

MONOTONICITY AND ASYMPTOTIC OF ZEROS OF LAGUERRE-SOBOLEV-TYPE ORTHOGONAL POLYNOMIALS OF HIGHER ORDER DERIVATIVES

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ABSTRACT. In this paper we analyze the location of zeros of polynomials orthogonal with respect to the inner product

$$(1) \quad \langle p, q \rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x} dx + Np^{(j)}(0)q^{(j)}(0),$$

where $\alpha > -1$, $N \geq 0$, and $j \in \mathbb{N}$. In particular, we focus our attention in their interlacing properties with respect to the zeros of Laguerre polynomials as well as in the monotonicity of each individual zero in terms of the mass N . Finally, we give necessary and sufficient conditions in terms of N in order the least zero of any Laguerre-Sobolev type orthogonal polynomial be negative.

1. INTRODUCTION AND STATEMENT OF RESULTS

Given a nontrivial probability measure μ supported in a infinite subset of the real line and a real number c , sequences of polynomials orthogonal with respect to the Sobolev inner product

$$(1) \quad \langle p, q \rangle = \int_{\mathbb{R}} p(x)q(x)d\mu(x) + Np^{(j)}(c)q^{(j)}(c),$$

have been introduced in [11]. There, the authors obtain a higher order recurrence relation that such polynomials satisfy as well as second order linear differential equation of holonomic type assuming that the measure μ is semi-classical. The zero distribution of such polynomials when the support of μ is located in the positive real semi-axis and $c = 0$ is analyzed in [12]. There the author proves that the zeros of the polynomials of degree n are real, simple, and at most one of them is outside $(0, +\infty)$. The location of these zeros is given in terms of the location of the zeros of the polynomials orthogonal with respect to the measures $d\mu$ and $x^2d\mu$.

Next, we will focus our attention in a canonical case of the above inner product related to the Laguerre weight. Indeed, let consider the sequence of polynomials $\{L_n^{(\alpha, N)}(x)\}_{n=0}^\infty$ orthogonal with respect to the Sobolev-type inner product

$$(2) \quad \langle p, q \rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x} dx + Np^{(j)}(0)q^{(j)}(0),$$

where $\alpha > -1$, $N \geq 0$, and $j \in \mathbb{N}$ as well as the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$, normalized as in Szegő's book [13], i.e., $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$.

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The case $j = 1$ has attracted the interest of many researchers from many points of view. First, the analysis of the asymptotic properties of such orthogonal polynomials is done (see [1] and the survey paper [10]). Second, the study of analytic properties of their zeros like interlacing with the zeros of Laguerre polynomials and monotonicity of each zero in terms of N , among others, are completely discussed in [4]. Furthermore, the limit behaviour of such zeros is also deduced in [10]. Third, the study of spectral properties of these orthogonal polynomials in the sense that they are eigenfunctions of an infinite order linear differential operator which reduces to order $2\alpha + 8$ if α is a nonnegative integer number and $N > 0$ is presented in [9]. Finally, a second order linear differential equation of holonomic type that such polynomials satisfy is deduced in [7]. As an immediate consequence, an electrostatic interpretation of these zeros is given.

In the last years, some attention was paid to the asymptotic properties of these orthogonal polynomials when $j > 1$. In particular, in [6] the authors obtain a Mehler-Heine formula for such polynomials as well as their outer relative asymptotics in terms of the standard Laguerre orthogonal polynomials. On the other hand, in [2] it is proved that these polynomials are eigenfunctions of an infinite order linear differential operator which reduces to order $2\alpha + 4 + 4j$ if α is a nonnegative integer number and $N > 0$. But, the analysis of their zeros in terms of N remains an open problem. This is the aim of our contribution.

The structure of the manuscript is as follows. In Section 2, a connection formula for the sequences of standard Laguerre orthogonal polynomials and the Laguerre-Sobolev type orthogonal polynomials is given. Notice that this connection formula is quite different of the connection formula given in Theorem 3.1 of [12]. It will very useful in the study of some analytic properties of the zeros of Laguerre-Sobolev type orthogonal polynomials. Indeed, In Section 3, we deduce interlacing properties between the zeros of such sequences as well as we locate the least zero of $L_n^{(\alpha, N)}(x)$ with respect to the interval $(0, +\infty)$ in terms of the mass N . Furthermore, we prove that each zero is a decreasing function of N and we analyze its limit when N tends to infinity as well as the speed of convergence. These results are new in the literature as far as we know. Some numerical examples allow to check our results.

2. CONNECTION FORMULA

In the following, we adopt the convention $\binom{i}{0} := 1$ and $\binom{i}{-j} := 0$, $i, j > 0$.

Theorem 1. *Let $\{L_n^{(\alpha, N)}(x)\}_{n=0}^{\infty}$ be the Laguerre-Sobolev-type orthogonal polynomials corresponding to the inner product (2). Then, for every $n > j$ and each $\alpha > -1$,*

$$(3) \quad L_n^{(\alpha, N)}(x) = \sum_{k=0}^{j+1} A_{n,k} L_{n-k}^{(\alpha+k)}(x),$$

where

$$(4) \quad \begin{aligned} A_{n,0} &= 1 + N \frac{\Gamma(j+1)}{\Gamma(\alpha+j+1)} \sum_{k=1}^{j+1} (-1)^{k+1} \binom{n+\alpha}{n-j-k} \binom{n-k}{j+1-k}, \\ A_{n,k} &= (-1)^k N \frac{\Gamma(j+1)}{\Gamma(\alpha+j+1)} \binom{n+\alpha}{n-j} \binom{n-k}{j+1-k}, \quad k = 1, \dots, j+1. \end{aligned}$$

Proof. Let $\{L_n^{(\alpha,N)}(x)\}_{n=0}^\infty$ be the sequence of polynomials defined by (3) and (4). We shall prove the orthogonality of these polynomials with respect to the inner product (2). Denote by π_n the set of polynomials of degree at most n . Let $p_{l+j+1}(x) = x^{j+1}q_l(x)$, $l = 0, 1, \dots, n-(j+2)$, where $q_l(x)$ is a polynomial of degree exactly l . Thus, the polynomials $1, x, \dots, x^j$ and $p_{l+j+1}(x)$, $l = 0, 1, \dots, n-(j+2)$, constitute a basis in π_{n-1} . Then, for $n \geq j+2$,

$$\begin{aligned} \langle p_{l+j+1}, L_n^{(\alpha,N)} \rangle &= \int_0^\infty x^{j+1} q_l(x) L_n^{(\alpha,N)}(x) x^\alpha e^{-x} dx \\ &= \sum_{k=0}^{j+1} A_{n,k} \int_0^\infty x^{j+1-k} q_l(x) L_{n-k}^{(\alpha+k)}(x) x^{\alpha+k} e^{-x} dx \\ &= 0, \end{aligned}$$

because of the orthogonality property of the classical Laguerre polynomials. Now, we define the integral $I_{k,i}$ by (see [8, (3.3)])

$$I_{k,i} = \int_0^\infty L_{n-i}^{(\alpha+i)}(x) x^{\alpha+k} e^{-x} dx = \binom{n-k-1}{n-i} \Gamma(\alpha+k+1).$$

Thus,

$$\begin{aligned} \langle 1, L_n^{(\alpha,N)} \rangle &= \int_0^\infty L_n^{(\alpha,N)}(x) x^\alpha e^{-x} dx \\ &= A_{n,0} \cdot 0 + A_{n,1} \cdot I_{0,1} + A_{n,2} \cdot I_{0,2} + \dots + A_{n,j} \cdot I_{0,j} + A_{n,j+1} \cdot I_{0,j+1}, \\ \langle x, L_n^{(\alpha,N)} \rangle &= \int_0^\infty x L_n^{(\alpha,N)}(x) x^\alpha e^{-x} dx \\ &= A_{n,0} \cdot 0 + A_{n,1} \cdot 0 + A_{n,2} \cdot I_{1,2} + \dots + A_{n,j} \cdot I_{1,j} + A_{n,j+1} \cdot I_{1,j+1}, \\ &\vdots \\ \langle x^{j-1}, L_n^{(\alpha,N)} \rangle &= \int_0^\infty x^{j-1} L_n^{(\alpha,N)}(x) x^\alpha e^{-x} dx \\ &= A_{n,0} \cdot 0 + A_{n,1} \cdot 0 + A_{n,2} \cdot 0 + \dots + A_{n,j} \cdot I_{j-1,j} + A_{n,j+1} \cdot I_{j-1,j+1}, \end{aligned}$$

and, finally,

$$\begin{aligned} \langle x^j, L_n^{(\alpha, N)} \rangle &= \int_0^\infty x^j L_n^{(\alpha, N)}(x) x^\alpha e^{-x} dx + Nj! \left[L_n^{(\alpha, N)}(x) \right]^{(j)} \Big|_{x=0} \\ &= A_{n,0} \cdot 0 + A_{n,1} \cdot 0 + A_{n,2} \cdot 0 + \cdots + A_{n,j} \cdot 0 + A_{n,j+1} \cdot I_{0,j+1} \\ &\quad + N\Gamma(j+1)(-1)^j \sum_{k=0}^{j+1} A_{n,k} L_{n-k-j}^{(\alpha+k+j)}(0). \end{aligned}$$

Using the formula of $I_{k,i}$, the identity (5.1.7) in Szegő's book [13], and the expression of $A_{n,k}$, $0 \leq k \leq j+1$, given in Theorem 1, our statement follows. \square

Notice that a different expression for the connection formula appears in Theorem 3.1 in [12]. Using the structure relation for Laguerre polynomials and after cumbersome computations (2.3) can be deduced. Our proof is more simple as well as here the connection coefficients are explicitly given.

Lemma 1. *For $n > j$ and each $\alpha > -1$, the inequalities*

$$A_{n,0} \geq 1, \quad (-1)^k A_{n,k} \geq 0, \quad k = 1, 2, \dots, j+1$$

hold.

Proof. The positivity of the numbers $(-1)^k A_{n,k}$, $1 \leq k \leq j+1$ is obvious. To prove that $A_{n,0} - 1 \geq 0$, notice that

$$\binom{n+\alpha}{n-j-\nu} \binom{n-\nu}{j+1-\nu} - \binom{n+\alpha}{n-j-\nu-1} \binom{n-\nu-1}{j-\nu} \geq 0$$

holds for $n > j$, $\alpha > -1$, and $\nu = 1, 3, 5, \dots, j$ or $j-1$ depending on the parity of j . \square

3. THE ZEROS

From (3), $L_n^{(\alpha, N)}(x)$ can be written

$$(5) \quad L_n^{(\alpha, N)}(x) = (1 + \tilde{A}_{n,0}N)L_n^{(\alpha)}(x) + \sum_{k=1}^{j+1} N\tilde{A}_{n,k}L_{n-k}^{(\alpha+k)}(x),$$

where $\tilde{A}_{n,0} = (A_{n,0} - 1)/N$ and $\tilde{A}_{n,k} = A_{n,k}/N$, $k = 1, \dots, j+1$. Now, we introduce the polynomial

$$(6) \quad F_{n,\alpha,l}(x) = - \sum_{k=0}^{j+1} \tilde{A}_{n,k} L_{n-k}^{(\alpha+k)}(x).$$

Then, (5) reads

$$(7) \quad L_n^{(\alpha, N)}(x) = L_n^{(\alpha)}(x) - NF_{n,\alpha,l}(x).$$

Let denote by $x_{n,k}^N(\alpha)$, $x_{n,k}(\alpha)$, and $\zeta_{n,k}(\alpha)$ the zeros of $L_n^{(\alpha, N)}(x)$, $L_n^{(\alpha)}(x)$, and $F_{n,\alpha,l}(x)$, respectively, arranged in a decreasing order. It was proved in [6], Theorem 3, that at most one of the zeros of $L_n^{(\alpha, N)}(x)$ is located outside $(0, \infty)$. Notice that this a particular case of Theorem 4.1. in [12].

In the present work, we will give explicitly the value N_0 of the mass such that for $N > N_0$ this situation occurs, i.e. the least zero is negative. Moreover, in [6] was showed that $x_{n,n}^N(\alpha) < x_{n,n}(\alpha) < \cdots < x_{n,1}^N(\alpha) < x_{n,1}(\alpha)$ (see also Theorem

4.3 in [12]). In such a sense, here we complete this result proving that the zeros $x_{n,k}^N(\alpha)$ and $x_{n,k}(\alpha)$ interlace with the zeros $\zeta_{n,k}(\alpha)$.

Theorem 2. *For every $n > j$ and each $\alpha > -1$, the inequalities*

$$(8) \quad \zeta_{n,n}(\alpha) < x_{n,n}^N(\alpha) < x_{n,n}(\alpha) < \cdots < \zeta_{n,1}(\alpha) < x_{n,1}^N(\alpha) < x_{n,1}(\alpha).$$

hold. Moreover, the smallest zero $x_{n,n}^N(\alpha)$ satisfies

$$x_{n,n}^N(\alpha) > 0, \quad \text{for } N < N_0,$$

$$x_{n,n}^N(\alpha) = 0, \quad \text{for } N = N_0,$$

$$x_{n,n}^N(\alpha) < 0, \quad \text{for } N > N_0,$$

where

$$N_0 = N_0(n, \alpha, j) = \frac{L_n^{(\alpha)}(0)}{F_{n,\alpha,l}(0)} = \frac{\binom{n+\alpha}{n}}{-\sum_{k=0}^{j+1} \tilde{A}_{n,k} \binom{n+\alpha}{n-k}}.$$

Proof. Observe that the leading coefficients of $L_n^{(\alpha,N)}(x)$, $L_n^{(\alpha)}(x)$, and $F_{n,\alpha,l}(x)$ are equal to

$$\frac{(-1)^n A_{n,0}}{n!}, \quad \frac{(-1)^n}{n!} \quad \text{and} \quad \frac{(-1)^{n-1} \tilde{A}_{n,0}}{n!},$$

respectively. Since (see [6, Thm 4]) $x_{n,n}^N(\alpha) < x_{n,n}(\alpha) < \cdots < x_{n,1}^N(\alpha) < x_{n,1}(\alpha)$ then, using the connection formula (7), we get

$$\text{sign}\left(L_n^{(\alpha,N)}(x_{n,k}(\alpha))\right) = -\text{sign}\left(F_{n,\alpha,l}(x_{n,k}(\alpha))\right) = (-1)^{n-k+1}$$

and

$$\text{sign}\left(L_n^{(\alpha)}(x_{n,k}^N(\alpha))\right) = \text{sign}\left(F_{n,\alpha,l}(x_{n,k}^N(\alpha))\right) = (-1)^{n-k},$$

for $k = 1, \dots, n$. Thus, the interlacing property (8) yields. In order to investigate the location of $x_{n,n}^N(\alpha)$ with respect to the origin, it suffices to observe that $L_n^{(\alpha,N)}(0) = 0$ if and only if $N = N_0$. \square

In order to show the behavior of the least zero $x_{n,n}^N(\alpha)$ of the Laguerre-Sobolev type orthogonal polynomials, we will do some numerical computations using the Mathematica software. We present, for n, j , and α fixed, a table that shows the behavior of $x_{n,n}^N(\alpha)$ with respect to N . In particular, for $N > N_0$, the least zero is negative.

For the case $j = 1$, N_0 reduces to

$$N_0 = N_0(n, \alpha, 1) = \frac{\Gamma(n-1)\Gamma(\alpha+2)\Gamma(\alpha+4)}{\Gamma(n+\alpha+2)}.$$

Then, for $n = 2, 3$ and $\alpha = -1/2, 1, 5$ we get

N	$x_{2,2}^N(-1/2)$	N	$x_{2,2}^N(1)$	N	$x_{2,2}^N(5)$
1/2	0.115964	3/2	0.271499	710	0.0419159
$N_0 = \sqrt{\pi}/2$	0	$N_0 = 2$	0	$N_0 = 720$	0
1	-0.0313955	5/2	-0.230139	730	-0.0414199

N	$x_{3,3}^N(-1/2)$	N	$x_{3,3}^N(1)$	N	$x_{3,3}^N(5)$
1/4	0.00211646	1/5	0.407703	79	0.0251697
$N_0 = \sqrt{\pi}/7$	0	$N_0 = 2/5$	0	$N_0 = 80$	0
1/2	-0.133233	3/5	-0.275762	81	-0.0248324

For the case $j = 2$, N_0 reduces to

$$N_0 = N_0(n, \alpha, 2) = \frac{\Gamma(n-2)\Gamma(\alpha+3)\Gamma(\alpha+6)}{2[n(\alpha+4) - (\alpha+2)]\Gamma(n+\alpha+2)}.$$

Thus, for $n = 3$, and $\alpha = -1/2, \alpha = 1, \alpha = 5$ we obtain

N	$x_{3,3}^N(-1/2)$	N	$x_{3,3}^N(1)$
1/4	0.0715899	1	0.415775
$N_0 = 3\sqrt{\pi}/16$	0	$N_0 = 3/2$	0
1/3	-0.00100291	2	-0.450761

N	$x_{3,3}^N(5)$
1259	0.00317496
$N_0 = 1260$	0
1261	-0.00317424

For the case $j = 3$, N_0 reduces to

$$\begin{aligned} N_0 &= N_0(n, \alpha, 3) \\ &= \frac{\Gamma(n-3)\Gamma(\alpha+4)\Gamma(\alpha+8)}{3[n^2(\alpha+5)(\alpha+6) - 3n(\alpha+4)(\alpha+5) + 2(\alpha+2)(\alpha+3)]\Gamma(n+\alpha+2)}. \end{aligned}$$

Thus, for $n = 4$, and $\alpha = -1/2, \alpha = 1, \alpha = 5$ we obtain

N	$x_{4,4}^N(-1/2)$	N	$x_{4,4}^N(1)$
1/10	0.102777	1	0.322548
$N_0 = 5\sqrt{\pi}/48$	0	$N_0 = 4/3$	0
2/10	-0.0346506	5/3	-0.41383

N	$x_{4,4}^N(5)$
2239	0.00200886
$N_0 = 2240$	0
2241	-0.00200899

In the next result, we obtain the monotonicity of the zeros $x_{n,k}^N(\alpha)$ with respect to N as well as their convergence when N tends to infinity to the zeros $\zeta_{n,k}(\alpha)$ with a speed of convergence of order $1/N$. For the proof of these statements, we need the following lemma concerning the behavior of the zeros of linear combinations of two polynomials with interlacing zeros .

Lemma 2. *Let $h_n(x) = (-1)^n a(x - x_1) \cdots (x - x_n)$ and $g_n(x) = (-1)^{n-1} b(x - \zeta_1) \cdots (x - \zeta_n)$ be polynomials with real and interlacing zeros,*

$$\zeta_n < x_n < \cdots < \zeta_1 < x_1,$$

where a and b are real positive constants. Then, for any real constant $c > 0$, the polynomial

$$f(x) = h_n(x) - cg_n(x)$$

has n real zeros $\eta_n < \eta_{n-1} < \dots < \eta_1$ which interlace with both the zeros of $h_n(x)$ and $g_n(x)$ in the following form

$$\zeta_n < \eta_n < x_n < \dots < \zeta_1 < \eta_1 < x_1.$$

Moreover, each η_k is a decreasing function of c and, for each $k = 1, \dots, n$,

$$(9) \quad \lim_{c \rightarrow \infty} \eta_k = \zeta_k \quad \text{and} \quad \lim_{c \rightarrow \infty} c[\eta_k - \zeta_k] = \frac{h_n(\zeta_k)}{g'_n(\zeta_k)}.$$

Recent results concerning zeros of linear combinations of orthogonal polynomials have been used in [4] and [5] in order to analyze monotonicity and the asymptotics for the zeros of some class of orthogonal polynomials. We omit here the proof of the above lemma (see [3, Lemma 1] and [5, Lemma 3]).

Theorem 3. For every $n > j$ and each $\alpha > -1$, the zeros $x_{n,k}^N(\alpha)$ are decreasing functions of N . Moreover,

$$\lim_{N \rightarrow \infty} x_{n,k}^N(\alpha) = \zeta_{n,k}(\alpha)$$

and

$$\lim_{N \rightarrow \infty} N[x_{n,k}^N(\alpha) - \zeta_{n,k}(\alpha)] = g_{n,k,l}(\alpha), \quad k = 1, \dots, n,$$

where

$$g_{n,k,l}(\alpha) = \frac{L_n^{(\alpha)}(\zeta_{n,k}(\alpha))}{F'_{n,\alpha,l}(\zeta_{n,k}(\alpha))}.$$

Proof. This is an immediate consequence of (7), the inequalities (8), and the Lemma 2. \square

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