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(1, 1)-q-coherent pairs

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Abstract In this paper, we introduce the concept of $(1, 1)$-q-coherent pair of linear functionals $(\mathcal{U}, \mathcal{V})$ as the q-analogue to the generalized coherent pair studied by Delgado and Marcellán in (Methods Appl Anal 11(2):273–266, 2004). This means that their corresponding sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$\frac{\left(D_q P_{n+1}\right)(x)}{[n+1]_q} + a_n \frac{\left(D_q P_n\right)(x)}{[n]_q} = R_n(x) + b_n R_{n-1}(x), \quad a_n \neq 0,\ n \geq 1,$$

$[n]_q = \frac{q^n-1}{q-1}$, $0 < q < 1$. We prove that if a pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is a $(1, 1)$-q-coherent pair, then at least one of them must be q-semiclassical of class at most 1, and these functionals are related by an expression $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$ where $\sigma(x)$ and $\rho(x)$ are polynomials of degrees ≤ 3 and 1, respectively. Finally, the q-classical case is studied.

Keywords Linear functionals · q-orthogonal polynomials · q-coherent pairs

Mathematics Subject Classifications (2010) 42C05

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1 Introduction

The notion of *coherent pair* was introduced by Iserles et al. in [12]. They stated that a pair of positive Borel measures (μ_0, μ_1) supported on the real line is, with our terminology, a $(1, 0)$-*coherent pair* if and only if there exist nonzero constants $\{a_n\}_{n\geq 1}$ such that their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$\frac{P'_{n+1}(x)}{n+1} + a_n \frac{P'_n(x)}{n} = R_n(x)\,, \quad a_n \neq 0,\ n \geq 1. \tag{1.1}$$

Moreover, this condition of coherence emerged as a sufficient condition for obtaining the relation

$$Q_{n+1}(x;\lambda) + c_n(\lambda) Q_n(x;\lambda) = P_{n+1}(x) + \frac{n+1}{n} a_n P_n(x)\,, \quad n \geq 1, \tag{1.2}$$

where $\{c_n(\lambda)\}_{n\geq 1}$ are rational functions in $\lambda > 0$ and $\{Q_n(x;\lambda)\}_{n\geq 0}$ is the SMOP associated with the Sobolev inner product

$$\langle p(x), q(x)\rangle_\lambda = \int_{-\infty}^{\infty} p(x)q(x)\ d\mu_0 + \lambda \int_{-\infty}^{\infty} p'(x)q'(x)\, d\mu_1\,, \quad \lambda > 0,\ p, q \in \mathbb{P}. \tag{1.3}$$

Also, they introduced the notion of symmetrically coherent pair, when the measures μ_0, μ_1 are symmetric and the subscripts in (1.1) are changed appropriately.

In [15] all $(1, 0)$-coherent pairs $(\mathcal{U}, \mathcal{V})$ of regular linear functionals, when one of them is classical are deduce. Later on, Meijer [20] determined all $(1, 0)$-coherent pairs $(\mathcal{U}, \mathcal{V})$ of regular linear functionals. He proved that at least one of them must be classical (Laguerre or Jacobi) and they are related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \leq 2$, $\deg(\rho(x)) = 1$. Indeed, he proved the above result by Marcellán and Petronilho is a necessary and sufficient condition fro the existence of a $(1, 0)$ coherent pair of linear functionals. Furthermore, Meijer also determined all symmetrically $(1, 0)$-coherent pairs (one of them must be Hermite or Gegenbauer).

Delgado and Marcellán [9] extended the notion of coherent pair to *generalized coherent pair* (in our terminology, $(1, 1)$-*coherent pair*), studying the relation

$$\frac{P'_{n+1}(x)}{n+1} + a_n \frac{P'_n(x)}{n} = R_n(x) + b_n R_{n-1}(x)\,, \quad a_n \neq 0,\ n \geq 1. \tag{1.4}$$

They proved that this is a necessary and sufficient condition for the relation (1.2). Also, they determined all $(1, 1)$-coherent pairs of linear functionals $(\mathcal{U}, \mathcal{V})$. Indeed, at least one of them must be semiclassical of class at most 1 and they are related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \leq 3$, $\deg(\rho(x)) = 1$, generalizing the results obtained by Meijer for $(1, 0)$-coherent pairs. Also, a generalization of this situation to symmetrically coherent pairs is stated by Delgado and Marcellán in [10].

On the other hand, Hahn [11] noted that characterizations of the classical SMOP based on derivatives and differential equations are too restrictive [7]. He used a more general operator, the so called *q-difference operator* given by

$$\text{for } q \neq 1, \quad \left(D_q f\right)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \; x \neq 0, \quad \text{and} \quad \left(D_q f\right)(0) = f'(0),$$

the latter by continuity and assuming $f'(0)$ exists. Note that $\lim_{q\uparrow 1} \left(D_q f\right)(x) = f'(x)$ if f is a differentiable function.

Thus, Area et al. [3, 4], extended the notion of coherent pair to *q-coherent pair* (in our terminology, $(1,0)$-*q-coherent pair*) of linear functionals $(\mathcal{U}, \mathcal{V})$ as follows. The corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$\frac{\left(D_q P_{n+1}\right)(x)}{[n+1]_q} + a_n \frac{\left(D_q P_n\right)(x)}{[n]_q} = R_n(x)\,, \quad a_n \neq 0, \; n \geq 1,$$

$0 < q < 1$, $[n]_q = \frac{q^n - 1}{q-1}$. They concluded that if $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-q-coherent pair of linear functionals then at least one of them must be q-classical and they are related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \leq 2$, $\deg(\rho(x)) = 1$. Also, they determined all q-coherent pairs of positive-definite linear functionals when $\mathcal{U}$ or $\mathcal{V}$ is either Big q-Jacobi, or Little q-Jacobi or Little q-Laguerre/Wall. Finally, by using limit processes ($q \uparrow 1$) in the Little q-Jacobi and Little q-Laguerre/Wall cases, they recovered the classification of all $(1,0)$-coherent pairs of linear functionals given by Meijer in [20].

In this work, we extend the concept of $(1,1)$-coherent pair of linear functionals $(\mathcal{U}, \mathcal{V})$ studied by Delgado and Marcellán in [9], to $(1,1)$-*q-coherent pair* of linear functionals as follows. The corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$\frac{\left(D_q P_{n+1}\right)(x)}{[n+1]_q} + a_n \frac{\left(D_q P_n\right)(x)}{[n]_q} = R_n(x) + b_n R_{n-1}(x)\,, \quad a_n \neq 0, \; n \geq 1.$$

Note that from the study of q-coherent pairs it is possible to recover the properties of coherent pairs in the continuous case ((1,0)-coherence and (1,1)-coherence) taking limits when $q \uparrow 1$.

An interesting direction in the work of (1,1)-q-coherent pairs is to study the orthogonal polynomials with respect to a Sobolev inner product where the regular linear functionals that determine this product constitutes a $(1,1)$ or (1,0)-q-coherent pair. For example, the Little q-Laguerre–Sobolev polynomials satisfy (1.2) as a consequence of the fact that the SMOP $\{P_n(x)\}_{n\geq 0}$ is orthogonal with respect to the Little q-Laguerre functional $\mathcal{U}$ taking into account $(\mathcal{U}, \mathcal{U})$ is a $(1,0)$-q-coherent pair [3, 5].

The structure of the manuscript is as follows. In Section 2 we give basic definitions and results which will be helpful in the following sections. In Section 3 we analyze $(1,1)$-q-coherent pairs (respectively, $(1,0)$-q-coherent pairs) of linear functionals. We prove that if $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-q-coherent pair (resp. $(1,0)$-q-coherent pair) of linear functionals then at least one of them must be q-semiclassical of class at most 1 (resp. q-classical) and they are related by $\sigma(x)\mathcal{U} = \rho(x)\mathcal{V}$, with $\deg(\sigma(x)) \le 3$, $\deg(\rho(x)) = 1$ (resp. ≤ 2 and 1), and therefore the other functional is q-semiclassical of class at most 5 (resp. at most 3). In Section 4 we study the case when $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-q-coherent pair (resp. $(1,0)$-q-coherent pair) of linear functionals such that $\mathcal{U}$ is q-classical and we get that $\mathcal{V}$ is q-semiclassical of class at most 2 (resp. at most 1).

2 Basic background

2.1 Linear functionals

We will denote by $\mathbb{P}$ the linear space of polynomials with complex coefficients and $\mathbb{P}^*$ is its algebraic dual space consisting of all linear functionals $\mathcal{U} : \mathbb{P} \to \mathbb{C}$. $\langle \mathcal{U}, p(x) \rangle$ will denote the image of polynomial $p(x)$ by $\mathcal{U}$.

Every sequence of monic polynomials $\{P_n(x)\}_{n\ge0}$, with $\deg(P_n(x)) = n$, is a basis of $\mathbb{P}$. Then, there exists a unique sequence $\{\wp_n\}_{n\ge0} \subset \mathbb{P}^*$, called the *dual basis of* $\{P_n(x)\}_{n\ge0}$, such that $\langle \wp_n, P_m(x) \rangle = \delta_{n,m}$, where $\delta_{n,m}$ denotes the Kronecker Delta. So, if $\mathcal{U} \in \mathbb{P}^*$ then $\mathcal{U} = \sum_n \lambda_n \wp_n$, where $\lambda_n = \langle \mathcal{U}, P_n(x) \rangle$.

Let $\mathcal{U} \in \mathbb{P}^*$, $q(x) \in \mathbb{P}$, and $a \in \mathbb{C}$. We define the linear functionals

$$\langle q(x)\mathcal{U}, p(x) \rangle = \langle \mathcal{U}, q(x)p(x) \rangle, \ \left\langle (x-a)^{-1}\mathcal{U}, p(x) \right\rangle = \left\langle \mathcal{U}, \frac{p(x)-p(a)}{x-a} \right\rangle, \ \ p \in \mathbb{P}.$$

Note that $(x-a)(x-a)^{-1}\mathcal{U} = \mathcal{U}$, but $(x-a)^{-1}(x-a)\mathcal{U} = \mathcal{U} - \langle \mathcal{U}, 1 \rangle \delta_a$, where δ_a is the *Dirac Delta linear functional at a*, defined by $\langle \delta_a, p(x) \rangle = p(a)$, $\forall p \in \mathbb{P}$.

The *q-difference of a polynomial* $p(x)$, with $q \in \mathbb{C} \setminus \{0, 1\}$, is defined by

$$\left(D_q p\right)(x) = \frac{p(qx) - p(x)}{(q-1)x} \text{ for } x \neq 0, \text{ and, by continuity, } \left(D_q p\right)(0) = p'(0). \tag{2.1}$$

Note that, for $a \in \mathbb{C} \setminus \{0\}$ and $p(x), r(x) \in \mathbb{P}$, we get

$$\Big(D_q\big[p(ax)\big]\Big)(x) = a\Big(D_q\big[p(x)\big]\Big)(ax), \tag{2.2}$$

$$\left(D_q\left[p\,r\right]\right)(x) = p(x)\left(D_q r\right)(x) + r(qx)\left(D_q p\right)(x), \tag{2.3}$$

$$\left(D_q p\right)(x) = \left(D_{q^{-1}} p\right)(qx). \tag{2.4}$$

From now, we assume that $0 < q < 1$ and we introduce the following notation

$$P_n^{[1,q]}(x) = \frac{\left(D_q P_{n+1}\right)(x)}{[n+1]_q}, \text{ where } P_n(x) \in \mathbb{P} \text{ and } \deg\left(P_n(x)\right) = n \in \mathbb{N},$$

$$[0]_q = 0, \quad [n]_q = \frac{q^n - 1}{q - 1}, \quad n \ge 1.$$

The *q-difference of a linear functional* $\mathcal{U}$ is the linear functional $D_q\mathcal{U}$ given by

$$\left\langle D_q\mathcal{U}, p(x) \right\rangle = -\left\langle \mathcal{U}, \left(D_q p\right)(x) \right\rangle, \quad \forall p \in \mathbb{P}. \tag{2.5}$$

Note that the q-difference operator D_q converges to the derivative operator $D = \frac{d}{dx}$ when $q \uparrow 1$. In fact, $(D_q p)(x)$ converges to $\frac{d}{dx} p(x)$ in $\mathbb{P}$ and $D_q \mathcal{U}$ converges to $D\mathcal{U}$ in $\mathbb{P}^*$, with $D\mathcal{U}$ given by $\langle D\mathcal{U}, p(x)\rangle = -\langle \mathcal{U}, p'(x)\rangle$, $\forall p(x) \in \mathbb{P}$.

Proposition 1 *Given $\mathcal{U} \in \mathbb{P}^*$ and $p(x) \in \mathbb{P}$, we get*

$$D_q \left[p(x)\mathcal{U}\right] = p\left(q^{-1}x\right) D_q\mathcal{U} + q^{-1}\left(D_q p\right)\left(q^{-1}x\right)\mathcal{U}. \tag{2.6}$$

Proof If $r(x) \in \mathbb{P}$, then

$$\begin{aligned}
\left\langle D_q\left[p(x)\mathcal{U}\right], r(x)\right\rangle &\overset{(2.5)}{=} -\left\langle p(x)\mathcal{U}, (D_q r)(x)\right\rangle \overset{s(qx):=p(x)}{=} -\left\langle \mathcal{U}, s(qx)(D_q r)(x)\right\rangle \\
&\overset{(2.3)}{=} -\left\langle \mathcal{U}, \left(D_q\left[sr\right]\right)(x) - r(x)\left(D_q s\right)(x)\right\rangle \\
&\overset{(2.5)}{=} \left\langle D_q\mathcal{U}, s(x)r(x)\right\rangle + \left\langle \mathcal{U}, r(x)\left(D_q s\right)(x)\right\rangle \\
&\overset{s(qx)=p(x)}{=} \left\langle p\left(q^{-1}x\right) D_q\mathcal{U} + \left(D_q\left[p\left(q^{-1}x\right)\right]\right)(x)\mathcal{U}, r(x)\right\rangle.
\end{aligned}$$

□

2.2 Orthogonal polynomials

$\mathcal{U} \in \mathbb{P}^*$ is said to be a *quasi-definite or regular* linear functional [8] if the leading principal submatrices of the Hankel matrix $H = (u_{i+j})_{i,j=0}^{\infty}$ associated with the *moments* of the functional ($u_k = \langle \mathcal{U}, x^k\rangle$, $k \in \mathbb{N}\bigcup\{0\}$) are nonsingular. Equivalently, there exists a unique sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ such that

$$\deg(P_n(x)) = n \text{ and } \langle \mathcal{U}, P_n(x)P_m(x)\rangle = k_n^P \delta_{n,m} \text{ with } k_n^P \neq 0,\ n, m \in \mathbb{N}\bigcup\{0\}.$$

$\{P_n(x)\}_{n\geq 0}$ is said to be a *sequence of monic orthogonal polynomials (SMOP)* with respect to the linear functional $\mathcal{U}$. Furthermore, if the principal leading submatrices of H are definite positive, then $\mathcal{U}$ is said to be *positive definite*.

Also, (*Favard Theorem* [8]), a sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ is a SMOP with respect to $\mathcal{U}$ if and only if there exist $\{\alpha_n^P\}_{n\geq 0}, \{\beta_n^P\}_{n\geq 0} \subset \mathbb{C}$ with $\beta_n^P \neq 0$, $n \geq 2$, such that the following *three-term recurrence relation (RRTT)* holds

$$P_n(x) = \left(x - \alpha_n^P\right) P_{n-1}(x) - \beta_n^P P_{n-2}(x), \quad n \geq 1, \quad P_0(x) = 1, \quad P_{-1}(x) = 0 \tag{2.7}$$

Moreover, $\mathcal{U}$ is positive definite if and only if $\alpha_n^P \in \mathbb{R}$ and $\beta_{n+1}^P > 0, n \geq 1$.

Now, we give some results relating the dual basis of a SMOP and its respective linear functional and q-differences.

Proposition 2 *Let $\{P_n(x)\}_{n\geq 0}$ be the SMOP with respect to $\mathcal{U}$, let $\{\wp_n\}_{n\geq 0}$ be its corresponding dual basis, and let $\{\wp_n^{[1,q]}\}_{n\geq 0}$ be the dual basis of the sequence of monic polynomials $\{P_n^{[1,q]}(x)\}_{n\geq 0}$. Then*

$$\mathcal{U} = \langle \mathcal{U}, 1\rangle \, \wp_0, \quad \wp_n = \frac{P_n(x)}{\left\langle \mathcal{U}, P_n^2(x)\right\rangle} \mathcal{U}, \quad n \in \mathbb{N}, \tag{2.8}$$

$$D_q \wp_n^{[1,q]} = -[n+1]_q \wp_{n+1}\,, \quad n \in \mathbb{N}. \tag{2.9}$$

Proof It is immediate that $\mathcal{U} = \sum_{n\geq 0} \langle \mathcal{U}, P_n(x)\rangle \wp_n = \langle \mathcal{U}, 1\rangle \wp_0$ (see [16, 17] for more details). □

Corollary 3 *Under the assumptions of the previous proposition we get*

$$D_q \wp_n^{[1,q]} = -[n+1]_q \frac{P_{n+1}(x)}{\left\langle \mathcal{U}, P_{n+1}^2(x)\right\rangle} \mathcal{U}, \quad n \in \mathbb{N}.$$

In particular, if $\{P_n^{[1,q]}(x)\}_{n\geq 0}$ is a SMOP with respect to $\mathcal{U}^{[1,q]} \in \mathbb{P}^*$, then

$$D_q \mathcal{U}^{[1,q]} = -\frac{\left\langle \mathcal{U}^{[1,q]}, 1\right\rangle}{\langle \mathcal{U}, P_1^2(x)\rangle} P_1(x)\mathcal{U}.$$

2.3 q-semiclassical and q-classical linear functionals

$\mathcal{U} \in \mathbb{P}^*$ is called *q-semiclassical* if it is regular and there exist polynomials $\sigma(x)$ and $\tau(x)$ such that $\mathcal{U}$ satisfies the distributional equation (q-Pearson equation)

$$D_q(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U},^1 \tag{2.10}$$

with $\deg(\tau(x)) \geq 1$. In these conditions, the *class of $\mathcal{U}$* is defined by the non-negative integer $s := \min\max\{\deg(\sigma(x)) - 2, \deg(\tau(x)) - 1\}$, where the minimum is taken among all pairs of polynomials $(\sigma(x), \tau(x))$ such that (2.10) holds. If $\sigma(x)$ is monic, the pair that determines the class is unique. We also say that the SMOP associated with $\mathcal{U}$ is a *q-semiclassical SMOP of class s* if the class of $\mathcal{U}$ is s. For some characterizations of the q-semiclassical linear functionals, see [18].

Proposition 4 *If the regular linear functionals $\mathcal{U}, \mathcal{V}$ are related by an expression of rational type, i.e., there exist nonzero polynomials $p(x)$ and $r(x)$ such that*

$$p(x)\mathcal{U} = r(x)\mathcal{V}, \tag{2.11}$$

then, $\mathcal{U}$ is q-semiclassical if and only if so is $\mathcal{V}$. Moreover, if the class of $\mathcal{U}$ is s, then the class of $\mathcal{V}$ is at most $s + \deg(p(x)) + \deg(r(x))$.

[1] This definition implies that $\sigma(x)$ can't be zero and $\tau(x)$ can't be a constant, otherwise, $\mathcal{U}$ would lose its regular condition.

Proof If $\mathcal{U}$ is q-semiclassical, then there exist nonzero polynomials $\sigma_u(x)$, $\tau_u(x)$, $\deg(\tau_u(x)) \geq 1$, such that $D_q[\sigma_u(x)\mathcal{U}] = \tau_u(x)\mathcal{U}$. We get

$$\begin{aligned} D_q\left[\sigma_v(x)\mathcal{V}\right] &= D_q\left[p(qx)r(x)\sigma_u(x)\mathcal{V}\right] \overset{(2.11)}{=} D_q\left[p(qx)p(x)\sigma_u(x)\mathcal{U}\right] \\ &= D_q[p(x)p\left(q^{-1}x\right)]\sigma_u(x)\mathcal{U} + p(x)p\left(q^{-1}x\right) D_q\left[\sigma_u(x)\mathcal{U}\right] \\ &\overset{(2.11)}{=} \left[\frac{p(qx)-p\left(q^{-1}x\right)}{(q-1)x}\sigma_u(x) + p\left(q^{-1}x\right)\tau_u(x)\right] r(x)\mathcal{V} = \tau_v(x)\mathcal{V}. \end{aligned}$$

Besides, since $\mathcal{V}$ is a regular linear functional, $\sigma_v(x) \neq 0$ and $\tau_v(x)$ is not a constant. The proof of the class follows from an easy computation. □

$\mathcal{U} \in \mathbb{P}^*$ is said to be *q-classical* if it is q-semiclassical of class $s = 0$, i.e.,

$$D_q[\sigma(x)\mathcal{U}] = \tau(x)\mathcal{U}, \quad \text{with } \deg(\sigma(x)) \leq 2, \ \deg(\tau(x)) = 1. \tag{2.12}$$

Its corresponding SMOP is said to be *q-classical SMOP*. For some characterizations of the q-classical SMOP and its classification see [2, 6, 7, 11, 13, 14, 19].

Theorem 5 [11] *Let $\mathcal{U}$ be a regular lineal functional with corresponding SMOP $\{P_n(x)\}_{n\geq 0}$. The following statements are equivalent*

(i) $\{P_n(x)\}_{n\geq 0}$ *is a q-classical SMOP and $\mathcal{U}$ satisfies* (2.12).
(ii) $\{P_n^{[1,q]}(x)\}_{n\geq 0}$ *is a SMOP with respect to $\mathcal{U}^{[1,q]} \in \mathbb{P}^*$.*

Moreover, $\mathcal{U}^{[1,q]} = \sigma(x)\mathcal{U}$ and $\{P_n^{[1,q]}(x)\}_{n\geq 0}$ is also a q-classical SMOP of the same type as $\{P_n(x)\}_{n\geq 0}$ because $D_q[\sigma(qx)\mathcal{U}^{[1,q]}] = [\tau(x) + (D_q\sigma)(x)]\mathcal{U}^{[1,q]}$ holds.

3 (1, 1)-q-coherent pairs of linear functionals

A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a $(1,1)$-*q-coherent pair* if their corresponding SMOP, $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$, satisfy

$$P_n^{[1,q]}(x) + a_n P_{n-1}^{[1,q]}(x) = R_n(x) + b_n R_{n-1}(x), \quad a_n \neq 0, \ n \geq 1. \tag{3.1}$$

We also say that $(\{P_n(x)\}_{n\geq 0}, \{R_n(x)\}_{n\geq 0})$ is a $(1,1)$-q-coherent pair. If $b_n = 0$ for all $n \geq 1$, the pair of either linear functionals or SMOP is a $(1,0)$-*q-coherent pair*.

Remark 6 If $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-q-coherent pair given by (3.1) then, $a_1 \neq b_1$ if and only if $P_n^{[1,q]}(x) \neq R_n(x)$ for all $n \geq 1$.

From now on we assume that $a_1 \neq b_1$.

Remark 7 If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-q-coherent pair given by (3.1), then for $n \geq 1$

$$P_n^{[1,q]}(x) = R_n(x) + (b_n - a_n)R_{n-1}(x) + \sum_{k=2}^{n}(-1)^{k-1}a_n a_{n-1}\cdots a_{n-(k-2)}\left(b_{n-(k-1)} - a_{n-(k-1)}\right)R_{n-k}(x). \tag{3.2}$$

Lemma 8 *If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-q-coherent pair given by (3.1), then there exists a monic polynomial $\gamma_n(x)$ of degree $n \geq 1$ such that*

$$\left\langle \gamma_n(x)\mathcal{V},\, P_m^{[1,q]}(x)\right\rangle = 0, \quad \textit{for } m \geq n+1.^2$$

Proof If $\gamma_n(x) = R_n(x) + \sum_{j=0}^{n-1} A_{j,n}R_j(x)$, then for $n \in \mathbb{N}$

$$\left\langle \gamma_n(x)\mathcal{V},\, P_{n+1}^{[1,q]}(x)\right\rangle \overset{(3.2)}{=} (b_{n+1} - a_{n+1})\left\langle \mathcal{V},\, R_n^2(x)\right\rangle + \sum_{k=2}^{n+1}(-1)^{k-1} a_{n+1}\cdots a_{n+1-(k-2)}\left(b_{n+1-(k-1)} - a_{n+1-(k-1)}\right) A_{n+1-k,n}\left\langle \mathcal{V},\, R_{n+1-k}^2(x)\right\rangle.$$

Since $a_1 \neq b_1$, we can choose real numbers $A_{0,n}, \ldots, A_{n-1,n}$, not all zero, such that $\langle \gamma_n(x)\mathcal{V},\, P_{n+1}^{[1,q]}(x)\rangle = 0$ for $n \geq 1$. Also, from (3.1) we get $\langle \gamma_n(x)\mathcal{V},\, P_{m+1}^{[1,q]}(x)\rangle = -a_{m+1}\langle \gamma_n(x)\mathcal{V},\, P_m^{[1,q]}(x)\rangle$ for $n \leq m-1$. Therefore, the lemma follows. □

Remark 9 In Lemma 8, we can choose $A_{1,n} = \cdots = A_{n-1,n} = 0$. Thus,

$$\gamma_n(x) = R_n(x) + A_{0,n} = R_n(x) + \frac{(-1)^{n+1}(b_{n+1} - a_{n+1})\left\langle \mathcal{V},\, R_n^2(x)\right\rangle}{a_{n+1}a_n\cdots a_3 a_2(b_1 - a_1)\langle \mathcal{V}, 1\rangle}, \quad n \geq 1. \tag{3.3}$$

Moreover, if $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-q-coherent pair, $\gamma_n(x)$ is given by

$$\gamma_n^{(1,0)}(x) = R_n(x) + \frac{(-1)^{n+1}\langle \mathcal{V},\, R_n^2(x)\rangle}{a_n\cdots a_3 a_2 a_1\langle \mathcal{V}, 1\rangle}, \quad n \geq 1. \tag{3.4}$$

Lemma 10 *If $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-q-coherent pair given by (3.1), then*

$$D_q[\gamma_n(x)\mathcal{V}] = -\varphi_{n+1}(x)\mathcal{U}, \quad n \geq 1, \tag{3.5}$$

where $\gamma_n(x)$ is the polynomial introduced in Lemma 8 and

$$\varphi_{n+1}(x) = \sum_{k=0}^{n}\frac{[k+1]_q\left\langle \gamma_n(x)\mathcal{V},\, P_k^{[1,q]}(x)\right\rangle}{\left\langle \mathcal{U},\, P_{k+1}^2(x)\right\rangle}P_{k+1}(x), \quad n \geq 1. \tag{3.6}$$

[2] If $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-q-coherent pair, this equality holds for $m \geq n$ and $\gamma_n(x)$ will be denoted by $\gamma_n^{(1,0)}(x)$.

If $(\mathcal{U}, \mathcal{V})$ *is a* $(1,0)$*-q-coherent pair, the polynomial* $\varphi_{n+1}(x)$ *is given by*

$$\varphi_{n+1}^{(1,0)}(x) = \sum_{k=0}^{n-1} \frac{[k+1]_q \left\langle \gamma_n^{(1,0)}(x)\mathcal{V}, P_k^{[1,q]}(x) \right\rangle}{\left\langle \mathcal{U}, P_{k+1}^2(x) \right\rangle} P_{k+1}(x), \quad n \geq 1. \tag{3.7}$$

Note that $\deg(\varphi_{n+1}(x)) \leq n+1$ *and* $\deg(\varphi_{n+1}^{(1,0)}(x)) \leq n$.

Proof Let $\{\wp_n\}_{n\geq 0}$ and $\{\wp_n^{[1,q]}\}_{n\geq 0}$ be the dual bases of $\{P_n(x)\}_{n\geq 0}$ and $\{P_n^{[1,q]}(x)\}_{n\geq 0}$, respectively. Since $\gamma_n(x)\mathcal{V} = \sum_{k\geq 0} \lambda_{k,n} \wp_k^{[1,q]}$ where $\lambda_{k,n} = \langle \gamma_n(x)\mathcal{V}, P_k^{[1,q]}(x)\rangle$ and, from Lemma 8, $\lambda_{k,n} = 0$ for $k \geq n+1$ and $n \geq 1$ (respectively, $k \geq n$, if $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-q-coherent pair), then $\gamma_n(x)\mathcal{V} = \sum_{k=0}^{n} \lambda_{k,n} \wp_k^{[1,q]}$. Thus,

$$D_q\left[\gamma_n(x)\mathcal{V}\right] \overset{(2.9)}{=} \sum_{k=0}^{n} \lambda_{k,n}\left(-[k+1]_q \wp_{k+1}\right) \overset{(2.8)}{=} -\sum_{k=0}^{n} \lambda_{k,n}[k+1]_q \frac{P_{k+1}(x)}{\left\langle \mathcal{U}, P_{k+1}^2(x)\right\rangle}\mathcal{U}.$$

□

Corollary 11 *If* $(\mathcal{U}, \mathcal{V})$ *is a* $(1,1)$*-q-coherent pair given by* (3.1)*, then there exist polynomials* $\alpha(x)$*,* $\beta(x)$*, and* $\phi(x)$ *such that*

$$\alpha(x)\mathcal{U} = \beta(x)\mathcal{V}, \tag{3.8}$$

$$\alpha(x) D_q\mathcal{V} = \phi(x)\mathcal{V}, \tag{3.9}$$

$$\phi(x)\mathcal{U} = \beta(x) D_q\mathcal{V}, \tag{3.10}$$

where

$$\alpha(x) = q\left[\gamma_2\left(q^{-1}x\right)\varphi_2(x) - \gamma_1\left(q^{-1}x\right)\varphi_3(x)\right], \tag{3.11}$$

$$\beta(x) = \gamma_1\left(q^{-1}x\right)\left(D_q\gamma_2\right)\left(q^{-1}x\right) - \gamma_2\left(q^{-1}x\right), \tag{3.12}$$

$$\phi(x) = \varphi_3(x) - \left(D_q\gamma_2\right)\left(q^{-1}x\right)\varphi_2(x), \tag{3.13}$$

with $\deg(\alpha(x)) \leq 4$, $\deg(\beta(x)) = 2$, $\deg(\phi(x)) \leq 3$.[3] *Besides,*

$$\phi(x)\gamma_n\left(q^{-1}x\right) + q^{-1}\alpha(x)\left(D_q\gamma_n\right)\left(q^{-1}x\right) = -\varphi_{n+1}(x)\beta(x), \quad n \geq 1, \tag{3.14}$$

where $\gamma_n(x)$ *and* $\varphi_{n+1}(x)$ *are the polynomials given in Lemma* 10.

Proof From (3.5) for $n=1$ and $n=2$ we get

$$\gamma_1\left(q^{-1}x\right) D_q\mathcal{V} + q^{-1}\mathcal{V} = -\varphi_2(x)\mathcal{U}, \tag{3.15}$$

$$\gamma_2\left(q^{-1}x\right) D_q\mathcal{V} + q^{-1}\left(D_q\gamma_2\right)\left(q^{-1}x\right)\mathcal{V} = -\varphi_3(x)\mathcal{U}. \tag{3.16}$$

[3] If $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-q-coherent pair, from Lemma 10, $\deg\left(\alpha^{(1,0)}(x)\right) \leq 3$, $\deg\left(\beta^{(1,0)}(x)\right) = 2$, and $\deg\left(\phi^{(1,0)}(x)\right) \leq 2$.

Then, by elimination of $D_q\mathcal{V}$, $\mathcal{U}$, and $\mathcal{V}$ we obtain (3.8)–(3.10), respectively. Besides, the degrees of $\alpha(x)$ and $\phi(x)$ follows, and $\deg(\beta(x)) = 2$ because its leading coefficient is $q^{-1}[2]_q q^{-1} - q^{-2} = q^{-1} \neq 0$. Finally,

$$
\begin{aligned}
-\varphi_{n+1}(x)\beta(x)\mathcal{V} &\overset{(3.8)}{=} -\alpha(x)\varphi_{n+1}(x)\mathcal{U} \overset{(3.5)}{=} \alpha(x) D_q[\gamma_n(x)\mathcal{V}] \\
&\overset{(3.9)}{=} \left[\gamma_n\left(q^{-1}x\right)\phi(x) + q^{-1}\alpha(x)\left(D_q\gamma_n\right)\left(q^{-1}x\right)\right]\mathcal{V}, \quad n \geq 1.
\end{aligned}
$$

□

Let $(\mathcal{U}, \mathcal{V})$ be a $(1, 1)$-q-coherent pair. The leading coefficients of $\varphi_2(x)$ and $\varphi_3(x)$ are, respectively

$$
[2]_q \frac{\left\langle \mathcal{V}, \gamma_1(x) P_1^{[1,q]}(x)\right\rangle}{\left\langle \mathcal{U}, P_2^2(x)\right\rangle} \underset{(3.2)}{\overset{(3.3)}{=}} \frac{[2]_q \left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_2 \left\langle \mathcal{U}, P_2^2(x)\right\rangle} b_2, \tag{3.17}
$$

$$
[3]_q \frac{\left\langle \gamma_2(x)\mathcal{V}, P_2^{[1,q]}(x)\right\rangle}{\left\langle \mathcal{U}, P_3^2(x)\right\rangle} \underset{(3.2)}{\overset{(3.3)}{=}} \frac{[3]_q \left\langle \mathcal{V}, R_2^2(x)\right\rangle}{a_3 \left\langle \mathcal{U}, P_3^2(x)\right\rangle} b_3. \tag{3.18}
$$

Thus, the leading coefficients of $\beta(x)$, $\alpha(x)$, and $\phi(x)$ are, respectively q^{-1},

$$
\left[q^{-1} \frac{[2]_q \left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_2 \left\langle \mathcal{U}, P_2^2(x)\right\rangle} b_2 - \frac{[3]_q \left\langle \mathcal{V}, R_2^2(x)\right\rangle}{a_3 \left\langle \mathcal{U}, P_3^2(x)\right\rangle} b_3 \right], \tag{3.19}
$$

$$
\left[\frac{[3]_q \left\langle \mathcal{V}, R_2^2(x)\right\rangle}{a_3 \left\langle \mathcal{U}, P_3^2(x)\right\rangle} b_3 - [2]_q q^{-1} \frac{[2]_q \left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_2 \left\langle \mathcal{U}, P_2^2(x)\right\rangle} b_2 \right]. \tag{3.20}
$$

If $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-q-coherent pair, then the leading coefficients of $\varphi_2^{(1,0)}(x)$ and $\varphi_3^{(1,0)}(x)$ are, respectively

$$
\frac{\left\langle \mathcal{V}, \gamma_1^{(1,0)}(x)\right\rangle}{\left\langle \mathcal{U}, P_1^2(x)\right\rangle} \overset{(3.4)}{=} \frac{\left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_1 \left\langle \mathcal{U}, P_1^2(x)\right\rangle}, \tag{3.21}
$$

$$
[2]_q \frac{\left\langle \mathcal{V}, \gamma_2^{(1,0)}(x) P_1^{[1,q]}(x)\right\rangle}{\left\langle \mathcal{U}, P_2^2(x)\right\rangle} \underset{(3.4)}{\overset{(3.2)}{=}} [2]_q \frac{\left\langle \mathcal{V}, R_2^2(x)\right\rangle}{a_2 \left\langle \mathcal{U}. P_2^2(x)\right\rangle}. \tag{3.22}
$$

Hence, $\deg\left(\varphi_2^{(1,0)}(x)\right) = 1$ and $\deg\left(\varphi_3^{(1,0)}(x)\right) = 2$. Besides, $\operatorname{leadcoeff}(\beta^{(1,0)}(x)) = \operatorname{leadcoeff}(\beta(x))$ and the leading coefficients of $\alpha^{(1,0)}(x)$ and $\phi^{(1,0)}(x)$ are, respectively

$$
\left[q^{-1} \frac{\left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_1 \left\langle \mathcal{U}, P_1^2(x)\right\rangle} - [2]_q \frac{\left\langle \mathcal{V}, R_2^2(x)\right\rangle}{a_2 \left\langle \mathcal{U}, P_2^2(x)\right\rangle} \right], \tag{3.23}
$$

$$
\left[[2]_q \frac{\left\langle \mathcal{V}, R_2^2(x)\right\rangle}{a_2 \left\langle \mathcal{U}, P_2^2(x)\right\rangle} - [2]_q q^{-1} \frac{\left\langle \mathcal{V}, R_1^2(x)\right\rangle}{a_1 \left\langle \mathcal{U}, P_1^2(x)\right\rangle} \right]. \tag{3.24}
$$

Now, we will prove that if $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-q-coherent (resp. $(1,0)$-q-coherent) pair of linear functionals, then at least one of them must be q-semiclassical of class at most 1 (resp. q-classical) and they are related by an expression of rational type. Indeed, we consider the zeros of the polynomial $\beta(x)$ given by (3.12). We have that

$$\left(D_q\beta\right)(x)=q^{-1}[2]_q\gamma_1(x), \tag{3.25}$$

and if ξ_1 and ξ_2 denote the zeros of $\beta(x)$

$$\beta(x)=q^{-1}(x-\xi_1)(x-\xi_2),\quad \left(D_q\beta\right)(x)=q^{-1}[2]_q\left[x-\frac{\xi_1+\xi_2}{q+1}\right]. \tag{3.26}$$

Thus, the possible cases to study are the following: (i) ξ and $q^{-1}\xi$ are the zeros of $\beta(x)$, i.e., $\beta(\xi)=0$ and $\left(D_q\beta\right)(q^{-1}\xi)=0$ (Theorem 12). (ii) ξ_1 and ξ_2 are the zeros such that $\xi_1\neq\xi_2$, $\xi_1\neq q\xi_2$, $\xi_2\neq q\xi_1$, i.e., $\beta(\xi_1)=0=\beta(\xi_2)$ and $\xi_1\neq\xi_2$, $\left(D_q\beta\right)(q^{-1}\xi_1)\neq 0$, $\left(D_q\beta\right)(q^{-1}\xi_2)\neq 0$ (Theorem 16). (iii) ξ is a double zero (Remark 17 and Theorem 18).

Theorem 12 *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-q-coherent pair given by (3.1). If ξ and $q^{-1}\xi$ are the zeros of $\beta(x)$, then there exist polynomials $\tilde{\alpha}_3(x)$, $\varphi_2(x)$, and $\gamma_1(x)$ of degrees ≤ 3, ≤ 2, and 1, respectively, such that*

$$D_q\left[\tilde{\alpha}_3(x)\mathcal{U}\right]=-\varphi_2(x)\mathcal{U}, \tag{3.27}$$

$$\tilde{\alpha}_3(x)\mathcal{U}=\gamma_1(x)\mathcal{V}. \tag{3.28}$$

Thus, $\mathcal{U}$ is q-semiclassical of class at most 1 and $\mathcal{V}$ is q-semiclassical of class at most 5. Moreover, if $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-q-coherent pair, then $\deg(\tilde{\alpha}_3^{(1,0)}(x))\leq 2$, $\deg(\varphi_2^{(1,0)}(x))=1$, $\deg(\gamma_1^{(1,0)}(x))=1$, hence $\mathcal{U}$ is q-classical and $\mathcal{V}$ is q-semiclassical of class at most 1.

Proof From (3.25) and (3.26), $\gamma_1(x)=x-q^{-1}\xi$. Then $\gamma_2(q^{-1}\xi)$, hence $\beta(x)=q^{-1}(x-\xi)\gamma_1(x)$, and so from (3.12) we get $\gamma_2(x)=\gamma_1(x)\nu_1(x)$, where $\nu_1(x)$ is a monic polynomial of degree 1. Also, from (3.11) we obtain $\alpha(\xi)=0$ and $\alpha(x)=q^{-1}(x-\xi)\tilde{\alpha}_3(x)$, where $\tilde{\alpha}_3(x)=q[\nu_1(q^{-1}x)\varphi_2(x)-\varphi_3(x)]$. Thus, $(D_q\gamma_2)(x)=\gamma_1(qx)+\nu_1(x)$ and, as a consequence, (3.15) and (3.16) become

$$\gamma_2\left(q^{-1}x\right)D_q\mathcal{V}+q^{-1}\nu_1\left(q^{-1}x\right)\mathcal{V}=-\nu_1\left(q^{-1}x\right)\varphi_2(x)\mathcal{U},$$
$$\gamma_2\left(q^{-1}x\right)D_q\mathcal{V}+q^{-1}\left[\gamma_1(x)+\nu_1\left(q^{-1}x\right)\right]\mathcal{V}=-\varphi_3(x)\mathcal{U}.$$

Thus we get (3.28). Now, if we take D_q in (3.28), from (3.5) we obtain (3.27). Besides, from Proposition 4, we get the desired result.

Finally, If $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-q-coherent pair, then from (3.21) and (3.23) we get $\deg(\varphi_2^{(1,0)}(x))=1$, $\deg(\tilde{\alpha}_3^{(1,0)}(x))\leq 2$. Besides, from Theorem 5 it follows that $\tilde{\alpha}_3^{(1,0)}(x)\mathcal{U}$ is also q-classical and from Proposition 4 we get the last result. □

Lemma 13 *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-q-coherent pair given by (3.1) and let $\alpha(x)$, $\beta(x)$, and $\phi(x)$ be the polynomials given in Corollary 11. If ξ is a zero of $\beta(x)$ such that $\beta(q^{-1}\xi)\neq 0$ and $\alpha(\xi)=0$, then $\phi(\xi)=0$ and $\gamma_1(q^{-1}\xi)\neq 0$.*

Proof From (3.25) it follows that $\gamma_1(q^{-1}\xi)\neq 0$ and, from (3.14) for $n=1$, we get $\phi(\xi)=0$. □

Lemma 14 *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-q-coherent pair given by (3.1) and let $\alpha(x)$, $\beta(x)$, $\phi(x)$, and $\gamma_n(x)$ be the polynomials given in Corollary 11. If ξ is a zero of $\beta(x)$ such that $\alpha(\xi)\neq 0$, then there exists a nonzero constant C, independent on n, such that*

$$\gamma_n\left(q^{-1}\xi\right)+C\left(D_q\gamma_n\right)\left(q^{-1}\xi\right)=0,\quad n\geq 1.$$

Proof From (3.14) for $n=1$, and since $\alpha(\xi)\neq 0$ and $\left(D_q\gamma_1\right)(q^{-1}x)=1$, we deduce $\phi(\xi)\neq 0$. □

Lemma 15 *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-q-coherent pair given by (3.1). If there exist constants ξ_1,ξ_2,C_1,C_2 independent on n, such that $\xi_1\neq q\xi_2$, $\xi_2\neq q\xi_1$, and*

$$\gamma_n\left(q^{-1}\xi_k\right)+C_k\left(D_q\gamma_n\right)\left(q^{-1}\xi_k\right)=0,\quad k=1,2,\quad n\geq 1,\tag{3.29}$$

where $\gamma_n(x)$ is given in Lemma 10, then $\xi_1=\xi_2$ and $C_1=C_2$.

Proof From (3.3) and (3.29) it follows that

$$R_n\left(q^{-1}\xi_1\right)+C_1\left(D_qR_n\right)\left(q^{-1}\xi_1\right)=R_n\left(q^{-1}\xi_2\right)+C_2\left(D_qR_n\right)\left(q^{-1}\xi_2\right),\quad n\geq 0.$$

Thus, since $\{R_n(x)\}_{n\geq 0}$ is a basis of $\mathbb{P}$, if $\tilde{\xi}_1=q^{-1}\xi_1$ and $\tilde{\xi}_2=q^{-1}\xi_2$, then we get

$$p(\tilde{\xi}_1)+C_1\left(D_qp\right)(\tilde{\xi}_1)=p(\tilde{\xi}_2)+C_2\left(D_qp\right)(\tilde{\xi}_2),\quad \forall p\in\mathbb{P}.\tag{3.30}$$

Case 1 $\xi_1\neq 0$ and $\xi_2\neq 0$. If $p(x)=(q^{-1}x-\tilde{\xi}_2)^n(x-\tilde{\xi}_2)^n$, $n\geq 1$, then (3.30) becomes

$$(\tilde{\xi}_1-\tilde{\xi}_2)^n\left[\left(q^{-1}\tilde{\xi}_1-\tilde{\xi}_2\right)^n+C_1\frac{\left(q\tilde{\xi}_1-\tilde{\xi}_2\right)^n-\left(q^{-1}\tilde{\xi}_1-\tilde{\xi}_2\right)^n}{(q-1)\tilde{\xi}_1}\right]=0,$$

$n\geq 1.$

If $\tilde{\xi}_1\neq\tilde{\xi}_2$ and $n=1$, then $C_1=(\tilde{\xi}_2-q^{-1}\tilde{\xi}_1)/(q^{-1}(q+1))$. If we replace this value in the above expression for $n=2$, we get $\tilde{\xi}_1^2-(q+q^{-1})\tilde{\xi}_2\tilde{\xi}_1+\tilde{\xi}_2^2=0$. Hence, either $\tilde{\xi}_1=q\tilde{\xi}_2$ or $\tilde{\xi}_2=q\tilde{\xi}_1$ which yields a contradiction. Thus, $\xi_1=\xi_2$ and from (3.30), $C_1=C_2$.

Case 2 $\xi_1\neq 0$ and $\xi_2=0$. In this case (3.30) becomes $p(\tilde{\xi}_1)+C_1(D_qp)(\tilde{\xi}_1)=p(0)+C_2p'(0),\forall p\in\mathbb{P}$. If $p(x)=x^n, n\geq 2$, then $C_1=\tilde{\xi}_1(1-q)/(q^n-1)$ for $n\geq 2$, which is a contradiction. Thus this case is not possible.

Case 3 $\xi_1=0$ and $\xi_2\neq 0$. Similar arguments apply as in the case 2.

Case 4 $\xi_1=0=\xi_2$. In this case, for $p(x)=x$ (3.30) yields $C_1=C_2$. □

Theorem 16 *Let $(\mathcal{U},\mathcal{V})$ be a $(1,1)$-q-coherent pair given by (3.1) and let $\beta(x)$ be the polynomial given by (3.12). If ξ_1,ξ_2 are the zeros of $\beta(x)$ such that $\xi_1\neq\xi_2$, $\xi_1\neq q\xi_2$, $\xi_2\neq q\xi_1$, then*

$$\tilde{\alpha}(x)\mathcal{U}=\tilde{\beta}(x)\mathcal{V}, \tag{3.31}$$

$$\tilde{\alpha}(x)D_q\mathcal{V}=\tilde{\phi}(x)\mathcal{V}, \tag{3.32}$$

$$\tilde{\phi}(x)\mathcal{U}=\tilde{\beta}(x)D_q\mathcal{V}, \tag{3.33}$$

where $\tilde{\beta}(x)=q^{-1}(x-\xi)$ for some $\xi\in\{\xi_1,\xi_2\}$, $\deg(\tilde{\alpha}(x))\leq 3$, and $\deg(\tilde{\phi}(x))\leq 2$. Moreover,

$$D_q\left[\tilde{\alpha}(x)\mathcal{V}\right]=\left(\tilde{\phi}\left(q^{-1}x\right)+q^{-1}\left(D_q\tilde{\alpha}\right)\left(q^{-1}x\right)\right)\mathcal{V}. \tag{3.34}$$

Therefore, $\mathcal{V}$ and $\mathcal{U}$ are q-semiclassical of class at most 1 and 5, respectively.

If $(\mathcal{U},\mathcal{V})$ is a $(1,0)$-q-coherent pair, then $\deg(\tilde{\alpha}^{(1,0)}(x))\leq 2$ and $\deg(\tilde{\phi}^{(1,0)}(x))\leq 1$. Thus, $\mathcal{V}$ is q-classical and $\mathcal{U}$ is q-semiclassical of class at most 3.

Proof Let $\alpha(x),\beta(x)$, and $\phi(x)$ be the polynomials given in Corollary 11 and let $\beta(x)=q^{-1}(x-\xi_1)(x-\xi_2)=(x-\xi_1)\tilde{\beta}(x)$. From Lemmas 14 and 15 it follows that either $\alpha(\xi_1)=0$ or $\alpha(\xi_2)=0$. Assuming $\alpha(\xi_1)=0$, then $\alpha(x)=(x-\xi_1)\tilde{\alpha}(x)$ and from Lemma 13 we get $\phi(\xi_1)=0$ and $\gamma_1(q^{-1}\xi_1)\neq 0$, so $\phi(x)=(x-\xi_1)\tilde{\phi}(x)$. Therefore, (3.8)–(3.10) and (3.14) become

$$\tilde{\alpha}(x)\mathcal{U}=\tilde{\beta}(x)\mathcal{V}+M_1\delta_{\xi_1}, \tag{3.35}$$

$$\tilde{\alpha}(x)D_q\mathcal{V}=\tilde{\phi}(x)\mathcal{V}+M_2\delta_{\xi_1}, \tag{3.36}$$

$$\tilde{\phi}(x)\mathcal{U}=\tilde{\beta}(x)D_q\mathcal{V}+M_3\delta_{\xi_1}, \tag{3.37}$$

$$\tilde{\phi}(x)\gamma_n\left(q^{-1}x\right)+q^{-1}\tilde{\alpha}(x)\left(D_q\gamma_n\right)\left(q^{-1}x\right)=-\varphi_{n+1}(x)\tilde{\beta}(x),\quad n\geq 1. \tag{3.38}$$

Hence, for $n\geq 1$,

$$\left(\tilde{\phi}(x)\gamma_n\left(q^{-1}x\right)+q^{-1}\tilde{\alpha}(x)\left(D_q\gamma_n\right)\left(q^{-1}x\right)\right)\mathcal{U}\overset{(3.38)}{\underset{(3.5)}{=}}\tilde{\beta}(x)D_q[\gamma_n(x)\mathcal{V}],$$

and, from (3.35) and (3.37), and using (2.6)

$$M_3\gamma_n\left(q^{-1}\xi_1\right)=-q^{-1}M_1\left(D_q\gamma_n\right)\left(q^{-1}\xi_1\right),\quad n\geq 1. \tag{3.39}$$

Thus, $M_1=0$ if and only if $M_3=0$. If $M_3=0$, then (3.31) and (3.33) hold. If $M_3\neq 0$ and $\tilde{\alpha}(\xi_2)\neq 0$, then $\alpha(\xi_2)\neq 0$ and from Lemma 14, there exists $C\neq 0$, which is independent on n, such that $\gamma_n(q^{-1}\xi_2)+C\left(D_q\gamma_n\right)(q^{-1}\xi_2)=0$, $n\geq 1$. But from Lemma 15 the previous identity or (3.39) can not hold, which is a contradiction. On the other hand, if $M_3\neq 0$ and $\tilde{\alpha}(\xi_2)=0$, we can do the same analysis as for ξ_1 and we obtain

$$\tilde{M}_3\gamma_n\left(q^{-1}\xi_2\right)=-q^{-1}\tilde{M}_1\left(D_q\gamma_n\right)\left(q^{-1}\xi_2\right),\quad n\geq 1. \tag{3.40}$$

Thus, $\tilde{M}_1 = 0$ if and only if $\tilde{M}_3 = 0$. If $\tilde{M}_3 = 0$, then we get (3.31) and (3.33). If $\tilde{M}_3 \neq 0$, then from Lemma 15 neither (3.39) nor (3.40) hold, which is a contradiction. So $\tilde{M}_3 = 0$.

We suppose that $M_3 = 0$ (if $\tilde{M}_3 = 0$, the following computations are true for ξ_2). Then,

$$-\varphi_2(x)\tilde{\beta}(x)\mathcal{V} \underset{(3.5)}{\overset{(3.31)}{=}} \tilde{\alpha}(x)D_q[\gamma_1(x)\mathcal{V}] \underset{(3.38)}{\overset{(3.36)}{=}} \gamma_1\left(q^{-1}x\right)M_2\delta_{\xi_1} - \varphi_2(x)\tilde{\beta}(x)\mathcal{V}.$$

Thus $M_2 = 0$ and, as a consequence, (3.32) holds. Finally, from (2.6) and (3.32) we obtain (3.34), and from Proposition 4 we deduce desired result. □

Remark 17 If $\beta(x)$ has a double zero ξ, then there are two possibilities: either $\left(D_q\beta\right)(q^{-1}\xi) = 0$ or $\left(D_q\beta\right)(q^{-1}\xi) \neq 0$. If $\left(D_q\beta\right)(q^{-1}\xi) = 0$, from (3.26) it follows that $\xi = 0$. Then, in this case Theorem 12 holds. In the other case, we have the following theorem.

Theorem 18 *Let $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-q-coherent pair given by (3.1) and let $\beta(x)$ be the polynomial given by (3.12). If ξ is a double zero of $\beta(x)$ such that $\left(D_q\beta\right)(q^{-1}\xi) \neq 0$, and there exists $N \geq 1$ such that $(q+1)\gamma_N(q^{-1}\xi) + (q-1)q^{-1}\xi\left(D_q\gamma_N\right)(q^{-1}\xi) \neq 0$, with $\gamma_N(x)$ the polynomial given in Lemma 10, then*

$$\tilde{\alpha}(x)\mathcal{U} = \tilde{\beta}(x)\mathcal{V}, \tag{3.41}$$

$$\tilde{\alpha}(x)D_q\mathcal{V} = \tilde{\phi}(x)\mathcal{V}, \tag{3.42}$$

$$\tilde{\phi}(x)\mathcal{U} = \tilde{\beta}(x)D_q\mathcal{V}, \tag{3.43}$$

where $\tilde{\beta}(x) = q^{-1}(x-\xi)$, $\deg(\tilde{\alpha}(x)) \leq 3$, and $\deg(\tilde{\phi}(x)) \leq 2$. Besides,

$$D_q\left[\tilde{\alpha}(x)\mathcal{V}\right] = \left(\tilde{\phi}\left(q^{-1}x\right) + q^{-1}\left(D_q\tilde{\alpha}\right)\left(q^{-1}x\right)\right)\mathcal{V}. \tag{3.44}$$

Therefore, $\mathcal{V}$ and $\mathcal{U}$ are q-semiclassical of class at most 1 and 5, respectively. If $(\mathcal{U}, \mathcal{V})$ is a $(1,0)$-q-coherent pair, then $\deg(\tilde{\alpha}^{(1,0)}(x)) \leq 2$ and $\deg(\tilde{\phi}^{(1,0)}(x)) \leq 1$. Thus, $\mathcal{V}$ is q-classical and $\mathcal{U}$ is q-semiclassical of class at most 3.

Proof Let $\alpha(x), \beta(x)$, and $\phi(x)$ be the polynomials given in Corollary 11. From (3.25) we get $\gamma_1(q^{-1}\xi) \neq 0$ and, as a consequence, $\gamma_1(q^{-1}\xi) + C\left(D_q\gamma_1\right)(q^{-1}\xi) \neq 0$ for all $C \neq -\gamma_1(q^{-1}\xi) = \frac{q-1}{q+1}q^{-1}\xi$. Hence, by hypothesis and from Lemma 14 we obtain $\alpha(\xi) = 0$, and from Lemma 13 it follows that $\phi(\xi) = 0$. Therefore, $\beta(x) = (x-\xi)\tilde{\beta}(x)$, $\alpha(x) = (x-\xi)\tilde{\alpha}(x)$, and $\phi(x) = (x-\xi)\tilde{\phi}(x)$. Thus (3.8)–(3.10) and (3.14) become

$$\tilde{\alpha}(x)\mathcal{U} = \tilde{\beta}(x)\mathcal{V} + \tilde{M}_1\delta_\xi, \tag{3.45}$$

$$\tilde{\alpha}(x)D_q\mathcal{V} = \tilde{\phi}(x)\mathcal{V} + \tilde{M}_2\delta_\xi, \tag{3.46}$$

$$\tilde{\phi}(x)\mathcal{U} = \tilde{\beta}(x)D_q\mathcal{V} + \tilde{M}_3\delta_\xi, \tag{3.47}$$

$$\tilde{\phi}(x)\gamma_n\left(q^{-1}x\right) + q^{-1}\tilde{\alpha}(x)\left(D_q\gamma_n\right)\left(q^{-1}x\right) = -\varphi_{n+1}(x)\tilde{\beta}(x), \quad n \geq 1. \tag{3.48}$$

Consequently, for $n \geq 1$,

$$\left(\tilde{\phi}(x)\gamma_n\left(q^{-1}x\right) + q^{-1}\tilde{\alpha}(x)\left(D_q\gamma_n\right)\left(q^{-1}x\right)\right)\mathcal{U} \underset{(3.5)}{\overset{(3.48)}{=}} \tilde{\beta}(x)D_q[\gamma_n(x)\mathcal{V}],$$

and from (3.45) and (3.47), and using (2.6) we get

$$\tilde{M}_3\gamma_n\left(q^{-1}\xi\right) + q^{-1}\tilde{M}_1\left(D_q\gamma_n\right)\left(q^{-1}\xi\right) = 0, \quad n \geq 1. \tag{3.49}$$

Thus, $\tilde{M}_1 = 0$ if and only if $\tilde{M}_3 = 0$. If $\tilde{M}_3 \neq 0$, from (3.49) for $n = 1$, it follows that $q^{-1}\tilde{M}_1/\tilde{M}_3 = -\gamma_1(q^{-1}\xi)$. Then $\gamma_n(q^{-1}\xi) + \frac{q-1}{q+1}q^{-1}\xi\left(D_q\gamma_n\right)(q^{-1}\xi) = 0$ for $n \geq 1$, which is a contradiction. So $\tilde{M}_3 = 0$ and, therefore, we get (3.41) and (3.43). Besides,

$$-\varphi_2(x)\tilde{\beta}(x)\mathcal{V} \underset{(3.5)}{\overset{(3.41)}{=}} \tilde{\alpha}(x)D_q[\gamma_1(x)\mathcal{V}] \underset{(3.48)}{\overset{(3.46)}{=}} \gamma_1\left(q^{-1}x\right)\tilde{M}_2\delta_\xi - \varphi_2(x)\tilde{\beta}(x)\mathcal{V}.$$

Hence $\tilde{M}_2 = 0$ and we obtain (3.42). Finally, from (2.6) and (3.42), (3.44) follows, and from Proposition 4 we deduce desired result. □

4 The case when $\mathcal{U}$ is q-classical

In this section, we will study the case when $(\mathcal{U}, \mathcal{V})$ is a $(1, 0)$-q-coherent or $(1, 1)$-q-coherent pair of linear functionals and $\mathcal{U}$ is q-classical.

Proposition 19 *Let $(\mathcal{U}, \mathcal{V})$ be a $(1, 0)$-q-coherent pair given by*

$$P_n^{[1,q]}(x) + a_n P_{n-1}^{[1,q]}(x) = R_n(x), \quad a_n \neq 0,\ n \geq 1. \tag{4.1}$$

If $\mathcal{U}$ is a q-classical linear functional given by (2.12), then

$$\sigma(x)\mathcal{U} = \frac{a_1\langle\mathcal{U}, \sigma(x)\rangle}{\left\langle\mathcal{V}, R_1^2(x)\right\rangle}\left[R_1(x) + \frac{\left\langle\mathcal{V}, R_1^2(x)\right\rangle}{a_1\langle\mathcal{V}, 1\rangle}\right]\mathcal{V}, \tag{4.2}$$

and therefore, $\mathcal{V}$ is a q-semiclassical linear functional of class at most 1.

Proof From Theorem 5, we have that $\{P_n^{[1,q]}(x)\}_{n\geq 0}$ is also a q-classical SMOP with respect to $\mathcal{U}^{[1,q]} = \sigma(x)\mathcal{U}$.

Let $\{\hbar_n\}_{n\geq 0}$ and $\{\wp_n^{[1,q]}\}_{n\geq 0}$ be the corresponding dual bases of the SMOP $\{R_n(x)\}_{n\geq 0}$ and $\{P_n^{[1,q]}(x)\}_{n\geq 0}$. From (4.1), we get

$$\wp_n^{[1,q]} = \sum_{k\geq 0}\left\langle\wp_n^{[1,q]}, R_k(x)\right\rangle\hbar_k = \hbar_n + a_{n+1}\hbar_{n+1}, \quad n \in \mathbb{N}. \tag{4.3}$$

Thus,

$$\frac{\mathcal{U}^{[1,q]}}{\left\langle\mathcal{U}^{[1,q]}, 1\right\rangle} \overset{(2.8)}{=} \wp_0^{[1,q]} \underset{n=0}{\overset{(4.3)}{=}} \hbar_0 + a_1\hbar_1 \overset{(2.8)}{=} \frac{\mathcal{V}}{\langle\mathcal{V}, 1\rangle} + a_1\frac{R_1(x)\mathcal{V}}{\left\langle\mathcal{V}, R_1^2(x)\right\rangle} \overset{(3.4)}{=} \frac{a_1\gamma_1^{(1,0)}(x)\mathcal{V}}{\left\langle\mathcal{V}, R_1^2(x)\right\rangle},$$

which is (4.2), and from Proposition 4 the statement follows. □

Theorem 20 [1, p. 314] *Let $\{T_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ be two SMOP with respect to $\mathcal{W}$ and $\mathcal{V}$, respectively. Then, the following statements are equivalent*

(i) *There exist complex sequences $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$, with $a_1 \neq b_1, a_n b_n \neq 0$, $n \geq 1$, and such that $T_n(x) + a_n T_{n-1}(x) = R_n(x) + b_n R_{n-1}(x)$, $n \geq 1$.*
(ii) *$T_n(x) \neq R_n(x)$, for $n \geq 1$, and there exist constants C^T, C^R, and η such that $(x - C^T)\mathcal{W} = \eta(x - C^R)\mathcal{V}$.*

Remark 21 Let $\{P_n^{[1,q]}(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ be SPOM with respect to $\mathcal{U}^{[1,q]}$ and $\mathcal{V}$, respectively (i.e., $\{P_n(x)\}_{n\geq 0}$ is a q-classical SMOP with respect to $\mathcal{U}$), and corresponding TTRR given as in (2.7). Then, from proof of Theorem 20 (see [1]) we have the following results:

In proof of (i) $\Rightarrow$ (ii) we get that the condition $b_n \neq 0, n \geq 1$, can be replaced by $b_2 \neq 0$. Besides, the constants $C^{P^{[1,q]}}$, C^R, and η are

$$C^{P^{[1,q]}} = \alpha_1^{P^{[1,q]}} - \frac{\beta_2^{P^{[1,q]}}(a_2 - b_2)}{b_2(a_1 - b_1)}, \quad C^R = \alpha_1^R - \frac{\beta_2^R(a_2 - b_2)}{a_2(a_1 - b_1)},$$
$$\eta = \frac{\beta_2^{P^{[1,q]}} a_2}{\beta_2^R b_2} \frac{\left\langle \mathcal{U}^{[1,q]}, 1\right\rangle}{\langle \mathcal{V}, 1\rangle}.$$

From proof of (ii) $\Rightarrow$ (i) it follows that

$$C^{P^{[1,q]}} = \alpha_{n+1}^R - b_{n+1} - \frac{\beta_{n+1}^R}{b_n}, \quad C^R = \alpha_{n+1}^{P^{[1,q]}} - a_{n+1} - \frac{\beta_{n+1}^{P^{[1,q]}}}{a_n}, \quad n \geq 2.$$

Finally, the following corollary is a straightforward consequence of Theorem 20, Theorem 5, and Proposition 4.

Corollary 22 *Let $\mathcal{U}$ be a q-classical linear functional given by (2.12), let $\mathcal{V}$ be a regular linear functional, and $\{P_n(x)\}_{n\geq 0}$, $\{R_n(x)\}_{n\geq 0}$ be their corresponding SMOP. The following statements are equivalent*

(i) *$(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-q-coherent pair given by (3.1), with $a_1 \neq b_1$ and $a_n b_n \neq 0$, for $n \geq 1$.*
(ii) *$P_n^{[1,q]}(x) \neq R_n(x)$, for $n \geq 1$, and there exist constants $C^{P^{[1,q]}}$, C^R, and η (see Remark 21) such that*

$$\left(x - C^{P^{[1,q]}}\right) \sigma(x)\mathcal{U} = \eta \left(x - C^R\right) \mathcal{V}.$$

Therefore, $\mathcal{V}$ is a q-semiclassical linear functional of class at most 2.

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