+2\tau)}{\rho}x^3 + 2\frac{(-1+\rho)(\alpha+1+\beta+2\tau)}{\rho}x^2 + 2\frac{(-\alpha-1-\beta+\rho+\rho\beta+\rho\tau-2\tau)(\alpha+1+\beta-\rho+2\tau-\rho\tau)}{(\alpha+1+\beta+2\tau)\rho}x - 2\frac{(-\alpha-1-\beta+\rho+\rho\beta+\rho\tau-2\tau)(\alpha+1+\beta-\rho+2\tau-\rho\tau)}{(\alpha+1+\beta+2\tau)\rho},$
$\gamma_1 = -\frac{\rho(1+\tau)(1+\tau+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+2)},$ $\gamma_{n+1} = -\frac{1}{4}\frac{\left(n+2\tau+2+(\alpha-\frac{1}{2})(1-(-1)^n)\right)\left(n+2\tau+2\beta+2+(\alpha-\frac{1}{2})(1-(-1)^n)\right)}{(n+2\tau+\alpha+\beta+1)(n+2\tau+\alpha+\beta+2)},$
$C_0(x) = -(\frac{-4+3\rho+4\rho\tau+2\rho\alpha+2\rho\beta-8\tau-4\alpha-4\beta}{\rho})x^2 - x + 2\frac{2\rho\tau-4\tau-2+2\rho-2\beta+\rho\beta-2\alpha}{\rho},$ $C_{n+1}(x) = (2n+2\beta+2\alpha+4\tau+3)x^2 + (-1)^n x - 2n - 4\tau - 2\beta - 4$
$D_0(x) = \frac{2(2\tau+\alpha+\beta+1)}{\rho}(x+1),$ $D_{n+1}(x) = 2(n+\alpha+\beta+2\tau+2)(x-(-1)^n),\ n \geq 0,\ n \geq 0.$

For $\lambda = \lambda_2$ we have

$$\Phi(x) = x^3 - x, \tag{5.42}$$

(5.43)

$$\Psi(x) = -x^3 + \frac{(-4+3\rho+2\rho\alpha+4\rho\tau+2\,\rho\beta-4\alpha-8\tau-4\beta)}{\rho}x^2 + 2x - 2\frac{-2\alpha+2\rho\alpha+\rho\beta+2\rho\tau-2+2\rho-2\beta-4\tau}{\rho},$$

$$B(x) = -2\frac{(\rho-1)\,(\alpha+1+\beta+2\tau)}{\rho}x^3 + 2\frac{(\rho-1)\,(\alpha+1+\beta+2\,\tau)}{\rho}x^2 - bx + b.$$

where

$$b = 2\frac{(-\beta-1+\rho-2\tau+\rho\tau+\rho\alpha-\alpha)\,(\rho\beta-\beta-1+\rho-2\tau+\rho\tau+\rho\alpha-\alpha)}{(\alpha+1+\beta+2\tau)\,\rho}$$

Using (5.2) and (5.3) we get

$$\gamma_1 = -\frac{\rho(1+\tau+\alpha)(1+\tau+\alpha+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+2)}, \tag{5.44}$$

$$\gamma_{2n+1} = -\frac{(n+1+\tau+\alpha)(n+1+\tau+\alpha+\beta)}{(2n+2\tau+\alpha+\beta+1)(2n+2\tau+\alpha+\beta+2)},\ n \geq 1, \tag{5.45}$$

$$\gamma_{2n+2} = -\frac{(n+1+\tau)(n+1+\tau+\beta)}{(2n+2\tau+\alpha+\beta+2)(2n+2\tau+\alpha+\beta+3)},\ n \geq 0. \tag{5.46}$$

From (2.37) and (2.39), the division by $x(x+1)$ leads to

$$C_0(x) = -\left(\frac{-4+3\rho+4\rho\tau+2\rho\alpha+2\rho\beta-8\tau-4\alpha-4\beta}{\rho}\right)x^2 - x + 2\frac{2\rho\tau-4\tau-2+2\rho-2\beta+\rho\beta-2\alpha}{\rho}, \tag{5.47}$$

$$D_0(x) = \frac{2(2\tau+\alpha+\beta+1)}{\rho}(x+1). \tag{5.48}$$

Using (2.52) – (2.55) we obtain, after simplification by $x(x+1)$,

$$C_{2n+1}(x) = (3+4n+2\alpha+4\tau+2\beta)x^2 + x - 4n - 4\alpha - 4 - 4\tau - 2\beta,\ n \geq 0, \tag{5.49}$$

$$C_{2n+2}(x) = (5+4n+4\tau+2\alpha+2\beta)x^2 - x - 4n - 2\beta - 4\tau - 4,\ n \geq 0, \tag{5.50}$$

$$D_{2n+1}(x) = 2(2n+2\tau+\alpha+\beta+2)(x-1),\ n \geq 0, \tag{5.51}$$

$$D_{2n+2}(x) = 2(2n+2\tau+\alpha+\beta+3)(x+1), n \geq 0. \tag{5.52}$$

Tabla 3
$\Phi(x) = x^3 - x,$
$\Psi(x) = -x^3 + \frac{(-4+3\rho+2\rho\alpha+4\rho\tau+2\rho\beta-4\alpha-8\tau-4\beta)}{\rho}x^2 + 2x$ $-2\frac{-2\alpha+2\rho\alpha+\rho\beta+2\rho\tau-2+2\rho-2\beta-4\tau}{\rho},$
$B(x) = -2\frac{(\rho-1)(\alpha+1+\beta+2\tau)}{\rho}x^3 + 2\frac{(\rho-1)(\alpha+1+\beta+2\tau)}{\rho}x^2$ $-2\frac{(-\beta-1+\rho-2\tau+\rho\tau+\rho\alpha-\alpha)(\rho\beta-\beta-1+\rho-2\tau+\rho\tau+\rho\alpha-\alpha)}{(\alpha+1+\beta+2\tau)\rho}x$ $+2\frac{(-\beta-1+\rho-2\tau+\rho\tau+\rho\alpha-\alpha)(\rho\beta-\beta-1+\rho-2\tau+\rho\tau+\rho\alpha-\alpha)}{(\alpha+1+\beta+2\tau)\rho},$
$\gamma_1 = -\frac{\rho(1+\tau+\alpha)(1+\tau+\alpha+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+2)},$
$\gamma_{n+1} = -\frac{1}{4}\frac{\left(n+2\tau+1+(\alpha+\frac{1}{2})(1+(-1)^n)\right)\left(n+2\tau+2\beta+1+(\alpha+\frac{1}{2})(1+(-1)^n)\right)}{(n+2\tau+\alpha+\beta+1)(n+2\tau+\alpha+\beta+2)},$ $n \geq 1,$
$C_0(x) = -(\frac{-4+3\rho+4\rho\tau+2\rho\alpha+2\rho\beta-8\tau-4\alpha-4\beta}{\rho})x^2 - x$
$C_{n+1}(x) = (2n + 2\beta + 2\alpha + 4\tau + 3)x^2$ $+(-1)^n x - 2n - 4\tau - 2\beta - 2$
$D_0(x) = \frac{2(2\tau+\alpha+\beta+1)}{\rho}(x + 1),$ $D_{n+1}(x) = 2(n + \alpha + \beta + 2\tau + 2)(x - (-1)^n),\ n \geq 0.$

The sequences described in Tables 1, 2, and 3 are the unique Laguerre-Hahn polynomial sequences satisfying (1.1). In fact, we will show that conditions in (2) are not compatible with the regularity of u. Indeed, let assume that there exists a regular linear functional u satisfying (2.47) with conditions in (2). Then, using (2.34) and (2.35), its shift $\tau_1 u$ satisfies (2.47) with

$$(2') \qquad \Phi^P(0) = 0,\ B^P(0) = 0,\ (B^P)'(0) = 0,\ \Psi^P(0) = 0,$$

We will analyze three cases.

1. First case: $\Phi^P(x) = x$.

Following [2] as well as [3] and [13], we distinguish two subcases.

1.1. $B^P(x) = x^2 + (2(1-\alpha) - \lambda)x + \alpha(\alpha - 1 - \lambda) - \rho$.

Here, we have $\Psi^P(x) = -x + \alpha - 1$, and $\gamma_1^P = \rho \neq 0$.

From (2') we get $\alpha = 1, \lambda = 0,\ \rho = 0$ which yields a contradiction.

1.2. $B^P(x) = (1 - \frac{1}{\rho})x^2 + \{2(\frac{1}{\rho} - 1)(2\tau + \alpha + 1) + \lambda(\frac{2}{\rho} - 1)\}x$
$+ (2\tau + \alpha + 1 + \lambda)\{(1 - \frac{1}{\rho})(2\tau + \alpha + 1) + 1 - \frac{\lambda}{\rho}\} - \rho(1+\tau)(1 + \tau + \alpha)$.

Here, $\Psi^P(x) = (\frac{2}{\rho} - 1)x + (1 - \frac{2}{\rho})(2\tau + \alpha + 1) - \frac{2}{\rho}\lambda$, and $\gamma_1^P = \rho(1 + \tau)(1 + \tau + \alpha), \quad \rho \neq 0$.

From (2') we have

$$\lambda = (\frac{\rho}{2} - 1)(2\tau + \alpha + 1), \tag{5.53}$$

$$\{2(\frac{1}{\rho} - 1)(2\tau + \alpha + 1) + \lambda(\frac{2}{\rho} - 1)\} = 0, \tag{5.54}$$

and

$$(2\tau+\alpha+1+\lambda)\{(1-\frac{1}{\rho})(2\tau+\alpha+1)+1-\frac{\lambda}{\rho}\}-\rho(1+\tau)(1+\tau+\alpha) = 0. \tag{5.55}$$

Replacing (5.53) in (5.54) we get

$$\rho(2\tau + \alpha + 1) = 0.$$

If $\rho = 0$, then $\gamma_1^P = 0$, that contradicts the regularity of the linear functional. If $2\tau + \alpha + 1 = 0$, then from (5.53) and (5.55) we get $\lambda = 0$ and $\rho(1+\tau)(1+\tau+\alpha) = 0$. Therefore, $\gamma_1^P = 0$, a contradiction with the regularity of the linear functional.

2. Second case: $\Phi^P(x) = x^2$.

Following [13] we distinguish two subcases.

2.1.

$$\begin{aligned} B^P(x) &= (\frac{1}{\rho} - 1)(\alpha - 1)x^2 + \{\frac{2}{\alpha} + [\alpha(1 - \frac{2}{\rho}) + \frac{2}{\rho}]\beta_0^P\}x + \frac{\alpha - 1}{\rho}(\beta_0^P)^2 \\ &\quad -\frac{2}{\alpha}\beta_0^P + \frac{\rho}{\alpha^2(\alpha - 1)} \end{aligned}$$

with $\beta_0^P = \frac{2}{\alpha(\alpha-2)} + \lambda$.

Here

$$\begin{aligned}\Psi^P(x) &= [(1-\frac{2}{\rho})(\alpha-1)-1]x-\frac{2}{\alpha}+2\frac{\alpha-1}{\rho}\beta_0^P, \text{ and}\\ \gamma_1^P &= \frac{\rho}{\alpha^2(\alpha-1)(\alpha+1)}, \qquad \alpha,\rho\neq 0.\end{aligned}$$

From (2') we have

(5.56) $$\beta_0^P=\frac{\rho}{\alpha(\alpha-1)}$$

and

(5.57) $$\frac{2}{\alpha}+(\alpha(1-\frac{2}{\rho})+\frac{2}{\rho})\beta_0^P=0.$$

Replacing (5.56) in (5.57) we get $\alpha\rho=0$, a contradiction.

2.2. $B^P(x)=(\frac{1}{\rho}-1)[2(\tau+\theta)-1]x^2+2\{[(\tau+\theta)(1-\frac{2}{\rho})+\frac{1}{\rho}]\beta_0^P+\frac{1-\theta}{\tau+\theta}\}x$ $+(\frac{2\tau+2\theta-1}{\rho})(\beta_0^P)^2+2\frac{\theta-1}{\tau+\theta}\beta_0^P+\frac{\rho(\tau+1)(\tau+2\theta-1)}{[2(\tau+\theta)-1](\tau+\theta)^2}$, with $\beta_0^P=\frac{1-\theta}{(\tau+\theta)(\tau+\theta-1)}+\lambda$.

Thus, $\Psi^P(x)=2[(\tau+\theta)(1-\frac{2}{\rho})+\frac{1}{\rho}-1]x+\frac{2}{\rho}(2\tau+2\theta-1)\beta_0^P+2\frac{\theta-1}{\tau+\theta}$, and $\gamma_1^P=-\frac{\rho(\tau+1)(\tau+2\theta-1)}{[2(\tau+\theta)+1][2(\tau+\theta)-1](\tau+\theta)^2}$, $\tau+\theta$, $\rho\neq 0$.

From (2')

(5.58) $$\beta_0^P=\rho\frac{1-\theta}{(\tau+\theta)(2\tau+2\theta-1)},$$

(5.59) $$(\frac{2\tau+2\theta-1}{\rho})(\beta_0^P)^2+2\frac{\theta-1}{\tau+\theta}\beta_0^P+\frac{\rho(\tau+1)(\tau+2\theta-1)}{(2(\tau+\theta)-1)(\tau+\theta)^2}=0,$$

and

(5.60) $$((\tau+\theta)(1-\frac{2}{\rho})+\frac{1}{\rho})\beta_0^P+\frac{1-\theta}{\tau+\theta}=0$$

Replacing (5.58) in (5.60) we get $\rho(\tau+\theta)(1-\theta)=0$. Since $\rho(\tau+\theta)\neq 0$ then $\theta=1$. Therefore, using (5.58) and (5.59), $\beta_0^P=0$ and $\frac{\rho(\tau+1)(\tau+2\theta-1)}{(2(\tau+\theta)-1)(\tau+\theta)^2}=0$. So, $\gamma_1^P=0$, a contradiction with the regularity of the linear functional.

3. Third case: $\Phi^P(x)=x^2-x$.

We distinguish two subcases.

3.1. $B^P(x)=\frac{1}{4}\{(1-\alpha)(-2x+1)^2-(2\mu\frac{\alpha-1}{\alpha-2}-\alpha\lambda)(-2x+1)+\frac{\mu^2}{2-\alpha}+\lambda\mu+\rho(\alpha+1)-1\}$.

In these conditions we get $\Psi^P(x) = -\frac{1}{2}\{(\alpha-2)(-2x+1)+\mu\}$ together with $\gamma_1^P = \frac{\rho}{4}$ and $\gamma_{n+1}^P = \frac{n(n+\alpha)(2n+\alpha-\mu)(2n+\alpha+\mu)}{4(2n+\alpha+1)(2n+\alpha-1)(2n+\alpha)^2}, \quad n \geq 1$.

From (2') we have

$$\alpha + \mu = 2, \tag{5.61}$$

$$(1-\alpha) - (2\mu\frac{\alpha-1}{\alpha-2} - \alpha\lambda) + \frac{\mu^2}{2-\alpha} + \lambda\mu + \rho(\alpha+1) - 1 = 0, \tag{5.62}$$

and

$$2(1-\alpha) - (2\mu\frac{\alpha-1}{\alpha-2} - \alpha\lambda) = 0. \tag{5.63}$$

Replacing (5.61) in (5.63) we get $\alpha\lambda = 0$. If $\alpha = 0$, then $\mu = 2$. Therefore $\gamma_2^P = 0$, in contradiction with the regularity of the linear functional. If $\lambda = 0$, then, from (5.62), we obtain $\rho(\alpha+1) = 0$. Thus $\gamma_1^P\gamma_2^P = 0$, in contradiction with the regularity of the linear functional.

3.2. $B^P(x) = \frac{1}{4}\{(\frac{1}{\rho}-1)(2\tau+\alpha+\beta+1)(-2x+1)^2 + \{[(\alpha+\beta+2\tau)(1-\frac{2}{\rho}) - 2(\frac{1}{\rho}-1)]K - \mu\}(-2x+1)$
$+\frac{\alpha+\beta+2\tau+1}{\rho}K^2 + \mu K + (2\tau+\alpha+\beta+3)4\frac{\rho(\tau+1)(\tau+1+\alpha)(\tau+1+\beta)(\tau+1+\alpha+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+3)(2\tau+\alpha+\beta+2)^2} - 1\}$
and $K = \frac{\alpha^2-\beta^2}{(\alpha+\beta+2(\tau+1))(\alpha+\beta+2\tau)} + \lambda$.

Thus, $\Psi^P(x) = -\frac{1}{2}\{[(1-\frac{2}{\rho})(2\tau+\alpha+\beta) - \frac{2}{\rho}](-2x+1) + 2\frac{\alpha+\beta+2\tau+1}{\rho}K + \mu\}$,

$\gamma_1^P = \frac{\rho(\tau+1)(\tau+1+\alpha)(\tau+1+\beta)(\tau+1+\alpha+\beta)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+3)(2\tau+\alpha+\beta+2)^2}$.

$\Psi^P(0) = 0$ yields a linear equation in λ. For such a value of λ the condition $B^P(0) = 0$ is equivalent to $-\frac{1}{4}\frac{\rho(\beta-1)(\beta+1)}{(2\tau+\alpha+\beta+1)} = 0$. If $\beta = 1$ then $(B^P)'(0) = \frac{\rho(\tau+1)(\tau+1+\alpha)}{(2\tau+\alpha+2)}$. If $\beta = -1$ then $(B^P)'(0) = \frac{\rho\tau(\tau+\alpha)}{(2\tau+\alpha)}$. In both cases the condition $(B^P)'(0) = 0$ is not compatible with the regularity of u.

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