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Higher Order Coherent Pairs

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Abstract In this paper, we study necessary and sufficient conditions for the relation

$$P_n^{[r]}(x) + a_{n-1,r} P_{n-1}^{[r]}(x) = R_{n-r}(x) + b_{n-1,r} R_{n-r-1}(x),$$
$$a_{n-1,r} \neq 0, \ n \geq r+1,$$

where $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are two sequences of monic orthogonal polynomials with respect to the quasi-definite linear functionals $\mathcal{U}, \mathcal{V}$, respectively, or associated with two positive Borel measures μ_0, μ_1 supported on the real line. We deduce the connection with Sobolev orthogonal polynomials, the relations between these functionals as well as their corresponding formal Stieltjes series. As sake of example, we find the coherent pairs when one of the linear functionals is classical.

Keywords Coherent pairs · Sobolev inner product · Stieltjes functions · Semiclassical linear functionals · Orthogonal polynomials

Mathematics Subject Classification (2000) 42C05

1 Introduction

The notion of *coherent pair* was introduced by A. Iserles, P.E. Koch, S.P. Nørsett and J.M. Sanz-Serna in 1991 [15]. They state that a pair of positive Borel measures (μ_0, μ_1) supported on the real line is, with our terminology, a $(1,0)$-coherent pair of order 1 if and only if there exist nonzero constants $\{a_{n,1}\}_{n\geq 1}$ such that their corresponding sequences of monic

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orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$R_n(x) = \frac{P'_{n+1}(x)}{n+1} + a_{n,1}\frac{P'_n(x)}{n}, \quad a_{n,1} \neq 0,\ n \geq 1. \tag{1.1}$$

Moreover, this condition of coherence is a sufficient condition for the existence of a relation

$$Q_{n+1}(x;\lambda) + c_{n,1}(\lambda)Q_n(x;\lambda) = P_{n+1}(x) + \frac{n+1}{n}a_{n,1}P_n(x), \quad n \geq 1, \tag{1.2}$$

where $\{c_{n,1}(\lambda)\}_{n\geq 1}$ are rational functions in $\lambda > 0$ and $\{Q_n(x;\lambda)\}_{n\geq 0}$ is the SMOP associated with the Sobolev inner product

$$\langle p(x), q(x)\rangle_{\lambda,1} = \int_{-\infty}^{\infty} p(x)q(x)\,d\mu_0 + \lambda\int_{-\infty}^{\infty} p'(x)q'(x)\,d\mu_1, \quad \lambda > 0,\ p,q \in \mathbb{P}. \tag{1.3}$$

Besides, they study the case when the measure μ_0 is classical (Laguerre and Jacobi). Furthermore, they introduce the notion of symmetrically coherent pair, when the two measures μ_0 and μ_1 are symmetric (i.e., invariant under the transformation $x \mapsto -x$) and the subscripts in (1.1) are changed appropriately.

In 1995, F. Marcellán, T. Pérez, J.C. Petronilho, and M. Piñar (see [20]) showed that if a pair of positive definite linear functionals $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-coherent pair of order 1, then both are semiclassical, $\mathcal{V}$ is of class at most 1 and $\mathcal{U}$ is of class at most 6. Moreover, they proved that there exist polynomials $\widetilde{\sigma}_2(x)$ and $\widetilde{\tau}_2(x)$ such that $\widetilde{\sigma}_2(x)\mathcal{V} = \widetilde{\tau}_2(x)\mathcal{U}$, with $\deg(\widetilde{\sigma}_2(x)) \leq 2$ and $\deg(\widetilde{\tau}_2(x)) \leq 3$.

On the other hand, F. Marcellán and J. Petronilho [18] studied (1.1) when $\mathcal{U}$ and $\mathcal{V}$, with respective SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$, are quasi-definite linear functionals and they solved the problem when one of the functionals is classical, i.e., Hermite, Laguerre, Jacobi or Bessel.

Finally, in 1997, H.G. Meijer [28] determined all $(1,0)$-coherent pairs $(\mathcal{U},\mathcal{V})$ of quasi-definite linear functionals of order 1. He proved that at least one of the functionals must be classical. Moreover, he showed that there are only two cases:

- If $\mathcal{U}$ is a classical linear functional, then there exist polynomials $\sigma_{\mathcal{U}}(x)$, $\tau_{\mathcal{U}}(x)$ and $\rho_{\mathcal{U}}(x)$, with $\deg(\sigma_{\mathcal{U}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{U}}(x)) = \deg(\rho_{\mathcal{U}}(x)) = 1$, such that $D(\sigma_{\mathcal{U}}(x)\mathcal{U}) = \tau_{\mathcal{U}}(x)\mathcal{U}$ and $\sigma_{\mathcal{U}}(x)\mathcal{U} = \rho_{\mathcal{U}}(x)\mathcal{V}$. Here $D\mathcal{U}$ denotes the derivative of a linear functional $\mathcal{U}$, which is the linear functional defined by $\langle D\mathcal{U}, p(x)\rangle = -\langle \mathcal{U}, p'(x)\rangle$, $\forall p \in \mathbb{P}$.
- If $\mathcal{V}$ is a classical linear functional, then there exist polynomials $\sigma_{\mathcal{V}}(x)$, $\tau_{\mathcal{V}}(x)$ and $\rho_{\mathcal{V}}(x)$, with $\deg(\sigma_{\mathcal{V}}(x)) \leq 2$ and $\deg(\tau_{\mathcal{V}}(x)) = \deg(\rho_{\mathcal{V}}(x)) = 1$, such that $D(\sigma_{\mathcal{V}}(x)\mathcal{V}) = \tau_{\mathcal{V}}(x)\mathcal{V}$ and $\sigma_{\mathcal{V}}(x)\mathcal{U} = \rho_{\mathcal{V}}(x)\mathcal{V}$.

H.G. Meijer also determined all symmetrically $(1,0)$-coherent pairs of order 1, providing similar results to those obtained in the non-symmetric case. Indeed, they correspond to Hermite and Gegenbauer cases. Some analytic properties of Sobolev orthogonal polynomials associated with such pairs of measures have been studied. In particular, their asymptotic behavior (see [26]) as well as the location of their zeros (see [27]). Later on, in 2004, A. Delgado and F. Marcellán [11] extended the notion of a coherent pair to *generalized coherent pairs* (in our terminology, (1, 1)-coherent pair of order 1), by studying the relation

$$R_n(x) + b_{n,1}R_{n-1}(x) = \frac{P'_{n+1}(x)}{n+1} + a_{n,1}\frac{P'_n(x)}{n}, \quad n \geq 1, \tag{1.4}$$

with $a_{n,1} \neq 0$ for all $n \geq 1$. They verified that this condition of generalized coherence is a necessary and sufficient condition for the relation (1.2). Also, they determined all $(1,1)$-coherent pairs of order 1 of linear functionals ($b_{n,1}$ can be zero). They proved that at least one of the quasi-definite linear functionals (either $\mathcal{U}$ or $\mathcal{V}$) must be semiclassical of class at most 1, generalizing the results obtained by H. G. Meijer for $(1,0)$-coherent pairs of order 1. Moreover, they showed that there are only two cases:

- If $\mathcal{U}$ is a semiclassical linear functional given by $D(\sigma_{\mathcal{U}}(x)\mathcal{U}) = \tau_{\mathcal{U}}(x)\mathcal{U}$ with $\deg(\sigma_{\mathcal{U}}(x)) \leq 3$ and $\deg(\tau_{\mathcal{U}}(x)) \leq 2$, then there exists a constant $C_{\mathcal{U}}$ such that $\sigma_{\mathcal{U}}(x)\mathcal{U} = (x - C_{\mathcal{U}})\mathcal{V}$.
- If $\mathcal{V}$ is a semiclassical linear functional given by $D(\sigma_{\mathcal{V}}(x)\mathcal{V}) = \tau_{\mathcal{V}}(x)\mathcal{V}$ with $\deg(\sigma_{\mathcal{V}}(x)) \leq 3$ and $\deg(\tau_{\mathcal{V}}(x)) \leq 2$, then there exists a constant $C_{\mathcal{V}}$ such that $\sigma_{\mathcal{V}}(x)\mathcal{U} = (x - C_{\mathcal{V}})\mathcal{V}$.

Finally, a generalization of this situation to symmetrically coherent pairs is stated by A. Delgado and F. Marcellán in 2005 (see [12]).

In 2001, another generalization of coherent pairs was introduced by F. Marcellán, A. Martínez-Finkelshtein, and J. Moreno-Balcázar in [22]. A pair of positive measures on the real line (μ_0, μ_1) is said to be a *k-coherent pair* (a $(k+1,0)$-coherent pair of order 1 according to our terminology), $k \in \mathbb{N}$, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$R_n(x) = \frac{P'_{n+1}(x)}{n+1} + \sum_{j=1}^{k+1} a_{n-j+1,n} \frac{P'_{n-j+1}(x)}{n-j+1}, \quad n \geq k+1, \tag{1.5}$$

with $a_{n-k,n} \neq 0$. Some nontrivial examples of k-coherent pairs are presented as well as the following relation between Sobolev polynomials $\{Q_n(x;\lambda)\}_{n\geq 0}$ associated with the inner product (1.3) and the polynomials $\{P_n(x)\}_{n\geq 0}$ associated with the first measure of this product is stated as a necessary condition for $(k+1,0)$-coherence of order 1

$$\begin{aligned} &Q_{n+1}(x;\lambda) + \sum_{j=1}^{k+1} c_{n-j+1,1}(\lambda) Q_{n-j+1}(x;\lambda) \\ &\quad = P_{n+1}(x) + \sum_{j=1}^{k+1} \frac{n+1}{n-j+1} a_{n-j+1,1} P_{n-j+1}(x), \quad n \geq k+1. \end{aligned}$$

We get (1.2) when $k = 0$.

K.H. Kwon, J.H. Lee, and F. Marcellán [17] studied k-coherent pairs for $k = 1$, but they called them *generalized coherent pairs*. They concluded that if $(\mathcal{U}, \mathcal{V})$ is a generalized coherent pair of (quasi-definite) linear functionals, then $\mathcal{U}$ and $\mathcal{V}$ must be semiclassical ($\mathcal{U}$ of class at most 6 and $\mathcal{V}$ of class at most 2). They also studied the case when either $\mathcal{U}$ or $\mathcal{V}$ is classical.

In 1999 and 2000, P. Maroni and R. Sfaxi ([24] and [25]) introduced the notion of *coherent pair associated with $\phi(x)$ with index s*, where $\phi(x)$ is a monic polynomial of degree t and $s \geq 0$. In such case, a pair $(\{R_n(x)\}_{n\geq 0}, \{P_n(x)\}_{n\geq 0})$ of SMOP with respect to the pair of quasi-definite linear functionals $(\mathcal{V}, \mathcal{U})$ satisfies

$$\phi(x) R_n(x) = \sum_{k=n-s}^{n+t} a_{n,k} \frac{P'_{k+1}(x)}{k+1}, \quad a_{n,n-s} \neq 0, \ n \geq s. \tag{1.6}$$

If $R_n(x) := P_n(x)$ for all $n \in \mathbb{N}$, $\{P_n(x)\}_{n\geq 0}$ is said to be a *diagonal sequence associated with* $\phi(x)$ *with index* s. They obtained necessary and sufficient conditions for (1.6) and they obtained results for the dual sequences of $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$. In 2006, Sfaxi and Alaya [30] continued this study from another point of view.

Notice that from the three-term recurrence relation that $\{R_n(x)\}_{n\geq 0}$ satisfies and expressing $\phi(x)R_n(x)$ as a linear combination of polynomials $R_{n-t-1}, \dots, R_{n+t}$, we get

$$R_{n+t}(x) + \sum_{k=n-t-1}^{n+t-1} b_{n,k} R_k(x) = \frac{P'_{n+t+1}(x)}{n+t+1} + \sum_{k=n-s}^{n+t-1} a_{n,k} \frac{P'_{k+1}(x)}{k+1}, \quad a_{n,n-s} \neq 0,\ n \geq s.$$

This is a relation of $(t+s, 2t+1)$-coherence of order 1.

Later on, in 2003 and 2004 M. Alfaro, F. Marcellán, A. Peña and M.L. Rezola (see [1] and [2]) analyzed the following algebraic relation between two SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$

$$R_n(x) + b_{n-1,0} R_{n-1}(x) = P_{n+1}(x) + a_{n-1,0} P_n(x), \quad n \geq 1,$$

that we will call $(1, 1)$-coherence of order 0. It yields the relation $(x - C^P)\mathcal{U} = \xi(x - C^R)\mathcal{V}$, with C^P and C^R constants. Under some conditions, a pair of quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$-coherent pair of order 0 if and only if these functionals satisfy the above relation of relational type.

Afterwards, in 2006 J. Petronilho [29] extended this problem as follows

$$R_n(x) + \sum_{i=1}^{N} b_{n-i,n,0} R_{n-i}(x) = P_n(x) + \sum_{i=1}^{M} a_{n-i,n,0} P_{n-i}(x), \quad n \geq \min\{M, N\},$$

where $a_{n-i,n,0} = 0$ and $b_{n-i,n,0} = 0$ if $n - i < 0$, (with our terminology, (M, N)-coherence of order 0). He proved that if a pair of quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherent pair of order 0 then, under some conditions, these functionals satisfy $\phi(x)\mathcal{U} = \psi(x)\mathcal{V}$, where $\phi(x)$ and $\psi(x)$ are polynomials of degree N and M, respectively. On the other hand, there the case $(M, N) = (2, 1)$ carefully studied. In 2010 and 2011, M. Alfaro, F. Marcellán, A. Peña and M.L. Rezola ([3] and [4]) studied the case $(M, N) = (2, 0)$, with $a_{n-2,n,0} \neq 0$ for $n \geq 2$, and stated the relation $\mathcal{U} = h(x)\mathcal{V}$ between their quasi-definite linear functionals, where $h(x)$ is a polynomial of degree 2.

To complete this historical overview, in 2008, M.N. de Jesus and J. Petronilho [10] proposed the more general case, the so called (M, N)-*coherent pairs of order* (r, s), where the derivatives of order r and s of two SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are related by

$$\sum_{i=0}^{M} a_{n+r-i,n,r} P^{(r)}_{n+r-i}(x) = \sum_{i=0}^{N} b_{n+s-i,n,s} R^{(s)}_{n+s-i}(x), \quad n \geq 0,$$

M and N are non-negative integers and $a_{n+r-i,n,r}$, $b_{n+r-i,n,r}$ are complex parameters satisfying some natural conditions. They proved that if $\mathcal{U}$ and $\mathcal{V}$ are the corresponding quasi-definite linear functionals associated with $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ then, for $0 \leq s \leq r$, there exist four polynomials $\gamma_{M+s+i}(x)$ and $\varphi_{N+r+i}(x)$ of degree $M+s+i$ and $N+r+i$, respectively, $i = 0, 1$, such that

$$D^{r-s}\big(\gamma_{M+s+i}(x)\mathcal{V}\big) = \varphi_{N+r+i}(x)\mathcal{U}, \quad i = 0, 1,$$

where $D\mathcal{U}$ denotes the distributional derivative of the linear functional $\mathcal{U}$. Therefore, if $r = s$ then $\mathcal{U}$ and $\mathcal{V}$ are related by a relation of rational type. Besides, they concluded that if $r = s + 1$ and $\{P_n(x)\}_{n\geq 0} \neq \{R_n(x)\}_{n\geq 0}$, then there exist polynomials $\gamma(x)$, $\phi(x)$, and $\psi(x)$ of degrees at most $2(M + s)$, $M + N + 2s + 2$ and $M + N + 2s + 1$, respectively, such that

$$\gamma(x)\mathcal{V} = \phi(x)\mathcal{U}, \qquad D\big(\phi(x)\mathcal{V}\big) = \psi(x)\mathcal{V},$$

$$D\big(\gamma(x)\phi(x)\mathcal{U}\big) = \big[2\gamma'(x)\phi(x) + \gamma(x)\big(\psi(x) - \phi'(x)\big)\big]\mathcal{U},$$

and hence, $\mathcal{U}$ and $\mathcal{V}$ are semiclassical linear functionals of classes at most $3M + N + 4s$ and $M + N + 2s$, respectively. When $r = s + 1$ and $\{P_n(x)\}_{n\geq 0} = \{R_n(x)\}_{n\geq 0}$, $\mathcal{U}$ and $\mathcal{V}$ coincide up to a constant factor and are semiclassical of class at most $\max\{M + s - 2, N + s\}$.

Finally, in 2010, A. Branquinho and M.N. Rebocho [7], stated that if a pair of positive Borel measures (μ_0, μ_1) is a $(1, 0)$-coherent pair of order 2, then each of them is semiclassical and μ_1 is a rational modification of μ_0. Also, they concluded that if $\{P_n(x)\}_{n\geq 0} = \{R_n(x)\}_{n\geq 0}$ then, under some conditions, $(\{P_n(x)\}_{n\geq 0}, \{P_n(x)\}_{n\geq 0})$ is a $(N + 2, N)$-coherent pair of order 1 if and only if $\mathcal{U}$ is a semiclassical linear functional of class at most N.

The concept of coherent pair of measures has been generalized to the case of measures supported on curves of the complex plane. In the pioneering contribution, [8] an approach to coherent pairs of measures supported on the unit circle has been done.

In this work, we focus our attention on $(1, 1)$-coherent pairs of order r of quasi-definite linear functionals $(\mathcal{U}, \mathcal{V})$, that is, their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$P_n^{[r]}(x) + a_{n-1,r} P_{n-1}^{[r]}(x) = R_{n-r}(x) + b_{n-1,r} R_{n-r-1}(x), \quad a_{n-1,r} \neq 0,\ n \geq r + 1. \quad (3.13)$$

If $b_{n,r} = 0$ for all $n \geq r + 1$, then $(\mathcal{U}, \mathcal{V})$ is said to be a $(1, 0)$-coherent pair of order r.

The structure of the manuscript is as follows. In Sect. 2 we review some of the standard facts on orthogonal polynomials. In Sect. 3 we introduce the notion of $(1, 0)$ and $(1, 1)$-coherent pair of order r of positive Borel measures supported on the real line and we state their relation with Sobolev orthogonal polynomials. In Sect. 4 we extend our study to $(1, 0)$ and $(1, 1)$ -coherent pairs of order r of quasi-definite linear functionals. From Lemma 22 which states the relation

$$D^r\big[\gamma_{n,r}(x)\mathcal{V}\big] = (-1)^r \varphi_{n+r,r}(x)\mathcal{U}, \quad n \geq 1, \quad (4.2)$$

for $(\mathcal{U}, \mathcal{V})$ a $(1, 1)$-coherent pair of order r of functionals, where $\gamma_{n,r}(x)$ is a monic polynomial of degree n and $\varphi_{n+r,r}(x)$ is a polynomial of degree at most $n + r$,[1] we obtain Theorem 23 and Theorem 28. The first one states that if $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$ (or $(1, 0)$)-coherent pair of order r, then for $r, k \in \mathbb{N}$ and $k \leq r$,

$$\tilde{\sigma}_{r+1}(x)\mathcal{V} = \tilde{\tau}_{r+1}(x)\mathcal{U},$$

$$\tilde{\sigma}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)D^k\mathcal{V} = \tilde{\tau}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)\mathcal{U}, \qquad \tilde{\tau}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)D^k\mathcal{V} = \tilde{\sigma}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)\mathcal{V},$$

[1] If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\varphi_{n+r,r}(x)) \leq n + r - 1$.

where $\tilde{\sigma}_{r+1}(x)$, $\tilde{\sigma}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)$, $\tilde{\sigma}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)$, $\tilde{\tau}_{r+1}(x)$, $\tilde{\tau}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)$ and $\tilde{\tau}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)$ are polynomials whose degree are given either explicitly or by an upper bound. The proof of this theorem is given for $k=0,1,2,3$ and $r \geq k$. On the other hand, we show that if $\mathcal{U}$ is a classical linear functional given by $D(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}$ with $\deg(\sigma(x)) \leq 2$ and $\deg(\tau(x)) = 1$, and $\mathcal{V}$ is a quasi-definite linear functional, such that $\langle \mathcal{U}, \sigma^r(x)\rangle = 1 = \langle \mathcal{V}, 1\rangle$, then $(x - C^{P^{[r]}})\sigma^r(x)\mathcal{U} = \xi(x - C^R)\mathcal{V}$ and $P_n^{[r]}(x) \neq R_{n-r}(x), n \geq r+1$, where ξ, $C^{P^{[r]}}$, and C^R are constants, is a necessary and sufficient condition for $(\mathcal{U}, \mathcal{V})$ to be a $(1,1)$-coherent pair of order r with $a_{r,r} \neq b_{r,r}$ and $a_{n,r}b_{n,r} \neq 0$, $n \geq r$. Besides, in this case $\mathcal{V}$ is a semiclassical linear functional of class at most 2.

In Sect. 5 we deduce some relations between the formal Stieltjes series associated with the linear functionals in a $(1,1)$ (or $(1,0)$)-coherent pair of order r.

2 Basic Background

2.1 Linear Functionals

We will denote by $\mathbb{P}$ the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}_n$ denotes the linear subspace of polynomials of degree at most n. Let $\mathcal{U}$ be a linear functional in $\mathbb{P}$. $\langle \mathcal{U}, p(x)\rangle$ will denote the image of polynomial $p(x)$ by $\mathcal{U}$.

Every sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$, with $\deg(P_n(x)) = n$, is a basis for $\mathbb{P}$. There exists a unique sequence of linear functionals $\{\wp_n\}_{n\geq 0}$, called *the dual basis of* $\{P_n(x)\}_{n\geq 0}$, such that $\langle \wp_n, P_m(x)\rangle = \delta_{n,m}$, $n,m \in \mathbb{N}$, where $\delta_{n,m}$ denotes the Kronecker Delta. So, each linear functional $\mathcal{U}$ in $\mathbb{P}$ can be expressed as

$$\mathcal{U} = \sum_{n\geq 0} \lambda_n \wp_n, \quad \text{where } \lambda_n = \langle \mathcal{U}, P_n(x)\rangle.$$

δ_a will denote *the Delta Dirac linear functional at* a, $a \in \mathbb{C}$, defined by

$$\langle \delta_a, p(x)\rangle = p(a), \quad \forall p \in \mathbb{P}.$$

Let $\mathcal{U}$ be a linear functional in $\mathbb{P}$ and $q(x) \in \mathbb{P} \setminus \{0\}$. We define the linear functionals $q(x)\mathcal{U}$ and $(q(x))^{-1}\mathcal{U}$ by

$$\langle q(x)\mathcal{U}, p(x)\rangle = \langle \mathcal{U}, q(x)p(x)\rangle, \qquad \langle (q(x))^{-1}\mathcal{U}, p(x)\rangle = \left\langle \mathcal{U}, \frac{p(x) - L_q(x;p)}{q(x)}\right\rangle, \quad (2.1)$$

for all $p \in \mathbb{P}$, where $L_q(x;p)$ denotes the interpolatory polynomial of $p(x)$ at the zeros of $q(x)$ taking into account their multiplicity. The above operations are not commutative. Indeed, $(x-a)(x-a)^{-1}\mathcal{U} = \mathcal{U}$ but

$$(x-a)^{-1}(x-a)\mathcal{U} = \mathcal{U} - \langle \mathcal{U}, 1\rangle \delta_a. \quad (2.2)$$

Proposition 1 *Let $\mathcal{U}$ be a linear functional and $q \in \mathbb{P}$, then for $r \in \mathbb{N}$*

$$D^r\left(q(x)\mathcal{U}\right) = \sum_{k=0}^{r} \binom{r}{k} q^{(k)}(x) D^{r-k}\mathcal{U}. \quad (2.3)$$

Finally, the *formal Stieltjes series* of the linear functional $\mathcal{U}$ is

$$S_{\mathcal{U}}(z) = -\sum_{n\geq 0} \frac{u_n}{z^{n+1}}. \tag{2.4}$$

Its rth derivative, $r \in \mathbb{N}$, is given by $S_{\mathcal{U}}^{(r)}(z) = (-1)^{r+1} \sum_{n\geq 0}(n+1)_r \frac{u_n}{z^{n+1+r}}$, where $(n+1)_r$ denotes the *Pochhammer symbol*,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \quad n \geq 1, \quad \text{and} \quad (a)_0 = 1.$$

The Pochhammer symbol satisfies $(a+b)_n = \sum_{k=0}^{n} \binom{n}{k} (a)_{n-k}(b)_k$ and $(-a)_n = (-1)^n(a-n+1)_n$.

2.2 Orthogonal Polynomials

A linear functional $\mathcal{U}$ is said to be *quasi-definite or regular* (see [9]) if the Hankel matrix $H = (u_{i+j})_{i,j=0}^{\infty}$ associated with the moments of the functional is quasi-definite, i.e., $\Delta_n = \det(H_n) = \det((u_{i+j})_{i,j=0}^{n}) \neq 0$, for all $n \in \mathbb{N}$. Hence, there exists a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ such that

(i) $\deg(P_n(x)) = n$, for all $n \in \mathbb{N}$,
(ii) $\langle \mathcal{U}, P_n(x)P_m(x)\rangle = k_n^P \delta_{n,m}$, $k_n^P \neq 0$ and $n, m \in \mathbb{N}$.

$\{P_n(x)\}_{n\geq 0}$ is said to be a *sequence of orthogonal polynomials* (*SOP*) with respect to the linear functional $\mathcal{U}$. This sequence is unique up to multiplicative constants. If all polynomials of the sequence are monic, $\{P_n(x)\}_{n\geq 0}$ is called the *sequence of monic orthogonal polynomials* (*SMOP*) with respect to the linear functional $\mathcal{U}$.

Moreover, if the leading principal submatrices of H are positive definite, then $\mathcal{U}$ is said to be *positive definite*. In this case there exists a positive Borel measure μ supported on the real line such that $\langle \mathcal{U}, p(x)\rangle = \int_{\mathbb{R}} p(x)d\mu(x)$, for all $p \in \mathbb{P}$.

Let $\{P_n(x)\}_{n\geq 0}$ be a SMOP with respect to a quasi-definite linear functional $\mathcal{U}$. Then,

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} u_0 & u_1 & \cdots & u_{n-1} & u_n \\ u_1 & u_2 & \cdots & u_n & u_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n-1} & u_n & \cdots & u_{2n-2} & u_{2n-1} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}, \quad n \geq 1, \qquad P_0(x) = 1. \tag{2.5}$$

Another characterization of orthogonal polynomials is given by the Favard Theorem: (see [9]). *A sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ is a SMOP with respect to a quasi-definite linear functional $\mathcal{U}$ (that is unique if $u_0 = 1$) if and only if there exist sequences of complex numbers $\{\alpha_n^P\}_{n\geq 0}$ and $\{\beta_n^P\}_{n\geq 0}$, $\beta_n^P \neq 0$, $n \geq 2$, such that they satisfy the* three-term recurrence relation (TTRR)

$$P_n(x) = \left(x - \alpha_n^P\right) P_{n-1}(x) - \beta_n^P P_{n-2}(x), \quad n \geq 2, \qquad P_1(x) = x - \alpha_1^P, \qquad P_0(x) = 1, \tag{2.6}$$

$\alpha_n^P = \frac{\langle \mathcal{U}, x P_{n-1}^2(x)\rangle}{\langle \mathcal{U}, P_{n-1}^2(x)\rangle}$, $\beta_{n+1}^P = \frac{\langle \mathcal{U}, P_n^2(x)\rangle}{\langle \mathcal{U}, P_{n-1}^2(x)\rangle} \neq 0$, $n \geq 1$. *Moreover, the linear functional $\mathcal{U}$ is positive definite if and only if α_n^P is real and $\beta_{n+1}^P > 0$, for $n \geq 1$.*

Given a SMOP $\{P_n(x)\}_{n\geq 0}$, for all $n \in \mathbb{N}$ we can define the *nth reproducing kernel associated with* $\mathcal{U}$

$$K_n(x, y; \mathcal{U}) = \sum_{j=0}^{n} \frac{P_j(y)}{\langle \mathcal{U}, P_j^2(x)\rangle} P_j(x), \tag{2.7}$$

that satisfy $\langle \mathcal{U}, K_n(x, y; \mathcal{U}) p(x)\rangle = p(y)$, for all $p \in \mathbb{P}_n$.

Finally, we give some results relating the dual basis of a SMOP and its respective linear functional and derivatives.

Proposition 2 [23] *Let* $\{P_n(x)\}_{n\geq 0}$ *be the SMOP with respect to a quasi-definite linear functional* $\mathcal{U}$ *and let* $\{\wp_n\}_{n\geq 0}$ *be its corresponding dual basis. Then*

$$\mathcal{U} = \langle \mathcal{U}, 1\rangle \wp_0, \qquad \wp_n = \frac{P_n(x)}{\langle \mathcal{U}, P_n^2(x)\rangle} \mathcal{U}, \quad \forall n \in \mathbb{N}, \quad \textit{and} \tag{2.8}$$

$$D^r \wp_n^{[r]} = (-1)^r \frac{n!}{(n-r)!} \wp_n, \quad n \geq r, \tag{2.9}$$

$r \in \mathbb{N}$, *where* $\{\wp_n^{[r]}\}_{n\geq r}$ *is the dual basis of the sequence of monic polynomials* $\{\frac{(n-r)!}{n!} P_n^{(r)}(x) = P_n^{[r]}(x)\}_{n\geq r}$.

Corollary 3 *Under the conditions of the previous proposition we get*

$$D^r \wp_n^{[r]} = (-1)^r \frac{n!}{(n-r)!} \frac{P_n(x)}{\langle \mathcal{U}, P_n^2(x)\rangle} \mathcal{U}, \quad n \geq r.$$

In particular, if $\{P_n^{[r]}(x)\}_{n\geq r}$ *is the SMOP with respect to the linear functional* $\mathcal{U}^{[r]}$, *then*

$$D^r \mathcal{U}^{[r]} = (-1)^r r! \frac{\langle \mathcal{U}^{[r]}, 1\rangle}{\langle \mathcal{U}, P_r^2(x)\rangle} P_r(x) \mathcal{U}.$$

2.3 Semiclassical and Classical Linear Functionals

Let $\sigma(x), \tau(x)$ be non-zero polynomials such that $\deg(\sigma(x)) = k \geq 0$ and $\deg(\tau(x)) = l \geq 1$, $\sigma(x) = a_k x^k + \dots$ and $\tau(x) = b_l x^l + \dots$. $(\sigma(x), \tau(x))$ is said to be an *admissible pair* if they satisfy either $k - 1 \neq l$ or, $k - 1 = l$ and $n a_{l+1} + b_l \neq 0$ for all $n \in \mathbb{N}$.

A linear functional $\mathcal{U}$ is said to be *semiclassical* if it is quasi-definite and there exists an admissible pair of polynomials $(\sigma(x), \tau(x))$ such that the following distributional Pearson equation holds

$$D\big(\sigma(x)\mathcal{U}\big) = \tau(x)\mathcal{U}. \tag{2.10}$$

Under these conditions, the *class of* $\mathcal{U}$ is defined by the non-negative integer

$$s := \min \max\Big\{\deg\big(\sigma(x)\big) - 2, \deg\big(\tau(x)\big) - 1\Big\}, \tag{2.11}$$

where the minimum is taken among all admissible pairs of polynomials $(\sigma(x), \tau(x))$ such that (2.10) holds. If $\sigma(x)$ is monic, the admissible pair that determines the class is unique. We also say that the SMOP associated with a semiclassical linear functional is a *semiclassical SMOP of class* s if the class of $\mathcal{U}$ is s (see [23]). Examples of semiclassical SMOP

Table 1 Classical SMOP

Classical MOP	Hermite	Laguerre	Jacobi
$P_n(x)$	$H_n(x)$	$L_n^{(\alpha)}(x)$	$P_n^{(\alpha,\beta)}(x)$
$\sigma(x)$	1	x	$1-x^2$
$\tau(x)$	$-2x$	$-x+\alpha+1$	$-(\alpha+\beta+2)x+\beta-\alpha$
$(a,b)\subset\mathbb{R}$	$(-\infty,\infty)$	$(0,\infty)$	$(-1,1)$
$w(x)$	e^{-x^2}	$x^{\alpha}e^{-x}$	$(1-x)^{\alpha}(1+x)^{\beta}$
Restriction	$-$	$\alpha>-1$	$\alpha>-1,\beta>-1$

have been extensively studied in the literature (see in [5, 6] and [16]). Its role in the theory of polynomials orthogonal with respect to weighted Sobolev inner products has been emphasized in [21].

Proposition 4 [24] *Let $\mathcal{U}$ and $\mathcal{V}$ be two quasi-definite linear functionals. If these functionals are related by a expression of rational type, i.e., there exist nonzero polynomials $p(x)$ and $q(x)$ such that*

$$p(x)\mathcal{U}=q(x)\mathcal{V}, \tag{2.12}$$

then, $\mathcal{U}$ is a semiclassical linear functional if and only if so is $\mathcal{V}$. Moreover, if the class of $\mathcal{U}$ is s, then the class of $\mathcal{V}$ is at most $s+\deg(p(x))+\deg(q(x))$.

A semiclassical linear functional $\mathcal{U}$ of class $s=0$, i.e.,

$$D\big(\sigma(x)\mathcal{U}\big)=\tau(x)\mathcal{U} \quad \text{with } \deg\big(\sigma(x)\big)\le 2 \text{ and } \deg\big(\tau(x)\big)=1, \tag{2.13}$$

is said to be a *classical linear functional* and its SMOP associated is called *classical SMOP*. Under linear transformations of the variable and some conditions on the parameters, we get the classical SMOP with respect to definite positive linear functionals, where the weight function $w(x)$ is positive and integrable. The associated linear functional is given by $\langle\mathcal{U},p(x)\rangle=\int_a^b p(x)w(x)dx$, for all $p\in\mathbb{P}$.

A characterization of classical orthogonal polynomials is the following

Theorem 5 [13] *Let $\mathcal{U}$ be a quasi-definite linear functional with SMOP $\{P_n(x)\}_{n\ge 0}$. The following statements are equivalent*

(i) $\{P_n(x)\}_{n\ge0}$ *is a classical SMOP and $\mathcal{U}$ satisfies* (2.13).
(ii) *The sequence of monic polynomials $\{P_n^{[1]}(x)=P_n'(x)/n\}_{n\ge1}$ is a SMOP with respect to a linear functional $\mathcal{U}^{[1]}$. Indeed, $\mathcal{U}^{[1]}=\sigma(x)\mathcal{U}$.*

Besides, $\{P_n^{[1]}(x)\}_{n\ge1}$ is also a classical SMOP of the same type as $\{P_n(x)\}_{n\ge0}$ because $\mathcal{U}^{[1]}$ satisfies the distributional differential equation

$$D\big(\sigma(x)\mathcal{U}^{[1]}\big)=\big(\tau(x)+\sigma'(x)\big)\mathcal{U}^{[1]}. \tag{2.14}$$

Corollary 6 [14] *In the conditions of Theorem 5, the following statements are equivalent*

(i) $\{P_n(x)\}_{n\ge0}$ *is a classical SMOP and $\mathcal{U}$ satisfies* (2.13).

Table 2 Basic notations

SMOP	Measure	Linear/Bilinear Functional	Moments
$P_n(x)$	μ_0	$\mathcal{U}$	u_n
$R_n(x)$	μ_1	$\mathcal{V}$	v_n
$Q_n(x;\lambda,r)$	$\langle,\rangle_{\lambda,r}$	$\mathcal{W}$	$w_{n,m}$

(ii) *For* $r\in\mathbb{N}$, *the sequence of monic polynomials* $\{P_n^{[r]}(x)=\frac{(n-r)!}{n!}P_n^{(r)}(x)\}_{n\geq r}$, *where* $P_n^{(r)}(x)$ *denotes the* r*th derivative of* $P_n(x)$, *is a SMOP associated to the linear functional*

$$\mathcal{U}^{[r]}=\sigma^r(x)\mathcal{U}. \tag{2.15}$$

Moreover, $\{P_n^{[r]}(x)\}_{n\geq r}$ *is also a SMOP of the same type as* $\{P_n(x)\}_{n\geq 0}$ *because* $\mathcal{U}^{[r]}$ *satisfies*

$$D\big(\sigma(x)\mathcal{U}^{[r]}\big)=\big(\tau(x)+r\sigma'(x)\big)\mathcal{U}^{[r]}. \tag{2.16}$$

Remark 7 For $r\in\mathbb{N}$, $r\leq n$, and for the classical monic orthogonal polynomials (Hermite $\{H_n(x)\}_{n\geq 0}$, Laguerre $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ and Jacobi $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$), we get

$$\begin{aligned}
H_n^{[r]}(x)&=\frac{(n-r)!}{n!}H_n^{(r)}(x)=H_{n-r}(x),\\
\big(L_n^{(\alpha)}(x)\big)^{[r]}&=\frac{(n-r)!}{n!}\big(L_n^{(\alpha)}(x)\big)^{(r)}=L_{n-r}^{(\alpha+r)}(x),\\
\big(P_n^{(\alpha,\beta)}(x)\big)^{[r]}&=\frac{(n-r)!}{n!}\big(P_n^{(\alpha,\beta)}(x)\big)^{(r)}=P_{n-r}^{(\alpha+r,\beta+r)}(x).
\end{aligned} \tag{2.17}$$

Remark 8 Other characterizations of classical orthogonal polynomials have been presented in [19]. Notice that the basic tool for their proofs is the algebraic theory of linear functionals in the linear space of polynomials with complex coefficients.

3 Sobolev Orthogonal Polynomials and Coherent Pairs of Order *r* of Measures

We will consider the Sobolev inner product

$$\big\langle p(x),q(x)\big\rangle_{\lambda,r}=\int_{-\infty}^{\infty}p(x)q(x)\,d\mu_0+\lambda\int_{-\infty}^{\infty}p^{(r)}(x)q^{(r)}(x)\,d\mu_1, \tag{3.1}$$

where μ_0, μ_1 are positive Borel measures supported on the real line, $\lambda\in\mathbb{R}^+$ and $p,q\in\mathbb{P}$, from now on, polynomials with real coefficients. In these conditions, we make the following assumptions and notations as in Table 2:

Thus,

$$w_{n,m}=\begin{cases}u_{n+m}+\lambda\frac{n!}{(n-r)!}\frac{m!}{(m-r)!}v_{n+m-2r} & n\geq r \text{ and } m\geq r,\\ u_{n+m} & n<r \text{ or } m<r.\end{cases}$$

Notice that $Q_n(x;\lambda,r)=P_n(x)$, for $n\leq r$. Besides, as the coefficients of $Q_n(x;\lambda,r)$ are rational functions of λ, with the degree of the denominator greater than or equal to the degree of the numerator, then

$$O_n^{(r)}(x;r)=\frac{n!}{(n-r)!}R_{n-r}(x),\quad n\geq r, \tag{3.2}$$

where

$$O_n(x;r)=\lim_{\lambda\to\infty}Q_n(x;\lambda,r)$$

is a monic polynomial of degree n.

Furthermore, for $n\geq r$,

$$\frac{(n-r)!}{n!}O_n(x;r)=\frac{(n-r)!}{n!}P_n(x)+\sum_{k=0}^{n-1}a_{k,n,r}\frac{(k-r)!}{k!}P_k(x),$$

with

$$a_{k,n,r}=\frac{(n-r)!}{n!}\frac{k!}{(k-r)!}\frac{\langle O_n(x;r),P_k(x)\rangle_{\mu_0}}{\|P_k(x)\|^2_{\mu_0}}.$$

Taking into account $\langle O_n(x;r),x^k\rangle_{\mu_0}=\int_{\mathbb{R}}O_n(x;r)x^k d\mu_0=0$, for $k<\min\{n,r\}$, then

$$\frac{(n-r)!}{n!}O_n(x;r)=\frac{(n-r)!}{n!}P_n(x)+\sum_{k=r}^{n-1}a_{k,n,r}\frac{(k-r)!}{k!}P_k(x),\quad n\geq r. \tag{3.3}$$

If the above expression is derived r times, we get

$$R_{n-r}(x)=O_n^{[r]}(x;r)=P_n^{[r]}(x)+\sum_{k=r}^{n-1}a_{k,n,r}P_k^{[r]}(x),\quad n\geq r, \tag{3.4}$$

where we use the notation

$$p^{[r]}(x)=\frac{(n-r)!}{n!}p^{(r)}(x),$$

where $p(x)$ a monic polynomial of degree n and $r\in\mathbb{N}$.

Definition 9 The pair of measures (μ_0,μ_1) is said to be a $(1,0)$**-coherent of order** r if in (3.4) $a_{k,n,r}=0$ for $k=r,\dots,n-2$ y $a_{n-1,r}=a_{n-1,n,r}\neq 0$, $n\geq r+1$, i.e., if their respective SMOP satisfy

$$R_{n-r}(x)=P_n^{[r]}(x)+a_{n-1,r}P_{n-1}^{[r]}(x)\quad\text{with } a_{n-1,r}\neq 0,\ n\geq r+1. \tag{3.5}$$

In this case, we say that either $(\mathcal{U},\mathcal{V})$ or $(\{P_n(x)\}_{n\geq 0},\{R_n(x)\}_{n\geq 0})$ is a $(1,0)$-coherent pair of order r.

Note that if (μ_0,μ_1) is a pair of $(1,0)$-coherent measures of order r, then (3.3) becomes

$$O_n(x;r)=P_n(x)+\frac{n}{n-r}a_{n-1,r}P_{n-1}(x)\quad a_{n-1,r}\neq 0,\ n\geq r+1. \tag{3.6}$$

On the other hand, for $n \in \mathbb{N}$

$$O_n(x;r) = Q_n(x;\lambda,r) + \sum_{k=0}^{n-1} c_{k,n,r}(\lambda) Q_k(x;\lambda,r),$$

where

$$\begin{aligned} c_{k,n,r}(\lambda) &= \frac{\langle O_n(x;r), Q_k(x;\lambda,r)\rangle_{\lambda,r}}{\|Q_k(x;\lambda,r)\|^2_{\lambda,r}} \\ &\overset{(3.2)}{\underset{(3.6)}{=}} \frac{1}{\|Q_k(x;\lambda,r)\|^2_{\lambda,r}} \Bigg[\int_{\mathbb{R}} \left(P_n(x) + \frac{n}{n-r} a_{n-1,r} P_{n-1}\right) Q_k(x;\lambda,r) d\mu_0 \\ &\quad + \lambda \int_{\mathbb{R}} \frac{n!}{(n-r)!} R_{n-r}(x) Q_k^{(r)}(x;\lambda,r) d\mu_1\Bigg]. \end{aligned}$$

So, $c_{k,n,r}(\lambda) = 0$ for $k = 0, \ldots, n-2$,

$$c_{n-1,r}(\lambda) := c_{n-1,n,r}(\lambda) = \frac{n}{n-r} a_{n-1,r} \frac{\|P_{n-1}(x)\|^2_{\mu_0}}{\|Q_{n-1}(x;\lambda,r)\|^2_{\lambda,r}} \neq 0, \quad n \geq r+1, \tag{3.7}$$

and

$$O_n(x;r) = Q_n(x;\lambda,r) + c_{n-1,r}(\lambda) Q_{n-1}(x;\lambda,r), \quad c_{n-1,r}(\lambda) \neq 0,\ n \geq r+1. \tag{3.8}$$

We have thus proved the following

Proposition 10 *If* (μ_0, μ_1) *is a pair of* $(1,0)$*-coherent measures of order* r *given by* (3.5), *then there exist constants* $c_{n-1,r}(\lambda) \neq 0$ *such that*

$$\begin{aligned} &Q_n(x;\lambda,r) + c_{n-1,r}(\lambda) Q_{n-1}(x;\lambda,r) = P_n(x) + \frac{n}{n-r} a_{n-1,r} P_{n-1}(x), \\ &a_{n-1,r} \neq 0,\ n \geq r+1, \end{aligned} \tag{3.9}$$

holds. Besides, $c_{n-1,r}(\lambda)$ *is given by* (3.7).

Conversely, if there exist constants $d_{n-1,r}(\lambda)$ and $a_{n-1,r} \neq 0$ such that

$$\begin{aligned} &Q_n(x;\lambda,r) + d_{n-1,r}(\lambda) Q_{n-1}(x;\lambda,r) = P_n(x) + \frac{n}{n-r} a_{n-1,r} P_{n-1}(x), \\ &a_{n-1,r} \neq 0,\ n \geq r+1, \end{aligned} \tag{3.10}$$

then, applying $\langle \cdot, q(x)\rangle_{\lambda,r}$ with $q \in \mathbb{P}_{n-2}$, we get

$$\int_{\mathbb{R}} \left(\frac{(n-r)!}{n!} P_n^{(r)}(x) + a_{n-1,r} \frac{(n-r-1)!}{(n-1)!} P_{n-1}^{(r)}(x)\right) q^{(r)}(x) d\mu_1 = 0, \quad \forall q \in \mathbb{P}_{n-2},$$

i.e.,

$$\int_{\mathbb{R}} \left(P_n^{[r]}(x) + a_{n-1,r} P_{n-1}^{[r]}(x)\right) p(x) d\mu_1 = 0, \quad \forall p \in \mathbb{P}_{n-r-2}. \tag{3.11}$$

So,

$$P_n^{[r]}(x) + a_{n-1,r} P_{n-1}^{[r]}(x) = R_{n-r}(x) + \sum_{k=0}^{n-r-1} b_{k,n,r} R_k(x),$$

with $b_{k,n,r} = 0$ for $k = 0, \dots, n-r-2$, and

$$b_{n-1,r} := b_{n-r-1,n,r} = \frac{\langle P_n^{[r]}(x), R_{n-r-1}(x)\rangle_{\mu_1}}{\|R_{n-r-1}(x)\|_{\mu_1}^2} + a_{n-1,r}, \quad n \geq r+1. \tag{3.12}$$

Therefore,

$$P_n^{[r]}(x) + a_{n-1,r} P_{n-1}^{[r]}(x) = R_{n-r}(x) + b_{n-1,r} R_{n-r-1}(x), \quad a_{n-1,r} \neq 0,\ n \geq r+1. \tag{3.13}$$

Finally, from (3.10) and (3.13), for $n \geq r+1$, we have

$$Q_n^{(r)}(x;\lambda,r) + d_{n-1,r}(\lambda) Q_{n-1}^{(r)}(x;\lambda,r) = \frac{n!}{(n-r)!}\big[R_{n-r}(x) + b_{n-1,r} R_{n-r-1}(x)\big]. \tag{3.14}$$

Definition 11 The pair of measures (μ_0, μ_1) is said to be $(1,1)$-**coherent of order** r if their respective SMOP satisfy (3.13). In this case, we say that either $(\mathcal{U}, \mathcal{V})$ or $(\{P_n(x)\}_{n\geq 0}, \{R_n(x)\}_{n\geq 0})$ is a $(1,1)$-coherent pair of order r.

Remark 12 $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-coherent pair of order r if and only if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order r and $b_{n-1,r} = 0$ for all $n \geq r+1$ (in (3.13)).

Proposition 13 *Let $n \geq r+1$. If there exist constants $a_{n-1,r} \neq 0$ and $d_{n-1,r}(\lambda) \neq 0$ such that* (3.10) *holds, then (μ_0, μ_1) is a $(1,1)$-coherent pair of order r given by* (3.13) *where $b_{n-1,r}$ is* (3.12).

Conversely, (μ_0, μ_1) is a $(1,1)$-coherent pair of order r given by (3.13), *then there are constants $d_{n-1,r}(\lambda) \neq 0$ given by*

$$d_{n-1,r}(\lambda) = \frac{\frac{n}{n-r} a_{n-1,r} \|P_{n-1}(x)\|_{\mu_0}^2 + \lambda \frac{n!}{(n-r)!} \frac{(n-1)!}{(n-r-1)!} b_{n-1,r} \|R_{n-r-1}(x)\|_{\mu_1}^2}{\|Q_{n-1}(x;\lambda,r)\|_{\lambda,r}^2}, \tag{3.15}$$

$n \geq r+1$, such that (3.10) *is satisfied.*

Therefore, (3.13) *and* (3.10) ($\approx$ (3.9)[2]) *are equivalent.*

Proof We already have shown (3.10) implies (3.13). Now, we going to see the converse. If (3.13) holds, then we have (3.11). So,

$$\left\langle P_n(x) + \frac{n}{n-r} a_{n-1,r} P_{n-1}(x), q(x) \right\rangle_{\lambda,r} = 0 \quad \text{for all } q \in \mathbb{P}_{n-2}.$$

On the other hand,

$$P_n(x) + \frac{n}{n-r} a_{n-1,r} P_{n-1}(x) = Q_n(x;\lambda,r) + \sum_{k=0}^{n-1} d_{k,n,r}(\lambda) Q_k(x;\lambda,r),$$

[2]In $(1,0)$-coherence of order r, (3.7) and (3.15) coincide.

where

$$d_{k,n,r}(\lambda)=\frac{\langle P_n(x)+\frac{n}{n-r}a_{n-1,r}P_{n-1}(x),\,Q_k(x;\lambda,r)\rangle_{\lambda,r}}{\|Q_k(x;\lambda,r)\|^2_{\lambda,r}}.$$

Thus $d_{k,n,r}(\lambda)=0$, $k=0,\dots,n-2$ and if we denote $d_{n-1,r}:=d_{n-1,n,r}$, we obtain (3.10). Besides, using (3.13) we have (3.15). □

Let come back to $(1,0)$-coherence of order r. We will see that the sequence $\{c_{n,r}(\lambda)\}_{n\geq r}$ in (3.9) has an additional property. From (3.9) and (3.5) we have

$$\begin{aligned}
&\|Q_{n-1}(x;\lambda,r)\|^2_{\lambda,r}\\
&=\left\langle Q_{n-1}(x;\lambda,r),\,P_{n-1}(x)+\frac{n-1}{n-r-1}a_{n-2,r}P_{n-2}(x)\right\rangle_{\mu_0}\\
&\quad+\lambda\left\langle Q^{(r)}_{n-1}(x;\lambda,r),\,P^{(r)}_{n-1}(x)+\frac{n-1}{n-r-1}a_{n-2,r}P^{(r)}_{n-2}(x)\right\rangle_{\mu_1}\\
&=\|P_{n-1}(x)\|^2_{\mu_0}+\frac{n-1}{n-r-1}a_{n-2,r}\left(\frac{n-1}{n-r-1}a_{n-2,r}-c_{n-2,r}(\lambda)\right)\|P_{n-2}(x)\|^2_{\mu_0}\\
&\quad+\lambda\left(\frac{(n-1)!}{(n-r-1)!}\right)^2\|R_{n-r-1}(x)\|^2_{\mu_1},\quad n\geq r+2.
\end{aligned}\tag{3.16}$$

Remark 14 If $(\{P_n(x)\}_{n\geq0},\{R_n(x)\}_{n\geq0})$ is a $(1,0)$-coherent pair of order r given by (3.5), then from (3.16), (3.7) and (3.9) we can obtain the Sobolev SMOP $\{Q_n(x;\lambda,r)\}_{n\geq0}$.

Finally, if we replace (3.16) in (3.7) we obtain

$$c_{n-1,r}(\lambda)=\frac{B_{n-1,r}}{E_{n-1,r}(\lambda)-c_{n-2,r}(\lambda)},\quad n\geq r+2,\tag{3.17}$$

where

$$B_{n-1,r}=\frac{\frac{n}{n-r}a_{n-1,r}\|P_{n-1}(x)\|^2_{\mu_0}}{\frac{n-1}{n-r-1}a_{n-2,r}\|P_{n-2}(x)\|^2_{\mu_0}},\tag{3.18}$$

$$\begin{aligned}
E_{n-1,r}(\lambda)&=\frac{(\frac{(n-1)!}{(n-r-1)!})^2\|R_{n-r-1}(x)\|^2_{\mu_1}}{\frac{n-1}{n-r-1}a_{n-2,r}\|P_{n-2}(x)\|^2_{\mu_0}}\lambda\\
&\quad+\frac{(\frac{n-1}{n-r-1}a_{n-2,r})^2\|P_{n-2}(x)\|^2_{\mu_0}+\|P_{n-1}(x)\|^2_{\mu_0}}{\frac{n-1}{n-r-1}a_{n-2,r}\|P_{n-2}(x)\|^2_{\mu_0}},
\end{aligned}\tag{3.19}$$

with initial condition

$$c_{r,r}(\lambda)=\frac{(r+1)a_{r,r}\|P_r(x)\|^2_{\mu_0}}{(r!)^2\|R_0(x)\|^2_{\mu_1}\lambda+\|P_r(x)\|^2_{\mu_0}}.\tag{3.20}$$

Note that $E_{n,r}(\lambda)$ is a polynomial in λ of degree 1, for $n\geq r+1$. Besides, if (μ_0,μ_1) is a $(1,0)$-coherent pair of measures of order r, then using (3.17)–(3.20) we can get the sequence $\{c_{n-1,r}(\lambda)\}_{n\geq r+1}$, and we can obtain the Sobolev orthogonal polynomials $Q_n(x;\lambda,r)$

(see Remark 14) taking into account the telescopic relation (3.9). For this, let remind that $Q_n(x;\lambda,r)=P_n(x)$, for $n\le r$.

Proposition 15 *In the conditions of Proposition 10, the sequence $\{c_{n,r}(\lambda)\}_{n\ge r}$ is given by*

$$c_{m+r,r}(\lambda)=\frac{f_{m,r}(\lambda)}{f_{m+1,r}(\lambda)},\quad m\ge 0, \tag{3.21}$$

where $\{f_{m,r}(\lambda)\}_{m\ge 0}$ is a sequence of orthogonal polynomials with respect to a positive Borel measure supported on $\mathbb{R}$.

Proof From (3.17)–(3.20) and by induction on m, it is easy to verify (3.21) and to check that $f_{m,r}(\lambda)$ is a polynomial in λ of degree m.

On the other hand, from (3.17) and (3.21)

$$\begin{aligned}\tilde{k}_m\tilde{f}_{m,r}(\lambda)=&\left(\frac{e_{m+r-1,r,1}}{B_{m+r-1,r}}\lambda+\frac{e_{m+r-1,r,2}}{B_{m+r-1,r}}\right)\tilde{k}_{m-1}\tilde{f}_{m-1,r}(\lambda)\\&-\frac{1}{B_{m+r-1,r}}\tilde{k}_{m-2}\tilde{f}_{m-2,r}(\lambda),\end{aligned}$$

for $m\ge 2$, where $f_{m,r}(\lambda)=\tilde{k}_m\tilde{f}_{m,r}(\lambda)$, $E_{m+r-1,r}(\lambda)=e_{m+r-1,r,1}\lambda+e_{m+r-1,r,2}$ and $\tilde{f}_{m,r}(\lambda)$ is a monic polynomial. As $\tilde{k}_m=\frac{e_{m+r-1,r,1}}{B_{m+r-1,r}}\tilde{k}_{m-1}\neq 0$, then

$$\begin{aligned}&\tilde{f}_{m,r}(\lambda)=(\lambda-s_m)\tilde{f}_{m-1,r}(\lambda)-t_m\tilde{f}_{m-2,r}(\lambda),\quad m\ge 2,\\&\tilde{f}_{0,r}(\lambda)=1,\qquad \tilde{f}_{1,r}(\lambda)=\lambda+\frac{\|P_r(x)\|^2_{\mu_0}}{(r!)^2\|R_0(x)\|^2_{\mu_1}},\end{aligned}$$

with

$$\begin{aligned}s_m&=-\frac{(\frac{m+r-1}{m-1}a_{m+r-2,r})^2\|P_{m+r-2}(x)\|^2_{\mu_0}+\|P_{m+r-1}(x)\|^2_{\mu_0}}{(\frac{(m+r-1)!}{(m-1)!})^2\|R_{m-1}(x)\|^2_{\mu_1}}\in\mathbb{R},\\t_m&=\frac{(a_{m+r-2,r})^2\|P_{m+r-2}(x)\|^4_{\mu_0}}{(\frac{(m+r-2)!}{(m-2)!})^4\|R_{m-1}(x)\|^2_{\mu_1}\|R_{m-2}(x)\|^2_{\mu_1}}>0.\end{aligned}$$

So, using the Favard Theorem, $\{\tilde{f}_{m,r}(\lambda)\}_{m\ge 0}$ is a SMOP with respect to a positive linear functional and therefore $\{f_{m,r}(\lambda)\}_{m\ge 0}$ is a sequence of orthogonal polynomials associated with a positive Borel measure supported on the real line. □

Now, in the same way, but for (μ_0,μ_1) a $(1,1)$-coherent pair of order r, we will compute an explicit expression for $\{d_{n,r}(\lambda)\}_{n\ge r}$. From (3.10), (3.13), and (3.14) we get

$$\begin{aligned}&\|Q_{n-1}(x;\lambda,r)\|^2_{\lambda,r}\\&=\left\langle Q_{n-1}(x;\lambda,r),P_{n-1}(x)+\frac{n-1}{n-r-1}a_{n-2,r}P_{n-2}(x)\right\rangle_{\mu_0}\\&\quad+\lambda\left\langle Q^{(r)}_{n-1}(x;\lambda,r),P^{(r)}_{n-1}(x)+\frac{n-1}{n-r-1}a_{n-2,r}P^{(r)}_{n-2}(x)\right\rangle_{\mu_1}\end{aligned}$$

$$= \left\| P_{n-1}(x) \right\|_{\mu_0}^2 + \frac{n-1}{n-r-1} a_{n-2,r} \left(\frac{n-1}{n-r-1} a_{n-2,r} - d_{n-2,r}(\lambda) \right) \left\| P_{n-2}(x) \right\|_{\mu_0}^2$$
$$+ \lambda \frac{(n-1)!}{(n-r-1)!} \left[\frac{(n-1)!}{(n-r-1)!} \left(\left\| R_{n-r-1}(x) \right\|_{\mu_1}^2 + b_{n-2,r}^2 \left\| R_{n-r-2}(x) \right\|_{\mu_1}^2 \right) \right.$$
$$\left. - b_{n-2,r} d_{n-2,r}(\lambda) \left\| R_{n-r-2}(x) \right\|_{\mu_1}^2 \right], \quad n \geq r+2. \tag{3.22}$$

Note that if $b_{n-2,r} = 0$, then (3.16) and (3.22) coincide and so $c_{n-2,r}(\lambda) = d_{n-2,r}(\lambda)$, which is the case when there is $(1,0)$-coherence of order r.

Remark 16 If $(\{P_n(x)\}_{n\geq 0}, \{R_n(x)\}_{n\geq 0})$ is a $(1,1)$-coherent pair of order r (respectively, a $(1,0)$-coherent pair of order r), then from $Q_n(x;\lambda,r) = P_n(x)$ for $n \leq r$, (3.22), (3.15) and (3.10) (respectively, (3.16), (3.7) and (3.9)) we can obtain the Sobolev SMOP $\{Q_n(x;\lambda,r)\}_{n\geq 0}$.

Finally, if we substitute (3.22) in (3.15), we get

$$d_{n-1,r}(\lambda) = \frac{F_{n-1,r,1}(\lambda)}{G_{n-1,r,1}(\lambda) - d_{n-2,r}(\lambda) H_{n-1,r,1}(\lambda)}, \quad n \geq r+2, \tag{3.23}$$

where

$$\begin{aligned} F_{n-1,r,1}(\lambda) &= \left[\frac{n!}{(n-r)!} \frac{(n-1)!}{(n-r-1)!} b_{n-1,r} \left\| R_{n-r-1}(x) \right\|_{\mu_1}^2 \right] \lambda \\ &\quad + \frac{n}{n-r} a_{n-1,r} \left\| P_{n-1}(x) \right\|_{\mu_0}^2, \\ G_{n-1,r,1}(\lambda) &= \left[\left(\frac{(n-1)!}{(n-r-1)!} \right)^2 \left(\left\| R_{n-r-1}(x) \right\|_{\mu_1}^2 + b_{n-2,r}^2 \left\| R_{n-r-2}(x) \right\|_{\mu_1}^2 \right) \right] \lambda \\ &\quad + \left\| P_{n-1}(x) \right\|_{\mu_0}^2 + \left(\frac{n-1}{n-r-1} a_{n-2,r} \right)^2 \left\| P_{n-2}(x) \right\|_{\mu_0}^2, \\ H_{n-1,r,1}(\lambda) &= \left[\frac{(n-1)!}{(n-r-1)!} b_{n-2,r} \left\| R_{n-r-2}(x) \right\|_{\mu_1}^2 \right] \lambda \\ &\quad + \frac{n-1}{n-r-1} a_{n-2,r} \left\| P_{n-2}(x) \right\|_{\mu_0}^2, \end{aligned} \tag{3.24}$$

with initial condition

$$d_{r,r}(\lambda) = \frac{[(r+1)!r!b_{r,r} \| R_0(x) \|_{\mu_1}^2]\lambda + (r+1)a_{r,r} \| P_r(x) \|_{\mu_0}^2}{[(r!)^2 \| R_0(x) \|_{\mu_1}^2]\lambda + \| P_r(x) \|_{\mu_0}^2}. \tag{3.25}$$

Note that $F_{n,r,1}(\lambda)$, $G_{n,r,1}(\lambda)$ and $H_{n,r,1}(\lambda)$, for $n \geq r+1$, are polynomials in λ of degree 1 and so $d_{n,r}(\lambda)$ is a rational function in λ for $n \geq r$. Besides, if (μ_0, μ_1) is a $(1,1)$-coherent pair of measures of order r given by (3.13), then from (3.23)–(3.25) we can obtain the sequence $\{d_{n,r}(\lambda)\}_{n\geq r}$ and using (3.10) we can compute the Sobolev orthogonal polynomials $Q_n(x;\lambda,r)$, $n \geq r+1$, without calculating their norms (see Remark 16). For this remember that $Q_n(x;\lambda,r) = P_n(x)$ for $n \leq r$.

Also note that when there is $(1,0)$-coherence of order r, then the decompositions (3.17) and (3.23) coincide.

Remark 17 The sequence $\{d_{n,r}(\lambda)\}_{n\geq r}$, given by Proposition 13 and (3.23), is such that

$$d_{m+r,r}(\lambda) = \frac{g_{m+1,r}(\lambda)}{h_{m+1,r}(\lambda)}, \quad m \geq 0, \tag{3.26}$$

where $g_{m+1,r}(\lambda)$ and $h_{m+1,r}(\lambda)$ are polynomials in λ of degree at most $m+1$. This is easy to check by induction on m.

4 Coherent Pairs of Order r of Linear Functionals

In this section we assume that $\mathcal{U}$ and $\mathcal{V}$ are quasi-definite linear functionals and we study necessary and sufficient conditions for $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-coherent pair of order r, i.e., their respective SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy the algebraic relation

$$P_n^{[r]}(x) + a_{n-1,r}P_{n-1}^{[r]}(x) = R_{n-r}(x) + b_{n-1,r}R_{n-r-1}(x), \quad a_{n-1,r} \neq 0,\ n \geq r+1. \tag{4.1}$$

Many of the results of this section remain valid if $b_{n-1,r} = 0$ for all $n \geq r+1$, i.e., if $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-coherent pair of order r.

From (4.1), we get $\langle \mathcal{V}, P_n^{[r]}(x)\rangle = -a_{n-1,r}\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle$ for $n \geq r+2$[3] and $\langle \mathcal{V}, P_{r+1}^{[r]}(x)\rangle = (b_{r,r} - a_{r,r})\langle \mathcal{V}, 1\rangle$. From now on we assume that $a_{r,r} \neq b_{r,r}$.

Remark 18 If $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order r given by (4.1) then, $a_{r,r} \neq b_{r,r}$ if and only if $P_n^{[r]}(x) \neq R_{n-r}(x)$ for all $n \geq r+1$.

Remark 19 If $(\{P_n(x)\}_{n\geq 0}, \{R_n(x)\}_{n\geq 0})$ is a $(1,1)$-coherent pair of order r given by (4.1), then for $n \geq r+1$

$$\begin{aligned}
P_n^{[r]}(x) &= R_{n-r}(x) + (b_{n-1,r} - a_{n-1,r})R_{n-r-1}(x) \\
&\quad + \sum_{k=2}^{n-r}(-1)^{k-1}a_{n-1,r}a_{n-2,r}\cdots a_{n-k+2,r}a_{n-k+1,r}(b_{n-k,r} - a_{n-k,r})R_{n-r-k}(x) \\
&= R_{n-r}(x) + (b_{n-1,r} - a_{n-1,r})R_{n-r-1}(x) \\
&\quad - a_{n-1,r}(b_{n-2,r} - a_{n-2,r})R_{n-r-2}(x) \\
&\quad + a_{n-1,r}a_{n-2,r}(b_{n-3,r} - a_{n-3,r})R_{n-r-3}(x) \\
&\quad + \cdots + (-1)^{n-r}a_{n-1,r}a_{n-2,r}\cdots a_{r+1,r}(b_{r,r} - a_{r,r})R_0(x).
\end{aligned}$$

Lemma 20 *If $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order r given by (4.1), then there exists a monic polynomial $\gamma_{n,r}(x)$ of degree $n \geq 1$ such that*

$$\left\langle \gamma_{n,r}(x)\mathcal{V}, P_{m+r}^{[r]}(x)\right\rangle = 0, \quad m \geq n+1.$$

If $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-coherent pair of order r, this equality holds for $m \geq n$.

[3] If $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-coherent pair of order r, this equality holds for $n \geq r+1$.

Proof Let

$$\gamma_{n,r}(x) = R_n(x) + \sum_{j=0}^{n-1} A_{j,n,r} R_j(x).$$

From Remark 19 we get

$$\begin{aligned}
&\langle \gamma_{n,r}(x)\mathcal{V}, P_{n+r+1}^{[r]}(x)\rangle \\
&\quad = (b_{n+r,r} - a_{n+r,r})\langle \mathcal{V}, R_n^2(x)\rangle \\
&\qquad - a_{n+r,r}(b_{n+r-1,r} - a_{n+r-1,r})A_{n-1,n,r}\langle \mathcal{V}, R_{n-1}^2(x)\rangle \\
&\qquad + \cdots + (-1)^n a_{n+r,r}a_{n+r-1,r}\cdots a_{r+2,r}a_{r+1,r}(b_{r,r} - a_{r,r})\, A_{0,n,r}\langle \mathcal{V}, R_0(x)\rangle.
\end{aligned}$$

Thus, since $b_{r,r} \neq a_{r,r}$, we can choose real numbers $A_{0,n,r}, \ldots, A_{n-1,n,r}$, not all zero, such that $\langle \gamma_{n,r}(x)\mathcal{V}, P_{n+r+1}^{[r]}(x)\rangle = 0$, for $n \geq 1$. Also, from (4.1)

$$\langle \gamma_{n,r}(x)\mathcal{V}, P_{m+r+1}^{[r]}(x)\rangle = -a_{m+r,r}\langle \gamma_{n,r}(x)\mathcal{V}, P_{m+r}^{[r]}(x)\rangle, \quad n \leq m-1.$$

Therefore, $\langle \gamma_{n,r}(x)\mathcal{V}, P_{m+r}^{[r]}(x)\rangle = 0$, for $m \geq n+1$. □

Remark 21 In Lemma 20, we can choose $A_{1,n,r} = A_{2,n,r} = \cdots = A_{n-1,n,r} = 0$ and

$$A_{0,n,r} = \frac{(-1)^{n+1}(b_{n+r,r} - a_{n+r,r})\langle \mathcal{V}, R_n^2(x)\rangle}{a_{n+r,r}a_{n+r-1,r}\cdots a_{r+2,r}a_{r+1,r}(b_{r,r} - a_{r,r})\langle \mathcal{V}, 1\rangle}.$$

So, $\gamma_{n,r}(x) = R_n(x) + A_{0,n,r}$, for $n \geq 1$.

Lemma 22 *If $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order r given by (4.1), then*

$$D^r[\gamma_{n,r}(x)\mathcal{V}] = (-1)^r \varphi_{n+r,r}(x)\mathcal{U}, \quad n \geq 1, \tag{4.2}$$

where $\gamma_{n,r}(x)$ is a monic polynomial of degree n and $\varphi_{n+r,r}(x)$ is a polynomial of degree at most $n+r$.[4]

Moreover,

$$\varphi_{n+r,r}(x) = \sum_{k=0}^{n} \frac{(k+r)!}{k!} \frac{\langle \gamma_{n,r}(x)\mathcal{V}, P_{k+r}^{[r]}(x)\rangle}{\langle \mathcal{U}, P_{k+r}^2(x)\rangle} P_{k+r}(x), \quad n \geq 1. \tag{4.3}$$

Proof Let $\gamma_{n,r}(x)$ be the polynomial introduced in Lemma 20. Let $\{\wp_n\}_{n\geq 0}$ be the dual basis of the SMOP $\{P_n(x)\}_{n\geq 0}$, and let $\{\wp_{n+r}^{[r]}\}_{n\geq 0}$ be the dual basis of the monic polynomials $\{P_{n+r}^{[r]}(x)\}_{n\geq 0}$. Since $\gamma_{n,r}(x)\mathcal{V} = \sum_{k\geq 0} \lambda_{k+r,n,r}\wp_{k+r}^{[r]}$, where $\lambda_{k+r,n,r} = \langle \gamma_{n,r}(x)\mathcal{V}, P_{k+r}^{[r]}(x)\rangle$, and, from Lemma 20, $\lambda_{k+r,n,r} = 0$ for $k \geq n+1$ and $n \geq 1$. Thus, $\gamma_{n,r}(x)\mathcal{V} = \sum_{k=0}^{n} \lambda_{k+r,n,r}\wp_{k+r}^{[r]}$ and, as a consequence,

$$D^r[\gamma_{n,r}(x)\mathcal{V}] = \sum_{k=0}^{n} \lambda_{k+r,n,r} D^r[\wp_{k+r}^{[r]}] \overset{(2.9)}{=} \sum_{k=0}^{n} \lambda_{k+r,n,r}(-1)^r \frac{(k+r)!}{k!}\wp_{k+r}$$

[4] If $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-coherent pair of order r, then the polynomial $\varphi_{n+r,r}(x)$ has degree at most $n+r-1$ and its expression corresponds to $\varphi_{n+r-1,r}(x)$ in (4.3).

$$\overset{(2.8)}{=} (-1)^r \sum_{k=0}^{n} \lambda_{k+r,n,r} \frac{(k+r)!}{k!} \frac{P_{k+r}(x)}{\langle \mathcal{U}, P_{k+r}^2(x) \rangle} \mathcal{U},$$

for $n \geq 1$. So, if we denote $\varphi_{n+r,r}(x) = \sum_{k=0}^{n} \lambda_{k+r,n,r} \frac{(k+r)!}{k!} \frac{P_{k+r}(x)}{\langle \mathcal{U}, P_{k+r}^2(x) \rangle}$, then the proof is completed. □

The following theorem states that if $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$-coherent pair of order r, then the linear functionals $\mathcal{U}$, $\mathcal{V}$, $D\mathcal{V}$ are related by a expression of rational type.

Theorem 23 *If $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$-coherent pair of order r given by (4.1), then for $r \in \mathbb{N}$*

(1) *There exist polynomials $\tilde{\sigma}_{r+1}(x)$ and $\tilde{\tau}_{r+1}(x)$ such that*

$$\tilde{\sigma}_{r+1}(x)\mathcal{V} = \tilde{\tau}_{r+1}(x)\mathcal{U}, \tag{4.4}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}(x)\big) = 2^r \quad \textit{and} \quad \deg\big(\tilde{\tau}_{r+1}(x)\big) \leq 2^r + 2r.$$

If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\tilde{\tau}_{r+1}(x)) \leq 2^r + 2r - 1$.

(2) *For $r \geq 1$, there exist polynomials $\tilde{\sigma}_{r+1}^{D\mathcal{V},\mathcal{U}}(x)$ and $\tilde{\tau}_{r+1}^{D\mathcal{V},\mathcal{U}}(x)$ such that*

$$\tilde{\sigma}_{r+1}^{D\mathcal{V},\mathcal{U}}(x) D\mathcal{V} = \tilde{\tau}_{r+1}^{D\mathcal{V},\mathcal{U}}(x)\mathcal{U}, \tag{4.5}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}^{D\mathcal{V},\mathcal{U}}(x)\big) = 2^{r-1} + 1 \quad \textit{and} \quad \deg\big(\tilde{\tau}_{r+1}^{D\mathcal{V},\mathcal{U}}(x)\big) \leq 2^{r-1} + 2r.$$

If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\tilde{\tau}_{r+1}^{D\mathcal{V},\mathcal{U}}(x)) \leq 2^{r-1} + 2r - 1$.

(3) *For $r \geq 2$, there exist polynomials $\tilde{\sigma}_{r+1}^{D^2\mathcal{V},\mathcal{U}}(x)$ and $\tilde{\tau}_{r+1}^{D^2\mathcal{V},\mathcal{U}}(x)$ such that*

$$\tilde{\sigma}_{r+1}^{D^2\mathcal{V},\mathcal{U}}(x) D^2\mathcal{V} = \tilde{\tau}_{r+1}^{D^2\mathcal{V},\mathcal{U}}(x)\mathcal{U}, \tag{4.6}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}^{D^2\mathcal{V},\mathcal{U}}(x)\big) = 2^{r-2} + 2 \quad \textit{and} \quad \deg\big(\tilde{\tau}_{r+1}^{D^2\mathcal{V},\mathcal{U}}(x)\big) \leq 2^{r-2} + 2r.$$

If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\tilde{\tau}_{r+1}^{D^2\mathcal{V},\mathcal{U}}(x)) \leq 2^{r-2} + 2r - 1$.

(4) *For $r \geq 3$, there exist polynomials $\tilde{\sigma}_{r+1}^{D^3\mathcal{V},\mathcal{U}}(x)$ and $\tilde{\tau}_{r+1}^{D^3\mathcal{V},\mathcal{U}}(x)$ such that*

$$\tilde{\sigma}_{r+1}^{D^3\mathcal{V},\mathcal{U}}(x) D^3\mathcal{V} = \tilde{\tau}_{r+1}^{D^3\mathcal{V},\mathcal{U}}(x)\mathcal{U}, \tag{4.7}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}^{D^3\mathcal{V},\mathcal{U}}(x)\big) = 2^{r-3} + 3 \quad \textit{and} \quad \deg\big(\tilde{\tau}_{r+1}^{D^3\mathcal{V},\mathcal{U}}(x)\big) \leq 2^{r-3} + 2r.$$

If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\tilde{\tau}_{r+1}^{D^3\mathcal{V},\mathcal{U}}(x)) \leq 2^{r-3} + 2r - 1$.

(5) *For $r \geq 1$, there exist polynomials $\tilde{\sigma}_{r+1}^{\mathcal{V},D\mathcal{V}}(x)$ and $\tilde{\tau}_{r+1}^{\mathcal{V},D\mathcal{V}}(x)$ such that*

$$\tilde{\tau}_{r+1}^{\mathcal{V},D\mathcal{V}}(x) D\mathcal{V} = \tilde{\sigma}_{r+1}^{\mathcal{V},D\mathcal{V}}(x)\mathcal{V}, \tag{4.8}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}^{\mathcal{V},D\mathcal{V}}(x)\big) \le 2^r + 2r - 1 \quad \text{and} \quad \deg\big(\tilde{\tau}_{r+1}^{\mathcal{V},D\mathcal{V}}(x)\big) \le 2^r + 2r.$$

If $(\mathcal{U}, \mathcal{V})$ *is a* $(1,0)$*-coherent pair of order* r, *then* $\deg(\tilde{\sigma}_{r+1}^{\mathcal{V},D\mathcal{V}}(x)) \le 2^r + 2r - 2$ *and* $\deg(\tilde{\tau}_{r+1}^{\mathcal{V},D\mathcal{V}}(x)) \le 2^r + 2r - 1$.

(6) *For* $r \ge 2$, *there exist polynomials* $\tilde{\sigma}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x)$ *and* $\tilde{\tau}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x)$ *such that*

$$\tilde{\tau}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x) D^2\mathcal{V} = \tilde{\sigma}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x)\mathcal{V}, \tag{4.9}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x)\big) \le 2^{r-1} + 2r - 1 \quad \text{and} \quad \deg\big(\tilde{\tau}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x)\big) \le 2^{r-1} + 2r + 1.$$

If $(\mathcal{U}, \mathcal{V})$ *is a* $(1,0)$*-coherent pair of order* r, *then* $\deg(\tilde{\sigma}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x)) \le 2^{r-1} + 2r - 2$ *and* $\deg(\tilde{\tau}_{r+1}^{\mathcal{V},D^2\mathcal{V}}(x)) \le 2^{r-1} + 2r$.

(7) *For* $r \ge 3$, *there exist polynomials* $\tilde{\sigma}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x)$ *and* $\tilde{\tau}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x)$ *such that*

$$\tilde{\tau}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x) D^3\mathcal{V} = \tilde{\sigma}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x)\mathcal{V}, \tag{4.10}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x)\big) \le 2^{r-2} + 2r - 1 \quad \text{and} \quad \deg\big(\tilde{\tau}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x)\big) \le 2^{r-2} + 2r + 2.$$

If $(\mathcal{U}, \mathcal{V})$ *is a* $(1,0)$*-coherent pair of order* r, *then* $\deg(\tilde{\sigma}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x)) \le 2^{r-2} + 2r - 2$ *and* $\deg(\tilde{\tau}_{r+1}^{\mathcal{V},D^3\mathcal{V}}(x)) \le 2^{r-2} + 2r + 1$.

Proof From Lemma 22 and Proposition 1 we get

$$\sum_{k=0}^{r} \binom{r}{k} \gamma_{n,r}^{(k)}(x) D^{r-k}\mathcal{V} = D^r\big[\gamma_{n,r}(x)\mathcal{V}\big] = (-1)^r \varphi_{n+r,r}(x)\mathcal{U}, \quad n \ge 1,$$

where $\gamma_{n,r}(x)$ is a monic polynomial of degree n and $\varphi_{n+r,r}(x)$ is a polynomial of degree at most $n+r$. In order to simplify the notation we write $\gamma_n^{(k)}$ and φ_{r+n} instead of $\gamma_{n,r}^{(k)}(x)$ and $\varphi_{r+n,r}(x)$, respectively. Then, taking $n = 1, 2, \ldots, r+1$ we obtain the system of linear equations

$$\Gamma \mathbf{d} = (-1)^r \Phi \mathcal{U}, \tag{4.11}$$

where Γ is a matrix of size $(r+1) \times (r+1)$, $\mathbf{d}$ and Φ are vectors of size $(r+1) \times 1$ such that

$$\mathbf{d} = \begin{pmatrix} D^r\mathcal{V} \\ D^{r-1}\mathcal{V} \\ \vdots \\ D\mathcal{V} \\ \mathcal{V} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \varphi_{r+1} \\ \varphi_{r+2} \\ \vdots \\ \varphi_{2r} \\ \varphi_{2r+1} \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} \gamma_1 & r1! & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \gamma_2 & r\gamma_2^{(1)} & \binom{r}{2}2! & 0 & \cdots & 0 & 0 & 0 & 0 \\ \gamma_3 & r\gamma_3^{(1)} & \binom{r}{2}\gamma_3^{(2)} & \binom{r}{3}3! & \cdots & 0 & 0 & 0 & 0 \\ \gamma_4 & r\gamma_4^{(1)} & \binom{r}{2}\gamma_4^{(2)} & \binom{r}{3}\gamma_4^{(3)} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{r-2} & r\gamma_{r-2}^{(1)} & \binom{r}{2}\gamma_{r-2}^{(2)} & \binom{r}{3}\gamma_{r-2}^{(3)} & \cdots & \binom{r}{r-3}\gamma_{r-2}^{(r-3)} & \binom{r}{r-2}(r-2)! & 0 & 0 \\ \gamma_{r-1} & r\gamma_{r-1}^{(1)} & \binom{r}{2}\gamma_{r-1}^{(2)} & \binom{r}{3}\gamma_{r-1}^{(3)} & \cdots & \binom{r}{r-3}\gamma_{r-1}^{(r-3)} & \binom{r}{r-2}\gamma_{r-1}^{(r-2)} & r(r-1)! & 0 \\ \gamma_{r} & r\gamma_{r}^{(1)} & \binom{r}{2}\gamma_{r}^{(2)} & \binom{r}{3}\gamma_{r}^{(3)} & \cdots & \binom{r}{r-3}\gamma_{r}^{(r-3)} & \binom{r}{r-2}\gamma_{r}^{(r-2)} & r\gamma_r^{(r-1)} & r! \\ \gamma_{r+1} & r\gamma_{r+1}^{(1)} & \binom{r}{2}\gamma_{r+1}^{(2)} & \binom{r}{3}\gamma_{r+1}^{(3)} & \cdots & \binom{r}{r-3}\gamma_{r+1}^{(r-3)} & \binom{r}{r-2}\gamma_{r+1}^{(r-2)} & r\gamma_{r+1}^{(r-1)} & \gamma_{r+1}^{(r)} \end{pmatrix}$$

(1) Solving (4.11) for $\mathcal{V}$ and $\mathcal{U}$ we get (4.4) where

$$\tilde{\sigma}_{r+1}(x) = \sigma_{r+1,r+1}(x) \quad \text{and} \quad \tilde{\tau}_{r+1}(x) = \tau_{r+1,r+1}(x),$$

with

$$\begin{aligned} \sigma_{n,k} &= \binom{r}{k-1}\sigma_{00}\sigma_{11}\sigma_{22}\cdots\sigma_{k-2,k-2}\big(\gamma_n^{(k-1)}\sigma_{k-1,k-1} - (k-1)!\,\sigma_{n,k-1}\big), \\ \tau_{n,k} &= \sigma_{k-1,k-1}\tau_{n,k-1} - \sigma_{n,k-1}\tau_{k-1,k-1}, \end{aligned} \tag{4.12}$$

for $k = 2, \ldots, r+1$ and $n = k, \ldots, r+1$, and initial conditions

$$\sigma_{00} = 1, \qquad \sigma_{11} = \gamma_1, \qquad \sigma_{n1} = \gamma_n, \qquad \tau_{11} = (-1)^r\varphi_{r+1}, \qquad \tau_{n1} = (-1)^r\varphi_{r+n}. \tag{4.13}$$

(2) For $r \geq 1$, solving (4.11) for $D\mathcal{V}$ and $\mathcal{U}$ we obtain (4.5) where

$$\tilde{\sigma}_{r+1}^{D\mathcal{V},\mathcal{U}} = r!\,\sigma_{r+1,r} - \gamma_{r+1}^{(r)}\sigma_{r,r} \quad \text{and} \quad \tilde{\tau}_{r+1}^{D\mathcal{V},\mathcal{U}} = r!\,\tau_{r+1,r} - \gamma_{r+1}^{(r)}\tau_{r,r},$$

with $\sigma_{n,k}$ and $\tau_{n,k}$ given by (4.13) and (4.12) for $k = 2, \ldots, r$ and $n = k, \ldots, r+1$.

(3) For $r \geq 2$, solving (4.11) for $D^2\mathcal{V}$ and $\mathcal{U}$ we get (4.6) where

$$\begin{aligned} \tilde{\sigma}_{r+1}^{D^2\mathcal{V},\mathcal{U}} &= r!\big((r-1)!\sigma_{r+1,r-1} - \gamma_{r+1}^{(r-1)}\sigma_{r-1,r-1}\big) \\ &\quad - \gamma_{r+1}^{(r)}\big((r-1)!\sigma_{r,r-1} - \gamma_r^{(r-1)}\sigma_{r-1,r-1}\big) \\ \tilde{\tau}_{r+1}^{D^2\mathcal{V},\mathcal{U}} &= r!\big((r-1)!\tau_{r+1,r-1} - \gamma_{r+1}^{(r-1)}\tau_{r-1,r-1}\big) \\ &\quad - \gamma_{r+1}^{(r)}\big((r-1)!\tau_{r,r-1} - \gamma_r^{(r-1)}\tau_{r-1,r-1}\big), \end{aligned}$$

with $\sigma_{n,k}$ and $\tau_{n,k}$ given by (4.13) and (4.12) for $k = 2, \ldots, r-1$ and $n = k, \ldots, r+1$.

(4) For $r \geq 3$, solving (4.11) for $D^3\mathcal{V}$ and $\mathcal{U}$ we obtain (4.7) where

$$\begin{aligned} \tilde{\sigma}_{r+1}^{D^3\mathcal{V},\mathcal{U}} &= r!\big((r-1)!\big[(r-2)!\sigma_{r+1,r-2} - \gamma_{r+1}^{(r-2)}\sigma_{r-2,r-2}\big] \\ &\quad - \gamma_{r+1}^{(r-1)}\big[(r-2)!\sigma_{r-1,r-2} - \gamma_{r-1}^{(r-2)}\sigma_{r-2,r-2}\big]\big) \\ &\quad - \gamma_{r+1}^{(r)}\big((r-1)!\big[(r-2)!\sigma_{r,r-2} - \gamma_r^{(r-2)}\sigma_{r-2,r-2}\big] \\ &\quad - \gamma_r^{(r-1)}\big[(r-2)!\sigma_{r-1,r-2} - \gamma_{r-1}^{(r-2)}\sigma_{r-2,r-2}\big]\big), \end{aligned}$$

$$\begin{aligned}\tilde{\tau}_{r+1}^{D^3\mathcal{V},\mathcal{U}} &= r!\big((r-1)!\big[(r-2)!\tau_{r+1,r-2}-\gamma_{r+1}^{(r-2)}\tau_{r-2,r-2}\big]\\&\quad-\gamma_{r+1}^{(r-1)}\big[(r-2)!\tau_{r-1,r-2}-\gamma_{r-1}^{(r-2)}\tau_{r-2,r-2}\big]\big)\\&\quad-\gamma_{r+1}^{(r)}\big((r-1)!\big[(r-2)!\tau_{r,r-2}-\gamma_{r}^{(r-2)}\tau_{r-2,r-2}\big]\\&\quad-\gamma_{r}^{(r-1)}\big[(r-2)!\tau_{r-1,r-2}-\gamma_{r-1}^{(r-2)}\tau_{r-2,r-2}\big]\big),\end{aligned}$$

with $\sigma_{n,k}$ and $\tau_{n,k}$ given by (4.13) and (4.12) for $k=2,\dots,r-2$ and $n=k,\dots,r+1$.

(5) For $r\geq 1$, solving (4.11) for $\mathcal{V}$ and $D\mathcal{V}$ we get (4.8) where

$$\begin{aligned}\tilde{\sigma}_{r+1}^{\mathcal{V},D\mathcal{V}} &= \sigma_{00}\sigma_{11}\sigma_{22}\cdots\sigma_{r-1,r-1}\big[\tau_{r,r}\gamma_{r+1}^{(r)}-r!\,\tau_{r+1,r}\big],\\ \tilde{\tau}_{r+1}^{\mathcal{V},D\mathcal{V}} &= \tau_{r+1,r}\sigma_{r,r}-\tau_{r,r}\sigma_{r+1,r},\end{aligned}$$

with $\sigma_{n,k}$ and $\tau_{n,k}$ given by (4.13) and (4.12) for $k=2,\dots,r$ and $n=k,\dots,r+1$.

(6) For $r\geq 2$, solving (4.11) for $\mathcal{V}$ and $D^2\mathcal{V}$ we obtain (4.9) where

$$\begin{aligned}\tilde{\sigma}_{r+1}^{\mathcal{V},D^2\mathcal{V}} &= (r-1)!\sigma_{00}\sigma_{11}\sigma_{22}\cdots\sigma_{r-2,r-2}\\&\quad\times\big(\gamma_{r+1}^{(r)}\big[\gamma_{r}^{(r-1)}\tau_{r-1,r-1}-(r-1)!\tau_{r,r-1}\big]\\&\quad-r!\big[\gamma_{r+1}^{(r-1)}\tau_{r-1,r-1}-(r-1)!\tau_{r+1,r-1}\big]\big),\\ \tilde{\tau}_{r+1}^{\mathcal{V},D^2\mathcal{V}} &= \big[\gamma_{r+1}^{(r-1)}\tau_{r-1,r-1}-(r-1)!\tau_{r+1,r-1}\big]\big[(r-1)!\sigma_{r,r-1}-\gamma_{r}^{(r-1)}\sigma_{r-1,r-1}\big]\\&\quad-\big[\gamma_{r}^{(r-1)}\tau_{r-1,r-1}-(r-1)!\tau_{r,r-1}\big]\big[(r-1)!\sigma_{r+1,r-1}-\gamma_{r+1}^{(r-1)}\sigma_{r-1,r-1}\big],\end{aligned}$$

with $\sigma_{n,k}$ and $\tau_{n,k}$ given by (4.13) and (4.12) for $k=2,\dots,r-1$ and $n=k,\dots,r+1$.

(7) For $r\geq 3$, solving (4.11) for $\mathcal{V}$ and $D^3\mathcal{V}$ we get (4.10) where

$$\begin{aligned}\tilde{\sigma}_{r+1}^{\mathcal{V},D^3\mathcal{V}} &= (r-1)!(r-2)!\sigma_{00}\cdots\sigma_{r-3,r-3}\big\{\gamma_{r+1}^{(r)}\big((r-1)!\big[\gamma_{r}^{(r-2)}\tau_{r-2,r-2}-(r-2)!\tau_{r,r-2}\big]\\&\quad-\gamma_{r}^{(r-1)}\big[\gamma_{r-1}^{(r-2)}\tau_{r-2,r-2}-(r-2)!\tau_{r-1,r-2}\big]\big)-r!\big((r-1)!\big[\gamma_{r+1}^{(r-2)}\tau_{r-2,r-2}\\&\quad-(r-2)!\tau_{r+1,r-2}\big]-\gamma_{r+1}^{(r-1)}\big[\gamma_{r-1}^{(r-2)}\tau_{r-2,r-2}-(r-2)!\tau_{r-1,r-2}\big]\big)\big\},\\ \tilde{\tau}_{r+1}^{\mathcal{V},D^3\mathcal{V}} &= \big((r-1)!\big[\gamma_{r}^{(r-2)}\tau_{r-2,r-2}-(r-2)!\tau_{r,r-2}\big]-\gamma_{r}^{(r-1)}\big[\gamma_{r-1}^{(r-2)}\tau_{r-2,r-2}\\&\quad-(r-2)!\tau_{r-1,r-2}\big]\big)\big((r-1)!\big[(r-2)!\sigma_{r+1,r-2}-\gamma_{r+1}^{(r-2)}\sigma_{r-2,r-2}\big]-\gamma_{r+1}^{(r-1)}\\&\quad\times\big[(r-2)!\sigma_{r-1,r-2}-\gamma_{r-1}^{(r-2)}\sigma_{r-2,r-2}\big]\big)-\big((r-1)!\big[\gamma_{r+1}^{(r-2)}\tau_{r-2,r-2}\\&\quad-(r-2)!\tau_{r+1,r-2}\big]-\gamma_{r+1}^{(r-1)}\big[\gamma_{r-1}^{(r-2)}\tau_{r-2,r-2}-(r-2)!\tau_{r-1,r-2}\big]\big)\big((r-1)!\\&\quad\times\big[(r-2)!\sigma_{r,r-2}-\gamma_{r}^{(r-2)}\sigma_{r-2,r-2}\big]\\&\quad-\gamma_{r}^{(r-1)}\big[(r-2)!\sigma_{r-1,r-2}-\gamma_{r-1}^{(r-2)}\sigma_{r-2,r-2}\big]\big),\end{aligned}$$

with $\sigma_{n,k}$ and $\tau_{n,k}$ given by (4.13) and (4.12) for $k=2,\dots,r-2$ and $n=k,\dots,r+1$.

Finally, the degrees of all polynomials are calculated using the respective recursive formulas and taking into account that $\gamma_{n,r}(x)$ is a monic polynomial of degree n and $\deg(\varphi_{n+r,r}(x))\leq n+r$. We obtain for $k=1,\dots,r+1$ and $n=k,\dots,r+1$

$$\deg\big(\sigma_{nk}(x)\big)=n+2^{k-1}-k, \tag{4.14}$$

$$\deg\bigl(\tau_{nk}(x)\bigr) \leq n + r + 2^{k-1} - 1. \tag{4.15}$$

Note that if $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\tau_{nk}(x)) \leq n + r + 2^{k-1} - 2$, because $\deg(\varphi_{n+r,r}(x)) \leq n + r - 1$. □

Corollary 24 *Let $r \in \mathbb{N}$. If $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$-coherent pair of order r given by (4.1), then there exist polynomials $\sigma_r(x)$, $\pi_r(x)$ and $\tau_r(x)$ such that*

$$\begin{aligned} \sigma_r(x)\mathcal{V} &= \tau_r(x)\mathcal{U}, \\ \sigma_r(x)D\mathcal{V} &= \pi_r(x)\mathcal{U}, \\ \tau_r(x)D\mathcal{V} &= \pi_r(x)\mathcal{V}, \end{aligned} \tag{4.16}$$

with

$$\deg\bigl(\sigma_r(x)\bigr) = 2^r, \qquad \deg\bigl(\pi_r(x)\bigr) \leq 2r + 2^r - 1, \qquad \deg\bigl(\tau_r(x)\bigr) \leq 2r + 2^r. \tag{4.17}$$

If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\pi_r(x)) \leq 2r + 2^r - 2$ and $\deg(\tau_r(x)) \leq 2r + 2^r - 1$.

Proof From the proof of Theorem 23 it is easy to check that (4.16) holds if

$$\sigma_r(x) = \sigma_{r+1,r+1}(x), \qquad \tau_r(x) = \tau_{r+1,r+1}(x),$$
$$\pi_r(x) = \sigma_{00}(x)\sigma_{11}(x)\sigma_{22}(x)\cdots\sigma_{r-1,r-1}(x)\bigl[\tau_{r,r}(x)\gamma_{r+1}^{(r)}(x) - r!\,\tau_{r+1,r}(x)\bigr],$$

where, for $k = 2, \dots, r$ and $n = k, \dots, r + 1$, $\sigma_{n,k}$ and $\tau_{n,k}$ are given by (4.13) and (4.12), and their degrees are given by (4.14) and (4.15). □

From the previous theorem we can state the following

Conjecture *If $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$-coherent pair of order r given by (4.1), then for $r, k \in \mathbb{N}$*

- *There exist polynomials $\tilde{\sigma}_{r+1}(x)$ and $\tilde{\tau}_{r+1}(x)$ such that*

$$\tilde{\sigma}_{r+1}(x)\mathcal{V} = \tilde{\tau}_{r+1}(x)\mathcal{U}, \tag{4.18}$$

  with

$$\deg\bigl(\tilde{\sigma}_{r+1}(x)\bigr) = 2^r \quad \text{and} \quad \deg\bigl(\tilde{\tau}_{r+1}(x)\bigr) \leq 2^r + 2r.$$

  If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\tilde{\tau}_{r+1}(x)) \leq 2^r + 2r - 1$.
- *For $k \leq r$, there exist polynomials $\tilde{\sigma}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)$ and $\tilde{\tau}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)$ such that*

$$\tilde{\sigma}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)D^k\mathcal{V} = \tilde{\tau}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)\mathcal{U}, \tag{4.19}$$

  with

$$\deg\bigl(\tilde{\sigma}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)\bigr) = 2^{r-k} + k \quad \text{and} \quad \deg\bigl(\tilde{\tau}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)\bigr) \leq 2^{r-k} + 2r.$$

  If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-coherent pair of order r, then $\deg(\tilde{\tau}_{r+1}^{D^k\mathcal{V},\mathcal{U}}(x)) \leq 2^{r-k} + 2r - 1$.

- *For $k \leq r$, there exist polynomials $\tilde{\sigma}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)$ and $\tilde{\tau}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)$ such that*

$$\tilde{\tau}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)D^k\mathcal{V} = \tilde{\sigma}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)\mathcal{V}, \tag{4.20}$$

with

$$\deg\big(\tilde{\sigma}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)\big) \leq 2^{r-(k-1)} + 2r - 1 \quad \textit{and} \quad \deg\big(\tilde{\tau}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)\big) \leq 2^{r-(k-1)} + 2r + k - 1.$$

If $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-coherent pair of order r, then $\deg(\tilde{\sigma}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)) \leq 2^{r-(k-1)} + 2r - 2$ and $\deg(\tilde{\tau}_{r+1}^{\mathcal{V},D^k\mathcal{V}}(x)) \leq 2^{r-(k-1)} + 2r + k - 2$.

When one of the quasi-definite linear functionals is classical, next we analyze its companion coherent measure. From Proposition 4, Corollary 6 and Theorem 25, we state in Corollary 27 that if $\mathcal{U}$ is a classical linear functional and $\mathcal{V}$ is a quasi-definite linear functional, then $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-coherent pair of order r (with $a_{r,r} \neq b_{r,r}$ and $a_{n,r}b_{n,r} \neq 0$ for $n \geq r$) if and only if $\mathcal{U}$ and $\mathcal{V}$ are related by a expression of rational type and, therefore, $\mathcal{V}$ is a semiclassical linear functional of class at most 2. The general result is as follows

Theorem 25 (See [1]) *Let $\{P_n^{[r]}(x)\}_{n\geq r}$ and $\{R_n(x)\}_{n\geq 0}$ be two SMOP with respect to the linear functionals $\mathcal{U}^{[r]}$ and $\mathcal{V}$, normalized by $\langle\mathcal{U}^{[r]}, 1\rangle = 1 = \langle\mathcal{V}, 1\rangle$, $r \in \mathbb{N}$. The following statements are equivalent*

(i) *There exist sequences $\{a_{n,r}\}_{n\geq r}$ and $\{b_{n,r}\}_{n\geq r}$ with $a_{r,r} \neq b_{r,r}$ and $a_{n,r}b_{n,r} \neq 0$, $n \geq r$, such that $\{P_n^{[r]}(x)\}_{n\geq r}$ and $\{R_n(x)\}_{n\geq 0}$ are connected by*

$$P_n^{[r]}(x) + a_{n-1,r}P_{n-1}^{[r]}(x) = R_{n-r}(x) + b_{n-1,r}R_{n-r-1}(x), \quad a_{n-1,r} \neq 0,\ n \geq r+1, \tag{4.1}$$

i.e., $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-coherent pair of order r, $a_{r,r} \neq b_{r,r}$ and $a_{n,r}b_{n,r} \neq 0$ for $n \geq r$.

(ii) *$P_n^{[r]}(x) \neq R_{n-r}(x)$ for $n \geq r+1$, and there exist constants ξ, $C^{P^{[r]}}$ and C^R such that*

$$\big(x - C^{P^{[r]}}\big)\mathcal{U}^{[r]} = \xi\big(x - C^R\big)\mathcal{V}. \tag{4.21}$$

Proof (i) $\Longrightarrow$ (ii): From (4.1) we get

$$\begin{aligned}\big\langle\mathcal{V}, P_n^{[r]}(x)\big\rangle &= (-1)a_{n-1,r}\big\langle\mathcal{V}, P_{n-1}^{[r]}(x)\big\rangle = (-1)^2 a_{n-1,r}a_{n-2,r}\big\langle\mathcal{V}, P_{n-2}^{[r]}(x)\big\rangle = \cdots \\ &= (-1)^{n-r}a_{n-1,r}a_{n-2,r}\cdots a_{r+1,r}(b_{r,r} - a_{r,r}), \quad n \geq r+2,\end{aligned}$$

and $\langle\mathcal{V}, P_{r+1}^{[r]}(x)\rangle = b_{r,r} - a_{r,r}$. Then, $\langle\mathcal{V}, P_n^{[r]}(x)\rangle \neq 0$ for $n \geq r+1$, and therefore, $P_n^{[r]}(x) \neq R_{n-r}(x)$ for $n \geq r+1$.

Now, we consider the respective TTRR of the SMOP $\{P_n^{[r]}(x)\}_{n\geq r}$ and $\{R_n(x)\}_{n\geq 0}$

$$\begin{aligned}&P_n^{[r]}(x) = \big(x - \alpha_n^{P^{[r]}}\big)P_{n-1}^{[r]}(x) - \beta_n^{P^{[r]}}P_{n-2}^{[r]}(x) \quad n \geq r+2, \\ &P_{r+1}^{[r]}(x) = x - \alpha_{r+1}^{P^{[r]}}, \qquad P_r^{[r]}(x) = 1,\end{aligned} \tag{4.22}$$

$$\begin{aligned}&R_n(x) = \big(x - \alpha_n^R\big)R_{n-1}(x) - \beta_n^R R_{n-2}(x) \quad n \geq 2, \\ &R_1(x) = x - \alpha_1^R, \qquad R_0(x) = 1.\end{aligned} \tag{4.23}$$

Let A be a constant given by

$$A = \frac{\beta_{r+2}^{P^{[r]}}(a_{r+1,r} - b_{r+1,r})}{b_{r+1,r}(a_{r,r} - b_{r,r})} - \alpha_{r+1}^{P^{[r]}}. \tag{4.24}$$

We can express the linear functional $(x + A)\mathcal{U}^{[r]}$ as $(x + A)\mathcal{U}^{[r]} = \sum_{j\geq 0} \rho_j \frac{R_j(x)}{\|R_j(x)\|_{\mathcal{V}}^2}\mathcal{V}$, where $\rho_j = \langle (x + A)\mathcal{U}^{[r]}, R_j(x)\rangle$. We will show by induction on j that

$$\rho_j = \big\langle (x + A)\mathcal{U}^{[r]}, R_j(x)\big\rangle = 0, \quad j \geq 2. \tag{4.25}$$

From (4.22) for $n = r + 1$ and $n = r + 2$ we get

$$\big\langle \mathcal{U}^{[r]}, \big(P_{r+1}^{[r]}(x)\big)^2\big\rangle = \big\| P_{r+1}^{[r]}(x)\big\|_{\mathcal{U}^{[r]}}^2 = \beta_{r+2}^{P^{[r]}}. \tag{4.26}$$

So, from (4.1), for $n = r + 1$ and $n = r + 2$, and (4.22), for $n = r + 1$, we obtain

$$\begin{aligned} \rho_2 &= \big\langle \mathcal{U}^{[r]}, \big[P_{r+1}^{[r]}(x) + \alpha_{r+1}^{P^{[r]}} + A\big] \\ &\quad \times \big[P_{r+2}^{[r]}(x) + a_{r+1,r}P_{r+1}^{[r]}(x) - b_{r+1,r}\big(P_{r+1}^{[r]}(x) + a_{r,r} - b_{r,r}\big)\big]\big\rangle \\ &\overset{(4.26)}{=} (a_{r+1,r} - b_{r+1,r})\beta_{r+2}^{P^{[r]}} - b_{r+1,r}(a_{r,r} - b_{r,r})\big(\alpha_{r+1}^{P^{[r]}} + A\big) \overset{(4.24)}{=} 0. \end{aligned}$$

Now let us assume that $\rho_i = 0$ for $2 \leq i \leq j - 1$. Then

$$\begin{aligned} \rho_j &\overset{(4.1)}{=} b_{j+r-1,r}\big\langle (x + A)\mathcal{U}^{[r]}, R_{j-1}(x)\big\rangle \\ &\quad - \big\langle \mathcal{U}^{[r]}, (x + A)\big(P_{j+r}^{[r]}(x) + a_{j+r-1,r}P_{j+r-1}^{[r]}(x)\big)\big\rangle \\ &\overset{j\geq 3}{=} b_{j+r-1,r}\rho_{j-1} - 0 = 0, \end{aligned}$$

completing the proof of (4.25). Therefore, $(x + A)\mathcal{U}^{[r]} = \sum_{j=0}^{1} \rho_j \frac{R_j(x)\mathcal{V}}{\|R_j(x)\|_{\mathcal{V}}^2}$, where

$$\rho_0 \overset{(4.22)}{=} \big\langle \mathcal{U}^{[r]}, P_{r+1}^{[r]}(x) + \alpha_{r+1}^{P^{[r]}} + A\big\rangle = \alpha_{r+1}^{P^{[r]}} + A \overset{(4.24)}{=} \frac{\beta_{r+2}^{P^{[r]}}(a_{r+1,r} - b_{r+1,r})}{b_{r+1,r}(a_{r,r} - b_{r,r})}$$

and, from (4.1) for $n = r + 1$ and (4.22) for $n = r + 1$,

$$\begin{aligned} \rho_1 &= \big\langle \mathcal{U}^{[r]}, \big(P_{r+1}^{[r]}(x) + \alpha_{r+1}^{P^{[r]}} + A\big)\big(P_{r+1}^{[r]}(x) + a_{r,r} - b_{r,r}\big)\big\rangle \\ &\underset{(4.26)}{\overset{(4.24)}{=}} \beta_{r+2}^{P^{[r]}} + \frac{\beta_{r+2}^{P^{[r]}}(a_{r+1,r} - b_{r+1,r})}{b_{r+1,r}} = \frac{\beta_{r+2}^{P^{[r]}}a_{r+1,r}}{b_{r+1,r}}. \end{aligned}$$

Therefore, since $R_1(x) = x - \alpha_1^R$ and $\|R_1(x)\|_{\mathcal{V}}^2 = \beta_2^R$, we obtain $(x - C^{P^{[r]}})\mathcal{U}^{[r]} = \xi(x - C^R)\mathcal{V}$ where

$$\begin{aligned} C^{P^{[r]}} &= \alpha_{r+1}^{P^{[r]}} - \frac{\beta_{r+2}^{P^{[r]}}(a_{r+1,r} - b_{r+1,r})}{b_{r+1,r}(a_{r,r} - b_{r,r})}, \\ C^R &= \alpha_1^R - \frac{\beta_2^R(a_{r+1,r} - b_{r+1,r})}{a_{r+1,r}(a_{r,r} - b_{r,r})}, \\ \xi &= \frac{\beta_{r+2}^{P^{[r]}}a_{r+1,r}}{\beta_2^R b_{r+1,r}}. \end{aligned} \tag{4.27}$$

(ii) $\Longrightarrow$ (i) We suppose that the linear functionals $\mathcal{U}^{[r]}$ and $\mathcal{V}$ satisfy (4.21). Then

$$\mathcal{U}^{[r]} - \langle \mathcal{U}^{[r]}, 1\rangle \delta_{C^R} + \left(C^R - C^{P^{[r]}}\right)\left(x - C^R\right)^{-1}\mathcal{U}^{[r]} = \xi\left[\mathcal{V} - \langle \mathcal{V}, 1\rangle \delta_{C^R}\right].$$

But since $\langle \mathcal{U}^{[r]}, 1\rangle = 1 = \langle \mathcal{V}, 1\rangle$,

$$\mathcal{V} = \frac{1}{\xi}\left\{\left[1 + \left(C^R - C^{P^{[r]}}\right)\left(x - C^R\right)^{-1}\right]\mathcal{U}^{[r]} + (\xi - 1)\delta_{C^R}\right\}. \tag{4.28}$$

Now we consider the Fourier expansion

$$P_n^{[r]}(x) = R_{n-r}(x) + \sum_{j=0}^{n-r-1} \xi_{j,n-r,r}^{P^{[r]}} R_j(x),$$

where

$$\begin{aligned}
\xi_{j,n-r,r}^{P^{[r]}} &= \frac{1}{\|R_j(x)\|^2}\left\langle \mathcal{V}, R_j(x)P_n^{[r]}(x)\right\rangle \\
&\underset{0\le j\le n-r-1}{\overset{(4.28)}{=}} \frac{1}{\|R_j(x)\|^2\xi}\left[\left(C^R - C^{P^{[r]}}\right)\left\langle \mathcal{U}^{[r]}, R_j\left(C^R\right)\frac{P_n^{[r]}(x) - P_n^{[r]}(C^R)}{x - C^R}\right\rangle\right. \\
&\qquad \left. + (\xi - 1)R_j\left(C^R\right)P_n^{[r]}\left(C^R\right)\right] \\
&= \frac{R_j(C^R)}{\|R_j(x)\|^2\xi}\left[(\xi - 1)P_n^{[r]}\left(C^R\right) + \left(C^R - C^{P^{[r]}}\right)P_{n-1}^{[r],(1)}\left(C^R\right)\right].
\end{aligned}$$

Here $\{P_n^{[r],(1)}(x)\}_{n\ge r}$ denotes the associated SMOP of the first kind for the SMOP $\{P_n^{[r]}(x)\}_{n\ge r}$. Then, for $n \ge r+1$

$$\begin{aligned}
P_n^{[r]}(x) = R_{n-r}(x) &+ \frac{1}{\xi}\left[(\xi - 1)P_n^{[r]}\left(C^R\right) + \left(C^R - C^{P^{[r]}}\right)P_{n-1}^{[r],(1)}\left(C^R\right)\right] \\
&\times K_{n-r-1}\left(x, C^R; \mathcal{V}\right),
\end{aligned}$$

where $K_{n-r-1}(x, C^R; \mathcal{V})$, $n \ge r+1$, denotes the reproducing kernel associated with $\mathcal{V}$ and, thus

$$\left\langle \mathcal{V}, P_n^{[r]}(x)\right\rangle = \frac{1}{\xi}\left[(\xi - 1)P_n^{[r]}\left(C^R\right) + \left(C^R - C^{P^{[r]}}\right)P_{n-1}^{[r],(1)}\left(C^R\right)\right], \quad n \ge r+1. \tag{4.29}$$

Therefore,

$$P_n^{[r]}(x) = R_{n-r}(x) + \left\langle \mathcal{V}, P_n^{[r]}(x)\right\rangle K_{n-r-1}\left(x, C^R; \mathcal{V}\right), \quad n \ge r+1. \tag{4.30}$$

In the same way

$$\left\langle \mathcal{U}^{[r]}, R_{n-r}(x)\right\rangle = (1 - \xi)R_{n-r}\left(C^{P^{[r]}}\right) + \xi\left(C^{P^{[r]}} - C^R\right)R_{n-r-1}^{(1)}\left(C^{P^{[r]}}\right), \quad n \ge r+1, \tag{4.31}$$

and

$$R_{n-r}(x) = P_n^{[r]}(x) + \left\langle \mathcal{U}^{[r]}, R_{n-r}(x)\right\rangle K_{n-1}\left(x, C^{P^{[r]}}; \mathcal{U}^{[r]}\right), \quad n \ge r+1, \tag{4.32}$$

where $K_{n-1}(x, C^{P^{[r]}}; \mathcal{U}^{[r]})$, for $n \geq r+1$, denotes the reproducing kernel associated with $\mathcal{U}^{[r]}$.

Moreover, since $P_n^{[r]}(x) \neq R_{n-r}(x)$ for $n \geq r+1$, from (4.30) and (4.32) we get $\langle \mathcal{V}, P_n^{[r]}(x)\rangle \neq 0$ and $\langle \mathcal{U}^{[r]}, R_{n-r}(x)\rangle \neq 0$ for $n \geq r+1$. Hence, from (4.30) for n and $n-1$ we obtain for $n \geq r+2$

$$P_n^{[r]}(x) - \frac{\langle \mathcal{V}, P_n^{[r]}(x)\rangle}{\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle} P_{n-1}^{[r]}(x)$$
$$= R_{n-r}(x) - \left(\frac{\langle \mathcal{V}, P_n^{[r]}(x)\rangle}{\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle} - \frac{\langle \mathcal{V}, P_n^{[r]}(x)\rangle R_{n-r-1}(C^R)}{\|R_{n-r-1}(x)\|^2} \right) R_{n-r-1}(x).$$

Applying the linear functional $\mathcal{U}^{[r]}$ on both hand sides, we get

$$\frac{\langle \mathcal{U}^{[r]}, R_{n-r}(x)\rangle}{\langle \mathcal{U}^{[r]}, R_{n-r-1}(x)\rangle} = \frac{\langle \mathcal{V}, P_n^{[r]}(x)\rangle}{\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle} - \frac{\langle \mathcal{V}, P_n^{[r]}(x)\rangle R_{n-r-1}(C^R)}{\|R_{n-r-1}(x)\|^2}.$$

Thus, for $n \geq r+2$,

$$P_n^{[r]}(x) - \frac{\langle \mathcal{V}, P_n^{[r]}(x)\rangle}{\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle} P_{n-1}^{[r]}(x) = R_{n-r}(x) - \frac{\langle \mathcal{U}^{[r]}, R_{n-r}(x)\rangle}{\langle \mathcal{U}^{[r]}, R_{n-r-1}(x)\rangle} R_{n-r-1}(x).$$

Therefore, (i) holds if we take

$$a_{n-1,r} = \frac{\langle \mathcal{V}, P_n^{[r]}(x)\rangle}{\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle} \neq 0 \quad \text{and} \quad b_{n-1,r} = \frac{\langle \mathcal{U}^{[r]}, R_{n-r}(x)\rangle}{\langle \mathcal{U}^{[r]}, R_{n-r-1}(x)\rangle} \neq 0,$$

for $n \geq r+2$, and since $P_{r+1}^{[r]}(x) \neq R_1(x)$ we can write $P_{r+1}^{[r]}(x) + a_{r,r} = R_1(x) + b_{r,r}$ with $a_{r,r} \neq b_{r,r}$ and $a_{r,r} b_{r,r} \neq 0$. □

Remark 26 Notice that

$$C^R = \alpha_{n+1}^{P^{[r]}} - a_{n,r} - \frac{\beta_{n+1}^{P^{[r]}}}{a_{n-1,r}} \quad \text{and} \quad C^{P^{[r]}} = \alpha_{n+1}^R - b_{n,r} - \frac{\beta_{n+1}^R}{b_{n-1,r}}, \quad \text{for } n \geq r+2.$$

Indeed, from (4.29) and the TTRR (4.22) and (4.23)), we obtain for $n \geq r+3$

$$\langle \mathcal{V}, P_n^{[r]}(x)\rangle = \left(C^R - \alpha_n^{P^{[r]}}\right)\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle - \beta_n^{P^{[r]}}\langle \mathcal{V}, P_{n-2}^{[r]}(x)\rangle.$$

Besides, if we apply $\mathcal{V}$ to (4.1), then we get for $n \geq r+2$

$$\langle \mathcal{V}, P_n^{[r]}(x)\rangle = -a_{n-1,r}\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle.$$

Thus,

$$0 = \left(a_{n-1,r} + C^R - \alpha_n^{P^{[r]}} + \frac{\beta_n^{P^{[r]}}}{a_{n-2,r}}\right)\langle \mathcal{V}, P_{n-1}^{[r]}(x)\rangle, \quad n \geq r+3,$$

but, $\langle \mathcal{V}, P_n^{[r]}(x)\rangle \neq 0$ for $n \geq r+1$, then $C^R = \alpha_{n+1}^{P^{[r]}} - a_{n,r} - \frac{\beta_{n+1}^{P^{[r]}}}{a_{n-1,r}}$ for $n \geq r+2$.

In the same way, but from (4.31), we obtain $C^{P^{[r]}} = \alpha_{n+1}^R - b_{n,r} - \frac{\beta_{n+1}^R}{b_{n-1,r}}$ for $n \geq r+2$.

As a straightforward consequence of Theorem 25, Corollary 6 and Proposition 4, we get

Corollary 27 *Let $\mathcal{U}$ be a classical linear functional given by (2.13) and let $\mathcal{V}$ be a quasi-definite linear functional, such that $\langle \mathcal{U}, \sigma^r(x)\rangle = 1 = \langle \mathcal{V}, 1\rangle$, $r \in \mathbb{N}$, and corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$. The following statements are equivalent*:

(i) *$(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order r given by (4.1), with $a_{r,r} \neq b_{r,r}$ and $a_{n,r}b_{n,r} \neq 0$ for $n \geq r$.*

(ii) *$P_n^{[r]}(x) \neq R_{n-r}(x)$ for $n \geq r+1$ and there exist constants $C^{P^{[r]}}$, C^R and ξ (see Remark 26) such that*

$$\big(x - C^{P^{[r]}}\big)\sigma^r(x)\mathcal{U} = \xi\big(x - C^R\big)\mathcal{V},$$

Therefore, $\mathcal{V}$ is a semiclassical linear functional of class at most 2, taking into account $\sigma^r(x)\mathcal{U}$ is a classical linear functional.

We will analyze every classical case. From (2.2), (2.17) and Table 1, for $r \in \mathbb{N}$ we get

- If $\mathcal{U}_{L^{(\alpha)}}$ is the Laguerre linear functional and $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ is its SMOP, then $\sigma(x) = x$. Assuming $\langle \mathcal{U}_{L^{(\alpha)}}, x^r\rangle = 1$ and

$$L_{n-r}^{(\alpha+r)}(x) + a_{n-1,r}^{L^{(\alpha)}} L_{n-r-1}^{(\alpha+r)}(x) = R_{n-r}(x) + b_{n-1,r}^{L^{(\alpha)}} R_{n-r-1}(x),$$
$$a_{n-1,r}^{L^{(\alpha)}} \neq 0,\ n \geq r+1,\ \ b_{r+1,r}^{L^{(\alpha)}} \neq 0,$$

then

$$\mathcal{V} = \xi^{-1}\big(x - C^R\big)^{-1}\big(x - C^{(L^{\alpha})^{[r]}}\big)x^r\mathcal{U}_{L^{(\alpha)}} + \delta_{C^R}.$$

- If $\mathcal{U}_{P^{(\alpha,\beta)}}$ is the Jacobi linear functional and $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ is its SMOP, then $\sigma(x) = 1 - x^2$. Assuming $\langle \mathcal{U}_{P^{(\alpha,\beta)}}, (1-x^2)^r\rangle = 1$ and

$$P_{n-r}^{(\alpha+r,\beta+r)}(x) + a_{n-1,r}^{P^{(\alpha,\beta)}} P_{n-r-1}^{(\alpha+r,\beta+r)}(x) = R_{n-r}(x) + b_{n-1,r}^{P^{(\alpha,\beta)}} R_{n-r-1}(x),$$
$$a_{n-1,r}^{P^{(\alpha,\beta)}} \neq 0,\ n \geq r+1,\ \ b_{r+1,r}^{P^{(\alpha,\beta)}} \neq 0,$$

then

$$\mathcal{V} = \xi^{-1}\big(x - C^R\big)^{-1}\big(x - C^{(P^{(\alpha,\beta)})^{[r]}}\big)\big(1-x^2\big)^r\mathcal{U}_{P^{(\alpha,\beta)}} + \delta_{C^R}.$$

5 The (Formal) Stieltjes Series and Coherent Pairs of Order *r*

The following theorem gives some relations for the formal Stieltjes Series associated with the linear functionals constituting either a $(1,0)$-coherent pair of order r or $(1,1)$-coherent pair of order r.

Theorem 28 *If $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair of order r given by (4.1), then for $r \in \mathbb{N}$,*

$$\varphi_{n+r,r}(z)S_{\mathcal{U}}(z) + (-1)^{r+1}\big[\gamma_{n,r}(z)S_{\mathcal{V}}(z)\big]^{(r)} = -A_{n,r}(z), \quad n \geq 1, \tag{5.1}$$

where

$$A_{n,r}(z) = (\mathcal{U}\theta_0\varphi_{n+r,r})(z) + (-1)^{r+1}[\mathcal{V}\theta_0\gamma_{n,r}]^{(r)}(z)$$

and

$$\deg\big(A_{n,r}(z)\big) \le n + r - 1,$$

If $(\mathcal{U},\mathcal{V})$ *is a* $(1,0)$*-coherent pair of order* r, *then* $\deg(A_{n,r}(z)) \le n+r-2$, *because* $\deg(\varphi_{n+r,r}(x)) \le n+r-1$. *with* $(\mathcal{U}\theta_0\varphi_{n+r,r})(z)$ *and* $[\mathcal{V}\theta_0\gamma_{n,r}]^{(r)}(z)$ *given in* [23].

Proof From Lemma 22, for $n \ge 1$ there exist polynomials

$$\gamma_{n,r}(x) = \sum_{j=0}^{n} a_j^{\gamma_{n,r}} x^j, \quad a_n^{\gamma_{n,r}} = 1 \quad \text{and} \quad \varphi_{n+r,r}(x) = \sum_{j=0}^{n+r} b_j^{\varphi_{n+r,r}} x^j$$

such that $\langle D^r[\gamma_{n,r}(x)\mathcal{V}], x^k\rangle = \langle (-1)^r\varphi_{n+r,r}(x)\mathcal{U}, x^k\rangle$, for $k \in \mathbb{N}$. So, for $n \ge 1$, $\frac{k!}{(k-r)!}\sum_{j=0}^{n} a_j^{\gamma_{n,r}} v_{k-r+j} = \sum_{j=0}^{n+r} b_j^{\varphi_{n+r,r}} u_{k+j}$, where $v_{k-r+j} = 0$ if $k-r+j<0$. Thus, multiplying by $z^{-(k+1)}$ and adding for $k = 0, 1, \dots,$ we get in each hand side

$$\begin{aligned}
&\sum_{k\ge r}\left[\frac{k!}{(k-r)!}\sum_{j=0}^{n} a_j^{\gamma_{n,r}} v_{k-r+j}\right] z^{-(k+1)} \\
&= \sum_{j=0}^{n} a_j^{\gamma_{n,r}} z^{j-r} \sum_{k\ge 0} (k+1)_r \frac{v_{k+j}}{z^{k+j+1}} \\
&= \sum_{j=0}^{n} a_j^{\gamma_{n,r}} z^{j-r} \sum_{l=0}^{r} \binom{r}{l} (-j)_{r-l}\left[(-1)^{l+1} z^l S_{\mathcal{V}}^{(l)}(z) - \sum_{k=0}^{j-1} (k+1)_l \frac{v_k}{z^{k+1}}\right] \\
&= (-1)^{r+1}\sum_{l=0}^{r}\binom{r}{l}\gamma_{n,r}^{(r-l)}(z) S_{\mathcal{V}}^{(l)}(z) - \sum_{j=0}^{n-1} a_{j+1}^{\gamma_{n,r}} \sum_{k=0}^{j} (-1)^r\big((j-k)-r+1\big)_r v_k z^{j-k-r} \\
&= (-1)^{r+1}\big(\gamma_{n,r}(z) S_{\mathcal{V}}(z)\big)^{(r)} + (-1)^{r+1}(\mathcal{V}\theta_0\gamma_{n,r})^{(r)}(z)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k\ge 0}\left[\sum_{j=0}^{n+r} b_j^{\varphi_{n+r,r}} u_{k+j}\right] z^{-(k+1)} &= \sum_{j=0}^{n+r} b_j^{\varphi_{n+r,r}} z^j \left[-S_{\mathcal{U}}(z) - \sum_{k=0}^{j-1}\frac{u_k}{z^{k+1}}\right] \\
&= -\varphi_{n+r,r}(z) S_{\mathcal{U}}(z) - (\mathcal{U}\theta_0\varphi_{n+r,r})(z).
\end{aligned}$$

Therefore, for $n \ge 1$

$$\begin{aligned}
&(-1)^{r+1}\big(\gamma_{n,r}(z) S_{\mathcal{V}}(z)\big)^{(r)} + (-1)^{r+1}(\mathcal{V}\theta_0\gamma_{n,r})^{(r)}(z) \\
&\quad = -\varphi_{n+r,r}(z) S_{\mathcal{U}}(z) - (\mathcal{U}\theta_0\varphi_{n+r,r})(z).
\end{aligned}$$

□

Theorem 29 *If* $(\mathcal{U},\mathcal{V})$ *is a* $(1,1)$*-coherent pair of order* r *given by* (4.1), *then for* $r \in \mathbb{N}$,

$$\sum_{k=0}^{r} B_{k,n,r}(z) S_{\mathcal{V}}^{(k)}(z) = C_{n,r}(z), \quad n \ge 1, \tag{5.2}$$

i.e., $S_{\mathcal{V}}(z)$ *is the* (*formal*) *solution of a non-homogeneous ordinary differential equation of order* r *with polynomial coefficients*

$$B_{k,n,r}(z) = \binom{r}{k}\big[\varphi_{n+r+1,r}(z)\gamma_{n,r}^{(r-k)}(z) - \varphi_{n+r,r}(z)\gamma_{n+1,r}^{(r-k)}(z)\big], \quad k = 0, 1, \ldots, n,$$
$$C_{n,r}(z) = (-1)^{r+1}\big[\varphi_{n+r,r}(z)A_{n+1,r}(z) - \varphi_{n+r+1,r}(z)A_{n,r}(z)\big],$$

with degree

$$\deg\big(B_{k,n,r}(z)\big) \le 2n + k + 1 \quad \textit{and} \quad \deg\big(C_{n,r}(z)\big) \le 2n + 2r.$$

If $(\mathcal{U}, \mathcal{V})$ *is a* $(1, 0)$*-coherent pair of order* r, *then* $\deg(B_{k,n,r}(z)) \le 2n + k$ *and* $\deg(C_{n,r}(z)) \le 2n + 2r - 2$.

Proof From (5.1), for $n \ge 1$, we obtain

$$\varphi_{n+r,r}(z)\varphi_{n+1+r,r}(z)S_{\mathcal{U}}(z) + (-1)^{r+1}\varphi_{n+r,r}(z)\big[\gamma_{n+1,r}(z)S_{\mathcal{V}}(z)\big]^{(r)}$$
$$= -\varphi_{n+r,r}(z)A_{n+1,r}(z)$$

and subtracting it, we get

$$\varphi_{n+r+1,r}(z)\big[\gamma_{n,r}(z)S_{\mathcal{V}}(z)\big]^{(r)} - \varphi_{n+r,r}(z)\big[\gamma_{n+1,r}(z)S_{\mathcal{V}}(z)\big]^{(r)} = C_{n,r}(z), \quad n \ge 1,$$

where

$$C_{n,r}(z) = (-1)^{r+1}\big[\varphi_{n+r,r}(z)A_{n+1,r}(z) - \varphi_{n+r+1,r}(z)A_{n,r}(z)\big].$$

Finally, using the Leibniz rule, we complete the proof. □

Remark 30 We can get $S_{\mathcal{V}}$ if we solve (formally) the differential equation (5.2). Hence, from (5.1) we can also obtain $S_{\mathcal{U}}$.

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References

1. Alfaro, M., Marcellán, F., Peña, A., Rezola, M.L.: On linearly related orthogonal polynomials and their functionals. J. Math. Anal. Appl. **287**, 307–319 (2003)
2. Alfaro, M., Marcellán, F., Peña, A., Rezola, M.L.: On rational transformations of linear functionals: direct problem. J. Math. Anal. Appl. **298**, 171–183 (2004)
3. Alfaro, M., Marcellán, F., Peña, A., Rezola, M.L.: When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials? J. Comput. Appl. Math. **233**, 1446–1452 (2010)
4. Alfaro, M., Marcellán, F., Peña, A., Rezola, M.L.: Orthogonal polynomials associated with an inverse quadratic spectral transform. Comput. Math. Appl. **61**, 888–900 (2011)
5. Bonan, S., Nevai, P.: Orthogonal polynomials and their derivatives, I. J. Approx. Theory **40**, 134–147 (1984)
6. Bonan, S., Lubinsky, D.S., Nevai, P.: Orthogonal polynomials and their derivatives, II. SIAM J. Math. Anal. **18**(4), 1163–1176 (1987)

7. Branquinho, A., Rebocho, M.N.: On the semiclassical character of orthogonal polynomials satisfying structure relations. J. Differ. Equ. Appl. **18**, 111–138 (2012)
8. Branquinho, A., Foulquié Moreno, A., Marcellán, F., Rebocho, M.N.: Coherent pairs of linear functionals on the unit circle. J. Approx. Theory **153**, 122–137 (2008)
9. Chihara, T.S.: An Introduction to Orthogonal Polynomials. Gordon & Breach, New York (1978)
10. de Jesus, M.N., Petronilho, J.: On linearly related sequences of derivatives of orthogonal polynomials. J. Math. Anal. Appl. **347**, 482–492 (2008)
11. Delgado, A.M., Marcellán, F.: Companion linear functionals and Sobolev inner products: a case study. Methods Appl. Anal. **11**(2), 237–266 (2004)
12. Delgado, A.M., Marcellán, F.: On an extension of symmetric coherent pairs of orthogonal polynomials. J. Comput. Appl. Math. **178**, 155–168 (2005)
13. Hahn, W.: Über die Jacobischen Polynome und Zwei Verwandte Polynomklassen. Math. Z. **39**, 634–638 (1935)
14. Hahn, W.: Über Höhere Ableitungen von Orthogonal Polynomen. Math. Z. **43**, 101 (1937)
15. Iserles, A., Koch, P.E., Nørsett, S.P., Sanz-Serna, J.M.: On polynomials orthogonal with respect to certain Sobolev inner products. J. Approx. Theory **65**, 151–175 (1991)
16. Ismail, M.E.H.: Classical and Quantum Orthogonal Polynomials in One Variable. Encyclopedia Math. Appl., vol. 98. Cambridge University Press, Cambridge (2005)
17. Kwon, K.H., Lee, J.H., Marcellán, F.: Generalized coherent pairs. J. Math. Anal. Appl. **253**, 482–514 (2001)
18. Marcellán, F., Petronilho, J.C.: Orthogonal polynomials and coherent pairs: the classical case. Indag. Math. **6**(3), 287–307 (1995)
19. Marcellán, F., Branquinho, A., Petronilho, J.: Classical Orthogonal polynomials: a functional approach. Acta Appl. Math. **34**, 283–303 (1994)
20. Marcellán, F., Pérez, T.E., Petronilho, J., Piñar, M.A.: What is beyond coherent pairs of orthogonal polynomials? J. Comput. Appl. Math. **65**, 267–277 (1995)
21. Marcellán, F., Pérez, T.E., Piñar, M.A.: Orthogonal polynomials on weighted Sobolev spaces: the semi-classical case. Ann. Numer. Math. **2**, 93–122 (1995)
22. Marcellán, F., Martínez-Finkelshtein, A., Moreno-Balcázar, J.: k-Coherence of measures with non-classical weights. In: Español, L., Varona, J.L. (eds.) Margarita Mathematica en Memoria de José Javier Guadalupe Hernández, pp. 77–83. Servicio de Publicaciones, Universidad de la Rioja, Logroño (2001)
23. Maroni, P.: Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. In: Brezinski, C., Gori, L., Ronveaux, A. (eds.) Orthogonal Polynomials and Their Applications. IMACS Annals Comput. Appl. Math, vol. 9, pp. 95–130 (1991)
24. Maroni, P.: Semi-classical character and finite-type relations between polynomial sequences. Appl. Numer. Math. **31**, 295–330 (1999)
25. Maroni, P., Sfaxi, R.: Diagonal orthogonal polynomial sequences. Methods Appl. Anal. **7**(4), 769–792 (2000)
26. Martínez-Finkelshtein, A., Moreno-Balcázar, J.J., Pérez, T.E., Piñar, M.A.: Asymptotics of Sobolev orthogonal polynomials for coherent pairs of measures. J. Approx. Theory **92**, 280–293 (1998)
27. Meijer, H.G.: Coherent pairs and zeros of Sobolev-type orthogonal polynomials. Indag. Math. **4**(2), 163–176 (1993)
28. Meijer, H.G.: Determination of all coherent pairs. J. Approx. Theory **89**, 321–343 (1997)
29. Petronilho, J.: On the linear functionals associated to linearly related sequences of orthogonal polynomials. J. Math. Anal. Appl. **315**, 379–393 (2006)
30. Sfaxi, R., Alaya, J.: On orthogonal polynomials with respect to the form $-(x-c)S'$. Period. Math. Hung. **52**(1), 67–99 (2006)