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Zeros of Sobolev orthogonal polynomials on the unit circle

K. Castillo · L. E. Garza · F. Marcellán

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Abstract In this contribution, we study the sequences of orthogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle_S := \int_{\mathbb{T}} f(z)\overline{g(z)}d\mu(z) + \lambda f^{(j)}(\alpha)\overline{g^{(j)}(\alpha)},$$

where μ is a nontrivial probability measure supported on the unit circle, $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{R}_+\setminus\{0\}$, and $j \in \mathbb{N}$. In particular, we analyze the behavior of their zeros when n and λ tend to infinity, respectively. We also provide some numerical examples to illustrate the behavior of these zeros with respect to α.

Keywords Probability measures on the unit circle · Orthogonal polynomials · Sobolev inner products · Hessenberg matrices · Zeros

Mathematics Subject Classifications (2010) 33C47 · 42C05

Dedicated to Professor Claude Brezinski and Professor Sebastiano Seatzu on the occasion of their 70th birthday.

K. Castillo (✉) · F. Marcellán
Departamento de Matemáticas, Universidad Carlos III de Madrid,
Ave. Universidad 30, Leganés, Spain
e-mail: kcastill@math.uc3m.es

F. Marcellán
e-mail: pacomarc@ing.uc3m.es

L. E. Garza
Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340,
Colima, México
e-mail: garzaleg@gmail.com

1 Introduction

Let μ be a nontrivial probability measure supported on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\{\phi_n\}_{n\geqslant 0}$ the sequence of polynomials with $\deg \phi_n = n$ such that

$$\int_{\mathbb{T}} \phi_m(z)\overline{\phi_n(z)}d\mu(z) = \delta_{m,n},$$

i.e., $\{\phi_n\}_{n\geqslant 0}$ is the sequence of orthonormal polynomials with respect to the measure μ (see [12, 23, 25], among others). The corresponding sequence of monic orthogonal polynomials will be denoted by $\{\Phi_n\}_{n\geqslant 0}$. The polynomial

$$K_n(z, y) = \sum_{k=0}^{n} \frac{\Phi_k(z)\overline{\Phi_k(y)}}{\|\Phi_k\|^2}$$

is called the reproducing kernel associated to $\{\phi_n\}_{n\geqslant 0}$. We will denote by $K_n^{(k,j)}(z, y)$ the $k-th$ and $j-th$ partial derivatives of $K_n(z, y)$ with respect to the variables z and y, respectively.

The measure μ can be decomposed into an absolutely continuous measure with respect to the Lebesgue measure and a singular part as follows

$$d\mu(\theta) = \mu'(\theta)\frac{d\theta}{2\pi} + d\mu_s(\theta).$$

It is said to belong to the Nevai class $\mathcal{N}$ (see [18] and [19]) if

$$\lim_{n\to\infty} |\phi_{n+1}(0)| = 0. \tag{1}$$

Furthermore, if $\mu \in \mathcal{N}$,

$$\left|\frac{\Phi_n(z)}{\Phi_{n-1}(z)} - z\right| \leqslant |\Phi_n(0)|, \qquad |z| \geqslant 1.$$

Thus,

$$\lim_{n\to\infty} \frac{\Phi_n(z)}{\Phi_{n-1}(z)} = z,$$

uniformly in compact subsets of $|z| > 1$. This result can be obtained under weaker conditions. A well known result (see [22]) says that any measure that satisfies a Lipschitz condition with some positive exponent i.e.,

$$\frac{d\mu(z)}{|dz|} = \mu'(z) \quad \text{with} \quad \mu'(z) > 0 \quad \text{a.e. for} \quad z \in \mathbb{T},$$

satisfies (1). Then, the Lipschitz condition is a sufficient condition for a measure in order to belong to the $\mathcal{N}$ class.

In this paper we will consider the following discrete Sobolev inner product associated with a nontrivial probability measure μ supported on the unit circle

$$\langle f, g\rangle_S := \int_{\mathbb{T}} f(z)\overline{g(z)}d\mu(z) + \lambda f^{(j)}(\alpha)\overline{g^{(j)}(\alpha)},\ \alpha \in \mathbb{C},\ \lambda \in \mathbb{R}^+\backslash\{0\},\ j \in \mathbb{N}, \tag{2}$$

where f, g belong to the Sobolev space

$$W^{j,2}[\mathbb{T};\mu] = \left\{ f \in \mathcal{C}_j(\mathbb{T}) \cap L^2[\mathbb{T};\mu] : f^{(j)} \in L^2[\mathbb{T};\mu] \right\}.$$

Here $\mathcal{C}_j(\mathbb{T})$ denotes the function space containing all functions $f : \mathbb{T} \to \mathbb{C}$ such that $f \in C^{j-2}$ and $f^{(j-1)}$ is absolutely continuous on $\mathbb{T}$. Notice that, since λ is a positive real number, there exists a family of polynomials orthonormal with respect to (2), which will be denoted by $\{\psi_n\}_{n\geqslant 0}$. The monic version will be denoted by $\{\Psi_n\}_{n\geqslant 0}$. More general Sobolev inner products have been analyzed (see, among others, [4, 5, 10, 11, 15] and [16]) focusing mainly in asymptotic properties of the corresponding families of orthogonal polynomials. The sequences of monic polynomials orthogonal with respect to (2) have been studied more recently in [6], where the authors obtained an expression that relates $\{\psi_n\}_{n\geqslant 0}$ and $\{\phi_n\}_{n\geqslant 0}$, as well as some asymptotic properties. The behavior of the zeros of ψ_n for the special cases $j = 0, 1$ has been analyzed. They turned out to be related to the zeros of a family of polynomials orthogonal with respect to a certain perturbation of the measure μ.

In this contribution, we obtain new results regarding the behavior of the zeros of $\{\psi_n\}_{n\geqslant 0}$ for a general j, when n or λ tend to infinity, respectively. We also show some numerical calculations concerning the zeros of $\{\psi_n\}_{n\geqslant 0}$ for some well known measures. Notice that, in contrast with the real line case (see [1, 7] and [17]), there is not a well developed theory for zeros of Sobolev orthogonal polynomials. In this manuscript a first approach to this topic is presented.

The structure of the manuscript is as follows. In Section 2, we deduce the connection between the Hessenberg matrices associated with the multiplication operator in terms of the bases $\{\Psi_n\}_{n\geqslant 0}$ and $\{\Phi_n\}_{n\geqslant 0}$, respectively. In Section 3, we study the location of the zeros of Ψ_n. Finally, in Section 4, some numerical computations are shown for particular choices of the measure μ and the location of α.

2 Hessenberg matrices

The matrix representation of the multiplication by the z operator with respect to the basis $\{\Phi_n\}_{n\geqslant 0}$ is given by $z\Phi = \mathbf{H}_\Phi \Phi$, where $\Phi = \{\Phi_0, \Phi_1, \ldots\}^t$ and $\mathbf{H}_\Phi$ is a semi-infinite Hessenberg matrix with ones on the upper diagonal and whose remaining entries are given in terms of the Verblunsky coeficients $\{\Phi_n(0)\}_{n\geqslant 1}$ (see Section 4.1, Chapter 4 in [23]). The study of the Hessenberg matrices is important because of their applications. As an example, it is well known (see [23]) that the zeros of the n-th orthogonal polynomial $\Phi_n(z)$ are the eigenvalues of the leading principal submatrix $n \times n$ of the Hessenberg matrix $\mathbf{H}_\Phi$, which we will denote by $(\mathbf{H}_\Phi)_n$. They are useful in quadrature formulas on the unit circle, see [13] and [9] as well as in the frequency analysis problem, see [14] and [21]. These matrices play an important role in some integrable systems related to the complex semi-discrete modified KdV equation, namely, the Schur flow, see [2, 20] and [24]. Our aim is to find a relation between

$\mathbf{H}_\Psi$, the Hessenberg matrix associated with the monic orthogonal polynomials $\{\Psi_n\}_{n\geqslant 0}$, and $\mathbf{H}_\Phi$. In particular, we get

$$z\begin{pmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{pmatrix} = (\mathbf{H}_\Phi)_n\begin{pmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{pmatrix} + \Phi_n(z)\begin{pmatrix}0\\ 0\\ \vdots\\ 1\end{pmatrix}, \tag{3}$$

and, on the other hand,

$$z\begin{pmatrix}\Psi_0(z)\\ \Psi_1(z)\\ \vdots\\ \Psi_{n-1}(z)\end{pmatrix} = (\mathbf{H}_\Psi)_n\begin{pmatrix}\Psi_0(z)\\ \Psi_1(z)\\ \vdots\\ \Psi_{n-1}(z)\end{pmatrix} + \Psi_n(z)\begin{pmatrix}0\\ 0\\ \vdots\\ 1\end{pmatrix}. \tag{4}$$

Furthermore, the relation between $\{\phi_n\}_{n\geqslant 0}$ and $\{\psi_n\}_{n\geqslant 0}$ is (see [6]).

Proposition 1 *Let $\phi_n(z) = \alpha_n z^n + \ldots$ and $\psi_n(z) = \beta_n z^n + \ldots$ with $\alpha_n, \beta_n > 0$. Then, $\{\psi_n\}_{n\geqslant 0}$ is the sequence of polynomials orthonormal with respect to* (2) if and only if

$$\psi_n(z) = \frac{\beta_n}{\alpha_n}\phi_n(z) - \lambda\psi_n^{(j)}(\alpha)K_{n-1}^{(0,j)}(z,\alpha), \qquad j = 0, 1, \ldots, \tag{5}$$

with

$$\frac{\beta_n}{\alpha_n} = \sqrt{\frac{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)}}, \tag{6}$$

and

$$\psi_n^{(j)}(\alpha) = \frac{\phi_n^{(j)}(\alpha)}{\sqrt{\left(1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)\right)\left(1+\lambda K_n^{(j,j)}(\alpha,\alpha)\right)}}. \tag{7}$$

Remark 1 The monic version of (5) is

$$\Psi_n(z) = \Phi_n(z) - \frac{\lambda\Phi_n^{(j)}(\alpha)}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}K_{n-1}^{(0,j)}(z,\alpha). \tag{8}$$

Thus,

$$\begin{pmatrix}\Psi_0(z)\\ \Psi_1(z)\\ \vdots\\ \Psi_{n-1}(z)\end{pmatrix} = \mathbf{L}_n\begin{pmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{pmatrix},$$

where $\mathbf{L}_n$ is a lower triangular matrix with 1 as entries in the main diagonal and the remaining entries are given by (8), i.e.

$$l_{m,k} = -\frac{\lambda \Phi_m^{(j)}(\alpha)\overline{\Phi_k^{(j)}(\alpha)}}{(1+\lambda K_{m-1}^{(j,j)}(\alpha,\alpha))\|\Phi_k\|^2}, \quad 1 \leqslant m \leqslant n, \quad 0 \leqslant k \leqslant m-1.$$

Substituting in (4), we obtain

$$z\mathbf{L}_n \begin{pmatrix} \Phi_0(z) \\ \Phi_1(z) \\ \vdots \\ \Phi_{n-1}(z) \end{pmatrix} = (\mathbf{H}_\Psi)_n \mathbf{L}_n \begin{pmatrix} \Phi_0(z) \\ \Phi_1(z) \\ \vdots \\ \Phi_{n-1}(z) \end{pmatrix} + [\Phi_n(z) + \sum_{k=0}^{n-1} l_{n,k}\Phi_k(z)] \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{9}$$

$$= (\mathbf{H}_\Psi)_n \mathbf{L}_n \begin{pmatrix} \Phi_0(z) \\ \Phi_1(z) \\ \vdots \\ \Phi_{n-1}(z) \end{pmatrix} + \Phi_n(z) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + \mathbf{A}_n \begin{pmatrix} \Phi_0(z) \\ \Phi_1(z) \\ \vdots \\ \Phi_{n-1}(z) \end{pmatrix}, \tag{10}$$

where

$$\mathbf{A}_n = \begin{pmatrix} 0 & \ldots\ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots\ldots & 0 \\ l_{n,0} & \ldots\ldots & l_{n,n-1} \end{pmatrix}.$$

As a consequence,

$$z \begin{pmatrix} \Phi_0(z) \\ \Phi_1(z) \\ \vdots \\ \Phi_{n-1}(z) \end{pmatrix} = [\mathbf{L}_n^{-1}(\mathbf{H}_\Psi)_n \mathbf{L}_n + \mathbf{L}_n^{-1}\mathbf{A}_n] \begin{pmatrix} \Phi_0(z) \\ \Phi_1(z) \\ \vdots \\ \Phi_{n-1}(z) \end{pmatrix} + \Phi_n(z) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

so

$$(\mathbf{H}_\Phi)_n = \mathbf{L}_n^{-1}(\mathbf{H}_\Psi)_n \mathbf{L}_n + \mathbf{L}_n^{-1}\mathbf{A}_n$$

and therefore, since

$$\mathbf{L}_n^{-1}\mathbf{A}_n = \mathbf{A}_n,$$

we have

Proposition 2 *Let* $(\mathbf{H}_\Phi)_n$ *and* $(\mathbf{H}_\Psi)_n$ *be the* $n \times n$ *truncated Hessenberg matrices associated with* $\{\Phi_n\}_{n\geqslant 0}$ *and* $\{\Psi_n\}_{n\geqslant 0}$*, respectively. Then,*

$$(\mathbf{H}_\Psi)_n = \mathbf{L}_n[(\mathbf{H}_\Phi)_n - \mathbf{A}_n]\mathbf{L}_n^{-1}.$$

As a consequence, the zeros of Ψ_n *are the eigenvalues of the matrix* $(\mathbf{H}_\Phi)_n - \mathbf{A}_n$*, a rank one perturbation of the matrix* $(\mathbf{H}_\Phi)_n$*.*

Remark 2 Notice that $\mathbf{A}_n = (0, \ldots, 0, 1)^t (l_{n,0}, l_{n-1}, \ldots, l_{n,n-1})$ and, since $l_{n,k} = 0$ for $k < j$, then

$$\mathbf{A}_n = \frac{\lambda \Phi_n^{(j)}(\alpha)}{1 + \lambda K_{n-1}^{(j,j)}(\alpha, \alpha)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \left(0, \ldots, 0, \frac{\overline{\Phi_j^{(j)}(\alpha)}}{\|\Phi_j\|^2}, \ldots, \frac{\overline{\Phi_{n-1}^{(j)}(\alpha)}}{\|\Phi_{n-1}\|^2} \right).$$

As an example, if μ_θ is the normalized Lebesgue measure, $\Phi_n(z) = \phi_n(z) = z^n$, $n \geqslant 0$ (see next section), and it is not difficult to see that in such a case, if $\alpha = 0$, then $\mathbf{A}_n = 0, n \neq j$, and

$$\mathbf{A}_j = \frac{\lambda (j!)^2}{1 + \lambda (j!)^2} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \left(0, \ldots, 0, 1, 0, \ldots, 0 \right),$$

where the one is in the position j. On the other hand, if $\alpha = 1$, then for $n \geqslant j$,

$$\mathbf{A}_n = \frac{\lambda \frac{n!}{(n-j)!}}{1 + \lambda \sum_{k=j}^{n-1} \left(\frac{k!}{(k-j)!} \right)^2} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \left(0, \ldots, 0, j!, (j+1)!, \ldots, \frac{(n-1)!}{(n-j-1)!} \right).$$

3 Asymptotic behavior of zeros

We begin by generalizing a Lemma that appears in [16].

Proposition 3 *Assume that* $p(z) = \prod_{i=1}^{n} (z - \zeta_i)$, *with* $\zeta_i \in \mathbb{D}$ *and* K *is an arbitrary compact subset in* $\mathbb{C} \backslash \mathbb{D}$. *Then, there exist two positive real numbers* $a = a(K)$ *and* $b = b(K)$, *such that*

$$0 < a \leq \left| \frac{p_n^{(j)}(z)}{n^j p_n(z)} \right| \leq b$$

for $z \in K$ *and* $j < n$.

Proof Generalizing the Leibniz rule for the j-th derivative of a product of functions $f_i(z) = z - \zeta_i$, $i = 0, \ldots, n$, we have

$$\frac{p_n^{(j)}(z)}{p_n(z)} = \sum_{k=1}^{[n]_j} \frac{1}{\prod_{i \in S_k} (z - \zeta_i)} = \sum_{k=1}^{[n]_j} \frac{\prod_{i \in S_k} (\overline{z} - \overline{\zeta}_i)}{\prod_{i \in S_k} |z - \zeta_i|^2}$$

where the index S_k runs through the whole list of $[n]_j = \dfrac{n!}{(n-j)!}$ subsets of j elements of $\{\zeta_1, \zeta_2, \dots \zeta_n\}$. Denoting $c = \min_{z\in K}\{|z|\}$ and $C = \max_{z\in K}\{|z|\}$ and taking into account that the convex hull of $\{\zeta_i\}_{i=1}^n$ lies on the disk $|z| \le R < 1$, we get

$$\left|\frac{p_n^{(j)}(z)}{p_n(z)}\right| \le \sum_{k=1}^{[n]_j} \frac{(C+R)^j}{(c-R)^{2j}} = b\,[n]_j,$$

and for the lower bound

$$\left|\frac{p_n^{(j)}(z)}{p_n(z)}\right| \ge \sum_{k=1}^{[n]_j} \frac{(c-R)^j}{(C+R)^{2j}} = a[n]_j.$$

□

According to Lucas's theorem (see [3]) we know that if the zeros of the polynomial p_n lie on the unit open disk, then the same is true for its derivatives and thus, in the conditions of the above lemma, the zeros of $p_n^{(j)}(z)$ lie on $|z| \le R$.

In this section, we analyze the behavior of the zeros of the polynomials orthogonal with respect to (2). First, denote by $\{\phi_n(z; d\mu_{j+1})\}_{n\geqslant 0}$ the corresponding sequence of monic orthogonal polynomial with respect to

$$d\mu_j = |z-\alpha|^{2(j+1)} d\mu, \qquad j \in \mathbb{N},$$

i.e., the product of $j+1$ Christoffel transformations [8]. For any $j \in \mathbb{N}$, the relation between $\phi_n(z; d\mu_{j+1})$ and $\phi_n(z, d\mu)$ is given by (see [16])

$$(z-\alpha)^{j+1}\phi_{n-j-1}(z, d\mu_{j+1}) = \frac{\eta_{n-j-1}}{\alpha_n}\left[\phi_n(z) - \sum_{k=0}^{j} \gamma_{n,k} K_{n-1}^{(0,k)}(z,\alpha)\right], \qquad (11)$$

where η_n is the leading coefficient of $\phi_n(z, d\mu_{j+1})$ and $\gamma_{n,k}$ is the k-th component of the vector

$$\begin{pmatrix}\phi_n(\alpha) & \phi_n'(\alpha) & \dots & \phi_n^{(j)}(\alpha)\end{pmatrix} \begin{pmatrix} K_{n-1}(\alpha,\alpha) & K_{n-1}^{(0,1)}(\alpha,\alpha) & \dots & K_{n-1}^{(0,j)}(\alpha,\alpha) \\ K_{n-1}^{(1,0)}(\alpha,\alpha) & K_{n-1}^{(1,1)}(\alpha,\alpha) & \dots & K_{n-1}^{(1,j)}(\alpha,\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(j,0)}(\alpha,\alpha) & K_{n-1}^{(j,1)}(\alpha,\alpha) & \dots & K_{n-1}^{(j,j)}(\alpha,\alpha) \end{pmatrix}^{-1}.$$

Let $\mu \in \mathcal{N}$, then (see [16])

$$\lim_{n\to\infty} \frac{\phi_n(z; d\mu_{j+1})}{\phi_{n+j+1}(z)} = \left(\frac{\overline{\alpha}}{|\alpha|}\frac{1}{\overline{\alpha}z - 1}\right)^{j+1}, \qquad (12)$$

holds uniformly in $|z| > 1$ if $|\alpha| \geqslant 1$, and in $|z| \geqslant 1$ if $|\alpha| > 1$.

On the other hand (see [6]), for $|\alpha| > 1$,

$$\lim_{n\to\infty} \frac{\psi_n(z)}{\phi_n(z)} = \frac{\overline{\alpha}}{|\alpha|}\frac{z-\alpha}{\overline{\alpha}z - 1}, \qquad (13)$$

uniformly on every compact subset of $|z| > 1$. From (12) and (13), we have

$$\left(\frac{|\alpha|}{\overline{\alpha}}(\overline{\alpha}z-1)\right)^j (z-\alpha)\lim_{n\to\infty}\frac{\phi_n(z;d\mu_{j+1})}{\phi_{n+j+1}(z)} = \lim_{n\to\infty}\frac{\psi_{n+j+1}(z)}{\phi_{n+j+1}(z)}.$$

Hence,

$$\lim_{n\to\infty}\frac{\psi_n(z)}{\phi_{n-j-1}(z;d\mu_{j+1})} = \left(\frac{|\alpha|}{\overline{\alpha}}(\overline{\alpha}z-1)\right)^j (z-\alpha),$$

uniformly $|z| \geqslant 1$. The following result follows immediately from Hurwitz's Theorem.

Proposition 4 *There is a positive integer n_0 such that, for $n \geqslant n_0$, the n-th Sobolev monic orthogonal polynomial $\psi_n(z)$ defined by (5), with $|\alpha| > 1$, has exactly 1 zero in $|z| > 1$ accumulating in α, while the remaining zeros belong to $|z| < 1$.*

Now, we turn our attention to the case when λ tends to infinity. For $j = 0$ (Uvarov's case) and $j = 1$, the behavior of the zeros was studied in [6], where the authors show that, for a fixed n, $j = 0$ and $\lambda \to \infty$, $n-1$ zeros of ψ_n tend to the zeros of $\phi_{n-1}(z, d\mu_1)$, and the remaining zero tends to $z = \alpha$. On the other hand, for $j = 1$, the zeros of ψ_n tend to the zeros of a linear combination of $\Phi_n(z)$, $(z-\alpha)\Phi_{n-1}(z, d\mu_1)$ and $(z-\alpha)^2\Phi_{n-2}(z, d\mu_2)$ when $\lambda \to \infty$. This result can be generalized for arbitrary j. Indeed, from (11), notice that

$$-\gamma_{n,j}K_{n-1}^{(0,j)}(z,\alpha) = (z-\alpha)^{j+1}\phi_{n-j-1}(z,d\mu_{j+1}) - \frac{\eta_{n-j-1}}{\alpha_n}\phi_n(z) + \sum_{k=0}^{j-1}\gamma_{n,k}K_{n-1}^{(0,k)}(z,\alpha).$$

Applying the last formula recursively for $k = 0, 1, \ldots, j-1$, we obtain

Proposition 5 *Let $\{\psi_n\}_{n\geqslant 0}$ be the sequence of orthonormal polynomials with respect to (2), with $j \in \mathbb{N}$. Then $\psi_n(z)$ can be expressed as a linear combination of $\phi_n(z)$, $(z-\alpha)\phi_{n-1}(z, d\mu_1)$, ..., $(z-\alpha)^{j+1}\phi_{n-j-1}(z, d\mu_{j+1})$. As a consequence, the zeros of $\psi_n(z)$ tend to the zeros of such a linear combination when $\lambda \to \infty$.*

Remark 3 Notice that, when λ tends to infinity in (8), we get the limit polynomial

$$S_n(z) = \Phi_n(z) - \frac{\Phi_n^{(j)}(\alpha)}{K_{n-1}^{(j,j)}(\alpha,\alpha)}K_{n-1}^{(0,j)}(z,\alpha). \tag{14}$$

It is easily seen that $S_n^{(j)}(\alpha) = 0$, as well as S_n is orthogonal to the linear space $span\{1, z-\alpha, \ldots, (z-\alpha)^{j-1}, (z-\alpha)^{j+1}, \ldots, (z-\alpha)^{n-1}\}$ in $\mathbb{P}_n$.

Now, assume that Q_n is a monic polynomial of degree n such that $Q_n^{(j)}(\alpha) = 0$. Then, we can put

$$Q_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} a_{n,k}\phi_k(z), \tag{15}$$

for some (unique) complex numbers $a_{n,k}$, and therefore

$$\Phi_n^{(j)}(\alpha) + \sum_{k=0}^{n-1} a_{n,k}\phi_k^{(j)}(\alpha) = 0.$$

On the other hand, from Cauchy-Schwarz inequality, we get

$$|\Phi_n^{(j)}(\alpha)|^2 \leqslant \sum_{k=0}^{n-1} |a_{n,k}|^2 \sum_{k=0}^{n-1} |\phi_k^{(j)}(\alpha)|^2 = K_{n-1}^{(j,j)}(\alpha,\alpha) \sum_{k=0}^{n-1} |a_{n,k}|^2,$$

and taking norms with respect to μ in (15), we obtain

$$\|Q_n\|^2 = \|\Phi_n\|^2 + \sum_{k=0}^{n-1} |a_{n,k}|^2.$$

Thus,

$$\|Q_n\|^2 = \|\Phi_n\|^2 + \sum_{k=0}^{n-1} |a_{n,k}|^2 \geqslant \|\Phi_n\|^2 + \frac{|\Phi_n^{(j)}(\alpha)|^2}{K_{n-1}^{(j,j)}(\alpha,\alpha)}.$$

But the term in the right is precisely $\|S_n\|^2$. As a consequence, we have proved the following extremal characterization for the limit polynomial S_n.

Proposition 6 *Let*

$$G_n = \min\left\{ \int_{\mathbb{T}} |Q_n(z)|^2 d\mu(z) \quad : \quad Q_n = z^n + \textit{lower terms}, \quad Q_n^{(j)}(\alpha) = 0 \right\}.$$

Then, $G_n = \|S_n\|^2$, where S_n is the limit polynomial defined by (14).

4 Examples

In order to analyze the behavior of the zeros according to the location of α, we present some numerical computations of such zeros for the orthogonal polynomials associated with perturbations of the form (2) for two special cases of probability measures on the unit circle: the Lebesgue and Bernstein-Szegő measures.

For the normalized Lebesgue measure μ_θ, it is very well known that its corresponding monic orthogonal polynomial sequence is $\Phi_n(z) = z^n$, $n \geqslant 0$.

Table 1 $n = 30$, $j = 2$, and $\lambda = 10$

α	Behavior of zeros
$0 < \lvert\alpha\rvert < 0.8202$	All zeros approximately aligned and increase with α
$\lvert\alpha\rvert \sim 0.8202$	One of the zeros (z_i) breaks the pattern
$0.8202 < \lvert\alpha\rvert < 1.3194$	z_i increases with α
$\lvert\alpha\rvert \sim 0.1.3194$	z_i changes sign
$1.1394 < \lvert\alpha\rvert < 1.6263$	z_i decreases with α
$\lvert\alpha\rvert \sim 1.6263$	z_i goes back to the aligned pattern
$\lvert\alpha\rvert > 1.6263$	All zeros approximately aligned and decrease with α

Thus,

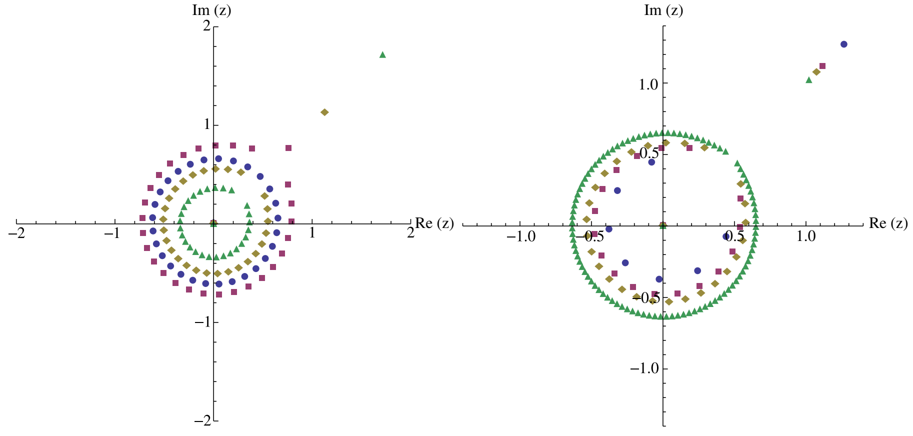
$$\Phi_n^{(j)}(\alpha) = \frac{n!}{(n-j)!}\alpha^{n-j},$$

$$K_{n-1}^{(0,j)}(z,\alpha) = \sum_{k=j}^{n-1} \frac{k!}{(k-j)!} z^k \bar{\alpha}^{k-j}, \quad \text{and}$$

$$K_n^{(j,j)}(\alpha,\alpha) = \sum_{k=j}^{n} \left(\frac{k!}{(k-j)!}\right)^2 |\alpha|^{2(k-j)}.$$

When n, j, and λ are fixed and α varies, we were able to identify some "critical" values of α that change the behavior of the zeros of ψ_n. Of course, such values of α will depend on n, j, and λ. The following table illustrates such a situation when $n = 30$, $j = 2$, and $\lambda = 10$.

Figure 1a illustrates the information in Table 1, showing the location of the zeros of ψ_{30} for several values of α. Namely, the zeros corresponding to $\alpha = 0.4 + 0.4i$ (blue), $\alpha = 0.7 + 0.7i$ (purple), $\alpha = 1.05 + 1.05i$ (yellow) and



(a) $n = 30, j = 2, \lambda = 10$ (b) $j = 2, \lambda = 10, \alpha = 1 + i$

Fig. 1 Lebesgue case: variation on α and n

$\alpha = 1.6 + 1.6i$ (green) were plotted. On the other hand, Fig 1b illustrates the behavior of the zeros of ψ_n as $n \to \infty$. The zeros corresponding to $n = 10$, $n = 20$, $n = 30$ and $n = 100$ were plotted. Notice the one of the zeros approaches the value of α as n increases, as stated in Proposition 4.

For the Bernstein-Szegő measure $|z-b|^{-2}\mu_\theta$, we have $\phi_n(z) = z^{n-1}(z-\bar{b})$, $n \geqslant 1$, where $|b| < 1$. Thus, we get

$$\phi_n^{(j)}(\alpha) = \frac{n!}{(n-j)!}\alpha^{n-j} - \bar{b}\,\frac{(n-1)!}{(n-j-1)!}\alpha^{n-j-1},$$

$$K_{n-1}^{(0,j)}(z,\alpha) = \sum_{k=j}^{n-1}(z^k - \bar{b}\,z^{k-1})\left(\frac{k!}{(k-j)!}\bar{\alpha}^{k-j} - \bar{b}\,\frac{(k-1)!}{(k-j-1)!}\bar{\alpha}^{k-j-1}\right),$$

$$K_n^{(j,j)}(\alpha,\alpha) = \sum_{k=j}^{n}\left|\frac{k!}{(k-j)!}\bar{\alpha}^{k-j} - \bar{b}\,\frac{(k-1)!}{(k-j-1)!}\bar{\alpha}^{k-j-1}\right|^2,$$

and, again, we obtain the expression of $\psi_n(z)$ using (5). We performed a similar numerical analysis of the zeros of ψ_{30} as a function of α as in the Lebesgue case. Figure 2a shows the behavior of the zeros of ψ_{30} for fixed j, λ, and b, and several values of α, namely $\alpha = 0.4 + 0.4i$ (blue), $\alpha = 0.7 + 0.7i$ (purple), $\alpha = 1.08 + 1.08i$ (yellow) and $\alpha = 2.2 + 2.2i$ (green). As before, the behavior of the zeros as $n \to \infty$ is illustrated in Fig. 2b, using the same values of n plotted in the Lebesgue case.

On the other hand, the behavior of the zeros of ψ_n when $\lambda \to \infty$ is shown in Fig. 3. According to Proposition 5, the zeros of ψ_n tend to a linear

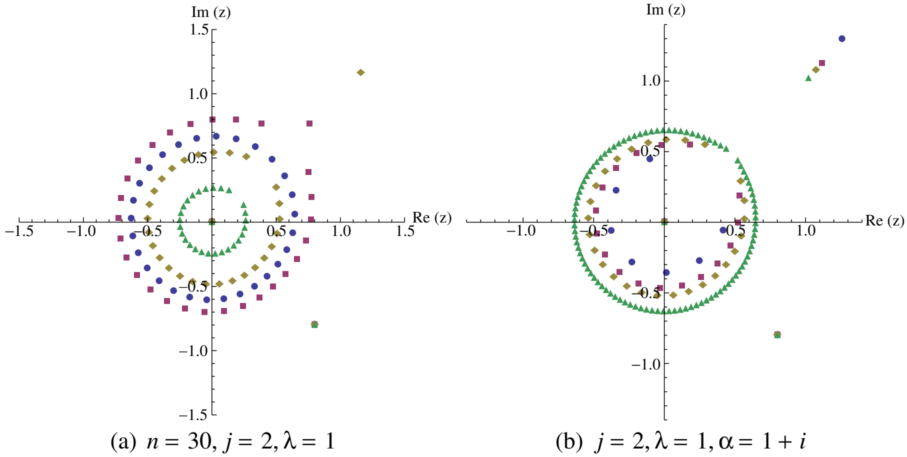
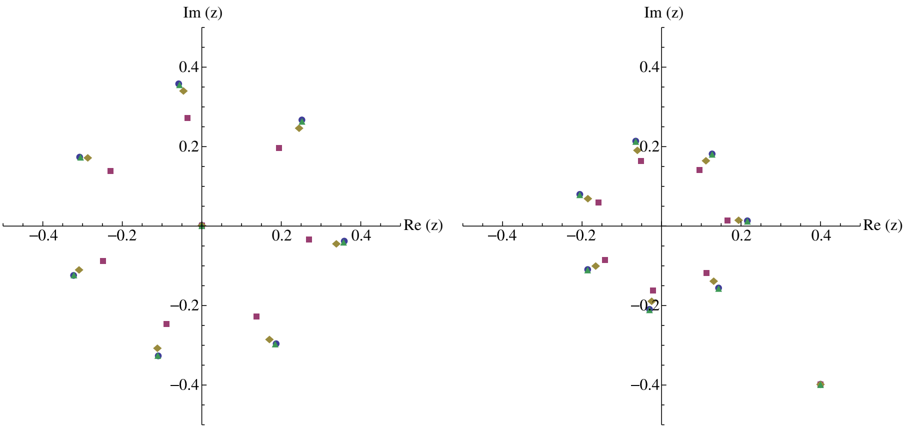


Fig. 2 Bernstein-Szegő case ($b = 0.8 + 0.8i$): variation on α and n



(a) $\alpha = 0.2 + 0.2i, \lambda = 0.1, 1, 10.$ (b) $b = 0.4 + 0.4i, \alpha = 0.1 + 0.1i, \lambda = 0.1, 1, 5.$

Fig. 3 Variation on λ (n=8, j=1)

combination of $\phi_n(z)$, $(z-\alpha)\phi_{n-1}(z, d\mu_1)$ and $(z-\alpha)\phi_{n-2}(z, d\mu_2)$, when $j = 1$. We computed the zeros of such polynomials for the Lebesgue (Fig. 3a) and Bernstein-Szegő (Fig. 3b) cases.

The points on the outer diameter correspond to the zeros of the above mentioned linear combination. Notice that, when λ increases, the zeros of ψ_n approach the outer diameter, according to Proposition 5.

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