Pacific Journal of Mathematics

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Volume 257 No. 1 May 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 257, No. 1, 2012

ON ORTHOGONAL POLYNOMIALS WITH RESPECT TO CERTAIN DISCRETE SOBOLEV INNER PRODUCT

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In this paper we deal with sequences of polynomials orthogonal with respect to the discrete Sobolev inner product

$$\langle f, g\rangle_S = \int_0^\infty \omega(x) f(x) g(x)\,dx + M f(\xi) g(\xi) + N f'(\xi) g'(\xi),$$

where ω is a weight function, $\xi \le 0$, and $M, N \ge 0$. The location of the zeros of discrete Sobolev orthogonal polynomials is given in terms of the zeros of standard polynomials orthogonal with respect to the weight function ω. In particular, for $\omega(x) = x^\alpha e^{-x}$ we obtain the asymptotics for discrete Laguerre–Sobolev orthogonal polynomials.

1. Introduction

Polynomials orthogonal with respect to an inner product

$$\langle f, g\rangle = \int_E \omega(x) f(x) g(x)\,dx + M f(\xi) g(\xi) + N f'(\xi) g'(\xi), \tag{1}$$

where ξ is a real number and $d\mu$ is a positive Borel measure supported on an infinite subset E of the real line have been considered by several authors (see, for instance, [Alfaro et al. 1992; López et al. 1995; Marcellán and Ronveaux 1990; Marcellán and Van Assche 1993] and the references therein). They are known in the literature as Sobolev-type or discrete Sobolev orthogonal polynomials. Special attention has been paid to their algebraic and analytic properties of these polynomials, in particular, the distribution of their zeros taking into account the location of the point ξ with respect to the set E.

When E is the interval $[0, +\infty)$ and $\xi = 0$, Meijer [1993a] analyzed some analytic properties of the zeros of the so called discrete Sobolev orthogonal polynomials (1). Some results of [Meijer 1993a] are direct generalizations of the results of [Koekoek and Meijer 1993], where the weight function is the Laguerre

MSC2010: primary 33C47; secondary 42C05.
Keywords: orthogonal polynomials, discrete Sobolev polynomials, Laguerre polynomials, asymptotics.

weight $\omega(x) = x^{\alpha} e^{-x}$. Koekoek and Meijer established properties of the discrete Laguerre–Sobolev polynomials such as their representation as a hypergeometric series, an holonomic second order linear differential equation associated with them, properties of the zeros, and a higher-order recurrence relation that such polynomials satisfy. The asymptotic properties of these discrete Laguerre–Sobolev polynomials have been studied in [Álvarez-Nodarse and Moreno-Balcázar 2004; Marcellán and Moreno-Balcázar 2006], while the analysis of convergence of the Fourier expansions in terms of such polynomials was done in [Fejzullahu and Marcellán 2009].

In this paper we consider the discrete Sobolev polynomials $\{\hat{S}_n\}_{n\geq 0}$ orthogonal with respect to (1) where $E = [0, +\infty)$ and $\xi < 0$. We show that these polynomials can be expressed as

$$\hat{S}_n(x) = \hat{P}_n(x) + A_{n,1}(x-\xi)\hat{P}^{[2]}_{n-1}(x) + A_{n,2}(x-\xi)^2 \hat{P}^{[4]}_{n-2}(x),$$

where $\{\hat{P}_n\}_{n\geq 0}$ and $\{\hat{P}^{[k]}_n\}_{n\geq 0}$, $k \in \mathbb{N}$, are the sequences of monic polynomials orthogonal with respect to the weight functions $\omega(\cdot)$ and $(\cdot - \xi)^k \omega(\cdot)$, respectively. Moreover, the behavior of the coefficients $A_{n,1}$ and $A_{n,2}$ is studied in more detail. In particular, when ω is the Laguerre weight, we obtain some asymptotic properties for the sequence of discrete Laguerre–Sobolev orthogonal polynomials.

The structure of the manuscript is as follows. In Section 2 we give some basic background concerning polynomial perturbations of a measure as well as interlacing properties for the zeros of the corresponding orthogonal polynomials. We point our that the results presented therein are of independent interest in terms of the core of our contribution. Indeed, they constitute an alternative approach in the subject. In Section 3, a representation of monic polynomials orthogonal with respect to the inner product (1) is given in terms of polynomial orthogonal with respect to polynomial perturbations of the weight function. Some results about their zeros are deduced. In Section 4 we focus our attention on the asymptotics of discrete Laguerre–Sobolev orthogonal polynomials. More precisely, we obtain outer relative asymptotics, a Mehler–Heine formula and the Plancherel–Rotach outer asymptotics for such orthogonal polynomials.

Throughout this paper positive constants are denoted by $c, c_1, \ldots$, and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence $\{u_n/v_n\}_n$ converges to 1. We will denote by $k(\pi_n)$ the leading coefficient of any polynomial π_n and $\hat{\pi}_n(x) = (k(\pi_n))^{-1}\pi_n(x)$.

2. Auxiliary results

Let ω denote a weight function on $(0, \infty)$, i.e., $\omega(x) \geq 0$ and all moments

$$c_n = \int_0^{\infty} \omega(x) x^n \, dx, \quad n = 0, 1, \ldots$$

exist. Let $\{\hat{P}_n\}_{n\geq 0}$ denote the sequence of monic polynomials orthogonal (SMOP, in short) with respect to the standard inner product

$$\langle f, g\rangle = \int_0^\infty \omega(x) f(x) g(x)\, dx.$$

In particular, from the moments we get an explicit expression of the SMOP. Indeed, we get

$$\hat{P}_0(x) = 1$$

and

$$\hat{P}_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & c_1 & c_2 & \dots & c_n \\ c_1 & c_2 & c_3 & \dots & c_{n+1} \\ c_2 & c_3 & c_4 & \dots & c_{n+2} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ c_{n-1} & c_n & c_{n+1} & \dots & c_{2n-1} \\ 1 & x & x^2 & \dots & x^n \end{vmatrix}, \quad n \geq 1, \tag{2}$$

where

$$\Delta_{n-1} = \begin{vmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_1 & c_2 & c_3 & \dots & c_n \\ c_2 & c_3 & c_4 & \dots & c_{n+1} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ c_{n-1} & c_n & c_{n+1} & \dots & c_{2n-2} \end{vmatrix}, \quad n \geq 1,$$

are the Gram determinants.

The n-th reproducing kernel for ω is

$$K_n(x, y) = \sum_{k=0}^{n} \frac{\hat{P}_k(x)\hat{P}_k(y)}{\|\hat{P}_k\|_\omega^2}.$$

Here, $\|\hat{P}_n\|_\omega^2 = \langle \hat{P}_n, \hat{P}_n\rangle$. Because of the Christoffel–Darboux formula, it may also be expressed as

$$K_n(x, y) = \frac{1}{\|\hat{P}_n\|_\omega^2} \frac{\hat{P}_{n+1}(x)\hat{P}_n(y) - \hat{P}_n(x)\hat{P}_{n+1}(y)}{x - y}.$$

The confluent formula reads as

$$K_n(x, x) = \sum_{k=0}^{n} \frac{(\hat{P}_k(x))^2}{\|\hat{P}_k\|_\omega^2} = \frac{1}{\|\hat{P}_n\|_\omega^2}\big(\hat{P}'_{n+1}(x)\hat{P}_n(x) - \hat{P}'_n(x)\hat{P}_{n+1}(x)\big). \tag{3}$$

In the same way we can describe the SMOP $\{\hat{P}_n^{[k]}\}_{n\geq 0}$, orthogonal with respect to the inner product

$$\langle f, g\rangle_k = \int_0^\infty (x - \xi)^k \omega(x) f(x) g(x)\, dx,$$

where $\xi \le 0$. For $n \ge 1$, they are given by the determinant (2) where c_i is replaced by d_i^k, $k \in \mathbb{N}$, where

$$d_n^k = \int_0^\infty (x-\xi)^k \omega(x) x^n \, dx = d_{n+1}^{k-1} - \xi d_n^{k-1}, \quad n = 0, 1, \dots, \tag{4}$$

and $c_n = d_n^0$. In the sequel, we will set

$$\| \hat{P}_n^{[k]} \|_{\omega,k}^2 = \int_0^\infty (x-\xi)^k \omega(x) \big(\hat{P}_n^{[k]}(x) \big)^2 \, dx.$$

Proposition 1. *Let $D_{n-1}^k = \det[a_{ij}^k]_{0 \le i,j \le n-1}$, where $a_{ij}^k = d_{i+j}^k$, $k \in \mathbb{N}$. Then*

$$D_{n-1}^k = (-1)^n D_{n-1}^{k-1} \hat{P}_n^{[k-1]}(\xi), \tag{5}$$

with $D_{n-1}^0 = \Delta_{n-1}$.

Proof. For $n \ge 1$ and $k \in \mathbb{N}$,

$$\hat{P}_n^{[k-1]}(x) = \frac{1}{D_{n-1}^{[k-1]}} \begin{vmatrix} d_0^{k-1} & d_1^{k-1} & \dots & d_n^{k-1} \\ d_1^{k-1} & d_2^{k-1} & \dots & d_{n+1}^{k-1} \\ . & . & \dots & . \\ d_{n-1}^{k-1} & d_n^{k-1} & \dots & d_{2n-1}^{k-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \tag{6}$$

with $\hat{P}_n = \hat{P}_n^{[0]}$. The determinant in (6) becomes [Szegő 1975, Formula (2.2.9)]

$$\hat{P}_n^{[k-1]}(x) = \frac{(-1)^n}{D_{n-1}^{k-1}} \begin{vmatrix} d_1^{k-1} - d_0^{k-1}x & d_2^{k-1} - d_1^{k-1}x & \dots & d_n^{k-1} - d_{n-1}^{k-1}x \\ d_2^{k-1} - d_1^{k-1}x & d_3^{k-1} - d_2^{k-1}x & \dots & d_{n+1}^{k-1} - d_n^{k-1}x \\ . & . & . & \dots \\ d_n^{k-1} - d_{n-1}^{k-1}x & d_{n+1}^{k-1} - d_n^{k-1}x & \dots & d_{2n-1}^{k-1} - d_{2n-2}^{k-1}x \end{vmatrix}.$$

Now, by using (4), (5) follows. $\square$

Next we will compute some integrals involving the polynomials $\hat{P}_n^{[k]}$.

Proposition 2. (i) *The integral $\int_0^\infty (x-\xi)^{k-1} \omega(x) \hat{P}_n^{[k]}(x) \, dx$ is given by*

$$\frac{\| \hat{P}_n^{[k-1]} \|_{\omega,k-1}^2}{\hat{P}_n^{[k-1]}(\xi)} = \begin{cases} \dfrac{\| \hat{P}_n \|_\omega^2}{\hat{P}_n(\xi)} & \text{if } k = 1, \\ \dfrac{(-1)^{k-1}}{\hat{P}_n^{[k-1]}(\xi)} \prod_{i=1}^{k-1} \dfrac{\hat{P}_{n+1}^{[i-1]}(\xi)}{\hat{P}_n^{[i-1]}(\xi)} \| \hat{P}_n \|_\omega^2 & \text{if } k \ge 2. \end{cases}$$

(ii) *The integral* $\int_0^\infty (x-\xi)^{k-2}\omega(x)\hat{P}_n^{[k]}(x)\,dx$ *is given by*

$$\frac{\left(\hat{P}_{n+1}^{[k-2]}(x)\right)'_{x=\xi}\|\hat{P}_n^{[k-2]}\|^2_{\omega,k-2}}{\hat{P}_n^{[k-1]}(\xi)\,\hat{P}_n^{[k-2]}(\xi)} = \begin{cases} \dfrac{\left(\hat{P}_{n+1}(x)\right)'_{x=\xi}\|\hat{P}_n\|^2_\omega}{\hat{P}_n(\xi)\,\hat{P}_n^{[1]}(\xi)} & \text{if } k=2,\\[2ex] \dfrac{(-1)^k\left(\hat{P}_{n+1}^{[k-2]}(x)\right)'_{x=\xi}}{\hat{P}_n^{[k-1]}(\xi)\,\hat{P}_n^{[k-2]}(\xi)}\prod_{i=1}^{k-2}\dfrac{\hat{P}_{n+1}^{[i-1]}(\xi)}{\hat{P}_n^{[i-1]}(\xi)}\|\hat{P}_n\|^2_\omega & \text{if } k\ge 3.\end{cases}$$

Proof. (i) Using (4) recursively as well as properties of determinants, we have

$$D_{n-1}^k\int_0^\infty (x-\xi)^{k-1}\omega(x)\hat{P}_n^{[k]}(x)\,dx = \begin{vmatrix} d_0^k & d_1^k & d_2^k & \dots & d_n^k\\ d_1^k & d_2^k & d_3^k & \dots & d_{n+1}^k\\ \cdot & \cdot & \cdot & \dots & \cdot\\ d_{n-1}^k & d_n^k & d_{n+1}^k & \dots & d_{2n-1}^k\\ d_0^{k-1} & d_1^{k-1} & d_2^{k-1} & \dots & d_n^{k-1}\end{vmatrix}$$

$$= \begin{vmatrix} d_1^{k-1} & d_2^{k-1} & d_3^{k-1} & \dots & d_{n+1}^{k-1}\\ d_2^{k-1} & d_3^{k-1} & d_4^{k-1} & \dots & d_{n+2}^{k-1}\\ \cdot & \cdot & \cdot & \dots & \cdot\\ d_n^{k-1} & d_{n+1}^{k-1} & d_{n+2}^{k-1} & \dots & d_{2n}^{k-1}\\ d_0^{k-1} & d_1^{k-1} & d_2^{k-1} & \dots & d_n^{k-1}\end{vmatrix}$$

$$= (-1)^n D_n^{k-1}.$$

On the other hand,

$$\|\hat{P}_n^{[k-1]}\|^2_{\omega,k-1} = \int_0^\infty (x-\xi)^{k-1}\omega(x)x^n\hat{P}_n^{[k-1]}(x)\,dx = \frac{D_n^{k-1}}{D_{n-1}^{k-1}},$$

and by using (5) we get

$$\int_0^\infty (x-\xi)^{k-1}\omega(x)\hat{P}_n^{[k]}(x)\,dx = \frac{(-1)^n D_{n-1}^{k-1}\|\hat{P}_n^{[k-1]}\|^2_{\omega,k-1}}{D_{n-1}^k} = \frac{\|\hat{P}_n^{[k-1]}\|^2_{\omega,k-1}}{\hat{P}_n^{[k-1]}(\xi)}. \tag{7}$$

On the other hand, we have from [Szegő 1975, Theorem 2.5]

$$(x-\xi)\hat{P}_n^{[k]}(x) = \hat{P}_{n+1}^{[k-1]}(x) - \frac{\hat{P}_{n+1}^{[k-1]}(\xi)}{\hat{P}_n^{[k-1]}(\xi)}\hat{P}_n^{[k-1]}(x). \tag{8}$$

Therefore,

$$\|\hat{P}_n^{[k]}\|^2_{\omega,k} = -\frac{\hat{P}_{n+1}^{[k-1]}(\xi)}{\hat{P}_n^{[k-1]}(\xi)}\|\hat{P}_n^{[k-1]}\|^2_{\omega,k-1}.$$

Using this relation recursively we obtain

$$\|\hat{P}_n^{[k]}\|^2_{\omega,k} = (-1)^k \prod_{i=1}^{k} \frac{\hat{P}_{n+1}^{[i-1]}(\xi)}{\hat{P}_n^{[i-1]}(\xi)}\|\hat{P}_n\|^2_{\omega}, \quad k \geq 2. \tag{9}$$

Combining (7) and (9), our statement follows.

(ii) We have

$$\left(\hat{P}_{n+1}^{[k-2]}(x)\right)' = \frac{1}{D_n^{k-2}} \begin{vmatrix} d_0^{k-2} & d_1^{k-2} & d_2^{k-2} & \dots & d_{n+1}^{k-2} \\ d_1^{k-2} & d_2^{k-2} & d_3^{k-2} & \dots & d_{n+2}^{k-2} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ d_n^{k-2} & d_{n+1}^{k-2} & d_{n+2}^{k-2} & \dots & d_{2n+1}^{k-2} \\ 0 & 1 & 2x & \dots & nx^{n-1} \end{vmatrix}, \quad n \geq 0. \tag{10}$$

Now, adding to the last column the n-th and $(n-1)$-th columns multiplied by $-2x$ and x^2, respectively, and repeating this operation for each of the preceding columns, we obtain

$$\begin{aligned} &\left(\hat{P}_{n+1}^{[k-2]}(x)\right)' \\ &= \frac{1}{D_n^{k-2}} \begin{vmatrix} d_0^{k-2} & d_1^{k-2} & d_2^{k-2}-2xd_1^{k-2}+x^2d_0^{k-2} & \dots & d_{n+1}^{k-2}-2xd_n^{k-2}+x^2d_{n-1}^{k-2} \\ d_1^{k-2} & d_2^{k-2} & d_3^{k-2}-2xd_2^{k-2}+x^2d_1^{k-2} & \dots & d_{n+2}^{k-2}-2xd_{n+1}^{k-2}+x^2d_n^{k-2} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ d_n^{k-2} & d_{n+1}^{k-2} & d_{n+2}^{k-2}-2xd_{n+1}^{k-2}+x^2d_n^{k-2} & \dots & d_{2n+1}^{k-2}-2xd_{2n}^{k-2}+x^2d_{2n-1}^{k-2} \\ 0 & 1 & 0 & \dots & 0 \end{vmatrix} \\ &= \frac{1}{D_n^{k-2}} \begin{vmatrix} d_2^{k-2}-2xd_1^{k-2}+x^2d_0^{k-2} & d_3^{k-2}-2xd_2^{k-2}+x^2d_1^{k-2} & \dots & d_{n+2}^{k-2}-2xd_{n+1}^{k-2}+x^2d_n^{k-2} \\ d_3^{k-2}-2xd_2^{k-2}+x^2d_1^{k-2} & d_4^{k-2}-2xd_3^{k-2}+x^2d_2^{k-2} & \dots & d_{n+3}^{k-2}-2xd_{n+2}^{k-2}+x^2d_{n+1}^{k-2} \\ \cdot & \cdot & \dots & \cdot \\ d_{n+1}^{k-2}-2xd_n^{k-2}+x^2d_{n-1}^{k-2} & d_{n+2}^{k-2}-2xd_{n+1}^{k-2}+x^2d_n^{k-2} & \dots & d_{2n+1}^{k-2}-2xd_{2n}^{k-2}+x^2d_{2n-1}^{k-2} \\ d_0^{k-2} & d_1^{k-2} & \dots & d_n^{k-2}. \end{vmatrix}. \end{aligned} \tag{11}$$

On the other hand,

$$D_{n-1}^k \int_0^\infty (x-\xi)^{k-2}\omega(x)\hat{P}_n^{[k]}(x)\,dx = \begin{vmatrix} d_0^k & d_1^k & \dots & d_n^k \\ d_1^k & d_2^k & \dots & d_{n+1}^k \\ \cdot & \cdot & \dots & \cdot \\ d_{n-1}^k & d_n^k & \dots & d_{2n-1}^k \\ d_0^{k-2} & d_1^{k-2} & \dots & d_n^{k-2} \end{vmatrix},$$

and by using (5), (9), and (11) we get

$$\begin{aligned}\int_0^\infty (x-\xi)^{k-2}\omega(x)\hat{P}_n^{[k]}(x)\,dx &= \frac{D_n^{k-2}\big(\hat{P}_{n+1}^{[k-2]}(x)\big)'_{x=\xi}}{D_{n-1}^{k-2}\hat{P}_n^{[k-1]}(\xi)\hat{P}_n^{[k-2]}(\xi)} = \frac{\big(\hat{P}_{n+1}^{[k-2]}(x)\big)'_{x=\xi}\|\hat{P}_n^{[k-2]}\|^2_{\omega,k-2}}{\hat{P}_n^{[k-1]}(\xi)\hat{P}_n^{[k-2]}(\xi)} \\ &= \frac{(-1)^k\big(\hat{P}_{n+1}^{[k-2]}(x)\big)'_{x=\xi}}{\hat{P}_n^{[k-1]}(\xi)\hat{P}_n^{[k-2]}(\xi)}\prod_{i=1}^{k-2}\frac{\hat{P}_{n+1}^{[i-1]}(\xi)}{\hat{P}_n^{[i-1]}(\xi)}\|\hat{P}_n\|^2_\omega. \qquad \square\end{aligned}$$

Denote by $x_{r,n}^{[k]}$, $r = 1, 2, \dots, n$, the zeros of $\hat{P}_n^{[k]}(x)$ in increasing order.

Proposition 3. (i) *The zeros of $\hat{P}_n^{[k]}(x)$ interlace with both the zeros of $\hat{P}_{n+1}^{[k-1]}(x)$ and $\hat{P}_n^{[k-1]}(x)$, i.e.,*

$$x_{r,n}^{[k-1]} < x_{r,n}^{[k]} < x_{r+1,n+1}^{[k-1]}, \qquad r = 1, 2, \dots, n.$$

(ii) *Between two consecutive zeros of $\hat{P}_{n+1}^{[k-2]}$, $k \ge 2$, there is exactly one zero of $\hat{P}_n^{[k]}$.*

(iii) $\operatorname{sgn} \hat{P}_n^{[k-2]}(x_{r,n-1}^{[k]}) = (-1)^{n-r} = -\operatorname{sgn} \hat{P}_{n-2}^{[k+2]}(x_{r,n-1}^{[k]})$ *for* $r = 1, 2, \dots, n-1$.

Proof. (i) Here we will use the same argument as in [Chihara 1978, page 65] (see also [Bracciali et al. 2002, Lemma 1]). It is well known that the zeros of $\hat{P}_{n+1}^{[k-1]}$ interlace with the zeros of $\hat{P}_n^{[k-1]}$, i.e.,

$$0 < x_{1,n+1}^{[k-1]} < x_{1,n}^{[k-1]} < x_{2,n+1}^{[k-1]} < \cdots < x_{n,n}^{[k-1]} < x_{n+1,n+1}^{[k-1]} < \infty.$$

From (5) $\hat{P}_{n+1}^{[k-1]}(\xi)/\hat{P}_n^{[k-1]}(\xi) < 0$ and taking (8) into account we have

$$\begin{aligned}\operatorname{sgn} \hat{P}_n^{[k]}(x_{r,n+1}^{[k-1]}) &= \operatorname{sgn} \hat{P}_n^{[k-1]}(x_{r,n+1}^{[k-1]}) = (-1)^{n-r+1} \quad \text{for } r = 1, 2, \dots, n+1,\\ \operatorname{sgn} \hat{P}_n^{[k]}(x_{r,n}^{[k-1]}) &= \operatorname{sgn} \hat{P}_{n+1}^{[k-1]}(x_{r,n}^{[k-1]}) = (-1)^{n-r+1} \quad \text{for } r = 1, 2, \dots, n.\end{aligned}$$

Thus, there exist zeros $x_{r,n}^{[k]}$, $r = 2, 3, \dots, n$, of $\hat{P}_n^{[k]}(x)$ satisfying

$$x_{r,n}^{[k-1]} < x_{r,n}^{[k]} < x_{r+1,n+1}^{[k-1]}, \qquad r = 1, 2, \dots, n.$$

(ii) By using (8) and the recurrence relation we obtain

$$(x-\xi)^2\hat{P}_n^{[k]}(x) = (d_{1,n}x + d_{2,n})\hat{P}_{n+1}^{[k-2]}(x) + d_{3,n}\hat{P}_n^{[k-2]}(x).$$

Since $\hat{P}_{n+1}^{[k-2]}(\xi) \neq 0$ we have $d_{3,n} \neq 0$. Now, the rest of the proof can be done in a similar way as in [Meijer 1993a, Lemma 6.1]; see also [Meijer 1993b, Lemma 4.1].

(iii) From (ii) we have $x_{r,n}^{[k-2]} < x_{r,n-1}^{[k]} < x_{r+1,n}^{[k-2]}$ for $r = 1, 2, \ldots, n-1$. Therefore,

$$\operatorname{sgn} \hat{P}_n^{[k-2]}(x_{r,n-1}^{[k]}) = (-1)^{n-r}.$$

Again, according to (ii), $x_{r-1,n-2}^{[k+2]} < x_{r,n-1}^{[k]} < x_{r,n-2}^{[k+2]}$ for $r = 1, 2, \ldots, n-2$, and $x_{n-2,n-2}^{[k+2]} < x_{n-1,n-1}^{[k]}$. Therefore,

$$\operatorname{sgn} \hat{P}_{n-2}^{[k+2]}(x_{r,n-1}^{[k]}) = (-1)^{n-r-1} \quad \text{and} \quad \operatorname{sgn} \hat{P}_{n-2}^{[k+2]}(x_{n-1,n-1}^{[k]}) = 1.$$

As a conclusion,

$$\operatorname{sgn} \hat{P}_n^{[k-2]}(x_{r,n-1}^{[k]}) = -\operatorname{sgn} \hat{P}_{n-2}^{[k+2]}(x_{r,n-1}^{[k]}), \quad r = 1, 2, \ldots, n-1. \qquad \square$$

3. Discrete Sobolev orthogonal polynomials

Connection formula. We consider the inner product

$$\langle f, g\rangle_S = \int_0^\infty \omega(x) f(x) g(x)\, dx + M f(\xi) g(\xi) + N f'(\xi) g'(\xi), \tag{12}$$

where $\xi \leq 0$, and $M, N \geq 0$. Let $\{\hat{S}_n\}_{n\geq 0}$ denote the SMOP with respect to the discrete Sobolev inner product (12)).

Theorem 1. *Let $M \geq 0$ and $N \geq 0$. There are real constants $A_{n,1}$ and $A_{n,2}$ such that*

$$\hat{S}_n(x) = \hat{P}_n(x) + A_{n,1}(x-\xi)\hat{P}_{n-1}^{[2]}(x) + A_{n,2}(x-\xi)^2 \hat{P}_{n-2}^{[4]}(x),$$

where

$$A_{n,1} = \frac{N I_{2,n}(\xi)\hat{P}_n'(\xi) - M I_{3,n}(\xi)\hat{P}_n(\xi)}{I_{1,n}(\xi) I_{3,n}(\xi) - N I_{2,n}(\xi)\hat{P}_{n-1}^{[2]}(\xi)},$$

$$A_{n,2} = \frac{M N \hat{P}_n(\xi)\hat{P}_{n-1}^{[2]}(\xi) - N I_{1,n}(\xi)\hat{P}_n'(\xi)}{I_{1,n}(\xi) I_{3,n}(\xi) - N I_{2,n}(\xi)\hat{P}_{n-1}^{[2]}(\xi)},$$

$$I_{1,n}(\xi) = -\frac{\hat{P}_n(\xi)}{K_{n-1}(\xi,\xi)},$$

$$I_{2,n}(\xi) = \frac{\hat{P}_{n-1}(\xi)\hat{P}_{n-1}^{[1]}(\xi)\hat{P}_{n-1}^{[2]\prime}(\xi)}{\hat{P}_{n-2}(\xi)\hat{P}_{n-2}^{[1]}(\xi)\hat{P}_{n-2}^{[2]}(\xi)\hat{P}_{n-2}^{[3]}(\xi)}\|\hat{P}_{n-2}\|_\omega^2,$$

$$I_{3,n}(\xi) = -\frac{\hat{P}_{n-1}(\xi)\hat{P}_{n-1}^{[1]}(\xi)\hat{P}_{n-1}^{[2]}(\xi)}{\hat{P}_{n-2}(\xi)\hat{P}_{n-2}^{[1]}(\xi)\hat{P}_{n-2}^{[2]}(\xi)\hat{P}_{n-2}^{[3]}(\xi)}\|\hat{P}_{n-2}\|_\omega^2.$$

Proof. We will prove that

$$\langle \hat{S}_n, (\cdot - \xi)^k \rangle_S = 0 \quad \text{for } k = 0, 1, \ldots, n-1.$$

For $k \geq 2$ and $n > k$,

$$\begin{aligned}
&\langle \hat{S}_n, (\cdot - \xi)^k \rangle_S \\
&= \int_0^\infty \omega(x) \hat{S}_n(x)(x-\xi)^k \, dx \\
&= \int_0^\infty \omega(x) \hat{P}_n(x)(x-\xi)^k \, dx + A_{n,1} \int_0^\infty (x-\xi)^2 \omega(x) \hat{P}^{[2]}_{n-1}(x)(x-\xi)^{k-1} \, dx \\
&\qquad + A_{n,2} \int_0^\infty (x-\xi)^4 \omega(x) \hat{P}^{[4]}_{n-2}(x)(x-\xi)^{k-2} \, dx \\
&= 0,
\end{aligned}$$

Now consider $k = 0$ and $n \geq 1$. We have

$$\begin{aligned}
\langle \hat{S}_n, 1 \rangle_S &= \int_0^\infty \omega(x) \hat{S}_n(x) \, dx + M \hat{S}_n(\xi) \\
&= A_{n,1} \int_0^\infty (x-\xi)\omega(x) \hat{P}^{[2]}_{n-1}(x) \, dx + A_{n,2} \int_0^\infty (x-\xi)^2 \omega(x) \hat{P}^{[4]}_{n-2}(x) \, dx \\
&\qquad + M \hat{P}_n(\xi).
\end{aligned}$$

On the other hand, by using Proposition 2(i),

$$I_{1,n}(\xi) = \int_0^\infty (x-\xi)\omega(x) \hat{P}^{[2]}_{n-1}(x) \, dx = -\frac{\hat{P}_n(\xi)}{\hat{P}_{n-1}(\xi) \hat{P}^{[1]}_{n-1}(\xi)} \| \hat{P}_{n-1} \|^2_\omega, \tag{13}$$

and taking derivatives in (8) and then substituting $x = \xi$ we get

$$\hat{P}^{[k]}_{n-1}(\xi) = \big(\hat{P}^{[k-1]}_n(x)\big)'_{x=\xi} - \frac{\hat{P}^{[k-1]}_n(\xi)}{\hat{P}^{[k-1]}_{n-1}(\xi)} \big(\hat{P}^{[k-1]}_{n-1}(x)\big)'_{x=\xi}. \tag{14}$$

Combining (3), (13), and (14), we get

$$I_{1,n}(\xi) = -\frac{\hat{P}_n(\xi)}{K_{n-1}(\xi, \xi)}.$$

Using Proposition 2(ii),

$$\begin{aligned}
I_{2,n}(\xi) &= \int_0^\infty (x-\xi)^2 \omega(x) \hat{P}^{[4]}_{n-2}(x) \, dx = \frac{\big(\hat{P}^{[2]}_{n-1}(x)\big)'_{x=\xi} \| \hat{P}^{[2]}_{n-2} \|^2_{\omega,2}}{\hat{P}^{[2]}_{n-2}(\xi) \hat{P}^{[3]}_{n-2}(\xi)} \\
&= \frac{\hat{P}_{n-1}(\xi) \hat{P}^{[1]}_{n-1}(\xi) \big(\hat{P}^{[2]}_{n-1}(x)\big)'_{x=\xi}}{\hat{P}_{n-2}(\xi) \hat{P}^{[1]}_{n-2}(\xi) \hat{P}^{[2]}_{n-2}(\xi) \hat{P}^{[3]}_{n-2}(\xi)} \| \hat{P}_{n-2} \|^2_\omega.
\end{aligned} \tag{15}$$

Therefore,

$$\langle \hat{S}_n, 1\rangle_S = A_{n,1} I_{1,n}(\xi) + A_{n,2} I_{2,n}(\xi) + M\hat{P}_n(\xi).$$

In the same way, for $k = 1$ and $n \geq 2$, we have

$$\begin{aligned}\langle \hat{S}_n, (\cdot - \xi)\rangle_S &= \int_0^\infty \omega(x)\hat{S}_n(x)(x-\xi)\,dx + N\hat{S}_n'(\xi)\\ &= A_{n,2} I_{3,n}(\xi) + N A_{n,1}\hat{P}_{n-1}^{[2]}(\xi) + N\hat{P}_n'(\xi),\end{aligned}$$

where

$$\begin{aligned}I_{3,n}(\xi) &= \int_0^\infty (x-\xi)^3\omega(x)\hat{P}_{n-2}^{[4]}(x)\,dx = \frac{\|\hat{P}_{n-2}^{[3]}\|_{\omega,3}^2}{\hat{P}_n^{[3]}(\xi)}\\ &= -\frac{\hat{P}_{n-1}(\xi)\hat{P}_{n-1}^{[1]}(\xi)\hat{P}_{n-1}^{[2]}(\xi)}{\hat{P}_{n-2}(\xi)\hat{P}_{n-2}^{[1]}(\xi)\hat{P}_{n-2}^{[2]}(\xi)\hat{P}_{n-2}^{[3]}(\xi)}\|\hat{P}_{n-2}\|_\omega^2.\end{aligned}$$

Finally, using the expressions of $A_{n,1}$ and $A_{n,2}$, our statement follows. □

Next, we will study the behavior of the coefficients $A_{n,1}$ and $A_{n,2}$.

Proposition 4.

(i) $I_{1,n}(\xi)I_{3,n}(\xi) - N I_{2,n}(\xi)\hat{P}_{n-1}^{[2]}(\xi) = -I_{2,n}(\xi)\hat{P}_{n-1}^{[2]}(\xi)(N + \alpha_n\beta_n)$, *where*

$$0 < \alpha_n = \frac{I_{1,n}(\xi)}{\hat{P}_{n-1}^{[2]}(\xi)} < d_0^1 \quad \text{and} \quad \frac{d_0^3}{d_0^2} < -\frac{\hat{P}_{n-1}^{[2]\prime}(\xi)}{\hat{P}_{n-1}^{[2]}(\xi)} = \frac{I_{2,n}(\xi)}{I_{3,n}(\xi)} = \frac{1}{\beta_n} < -\frac{n}{\xi}.$$

(ii) $N I_{2,n}(\xi)\hat{P}_n'(\xi) - M I_{3,n}(\xi)\hat{P}_n(\xi) = I_{2,n}(\xi)\hat{P}_n'(\xi)(N + M\beta_n\gamma_n)$, *where*

$$\frac{d_0^1}{c_0} < -\frac{\hat{P}_n'(\xi)}{\hat{P}_n(\xi)} = \frac{1}{\gamma_n} < -\frac{n}{\xi}.$$

(iii) $MN\hat{P}_n(\xi)\hat{P}_{n-1}^{[2]}(\xi) - N I_{1,n}(\xi)\hat{P}_n'(\xi) = N\hat{P}_n(\xi)\hat{P}_{n-1}^{[2]}(\xi)\Big(M + \dfrac{\alpha_n}{\gamma_n}\Big).$

Proof. (i) From the Christoffel–Darboux formula for polynomials $\{\hat{P}_n^{[2]}\}_{n\geq 0}$ we have

$$\begin{aligned}(16)\quad (x-\xi)\sum_{k=0}^n \frac{\hat{P}_k^{[2]}(x)\hat{P}_k^{[2]}(y)}{\|\hat{P}_k^{[2]}\|_{\omega,2}^2} - \sum_{k=0}^n \frac{\hat{P}_k^{[2]}(x)}{\|\hat{P}_k^{[2]}\|_{\omega,2}^2}(y-\xi)\hat{P}_k^{[2]}(y)&\\ = \frac{1}{\|\hat{P}_n^{[2]}\|_{\omega,2}^2}\big(\hat{P}_{n+1}^{[2]}(x)\hat{P}_n^{[2]}(y) - \hat{P}_n^{[2]}(x)\hat{P}_{n+1}^{[2]}(y)\big).&\end{aligned}$$

If we multiply (16) by $(y-\xi)\omega(y)$ and integrate over $(0,\infty)$, evaluation at $x=\xi$ yields

$$-\sum_{k=0}^{n}\frac{\hat{P}_k^{[2]}(\xi)}{\|\hat{P}_k^{[2]}\|_{\omega,2}^2}\int_0^\infty (y-\xi)^2\omega(y)\hat{P}_k^{[2]}(y)\,dy = \frac{1}{\|\hat{P}_n^{[2]}\|_{\omega,2}^2}\big(\hat{P}_{n+1}^{[2]}(\xi)I_{1,n+1}(\xi)-\hat{P}_n^{[2]}(\xi)I_{1,n+2}(\xi)\big).$$

Since

$$\int_0^\infty (y-\xi)^2\omega(y)\hat{P}_k^{[2]}(y)\,dy = 0 \quad \text{for } k=1,2,\dots,n$$

and $\hat{P}_0^{[2]}=1$, the left-hand side is negative. Therefore,

$$\hat{P}_{n+1}^{[2]}(\xi)I_{1,n+1}(\xi)-\hat{P}_n^{[2]}(\xi)I_{1,n+2}(\xi)<0.$$

From (5) we have

$$\operatorname{sgn}\hat{P}_{n+1}^{[2]}(\xi)=(-1)^{n+1} \quad \text{and} \quad \operatorname{sgn}\hat{P}_n^{[2]}(\xi)=(-1)^n.$$

Thus, $\hat{P}_{n+1}^{[2]}(\xi)\hat{P}_n^{[2]}(\xi)$ is negative and, as a consequence,

$$\frac{I_{1,n+2}(\xi)}{\hat{P}_{n+1}^{[2]}(\xi)}<\frac{I_{1,n+1}(\xi)}{\hat{P}_n^{[2]}(\xi)}.$$

Using this relation recursively, we get

$$\frac{I_{1,n}(\xi)}{\hat{P}_{n-1}^{[2]}(\xi)}<I_{1,1}(\xi)=d_0^1.$$

On the other hand, (5) and (13) imply that $\operatorname{sgn} I_{1,n}(\xi)=(-1)^{n+1}$; therefore,

$$0<\frac{I_{1,n}(\xi)}{\hat{P}_{n-1}^{[2]}(\xi)}<d_0^1.$$

From (16)

$$0<\sum_{k=0}^{n}\frac{\big(\hat{P}_k^{[2]}(\xi)\big)^2}{\|\hat{P}_k^{[2]}\|_{\omega,2}^2}=\frac{1}{\|\hat{P}_n^{[2]}\|_{\omega,2}^2}\big(\hat{P}_{n+1}^{[2]\prime}(\xi)\hat{P}_n^{[2]}(\xi)-\hat{P}_n^{[2]\prime}(\xi)\hat{P}_{n+1}^{[2]}(\xi)\big).$$

Since $\hat{P}_{n+1}^{[2]}(\xi)\hat{P}_n^{[2]}(\xi)$ is negative this yields

$$\frac{\hat{P}_{n+1}^{[2]\prime}(\xi)}{\hat{P}_{n+1}^{[2]}(\xi)}<\frac{\hat{P}_n^{[2]\prime}(\xi)}{\hat{P}_n^{[2]}(\xi)}.$$

Using this relation recursively, we obtain

$$\frac{\hat{P}_{n+1}^{[2]\prime}(\xi)}{\hat{P}_{n+1}^{[2]}(\xi)} < \frac{\hat{P}_{1}^{[2]\prime}(\xi)}{\hat{P}_{1}^{[2]}(\xi)} = -\frac{d_0^3}{d_0^2}.$$

Let $0 < x_{1,n}^{[2]} < x_{2,n}^{[2]} < \cdots < x_{n,n}^{[2]}$ denote the zeros of $\hat{P}_n^{[2]}$. Then

$$-\frac{\hat{P}_{n}^{[2]\prime}(\xi)}{\hat{P}_{n}^{[2]}(\xi)} = \frac{1}{x_{1,n}^{[2]}-\xi} + \frac{1}{x_{2,n}^{[2]}-\xi} + \cdots + \frac{1}{x_{n,n}^{[2]}-\xi} < -\frac{n}{\xi}.$$

Statements (ii) and (iii) can be proved in a similar way as (i). □

Proposition 5. *Let $M, N \geq 0$ and not both zero. Then*

$$\operatorname{sgn} A_{n,1} = -1 \quad \textit{and} \quad \operatorname{sgn} A_{n,2} = -\operatorname{sgn} N.$$

Proof. From (5) and Proposition 4

$$\operatorname{sgn} A_{n,1} = -\operatorname{sgn} \frac{\hat{P}_n'(\xi)}{\hat{P}_{n-1}^{[2]}(\xi)} = \operatorname{sgn}\left(-\frac{\hat{P}_n'(\xi)}{\hat{P}_n(\xi)}\right) \operatorname{sgn} \frac{\hat{P}_n(\xi)}{\hat{P}_{n-1}^{[2]}(\xi)} = -1.$$

In a similar way,

$$\begin{aligned} \operatorname{sgn} A_{n,2} &= -\operatorname{sgn} N \ \operatorname{sgn} \frac{\hat{P}_n(\xi)}{I_{2,n}} \\ &= \operatorname{sgn} N \ \operatorname{sgn}\left(-\frac{\hat{P}_{n-1}^{[2]}(\xi)}{\hat{P}_{n-1}^{[2]\prime}(\xi)}\right) \operatorname{sgn} \frac{\hat{P}_n(\xi)\hat{P}_{n-2}(\xi)\hat{P}_{n-2}^{[1]}(\xi)\hat{P}_{n-2}^{[2]}(\xi)\hat{P}_{n-2}^{[3]}(\xi)}{\hat{P}_{n-1}(\xi)\hat{P}_{n-1}^{[1]}(\xi)\hat{P}_{n-1}^{[2]}(\xi)} \\ &= -\operatorname{sgn} N. \end{aligned}$$

□

The zeros. We now analyze the zeros of the polynomials $\hat{S}_n$. The techniques are the same as those used by Meijer [1993a; 1993b].

Theorem 2. *The discrete Sobolev orthogonal polynomial $\hat{S}_n$ has n real simple zeros and at most one of them is outside of $[\xi, \infty)$.*

Proof. Since for $N = 0$, $\hat{S}_n$ is a standard orthogonal polynomial, in the sequel we will consider the cases when $N > 0$ and $M \geq 0$. Let $\nu_1 < \nu_2 < \cdots < \nu_k$ be the zeros of $\hat{S}_n(x)$ on (ξ, ∞) with odd multiplicity. Let us introduce the polynomial

$$\phi(x) = (x-\nu_1)(x-\nu_2)\cdots(x-\nu_k).$$

Notice that $\phi(\xi)$ and $\phi'(\xi)$ have opposite signs and $\phi(x)\hat{S}_n(x)$ does not change sign on $[\xi, \infty)$. If $\deg \phi \leq n-2$, then

$$0 = \langle \phi, \hat{S}_n \rangle_S = \int_0^\infty \omega(x)\phi(x)\hat{S}_n(x)\,dx + M\phi(\xi)\hat{S}_n(\xi) + N\phi'(\xi)\hat{S}_n'(\xi)$$

and

$$0 = \langle(\cdot - \xi)\phi, \hat{S}_n\rangle_S = \int_0^\infty \omega(x)(x-\xi)\phi(x)\hat{S}_n(x)\,dx + N\phi(\xi)\hat{S}_n'(\xi).$$

This means that $\phi'(\xi)\hat{S}_n'(\xi)$ and $\phi(\xi)\hat{S}_n'(\xi)$ have the same sign, and therefore $\phi'(\xi)$ and $\phi(\xi)$ have the same sign. This yields a contradiction.

As a conclusion, $\deg\phi = n-1$ or $\deg\phi = n$, which proves our statement. □

Next, we prove that the zeros of $\hat{S}_n(x)$ interlace with the zeros of $\hat{P}_{n-1}^{[2]}(x)$ if $\hat{S}_n(x)$ has a zero outside $[\xi, \infty)$. Notice that, by Theorem 1, $\hat{S}_n(\xi) \neq 0$.

Theorem 3. *Denote by $\nu_{r,n}$, $r = 1, 2, \ldots, n$, the zeros of $\hat{S}_n(x)$ in increasing order. Suppose that $\nu_{1,n} < \xi$. Then $2\xi - x_{1,n-1}^{[2]} < \nu_{1,n} < \xi$ and*

$$\xi < \nu_{2,n} < x_{1,n-1}^{[2]} < \cdots < \nu_{n,n} < x_{n-1,n-1}^{[2]}.$$

Proof. From Theorem 1 we have

$$\hat{S}_n(x_{r,n-1}^{[2]}) = \hat{P}_n(x_{r,n-1}^{[2]}) + A_{n,2}(x_{r,n-1}^{[2]} - \xi)^2 \hat{P}_{n-2}^{[4]}(x_{r,n-1}^{[2]}), \quad r = 1, 2, \ldots, n-1.$$

Then from Proposition 3(iii) and Proposition 5 we get

$$\operatorname{sgn} \hat{S}_n(x_{r,n-1}^{[2]}) = (-1)^{n-r}, \quad r = 1, 2, \ldots, n-1,$$

On the other hand, from (5) and Theorem 1,

$$\operatorname{sgn} \hat{S}_n(\xi) = \operatorname{sgn} \hat{P}_n(\xi) = (-1)^n.$$

Therefore, every interval $(\xi, x_{1,n-1}^{[2]})$ and $(x_{r,n-1}^{[2]}, x_{r+1,n-1}^{[2]})$, for $r = 1, \ldots, n-2$, contains an odd number of zeros of $\hat{S}_n(x)$. Since $\hat{S}_n$ has n real zeros and at most one of them is outside of (ξ, ∞), then

$$\xi < \nu_{2,n} < x_{1,n-1}^{[2]} < \cdots < \nu_{n,n} < x_{n-1,n-1}^{[2]}.$$

Now, we will prove that $2\xi - x_{1,n-1}^{[2]} < \nu_{1,n} < \xi$. Let

$$\hat{S}_n(x) = (x - \nu_{1,n})(x - \nu_{2,n}) \cdots (x - \nu_{n,n}).$$

By Theorem 1 and Proposition 4,

$$\hat{S}_n'(\xi) = \hat{P}_n'(\xi) + A_{n,1}\hat{P}_{n-2}^{[2]}(\xi) = \frac{\beta_n \hat{P}_n(\xi)(M + \alpha_n/\gamma_n)}{N + \alpha_n\beta_n}.$$

Therefore,

$$\operatorname{sgn} \hat{S}_n'(\xi) = \operatorname{sgn} \hat{P}_n(\xi) = \operatorname{sgn} \hat{S}_n(\xi)$$

and

$$0 < \frac{\hat{S}_n'(\xi)}{\hat{S}_n(\xi)} = \frac{1}{\xi - \nu_{1,n}} - \frac{1}{\nu_{2,n} - \xi} - \cdots - \frac{1}{\nu_{n,n} - \xi}.$$

Hence $\dfrac{1}{\xi-\nu_{1,n}} > \dfrac{1}{\nu_{2,n}-\xi}$, which implies successively

$$x_{1,n-1}^{[2]}-\xi > \nu_{2,n}-\xi > \xi-\nu_{1,n} \quad \text{and} \quad 2\xi - x_{1,n-1}^{[2]} < \nu_{1,n}.$$

Our statement follows. □

4. Discrete Laguerre–Sobolev orthogonal polynomials: asymptotics

Laguerre polynomials. For $\alpha \in \mathbb{R}$, the Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}.$$

For $\alpha > -1$, the $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ are orthogonal on $[0, +\infty)$ with respect to the weight function $\omega(x) = x^\alpha e^{-x}$ [Szegő 1975, Chapter V]. Let $\{L_n^{(\alpha,k)}\}_{n=0}^{\infty}$, $k \in \mathbb{N}$, denote the sequence of polynomials orthogonal with respect to the modified Laguerre weight $(x-\xi)^k\omega(x)$, $\xi < 0$, normalized by the condition that $L_n^{(\alpha,k)}$ has the same leading coefficient as the classical Laguerre orthogonal polynomial $L_n^{(\alpha)} = L_n^{(\alpha,0)}$. That is, $k(L_n^{(\alpha,k)}) = (-1)^n/n!$.

We summarize some properties of the $L_n^{(\alpha,k)}(x)$, $k \in \mathbb{N}\cup\{0\}$, to be used later.

Proposition 6 [Fejzullahu 2011]. (i) *For* $\alpha > -1$,

$$\|L_n^{(\alpha)}\|_\alpha^2 = \int_0^\infty (L_n^{(\alpha)}(x))^2 x^\alpha e^{-x} d(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}.$$

(ii) *For every* $n \in \mathbb{N}$,

$$(L_n^{(\alpha)}(x))' = -L_{n-1}^{(\alpha+1)}(x).$$

(iii) (Perron's formula) *Let* $\alpha \in \mathbb{R}$. *Then*

$$L_n^{(\alpha)}(x) = 2^{-1}\pi^{-1/2}e^{x/2}(-x)^{-\alpha/2-1/4}n^{\alpha/2-1/4}e^{2\sqrt{-nx}}\big(1+O(n^{-1/2})\big).$$

This relation holds for x in the complex plane cut along the positive real semiaxis; *both* $(-x)^{-\alpha/2-1/4}$ *and* $\sqrt{-x}$ *must be taken real and positive if* $x < 0$. *The bound of the remainder holds uniformly in every closed domain which does not overlap the positive real semiaxis.*

Moreover, *we get the outer ratio asymptotics*

$$\lim_{n\to\infty} n^{(l-j)/2}\frac{L_{n+k}^{(\alpha+j)}(x)}{L_{n+h}^{(\alpha+l)}(x)} = (-x)^{(l-j)/2}, \quad j, l \in \mathbb{R},\ h, k \in \mathbb{Z},$$

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,k)}(x)}{n^{k/2}L_n^{(\alpha)}(x)} = \frac{1}{(\sqrt{-x}+\sqrt{-\xi}\,)^k},$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

(iv) (Mehler–Heine formula) *Uniformly on compact subsets of* $\mathbb{C}$, *we have*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x/(n+j))}{n^{\alpha}} = x^{-\alpha/2} J_{\alpha}(2\sqrt{x}\,)$$

and

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,k)}(x/(n+j))}{n^{\alpha+k/2}} = \frac{1}{(\sqrt{-\xi}\,)^k} x^{-\alpha/2} J_{\alpha}(2\sqrt{x})$$

where $j \in \mathbb{N} \cup 0$ *and* J_α *is the Bessel function of the first kind.*

(v) (Plancherel–Rotach type outer asymptotics for $L_n^{(\alpha,N)}$) *Uniformly on compact subsets of* $\mathbb{C} \setminus [0, 4]$ *and uniformly on* $j \in \mathbb{N} \cup \{0\}$, *we have*

$$\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = -\frac{1}{\phi((x-2)/2)}$$

and

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,N)}((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = \left(\frac{\phi((x-2)/2)+1}{x}\right)^N,$$

where ϕ *is the conformal mapping of* $\mathbb{C} \setminus [-1, 1]$ *onto the exterior of the unit circle given by*

$$\phi(x) = x + \sqrt{x^2-1}, \quad x \in \mathbb{C} \setminus [-1, 1],$$

with $\sqrt{x^2-1} > 0$ *when* $x > 1$.

Proposition 7. $$L_n^{(\alpha,2)\prime}(\xi) \cong \frac{n}{4\xi} L_n^{(\alpha+1)}(\xi).$$

Proof. Using integration by parts we have

$$\int_0^\infty \big(L_n^{(\alpha,2)}(x)\big)' L_k^{(\alpha+1,3)}(x)(x-\xi)^3 x^{\alpha+1} e^{-x}\,dx = \begin{cases} 0 & \text{if } k \le n-3, \\ n(n-1)\|\hat{L}_n^{(\alpha,2)}\|_{\alpha,2}^2 & \text{if } k = n-2. \end{cases}$$

Therefore,

$$\big(L_n^{(\alpha,2)}(x)\big)' = -L_{n-1}^{(\alpha+1,3)}(x) + H_n L_{n-2}^{(\alpha+1,3)}(x),$$

where

$$H_n = \frac{n(n-1)\|\hat{L}_n^{(\alpha,2)}\|_{\alpha,2}^2}{\|\hat{L}_{n-2}^{(\alpha+1,3)}\|_{\alpha+1,3}^2}.$$

Using (8) and Proposition 6(iii),

$$\begin{aligned} H_n &= \frac{(n+1)^2(n+\alpha)}{(n-1)^3} \frac{L_{n-2}^{(\alpha+1,2)}(\xi)}{L_{n-1}^{(\alpha+1,2)}(\xi)} \prod_{i=1}^{2} \frac{L_{n-2}^{(\alpha+1,i-1)}(\xi)}{L_{n-1}^{(\alpha+1,i-1)}(\xi)} \frac{L_{n+1}^{(\alpha,i-1)}(\xi)}{L_{n}^{(\alpha,i-1)}(\xi)} \\ &= \frac{L_{n-2}^{(\alpha+1,2)}(\xi)}{L_{n-1}^{(\alpha+1,2)}(\xi)} \prod_{i=1}^{2} \frac{L_{n-2}^{(\alpha+1,i-1)}(\xi)}{L_{n-1}^{(\alpha+1,i-1)}(\xi)} \frac{L_{n+1}^{(\alpha,i-1)}(\xi)}{L_{n}^{(\alpha,i-1)}(\xi)} + O\Big(\frac{1}{n}\Big). \end{aligned}$$

On the other hand, [Fejzullahu 2011, Proposition 2.2] gives

$$\big(L_n^{(\alpha,2)}(x)\big)' = -L_{n-1}^{(\alpha,3)}(x) + G_n L_{n-2}^{(\alpha+1,3)}(x), \tag{17}$$

where

$$\begin{aligned} G_n &= H_n - \frac{n^3}{(n-1)^3} \prod_{i=1}^{3} \frac{L_{n-2}^{(\alpha+1,i-1)}(\xi)}{L_{n-1}^{(\alpha+1,i-1)}(\xi)} \frac{L_{n}^{(\alpha,i-1)}(\xi)}{L_{n-1}^{(\alpha,i-1)}(\xi)} \\ &= \prod_{i=1}^{3} \frac{L_{n-2}^{(\alpha+1,i-1)}(\xi)}{L_{n-1}^{(\alpha+1,i-1)}(\xi)} \left(\frac{L_{n+1}^{(\alpha)}(\xi) L_{n+1}^{(\alpha,1)}(\xi)}{L_{n}^{(\alpha)}(\xi) L_{n}^{(\alpha,1)}(\xi)} - \frac{L_{n}^{(\alpha)}(\xi) L_{n}^{(\alpha,1)}(\xi) L_{n}^{(\alpha,2)}(\xi)}{L_{n-1}^{(\alpha)}(\xi) L_{n-1}^{(\alpha,1)}(\xi) L_{n-1}^{(\alpha,2)}(\xi)} \right) + O\Big(\frac{1}{n}\Big). \end{aligned}$$

Again, from [Fejzullahu 2011, Proposition 2.2],

$$\frac{L_{n+1}^{(\alpha)}(\xi) L_{n+1}^{(\alpha,1)}(\xi)}{L_{n}^{(\alpha)}(\xi) L_{n}^{(\alpha,1)}(\xi)} = \frac{L_{n+1}^{(\alpha)}(\xi) L_{n+1}^{(\alpha-1,1)}(\xi)}{L_{n}^{(\alpha)}(\xi) L_{n}^{(\alpha,1)}(\xi)} + \frac{L_{n+2}^{(\alpha-1)}(\xi)}{L_{n+1}^{(\alpha-1)}(\xi)} + O\Big(\frac{1}{n}\Big),$$

$$\begin{aligned} \frac{L_{n}^{(\alpha)}(\xi) L_{n}^{(\alpha,1)}(\xi) L_{n}^{(\alpha,2)}(\xi)}{L_{n-1}^{(\alpha)}(\xi) L_{n-1}^{(\alpha,1)}(\xi) L_{n-1}^{(\alpha,2)}(\xi)} &= \frac{L_{n}^{(\alpha)}(\xi) L_{n}^{(\alpha,1)}(\xi) L_{n}^{(\alpha-1,2)}(\xi)}{L_{n-1}^{(\alpha)}(\xi) L_{n-1}^{(\alpha,1)}(\xi) L_{n-1}^{(\alpha,2)}(\xi)} \\ &\quad + \frac{L_{n+1}^{(\alpha-1)}(\xi) L_{n+1}^{(\alpha-1,1)}(\xi)}{L_{n}^{(\alpha-1)}(\xi) L_{n}^{(\alpha-1,1)}(\xi)} + O\Big(\frac{1}{n}\Big), \end{aligned}$$

and

$$\begin{aligned} \frac{L_{n+2}^{(\alpha-1)}(\xi)}{L_{n+1}^{(\alpha-1)}(\xi)} - \frac{L_{n+1}^{(\alpha-1)}(\xi) L_{n+1}^{(\alpha-1,1)}(\xi)}{L_{n}^{(\alpha-1)}(\xi) L_{n}^{(\alpha-1,1)}(\xi)} &= \frac{L_{n+2}^{(\alpha-2)}(\xi)}{L_{n+1}^{(\alpha-1)}(\xi)} + 1 \\ &\quad - \frac{L_{n+1}^{(\alpha-1)}(\xi) L_{n+1}^{(\alpha-2,1)}(\xi)}{L_{n}^{(\alpha-1)}(\xi) L_{n}^{(\alpha-1,1)}(\xi)} - \frac{L_{n+2}^{(\alpha-2)}(\xi)}{L_{n+1}^{(\alpha-2)}(\xi)} + O\Big(\frac{1}{n}\Big) \\ &= \frac{L_{n+2}^{(\alpha-2)}(\xi)}{L_{n+1}^{(\alpha-1)}(\xi)} - \frac{L_{n+1}^{(\alpha-1)}(\xi) L_{n+1}^{(\alpha-2,1)}(\xi)}{L_{n}^{(\alpha-1)}(\xi) L_{n}^{(\alpha-1,1)}(\xi)} - \frac{L_{n+2}^{(\alpha-3)}(\xi)}{L_{n+1}^{(\alpha-2)}(\xi)} + O\Big(\frac{1}{n}\Big). \end{aligned}$$

Therefore, by using Proposition 6(iii),

$$\sqrt{n}G_n \cong -\sqrt{-\xi}.$$

and taking into account (17) the result follows. $\square$

Discrete Laguerre–Sobolev orthogonal polynomials. Let $\{S_n\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the discrete Sobolev inner product (12), where $\omega(x) = x^\alpha e^{-x}$ and $\xi < 0$, normalized by the condition that S_n has the same leading coefficient as the classical Laguerre orthogonal polynomial $L_n^{(\alpha)}$, i.e., $k(S_n) = (-1)^n/n!$.

Theorem 4. *Let $M \geq 0$ and $N \geq 0$. There are real constants $B_{n,0}$, $B_{n,1}$, and $B_{n,2}$ such that*

$$S_n(x) = B_{n,0}L_n^{(\alpha)}(x) + B_{n,1}(x-\xi)L_{n-1}^{(\alpha,2)}(x) + B_{n,2}(x-\xi)^2 L_{n-2}^{(\alpha,4)}(x), \tag{18}$$

where $B_{n,0} = \dfrac{1}{1+A_{n,1}+A_{n,2}}$, $B_{n,1} = -\dfrac{A_{n,1}}{n(1+A_{n,1}+A_{n,2})}$, *and*

$$B_{n,2} = \frac{A_{n,2}}{n(n-1)(1+A_{n,1}+A_{n,2})}.$$

Moreover:

(i) *If $M > 0$ and $N > 0$, then*

$$B_{n,0} \cong \frac{8\xi n^\alpha}{M\big(L_n^{(\alpha)}(\xi)\big)^2}, \qquad B_{n,1} \cong -\frac{32\xi\sqrt{-\xi}\,n^{\alpha-1/2}}{M\big(L_n^{(\alpha)}(\xi)\big)^2}, \qquad B_{n,2} \cong \frac{1}{n^2}. \tag{19}$$

(ii) *If $M = 0$ and $N > 0$, then*

$$B_{n,0} \cong \frac{1}{4\sqrt{-\xi n}}, \qquad B_{n,1} \cong -\frac{1}{n}, \qquad B_{n,2} \cong \frac{1}{4n^2\sqrt{-\xi n}}.$$

(iii) *If $M > 0$ and $N = 0$, then*

$$B_{n,0} \cong \frac{\sqrt{-\xi}}{Mn^{1/2-\alpha}\big(L_{n-1}^{(\alpha)}(\xi)\big)^2}, \qquad B_{n,1} \cong -\frac{1}{n}, \qquad B_{n,2} = 0.$$

Proof. From Theorem 1,

$$S_n(x) = \frac{(-1)^n \hat{S}_n(x)}{n!\,(1+A_{n,1}+A_{n,2})}$$

and, as a consequence,

$$S_n(x) = B_{n,0}L_n^{(\alpha)}(x) + B_{n,1}(x-\xi)L_{n-1}^{(\alpha,2)}(x) + B_{n,2}(x-\xi)^2 L_{n-2}^{(\alpha,4)}(x),$$

where $B_{n,0}$, $B_{n,1}$, and $B_{n,2}$ are as in the statement of the theorem.

Now, from Proposition 4 we can obtain the behavior of the coefficients $B_{n,0}$, $B_{n,1}$ and $B_{n,2}$ for n large enough. In order to estimate $A_{n,1}$ and $A_{n,2}$, first we compute $\alpha_n\beta_n$, α_n/γ_n, $\beta_n\gamma_n$ and $I_{2,n}(\xi)$. From (13) and Proposition 6, we can write

$$\alpha_n\beta_n = -\frac{I_{1,n}(\xi)}{\hat{L}_{n-1}^{(\alpha,2)'}(\xi)} = \frac{\hat{L}_n^{(\alpha)}(\xi)}{\hat{L}_{n-1}^{(\alpha)}(\xi)\hat{L}_{n-1}^{(\alpha,1)}(\xi)\hat{L}_{n-1}^{(\alpha,2)'}(\xi)}\|\hat{L}_{n-1}^{(\alpha)}\|_\alpha^2$$
$$= -\frac{\Gamma(n+\alpha)}{\Gamma(n)}\frac{nL_n^{(\alpha)}(\xi)}{L_{n-1}^{(\alpha)}(\xi)L_{n-1}^{(\alpha,1)}(\xi)L_{n-1}^{(\alpha,2)'}(\xi)} \cong \frac{8(-\xi)^{3/2}n^{\alpha-1/2}}{L_n^{(\alpha)}(\xi)L_n^{(\alpha+1)}(\xi)},$$

$$\frac{\alpha_n}{\gamma_n} = -\frac{I_{1,n}(\xi)\hat{L}_n^{(\alpha)'}(\xi)}{\hat{L}_n^{(\alpha)}(\xi)\hat{L}_{n-1}^{(\alpha,2)}(\xi)} = \frac{\hat{L}_n^{(\alpha)'}(\xi)}{\hat{L}_{n-1}^{(\alpha)}(\xi)\hat{L}_{n-1}^{(\alpha,1)}(\xi)\hat{L}_{n-1}^{(\alpha,2)}(\xi)}\|\hat{L}_{n-1}^{(\alpha)}\|_\alpha^2$$
$$= \frac{\Gamma(n+\alpha)}{\Gamma(n)}\frac{nL_{n-1}^{(\alpha+1)}(\xi)}{L_{n-1}^{(\alpha)}(\xi)L_{n-1}^{(\alpha,1)}(\xi)L_{n-1}^{(\alpha,2)}(\xi)} \cong \frac{8(-\xi)^{3/2}n^{\alpha-1/2}L_n^{(\alpha+1)}(\xi)}{\left(L_n^{(\alpha)}(\xi)\right)^3},$$

$$\beta_n\gamma_n = \alpha_n\beta_n\frac{\gamma_n}{\alpha_n} \cong \left(\frac{L_n^{(\alpha)}(\xi)}{L_n^{(\alpha+1)}(\xi)}\right)^2 \cong -\frac{\xi}{n},$$

$$I_{2,n}(\xi) \cong (-1)^{n-1}(n-2)!\,n^{\alpha+3}\frac{L_{n-1}^{(\alpha)}(\xi)L_{n-1}^{(\alpha,1)}(\xi)L_{n-1}^{(\alpha,2)'}(\xi)}{L_{n-2}^{(\alpha)}(\xi)L_{n-2}^{(\alpha,1)}(\xi)L_{n-2}^{(\alpha,2)}(\xi)L_{n-3}^{(\alpha,3)}(\xi)}$$
$$\cong \frac{8\xi(-1)^{n-1}(n-2)!\,n^{\alpha+2}}{L_n^{(\alpha)}(\xi)}.$$

Next, we will analyze the following three situations.

(i) Let $M>0$ and $N>0$. Then,

$$A_{n,1} \cong -\frac{\hat{L}_n^{(\alpha)'}(\xi)}{\hat{L}_{n-1}^{(\alpha,2)}(\xi)} = \frac{nL_n^{(\alpha)'}(\xi)}{L_{n-1}^{(\alpha,2)}(\xi)} = -\frac{nL_{n-1}^{(\alpha+1)}(\xi)}{L_{n-1}^{(\alpha,2)}(\xi)} \cong -4\sqrt{-\xi n}$$

and

$$A_{n,2} \cong -\frac{M\hat{L}_n^{(\alpha)}(\xi)}{I_{n,2}(\xi)} \cong \frac{M\left(L_n^{(\alpha)}(\xi)\right)^2}{8\xi n^\alpha}.$$

Therefore,

$$B_{n,0} \cong \frac{8\xi n^\alpha}{M\left(L_n^{(\alpha)}(\xi)\right)^2}, \quad B_{n,1} \cong \frac{32\xi\sqrt{-\xi}n^{\alpha-1/2}}{M\left(L_n^{(\alpha)}(\xi)\right)^2}, \quad B_{n,2} \cong \frac{1}{n^2}.$$

(ii) Let $M=0$ and $N>0$. Then,

$$A_{n,1} \cong -4\sqrt{-\xi n} \quad \text{and} \quad A_{n,2} = -\frac{\hat{L}_n^{(\alpha)}(\xi)}{I_{n,2}(\xi)}\frac{\alpha_n}{\gamma_n} \cong -1.$$

Therefore,

$$B_{n,0} \cong -\frac{1}{4\sqrt{-\xi n}}, \quad B_{n,1} \cong -\frac{1}{n}, \quad B_{n,2} \cong \frac{1}{4n^2\sqrt{-\xi n}}.$$

(iii) Let $M > 0$ and $N = 0$. Then,

$$A_{n,1} = \frac{M\hat{L}_n^{(\alpha)}(\xi)}{I_{n,1}(\xi)} = -\frac{M\hat{L}_{n-1}^{(\alpha)}(\xi)\hat{L}_{n-1}^{(\alpha,1)}(\xi)}{\|L_{n-1}^{(\alpha)}\|_\alpha^2} \cong -\frac{Mn^{1/2-\alpha}}{\sqrt{-\xi}}\big(L_{n-1}^{(\alpha)}(\xi)\big)^2, \quad A_{n,2} = 0.$$

Therefore,

$$B_{n,0} \cong -\frac{\sqrt{-\xi}}{Mn^{1/2-\alpha}\big(L_{n-1}^{(\alpha)}(\xi)\big)^2}, \quad B_{n,1} \cong -\frac{1}{n}, \quad B_{n,2} = 0. \qquad \square$$

Next we deduce several asymptotic properties for discrete Laguerre–Sobolev polynomials when $M, N \geq 0$. (For $M > 0$ and $N = 0$, the same asymptotic results for corresponding Laguerre-type polynomials has been deduced in [Dueñas et al. 2011] and [Fejzullahu and Zejnullahu 2010].)

Theorem 5. (i) (Outer relative asymptotics) *Uniformly on compact subsets of* $\mathbb{C} \setminus [0, \infty)$ *we have*:

- *If* $M > 0$ *and* $N > 0$, *then*

$$\lim_{n\to\infty} \frac{S_n(x)}{L_n^{(\alpha)}(x)} = \left(\frac{\sqrt{-x} - \sqrt{-\xi}}{\sqrt{-x} + \sqrt{-\xi}}\right)^2.$$

  Notice that, according to the Hurwitz's Theorem, the point ξ *attracts two negative zeros of* $S_n(x)$ *for n large enough.*

- *If* $M = 0$ *and* $N > 0$ *or* $M > 0$ *and* $N = 0$, *then*

$$\lim_{n\to\infty} \frac{S_n(x)}{L_n^{(\alpha)}(x)} = \frac{\sqrt{-x} - \sqrt{-\xi}}{\sqrt{-x} + \sqrt{-\xi}}.$$

  Notice that, according to the Hurwitz's Theorem, the point ξ *attracts one negative zero of* $S_n(x)$ *for n large enough.*

(ii) (Mehler–Heine formula)

- *If* $M > 0$ *and* $N > 0$

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^\alpha} = x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

- *If* $M = 0$ *and* $N > 0$ *or* $M > 0$ *and* $N = 0$

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^\alpha} = -x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

  uniformly on compact subsets of $\mathbb{C}$.

(iii) (Plancherel–Rotach type outer asymptotics for S_n)

- *If* $M \geq 0$ *and* $N \geq 0$, *then*

$$\lim_{n\to\infty} \frac{S_n(nx)}{L_n^{(\alpha)}(nx)} = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, 4]$.

Proof. We will prove the theorem when $M > 0$ and $N > 0$. The proofs of the other cases can be done in a similar way.

(i) From (18)

$$\frac{S_n(x)}{L_n^{(\alpha)}(x)} = B_{n,0} + nB_{n,1}(x-\xi)\frac{L_{n-1}^{(\alpha,2)}(x)}{nL_n^{(\alpha)}(x)} + n^2B_{n,2}(x-\xi)^2\frac{L_{n-2}^{(\alpha,4)}(x)}{n^2L_n^{(\alpha)}(x)}.$$

Now, Proposition 6(iii) and (19) yield

$$\lim_{n\to\infty} \frac{S_n(x)}{L_n^{(\alpha)}(x)} = (x-\xi)^2 \lim_{n\to\infty} \frac{L_{n-2}^{(\alpha,4)}(x)}{n^2L_n^{(\alpha)}(x)} = \left(\frac{\sqrt{-x}-\sqrt{-\xi}}{\sqrt{-x}+\sqrt{-\xi}}\right)^2.$$

(ii) Scaling the variable as $x \to x/n$ in (18) then dividing by n^α we get

$$\begin{aligned}\frac{S_n(x/n)}{n^\alpha}&\\ = B_{n,0}\frac{L_n^{(\alpha)}(x/n)}{n^\alpha} &+ nB_{n,1}(x/n-\xi)\frac{L_{n-1}^{(\alpha,2)}(x/n)}{n^{\alpha+1}} + n^2B_{n,2}(x/n-\xi)^2\frac{L_{n-2}^{(\alpha,4)}(x/n)}{n^{\alpha+2}}.\end{aligned}$$

Now, Proposition 6(iv) and (19) yield

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^\alpha} = (-\xi)^2 \lim_{n\to\infty} \frac{L_{n-2}^{(\alpha,4)}(x)}{n^{\alpha+2}} = x^{-\alpha/2}J_\alpha(2\sqrt{x}).$$

(iii) Dividing (18) by $L_n^\alpha(x)$ then scaling the variable as $x \to nx$ we get

$$\begin{aligned}\frac{S_n(nx)}{L_n^{(\alpha)}(nx)} = B_{n,0} + nB_{n,1}\frac{nx-\xi}{n}\frac{L_{n-1}^{(\alpha,2)}(nx)}{L_{n-1}^{(\alpha)}(nx)}\frac{L_{n-1}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}&\\ + n^2B_{n,2}\frac{(nx-\xi)^2}{n^2}\frac{L_{n-2}^{(\alpha,4)}(nx)}{L_{n-2}^{(\alpha)}(nx)}\frac{L_{n-2}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}&.\end{aligned}$$

From Proposition 6(v) and (19)

$$\lim_{n\to\infty} \frac{S_n(nx)}{L_n^{(\alpha)}(nx)} = x^2\left(\frac{\phi((x-2)/2)+1}{x}\right)^4 \frac{1}{(\phi((x-2)/2))^2}.$$

Now, using the fact that $(\phi(z)+1)^2 = 2(z+1)\phi(z)$ if $|z| > 1$, we get our result. □

Acknowledgements

The third author (BF) acknowledges the kind reception of the Departamento de Matemáticas of Universidad Carlos III de Madrid during his visit from December 2010 to March 2011. In this period the content of manuscript was discussed in several seminars. The work of the first (FM) and fourth (EH) authors has been supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain, grant MTM2009-12740-C03-01. The authors thank the valuable comments by the referees which have contributed to improve the presentation of the manuscript.

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Received June 20, 2011. Revised January 23, 2012.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 257 No. 1 May 2012

0030-8730(201205)257:1;1-7