Sieved para-orthogonal polynomials on the unit circle*

F. Marcellán[a] and A. Sri Ranga[b†]

Instituto de Ciencias Matemáticas (ICMAT) and Departamento de Matemáticas
Universidad Carlos III de Madrid, 28911, Leganés, Spain

[b]Departamento de Matemática Aplicada,
IBILCE, UNESP - Universidade Estadual Paulista
15054-000, São José do Rio Preto, SP, Brazil

April 30, 2014

Abstract

Sieved orthogonal polynomials on the unit circle were introduced independently by Ismail and Li [14] and Marcellán and Sansigre [17]. We look at the para-orthogonal polynomials, chain sequences and quadrature formulas that follow from the kernel polynomials of sieved orthogonal polynomials on the unit circle.

Keywords: Sieved Orthogonal polynomials on the unit circle, Para-orthogonal polynomials, Chain sequences.

2010 Mathematics Subject Classification: 42C05, 33C47.

1 Introduction

Given a nontrivial positive measure $\mu(\zeta) = \mu(e^{i\theta})$ supported on the unit circle $\mathcal{C} = \{\zeta = e^{i\theta}\colon\ 0 \leq \theta \leq 2\pi\}$, it is well known that the associated sequence of monic OPUC (Orthogonal Polynomials on the Unit Circle) $\{S_n(z)\}_{n=0}^{\infty}$ can be defined by

$$\int_{\mathcal{C}} \bar{\zeta}^j S_n(\zeta) d\mu(\zeta) = \int_0^{2\pi} e^{-ij\theta} S_n(e^{i\theta}) d\mu(e^{i\theta}) = 0, \quad 0 \leq j \leq n-1, \quad n \geq 1.$$

Letting $\kappa_n^{-2} = \|S_n\|^2 = \int_{\mathcal{C}} |S_n(\zeta)|^2 d\mu(\zeta)$, the orthonormal polynomials on the unit circle are $s_n(z) = \kappa_n S_n(z)$, $n \geq 0$.

OPUC were introduced by Gábor Szegő in the first half of the 20th century (see the monograph [26]). Thus, they are also referred to as Szegő polynomials. These polynomials, which have received a lot of attention in recent years (see, for example, [1, 4, 8, 7, 16, 19, 20, 21, 25, 27]), have applications in quadrature rules, signal processing, operator and spectral theory and many other topics.

Chapter 8 of Ismail's recent book [13] on these polynomials and the two recent volumes [22] and [23] by Barry Simon, specifically under the title "Orthogonal Polynomials on the Unit Circle", have provided us with many useful tools for further research in this area.

*The first author has been supported by Dirección general de Política Científica y Tecnológica, Ministerio de Economía y Competitividad of Spain, grant MTM2012-36732-C03-01 as well as by CAPES, Brazil. The second author's work was support by funds from CAPES, CNPq and FAPESP of Brazil.

†ranga@ibilce.unesp.br (corresponding author)

The monic OPUC satisfy the so called forward and backward recurrence relations, respectively,

$$\begin{aligned} S_n(z) &= zS_{n-1}(z) - \overline{\alpha}_{n-1}\, S_{n-1}^*(z), \\ S_n(z) &= (1-|\alpha_{n-1}|^2)zS_{n-1}(z) - \overline{\alpha}_{n-1}S_n^*(z), \end{aligned} \quad n \geq 1, \tag{1.1}$$

where $\alpha_{n-1} = -\overline{S_n(0)}$ and $S_n^*(z) = z^n\overline{S_n(1/\bar{z})}$ denotes the reversed (reciprocal) polynomial of $S_n(z)$. Following Simon [22], we refer to the numbers α_n as Verblunsky coefficients. It is known that these coefficients are such that $|\alpha_n| < 1$, $n \geq 0$, as well as that OPUC and the associated measure are completely characterized by the Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ as stated by the following theorem.

Theorem A. *Given an arbitrary sequence of complex numbers $\{\alpha_n\}_{n=0}^{\infty}$, where $|\alpha_n| < 1$, $n \geq 0$, then associated with this sequence there exists a unique nontrivial probability measure μ on the unit circle such that the polynomials $\{S_n(z)\}_{n=0}^{\infty}$ generated by* (1.1) *are the respective OPUC.*

Here, μ is a nontrivial positive measure if its support is infinite and it is a nontrivial probability measure if $\mu_0 = \int_{\mathcal{C}} d\mu(\zeta) = 1$. The above theorem, known as Favard's theorem for the unit circle, has been referred to as Verblunsky's theorem in Simon [22].

Given the sequence of Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ let μ be the associated nontrivial probability measure and let $\{S_n(z)\}_{n=0}^{\infty}$ be the corresponding OPUC. For a positive integer ℓ the sieved OPUC $\{S_n^{(\ell)}(z)\}_{n=0}^{\infty}$ are defined as those orthogonal polynomials associated with the Verblunsky coefficients $\{\alpha_n^{(\ell)}\}_{n=0}^{\infty}$ given by

$$\alpha_n^{(\ell)} = \begin{cases} 0, & \text{if} \quad (n+1) \neq 0 \mod \ell, \\ \alpha_{\lfloor n/\ell \rfloor}, & \text{if} \quad (n+1) = 0 \mod \ell\ , \end{cases} \tag{1.2}$$

for $n \geq 0$. We also denote by $\mu^{(\ell)}$ the nontrivial probability measure on the unit circle associated with $\{\alpha_n^{(\ell)}\}_{n=0}^{\infty}$.

Note that $\{S_n^{(1)}(z)\}_{n=0}^{\infty}$ are the polynomials $\{S_n(z)\}_{n=0}^{\infty}$. The earliest treatment of the sieved orthogonal polynomials $\{S_n^{(\ell)}(z)\}_{n=0}^{\infty}$ for $\ell \geq 2$ is found in Ismail and Li [14]. However, the sieved orthogonal polynomials $\{S_n^{(2)}(z)\}_{n=0}^{\infty}$ have been studied earlier than [14] by Marcellán and Sansigre (see [17] and [18]). The following results, established in [14], will be the basic requirement for the results obtained in the present manuscript.

$$S_{r\ell+j}^{(\ell)}(z) = z^j S_r(z^\ell), \quad j = 0, 1, \ldots, \ell-1, \quad r \geq 0, \tag{1.3}$$

and

$$d\mu^{(\ell)}(e^{i\theta}) = \ell^{-1} d\mu(e^{i\ell\theta}), \quad 0 \leq \theta \leq 2\pi.$$

For the reversed polynomials $S_n^{(\ell)*}(z)$ there hold

$$S_{r\ell+j}^{(\ell)*}(z) = z^{r\ell+j}\overline{S_{r\ell+j}^{(\ell)}(1/\bar{z})} = (z^\ell)^r\,\overline{S_r(1/\bar{z}^\ell)} = S_r^*(z^\ell), \quad j = 0, 1, \ldots, \ell-1, \quad r \geq 0. \tag{1.4}$$

The aim of the present manuscript is to explore the connection between para-orthogonal polynomials, chain sequences and quadrature formulas that follow from the kernel polynomials of the sieved OPUC. The structure of the manuscript is as follows. In Section 2 we present a basic background concerning para-orthogonal polynomials associated with a nontrivial probability measure on the unit circle, the three term recurrence relation they satisfy, the representation of the coefficients of such a recurrence relation as chain sequences and the role of zeros of para-orthogonal polynomials in Gaussian quadrature rules on the unit circle. In Section 3, we consider para-orthogonal polynomials associated with the sieved measure on the unit circle.

Then we obtain its corresponding three term recurrence relation as well as the chain sequences for the coefficients of such a recurrence relation. Finally, in Section 4 we deal with the Gaussian quadrature rules for such sieved para-orthogonal polynomials. The nodes and the corresponding weights are deduced.

2 Para-orthogonal polynomials from kernel polynomials

Recall that the Christoffel-Darboux formula of order n associated with the sequence $\{S_n(z)\}_{n=0}^{\infty}$ of OPUC is such that

$$K_n(z,\, w) = \sum_{j=0}^{n} \overline{s_j(w)}\, s_j(z) = \frac{\overline{s^*_{n+1}(w)}\, s^*_{n+1}(z) - \overline{s_{n+1}(w)}\, s_{n+1}(z)}{1 - \bar{w}\, z}.$$

Here, $s_n(z) = \kappa_n S_n(z)$ are the normalized OPUC. See, for example [22, Thm. 2.2.7], where $K_n(z,\, w)$ is referred to as a CD kernel, meaning a Christoffel-Darboux kernel.

With $|w| = 1$, we consider the sequence $\{P_n(w;\, z)\}_{n=0}^{\infty}$ of polynomials in z given by

$$P_n(w;\, z) = \frac{\kappa_{n+1}^{-2}\, \overline{w}}{\overline{S_{n+1}(w)}} \frac{K_n(z,\, w)}{1 + \tau_{n+1}(w)\alpha_n}, \quad n \geq 0,$$

where $\tau_n = S_n(w)/S^*_n(w)$, $n \geq 0$. It is easily verified that $P_n(w;\, z)$ is a monic polynomial of degree n in z, which can be simply written as

$$P_n(w;\, z) = \frac{1}{z - w} \frac{S_{n+1}(z) - \tau_{n+1}(w) S^*_{n+1}(z)}{1 + \tau_{n+1}(w)\alpha_n} = \frac{1}{z - w}[z S_n(z) - \tau_n(w) S^*_n(z)], \quad n \geq 0. \tag{2.1}$$

Since $|w| = 1$ we have $|\tau_n(w)| = 1$ for $n \geq 0$. Hence, $S_{n+1}(z) - \tau_{n+1}(w) S^*_{n+1}(z)$ is known as a para-orthogonal polynomial associated with S_{n+1}. From known properties of para-orthogonal polynomials (see [15]), $S_{n+1}(z) - \tau_{n+1}(w) S^*_{n+1}(z)$ has $n+1$ simple zeros on the unit circle $|z| = 1$. In particular, w is one of the zeros of $S_{n+1}(z) - \tau_{n+1}(w) S^*_{n+1}(z)$. Consequently, the polynomial $P_n(w;\, z)$ has all its n zeros simple and lying on the unit circle $|z| = 1$. However, none of the zeros of $P_n(w;\, z)$ can be equal to the value w.

Perhaps the first reference that explicitly brings the connection between CD kernels and para-orthogonal polynomials is González-Vera, Santos-León and Njåstad [11] (see also [2, Thm. 2.1]). Such results in a setting based on linear algebra and also without the use of the name para-orthogonal appear, even earlier than [11], in Gragg [12]. However, the name para-orthogonal polynomials for $S_n(z) - \tau_n S^*_n(z)$, where $|\tau_n| = 1$ and S_n are OPUC, is due to Jones, Njåstad and Thron [15]. We may refer to the polynomials $(z - w)P_n(w; z)$ as the CD kernel POPUC.

More on studies that use the connection between CD kernels and para-orthogonal polynomials we refer to [3], [10] and [28]. We also cite [24], where there is a nice section on CD kernels and para-orthogonal polynomials.

Some of the results shown recently in [7] that are relevant for the present manuscript can be summarized as follows.

For the (special and appropriately scaled) CD kernel POPUC $(z - 1)P_n(1; z)$, given by

$$R_n(z) = \sigma_n \frac{z S_n(z) - \tau_n S^*_n(z)}{z - 1}, \quad n \geq 0, \tag{2.2}$$

where

$$\tau_0 = \frac{S_0(1)}{S^*_0(1)} = 1 \quad \text{and} \quad \tau_n = \frac{S_n(1)}{S^*_n(1)} = \frac{\tau_{n-1} - \overline{\alpha}_{n-1}}{1 - \tau_{n-1}\alpha_{n-1}}, \quad n \geq 1,$$

$$\sigma_0 = 1 \quad \text{and} \quad \sigma_n = \frac{1 - \tau_{n-1}\alpha_{n-1}}{1 - \mathcal{R}e(\tau_{n-1}\alpha_{n-1})}\, \sigma_{n-1}, \quad n \geq 1,$$

the following three term recurrence formula holds.

$$R_{n+1}(z) = \big[(1+ic_{n+1})z + (1-ic_{n+1})\big] R_n(z) - 4\, d_{n+1} z R_{n-1}(z), \quad n \geq 1, \tag{2.3}$$

with $R_0(z) = 1$ and $R_1(z) = (1+ic_1)z + (1-ic_1)$, where the real sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_{n+1}\}_{n=1}^{\infty}$ are such that

$$\begin{aligned} c_n &= \frac{-\mathcal{I}m(\tau_{n-1}\alpha_{n-1})}{1-\mathcal{R}e(\tau_{n-1}\alpha_{n-1})} = i\,\frac{\tau_n - \tau_{n-1}}{\tau_n + \tau_{n-1}}, \\ d_{n+1} &= \frac{1}{4}\frac{\big[1-|\tau_{n-1}\alpha_{n-1}|^2\big]\,\big|1-\tau_n\,\alpha_n\big|^2}{\big[1-\mathcal{R}e(\tau_{n-1}\alpha_{n-1})\big]\big[1-\mathcal{R}e(\tau_n\alpha_n)\big]}, \end{aligned} \qquad n \geq 1. \tag{2.4}$$

Moreover, $\{d_{n+1}\}_{n=1}^{\infty}$, where $d_{n+1} = (1-m_n)m_{n+1}$, is a positive chain sequence with the parameter sequence $\{m_{n+1}\}_{n=0}^{\infty}$ given by

$$m_{n+1} = \frac{1}{2}\frac{\big|1-\tau_n\alpha_n\big|^2}{\big[1-\mathcal{R}e(\tau_n\alpha_n)\big]}, \quad n \geq 0.$$

Since $0 < m_1 < 1$, setting

$$d_1 = m_1 = \frac{1}{2}\frac{\big|1-\tau_0\alpha_0\big|^2}{\big[1-\mathcal{R}e(\tau_0\alpha_0)\big]}, \tag{2.5}$$

then $\{m_n\}_{n=0}^{\infty}$, with $m_0 = 0$, is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^{\infty}$. The sequence $\{m_n\}_{n=0}^{\infty}$, together with the sequence $\{c_n\}_{n=1}^{\infty}$, can be used to characterize the above measure μ. For example, the associated monic OPUC $\{S_n(z)\}_{n=0}^{\infty}$ can be given as

$$S_n(z)\prod_{k=1}^{n}(1+ic_k) = R_n(z) - 2(1-m_n)R_{n-1}(z), \quad n \geq 1,$$

and that the corresponding Verblunsky coefficients can be derived from

$$\tau_n = \frac{1-ic_n}{1+ic_n}\,\tau_{n-1} \quad \text{and} \quad \alpha_{n-1} = \frac{1}{\tau_n}\,\frac{1-2m_n-ic_n}{1+ic_n}, \quad n \geq 1,$$

with $\tau_0 = 1$. Moreover, if the maximal parameter sequence $\{M_n\}_{n=0}^{\infty}$ of the positive chain sequence $\{d_n\}_{n=1}^{\infty}$ is different from its minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$, then the measure μ has a positive mass (pure point) of size M_0 at $z = 1$.

Observe from (2.2) that $\sigma_n = \prod_{k=1}^{n}(1+ic_k)$, $n \geq 1$, and that $\sigma_n/\overline{\sigma}_n = 1/\tau_n$, $n \geq 1$. Thus, we can also write

$$\begin{aligned} \sigma_n S_n(z) &= R_n(z) - 2(1-m_n)R_{n-1}(z), \\ \tau_n\sigma_n S_n^{*}(z) &= R_n(z) - 2(1-m_n)zR_{n-1}(z), \end{aligned} \qquad n \geq 1. \tag{2.6}$$

With respect to the measure μ the polynomials R_n also satisfy the following so called L-orthogonality property

$$\int_{\mathcal{C}} \zeta^{-n+j} R_n(\zeta)\,(1-\zeta)d\mu(\zeta) = 0, \quad 0 \leq j \leq n-1.$$

If we also consider the polynomials $Q_n(z)$, $n \geq 0$, given by

$$Q_{n+1}(z) = \big[(1+ic_{n+1})z + (1-ic_{n+1})\big] Q_n(z) - 4\, d_{n+1} z Q_{n-1}(z), \quad n \geq 1, \tag{2.7}$$

with $Q_0(z) = 0$ and $Q_1(z) = 2d_1$, where $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are as in (2.3) and (2.5), then we can state the following lemma.

Lemma 2.1

$$R_n(1) - 2(1-m_n)R_{n-1}(1) = Q_n(1) - 2(1-m_n)Q_{n-1}(1) = 2^n m_1 m_2 \cdots m_n, \quad n \geq 1.$$

Proof. Easily verified from the three term recurrence relations for R_n and Q_n. ■

Gaussian type quadrature rules on the unit circle are based on the zeros of POPUC. We can derive information about the Gaussian type quadrature rule based on the zeros of $(z-1)R_n(z)$ from the following results given very recently in [5].

We get

$$\frac{R_n(z) - Q_n(z)}{(z-1)R_n(z)} = \frac{\lambda_{n,0}}{z-1} + \sum_{k=1}^{n} \frac{\lambda_{n,k}}{z - z_{n,k}}, \quad n \geq 1,$$

where $z_{n,k}$, $k = 1, 2, \ldots, n$, are the zeros of $R_n(z)$ and the quantities $\lambda_{n,k}$, $k = 0, 1, \ldots, n$, shown in [5] to be positive, satisfy

$$\lambda_{n,0} = 1 - \frac{Q_n(1)}{R_n(1)} \quad \text{and} \quad \lambda_{n,k} = \frac{Q_n(z_{n,k})}{(1 - z_{n,k})R_n'(z_{n,k})}, \quad k = 1, 2, \ldots, n. \tag{2.8}$$

Moreover,

$$-\sum_{k=0}^{\infty} \mu_{k+1} z^k - \frac{R_n(z) - Q_n(z)}{(z-1)R_n(z)} = O\big(z^n\big), \quad n \geq 1,$$

and

$$\sum_{k=0}^{\infty} \mu_{-k} z^{-k-1} - \frac{R_n(z) - Q_n(z)}{(z-1)R_n(z)} = O\big((1/z)^{n+2}\big), \quad n \geq 1,$$

where $\mu_k = \int_{\mathcal{C}} \zeta^{-k} d\mu(\zeta)$, $k = 0, \pm 1, \pm 2, \ldots$.

Thus, by the same idea used by Gauss to discover the Gaussian Quadrature formulas, we obtain the following.

Theorem 2.2 *Let $z_{n,k} = e^{i\theta_{n,k}}$, $k = 1, 2, \ldots, n$, be the zeros of R_n and let $z_{n,0} = 1$. Then the following quadrature formula holds.*

$$\int_{\mathcal{C}} \zeta^p d\mu(\zeta) = \sum_{k=0}^{n} \lambda_{n,k} (z_{n,k})^p, \quad p = 0, \pm 1, \ldots, \pm n, \tag{2.9}$$

where the weights $\lambda_{n,k}$, $k = 0, 1, \ldots, n$, are all positive and can be given in terms of $\{R_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ by

$$\begin{aligned} \lambda_{n,0} &= \frac{(1-m_n)2^{2n} d_1 d_2 \cdots d_n}{R_n(1)\big[R_n(1) - 2(1-m_n)R_{n-1}(1)\big]}, \\ \lambda_{n,k} &= \frac{2^{2n-1} d_1 d_2 \cdots d_n z_{n,k}^{n-1}}{(1 - z_{n,k})R_n'(z_{n,k})R_{n-1}(z_{n,k})}, \quad k = 1, 2, \ldots n. \end{aligned} \tag{2.10}$$

Proof. From the recurrence formulas for $\{R_n(z)\}_{n=0}^{\infty}$ and $\{Q_n(z)\}_{n=0}^{\infty}$ together with the theory of continued fractions

$$\begin{aligned} \frac{Q_n(1)}{R_n(1)} &= \frac{d_1}{\lceil 1} - \frac{d_2}{\lceil 1} - \frac{d_3}{\lceil 1} - \cdots - \frac{d_n}{\lceil 1}, \\ &= (1-m_0)\frac{m_1}{\lceil 1} - \frac{(1-m_1)m_2}{\lceil 1} - \cdots - \frac{(1-m_{n-1})m_n}{\lceil 1}, \end{aligned}$$

for $n \geq 1$, where $\{m_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^{\infty}$. Hence, one can write (see the proof of Lemma 3.2 in [6, p. 83])

$$\lambda_{n,0} = 1 - \frac{Q_n(1)}{R_n(1)} = \frac{1}{1 + \displaystyle\sum_{k=1}^{n} \frac{m_1 m_2 \cdots m_k}{(1-m_1)(1-m_2)\cdots(1-m_k)}}, \quad n \geq 1, \tag{2.11}$$

which shows the positiveness of $\lambda_{n,0}$. The expression for $\lambda_{n,0}$ in the theorem, which we use in a latter part of this manuscript, can be obtained as follows.

From Lemma 2.1,

$$\begin{aligned} 1 - \frac{Q_n(1)}{R_n(1)} &= \frac{Q_n(1) - 2(1-m_n)Q_{n-1}(1)}{R_n(1) - 2(1-m_n)R_{n-1}(1)} - \frac{Q_n(1)}{R_n(1)} \\ &= \frac{(1-m_n)\big[Q_n(1)R_{n-1}(1) - Q_{n-1}(1)R_n(1)\big]}{\big[R_n(1) - 2(1-m_n)R_{n-1}(1)\big]R_n(1)}, \end{aligned}$$

from which using the determinant formulas

$$Q_n(z)R_{n-1}(z) - Q_{n-1}(z)R_n(z) = 2^{2n-1} d_1 d_2 \cdots d_n z^{n-1}, \quad n \geq 1,$$

the required value for $\lambda_{n,0}$ follows.

Now for the expressions concerning the coefficients $\lambda_{n,k}$, $k \geq 1$, we first write (2.8) as

$$\lambda_{n,k} = \frac{Q_n(z_{n,k})R_{n-1}(z_{n,k}) - Q_{n-1}(z_{n,k})R_n(z_{n,k})}{(1-z_{n,k})R_n'(z_{n,k})R_{n-1}(z_{n,k})}.$$

Thus, the determinant formula gives the required expressions for the coefficients $\lambda_{n,k}$. The positiveness of $\lambda_{n,k}$ has been shown in [5] using the expression

$$\lambda_{n,k} = \frac{Q_n(z_{n,k})R_{n-1}(z_{n,k}) - Q_{n-1}(z_{n,k})R_n(z_{n,k})}{(1-z_{n,k})[R_n'(z_{n,k})R_{n-1}(z_{n,k}) - R_{n-1}'(z_{n,k})R_n(z_{n,k})]}.$$

This completes the proof of the Theorem. ■

3 Sieved para-orthogonal polynomials

With $\ell \geq 2$, we consider the CD kernel POPUC $(z-1)R_n^{(\ell)}(z)$ associated with the sieved OPUC $\{S_n^{(\ell)}(z)\}_{n=0}^{\infty}$ given in section 1. We have

$$R_n^{(\ell)}(z) = \sigma_n^{(\ell)} \frac{zS_n^{(\ell)}(z) - \tau_n^{(\ell)} S_n^{(\ell)*}(z)}{z-1}, \quad n \geq 1,$$

where

$$\begin{aligned} &\tau_0^{(\ell)} = \frac{S_0^{(\ell)}(1)}{S_0^{(\ell)*}(1)} = 1 \quad \text{and} \quad \tau_n^{(\ell)} = \frac{S_n^{(\ell)}(1)}{S_n^{(\ell)*}(1)} = \frac{\tau_{n-1}^{(\ell)} - \overline{\alpha}_{n-1}^{(\ell)}}{1 - \tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}}, \quad n \geq 1, \\ &\sigma_0^{(\ell)} = 1 \quad \text{and} \quad \sigma_n^{(\ell)} = \frac{1 - \tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}}{1 - \mathcal{R}e(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})}\, \sigma_{n-1}^{(\ell)}, \quad n \geq 1, \end{aligned} \tag{3.1}$$

As in (2.3), the polynomials $\{R_n^{(\ell)}\}_{n=0}^{\infty}$ satisfy the recurrence formulas

$$R_{n+1}^{(\ell)}(z) = \big[(1 + ic_{n+1}^{(\ell)})z + (1 - ic_{n+1}^{(\ell)})\big]R_n^{(\ell)}(z) - 4\, d_{n+1}^{(\ell)} z R_{n-1}^{(\ell)}(z), \quad n \geq 1, \tag{3.2}$$

with $R_0^{(\ell)}(z) = 1$ and $R_1^{(\ell)}(z) = (1 + ic_1^{(\ell)})z + (1 - ic_1^{(\ell)})$, where the real sequences $\{c_n^{(\ell)}\}_{n=1}^{\infty}$ and $\{d_{n+1}^{(\ell)}\}_{n=1}^{\infty}$ are such that

$$\left.\begin{aligned} c_n^{(\ell)} &= \frac{-\mathcal{I}m(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})}{1 - \mathcal{R}e(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})} = i\,\frac{\tau_n^{(\ell)} - \tau_{n-1}^{(\ell)}}{\tau_n^{(\ell)} + \tau_{n-1}^{(\ell)}}, \\ d_{n+1}^{(\ell)} &= \frac{1}{4}\frac{\big[1 - |\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}|^2\big]\,\big|1 - \tau_n^{(\ell)}\,\alpha_n^{(\ell)}\big|^2}{\big[1 - \mathcal{R}e(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})\big]\big[1 - \mathcal{R}e(\tau_n^{(\ell)}\alpha_n^{(\ell)})\big]}, \end{aligned}\right. \quad n \geq 1. \tag{3.3}$$

Moreover, $\{d_{n+1}^{(\ell)}\}_{n=1}^{\infty}$, where $d_{n+1}^{(\ell)} = (1 - m_n^{(\ell)})m_{n+1}^{(\ell)}$, is a positive chain sequence with the parameter sequence $\{m_{n+1}^{(\ell)}\}_{n=0}^{\infty}$ given by

$$m_{n+1}^{(\ell)} = \frac{1}{2}\frac{\big|1 - \tau_n^{(\ell)}\alpha_n^{(\ell)}\big|^2}{\big[1 - \mathcal{R}e(\tau_n^{(\ell)}\alpha_n^{(\ell)})\big]}, \quad n \geq 0.$$

Setting $d_1^{(\ell)} = m_1^{(\ell)}$, we can also say that $\{m_n^{(\ell)}\}_{n=0}^{\infty}$, with $m_0^{(\ell)} = 0$, is the minimal parameter sequence of the positive chain sequence $\{d_n^{(\ell)}\}_{n=1}^{\infty}$ and that if $\{M_n^{(\ell)}\}_{n=0}^{\infty}$ is the maximal parameter sequence of $\{d_n^{(\ell)}\}_{n=1}^{\infty}$, then $M_0^{(\ell)}$ (if $M_0^{(\ell)} > 0$) is the mass size of the pure point at $z = 1$ in the measure $\mu^{(\ell)}$.

From (1.2), since

$$\alpha_{r\ell+j}^{(\ell)} = 0, \;\; 0 \leq j \leq \ell - 2, \quad \text{and} \quad \alpha_{r\ell+\ell-1}^{(\ell)} = \alpha_r,$$

for $r \geq 0$, from (3.1) we have

$$\tau_{r\ell+j}^{(\ell)} = \tau_r, \;\; 0 \leq j \leq \ell - 1, \quad r \geq 0.$$

Thus, from (2.2) and (3.1),

$$\sigma_{r\ell+j}^{(\ell)} = \sigma_r, \;\; 0 \leq j \leq \ell - 1, \quad r \geq 0.$$

Moreover, from (2.4) and (3.3),

$$\begin{aligned} &m_{r\ell+j+1}^{(\ell)} = \tfrac{1}{2}, \; 0 \leq j \leq \ell - 2, \quad \text{and} \quad m_{r\ell+\ell}^{(\ell)} = m_{r+1}, \\ &c_{r\ell+j+1}^{(\ell)} = 0, \; 0 \leq j \leq \ell - 2, \quad \text{and} \quad c_{r\ell+\ell}^{(\ell)} = c_{r+1}, \\ &d_{r\ell+1}^{(\ell)} = \tfrac{1}{2}(1 - m_r), \quad d_{r\ell+j+1}^{(\ell)} = \tfrac{1}{4}, \; 1 \leq j \leq \ell - 2, \quad \text{and} \quad d_{r\ell+\ell}^{(\ell)} = \tfrac{1}{2}m_{r+1}, \end{aligned}$$

for $r \geq 0$.

Using these results together with $d\mu^{(\ell)}(z) = \ell^{-1}d\mu(z^{\ell})$, we can state the following theorem.

Theorem 3.1 *Let $\{d_n\}_{n=1}^{\infty}$ be a positive chain sequence with the minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ and the maximal parameter sequence $\{M_n\}_{n=0}^{\infty}$. For $\ell \geq 2$, set $\{d_n^{(\ell)}\}_{n=1}^{\infty}$ to be the positive chain sequence with its minimal parameter sequence given by*

$$m_{r\ell+j+1}^{(\ell)} = \frac{1}{2}, \;\; 0 \leq j \leq \ell - 2, \quad \textit{and} \quad m_{(r+1)\ell}^{(\ell)} = m_{r+1} \quad \textit{for} \quad r \geq 0.$$

If $g_0 = \ell^{-1}M_0$ and $g_n = d_n^{(\ell)}/(1 - g_{n-1})$, $n \geq 1$, then $\{g_n\}_{n=0}^{\infty} = \{M_n^{(\ell)}\}_{n=0}^{\infty}$ is the maximal parameter sequence of $\{d_n^{(\ell)}\}_{n=1}^{\infty}$.

Since

$$R^{(\ell)}_{r\ell+j}(z) = \sigma^{(\ell)}_{r\ell+j} \frac{zS^{(\ell)}_{r\ell+j}(z) - \tau^{(\ell)}_{r\ell+j} S^{(\ell)*}_{r\ell+j}(z)}{z-1}, \quad 0 \le j \le \ell-1, \quad r \ge 0,$$

we obtain from (1.3), (1.4) that

$$R^{(\ell)}_{r\ell+j}(z) = \sigma_r \frac{z^{j+1} S_r(z^\ell) - \tau_r S^*_r(z^\ell)}{z-1}, \quad 0 \le j \le \ell-1, \quad r \ge 0. \tag{3.4}$$

With the above results we can state the following.

Theorem 3.2 *Let the real sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be such that $\{d_n\}_{n=1}^{\infty}$ is also a positive chain sequence with its minimal parameter sequence denoted by $\{m_n\}_{n=0}^{\infty}$. Let the polynomials $\{R_n(z)\}_{n=0}^{\infty}$ be given by the three term recurrence formula*

$$R_{n+1}(z) = \big[(1+ic_{n+1})z + (1-ic_{n+1})\big] R_n(z) - 4\, d_{n+1} z R_{n-1}(z), \quad n \ge 1,$$

with $R_0(z) = 1$ and $R_1(z) = (1+ic_1)z + (1-ic_1)$. Then for any $r \ge 1$, $\ell \ge 1$ and $0 \le j \le \ell-1$, the polynomial $R^{(\ell)}_{r\ell+j}(z)$, of degree $r\ell+j$, given by

$$R^{(\ell)}_{r\ell+j}(z) = \frac{z^{j+1}-1}{z-1} R_r(z^\ell) + 2(1-m_r) \frac{z^\ell - z^{j+1}}{z-1} R_{r-1}(z^\ell), \tag{3.5}$$

has $r\ell+j$ simple zeros on the unit circle $|z|=1$ with $z \ne 1$. Moreover, if $N = n+1$, then between any two zeros of $R^{(\ell)}_N$ (on the unit circle) there is a zero of $R^{(\ell)}_n$.

Proof. The expression for $R^{(\ell)}_{r\ell+j}(z)$ is immediate taking into account (2.6) in (3.4). Properties of the zeros of these polynomials follow from the three term recurrence formula (3.2) and results given in [9]. ∎

Note that for $\ell = 1$ then the polynomials $R^{(\ell)}_n(z)$ become the polynomials $R_n(z)$. For $\ell \ge 2$, considering only the polynomials $R^{(\ell)}_{r\ell+\ell-1}$ we have

$$R^{(\ell)}_{\ell-1}(z) = \frac{z^\ell S_0(z^\ell) - \tau_0 S^*_0(z^\ell)}{z-1} = \frac{z^\ell - 1}{z-1},$$

$$R^{(\ell)}_{r\ell-1}(z) = \frac{\prod_{j=0}^{r-2}\left[1-\tau_j\alpha_j\right]}{\prod_{j=0}^{r-2}\left[1-\mathcal{R}e(\tau_j\alpha_j)\right]} \frac{z^\ell S_{r-1}(z^\ell) - \tau_{r-1} S^*_{r-1}(z^\ell)}{z-1} = \frac{z^\ell-1}{z-1} R_{r-1}(z^\ell), \quad r \ge 2. \tag{3.6}$$

Thus, we also have the following three term recurrence formula

$$R^{(\ell)}_{(r+2)\ell-1}(z) = \big[(1+ic_{r+1})z^\ell + (1-ic_{r+1})\big] R^{(\ell)}_{(r+1)\ell-1}(z) - 4\, d_{r+1} z^\ell R^{(\ell)}_{r\ell-1}(z), \quad r \ge 1,$$

with

$$R^{(\ell)}_{\ell-1}(z) = \frac{z^\ell-1}{z-1} \quad \text{and} \quad R^{(\ell)}_{2\ell-1}(z) = \frac{z^\ell-1}{z-1}\big[(1+ic_1)z^\ell + (1-ic_1)\big].$$

4 Some remarks on sieved Gaussian quadrature

As given by Theorem 2.2, the quadrature rules based on the zeros of the sieved para-orthogonal polynomials $(z-1)R^{(\ell)}_n(z)$ are

$$\int_{\mathcal{C}} \zeta^p d\mu^{(\ell)}(\zeta) = \sum_{m=0}^{n} \lambda^{(\ell)}_{n,m}\, (z^{(\ell)}_{n,m})^p, \quad p = 0, \pm1, \ldots, \pm n, \tag{4.1}$$

where $z_{n,0}^{(\ell)} = 1$, $z_{n,m}^{(\ell)}$, $m = 1, 2, \ldots, n$, are the zeros of $R_n^{(\ell)}(z)$, and the positive weights $\lambda_{n,m}^{(\ell)}$, $m = 0, 1, \ldots, n$, can be given by

$$\begin{aligned} \lambda_{n,0}^{(\ell)} &= \frac{(1-m_n^{(\ell)})2^{2n}d_1^{(\ell)}d_2^{(\ell)}\cdots d_n^{(\ell)}}{R_n^{(\ell)}(1)[R_n^{(\ell)}(1)-2(1-m_n^{(\ell)})R_{n-1}^{(\ell)}(1)]}, \\ \lambda_{n,m}^{(\ell)} &= \frac{2^{2n-1}d_1^{(\ell)}d_2^{(\ell)}\cdots d_n^{(\ell)}(z_{n,m}^{(\ell)})^{n-1}}{(1-z_{n,m}^{(\ell)})\,R_n^{(\ell)\prime}(z_{n,m}^{(\ell)})\,R_{n-1}^{(\ell)}(z_{n,m}^{(\ell)})}, \quad m = 1, 2, \ldots, n. \end{aligned} \tag{4.2}$$

However, we can say a bit more about the Gaussian quadrature formula based on the zeros of $(z-1)R_{r\ell+\ell-1}^{(\ell)}(z)$, $r \geq 1$. We write this quadrature formula in the form

$$\int_{\mathcal{C}} \zeta^p d\mu^{(\ell)}(\zeta) = \sum_{k=0}^{r}\sum_{j=0}^{\ell-1} \rho_{r,k,j}(w_{r,k,j})^p,$$

where $w_{r,0,0} = z_{r\ell+\ell-1,\,0}^{(\ell)} = 1$ and the remaining $w_{r,k,j} = z_{r\ell+\ell-1,\,k\ell+j}^{(\ell)}$ are the zeros of $R_{r\ell+\ell-1}^{(\ell)}(z)$ and $\rho_{r,k,j} = \lambda_{r\ell+\ell-1,\,k\ell+j}^{(\ell)}$, $j = 0, 1, \ldots, \ell-1$, $k = 0, 1, \ldots, r$.

Theorem 4.1 *The nodes $w_{r,k,j}$ and the weights $\rho_{r,k,j}$ of the quadrature formula based on the zeros of the polynomial $(z-1)R_{r\ell+\ell-1}^{(\ell)}(z)$ are such that*

$$\begin{aligned} w_{r,0,j} &= e^{i2\pi j/\ell}, \quad j = 0, 1, \ldots, \ell-1, \\ w_{r,k,j} &= e^{i(\theta_{r,k}+2\pi j)/\ell}, \quad j = 0, 1, \ldots, \ell-1, \quad k = 1, 2, \ldots, r, \end{aligned}$$

and

$$\rho_{r,k,j} = \frac{1}{\ell}\lambda_{r,k}, \qquad j = 0, 1, \ldots, \ell-1, \quad k = 0, 1, \ldots, r.$$

Proof. The formulas for the nodes simply follow from (3.6). The results for weights can be verified in two different ways. The first one is the direct substitution of the zeros in the expressions given by (4.2). For example, using information such as $(w_{r,0,j})^\ell = 1$ and $(w_{r,k,j})^\ell = z_{r,k}$, we have

$$\begin{aligned} d_1^{(\ell)}\cdots d_{\ell-1}^{(\ell)}d_\ell^{(\ell)}d_{\ell+1}^{(\ell)}&\cdots d_{(r-1)\ell+\ell-1}d_{r\ell}d_{r\ell+1}\cdots d_{r\ell+\ell-1}^{(\ell)} \\ &= \tfrac{1}{2}(1-m_0)\cdots\tfrac{1}{4}\tfrac{1}{2}m_1\tfrac{1}{2}(1-m_1)\cdots\tfrac{1}{4}\tfrac{1}{2}m_r\tfrac{1}{2}(1-m_r)\cdots\tfrac{1}{4}, \\ &= \frac{2}{4^{(r+1)\ell-1-r}}(1-m_r)d_1d_2\cdots d_r, \end{aligned}$$

$$\begin{aligned} (1-w_{r,0,j})R_{r\ell+\ell-1}^{(\ell)\prime}(w_{r,0,j}) &= -\ell\,(w_{r,0,j})^{\ell-1}R_r(1), \\ (1-w_{r,k,j})R_{r\ell+\ell-1}^{(\ell)\prime}(w_{r,k,j}) &= \ell\,(1-z_{r,k})(w_{r,k,j})^{\ell-1}R_r'(z_{r,k}). \end{aligned}$$

and

$$\begin{aligned} R_{r\ell+\ell-2}^{(\ell)}(w_{r,0,j}) &= -(w_{r,0,j})^{-1}\big[R_r(1)-2(1-m_r)R_{r-1}(1)\big], \\ R_{r\ell+\ell-2}^{(\ell)}(w_{r,k,j}) &= 2(1-m_r)(w_{r,k,j})^{\ell-1}R_{r-1}(z_{r,k}). \end{aligned}$$

Thus the relations $\rho_{r,k,j} = \ell^{-1}\,\lambda_{r,k}$ follow from (2.10) and (4.2).

The other way to verify these results is to obtain the new quadrature formula (4.1) directly from the quadrature formula (2.9). Since $(w_{r,k,j})^\ell = z_{r,k}$, we have from (2.10), with $n = r$,

$$\int_{2\pi j/\ell}^{2\pi(j+1)/\ell} (e^{i\ell\theta})^p d\mu(e^{i\ell\theta}) = \sum_{k=0}^{r} \lambda_{r,k}(w_{r,k,j})^{\ell p}, \quad p = 0, \pm 1, \ldots, \pm r,$$

for $j = 0, 1, \ldots, \ell - 1$. Hence,

$$\int_0^{2\pi} (e^{i\ell\theta})^p d\mu(e^{i\ell\theta}) = \sum_{j=0}^{\ell-1} \sum_{k=0}^{r} \lambda_{r,k} (w_{r,k,j})^{\ell p}, \quad p = 0, \pm 1, \ldots, \pm r,$$

As a consequence, we can write

$$\int_0^{2\pi} (e^{i\theta})^{\ell p} d\mu^{(\ell)}(e^{i\theta}) = \sum_{k=0}^{r} \Big[\ell^{-1} \lambda_{r,k} \sum_{j=0}^{\ell-1} (w_{r,k,j})^{\ell p}\Big], \quad p = 0, \pm 1, \ldots, \pm r.$$

Note that the weights and nodes are exactly what we wanted. Only thing we still need to verify is the validity of the above quadrature rule for the powers $\ell p + q$, $q = 1, 2, \ldots, \ell - 1$.

The left hand side can be written as

$$\begin{aligned} \int_0^{2\pi} (e^{i\theta})^{\ell p+q} d\mu^{(\ell)}(e^{i\theta}) \ &= \sum_{j=0}^{\ell-1} \int_0^{2\pi/\ell} (e^{i\theta + i2\pi j/\ell})^{\ell p+q} d\mu^{(\ell)}(e^{i\theta}) \\ &= \int_0^{2\pi/\ell} (e^{i\theta})^{\ell p+q} \Big[\sum_{j=0}^{\ell-1} (e^{i2\pi j/\ell})^q\Big] d\mu^{(\ell)}(e^{i\theta}). \end{aligned}$$

For $q = 1, 2, \ldots, \ell - 1$, since the sum within the integral is zero the resulting integral is zero.

Since $w_{r,k,j}$ are the ℓ^{th} roots of $z_{r,k}$, it follows that

$$\sum_{j=0}^{\ell-1} (w_{r,k,j})^{\ell p+q} \ = \sum_{j=0}^{\ell-1} e^{i(\theta_{r,k} + 2\pi j)(\ell p+q)/\ell} = 0, \quad q = 1, 2, \ldots, \ell - 1.$$

So the right hand side is also zero for such powers. ∎

References

[1] J. Breuer, E. Ryckman and B. Simon, Equality of the Spectral and Dynamical Definitions of Reflection, *Commun. Math. Phys.*, **295** (2010), 531-550.

[2] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, Quadrature and orthogonal rational functions, *J. Comput. Appl. Math.*, **127** (2001), 67-91.

[3] M. J. Cantero, L. Moral and L. Velázquez, Measures and para-orthogonal polynomials on the unit circle, *East J. Approx.*, **8** (2002), 447-464.

[4] K. Castillo, L. Garza and F. Marcellán, Perturbations on the subdiagonals of Toeplitz matrices, *Linear Alg. Appl.*, **434** (2011), 1563-1579.

[5] K. Castillo, M. S. Costa, A. Sri Ranga and D. O. Veronese, A favard type theorem for orthogonal polynomials on the unit circle from a three term recurrence formula, submitted.

[6] T. S. Chihara, "An Introduction to Orthogonal Polynomials", Mathematics and its Applications Series, Gordon and Breach, New York, 1978.

[7] M. S. Costa, H. M. Felix and A. Sri Ranga, Orthogonal polynomials on the unit circle and chain sequences, *J. Approx. Theory*, **173** (2013), 14-32.

[8] M. S. Costa, E. Godoy, R. L. Lamblém and A. Sri Ranga, Basic hypergeometric functions and orthogonal Laurent polynomials, *Proc. Amer. Math. Soc.*, **140** (2012), 2075-2089.

[9] D. K. Dimitrov and A. Sri Ranga, Zeros of a family of hypergeometric para-orthogonal polynomials on the unit circle, *Math. Nachr.*, **286** (2013), 1778-1791.

[10] L. Golinskii, Quadrature formula and zeros of para-orthogonal polynomials on the unit circle, *Acta Math. Hungar.* **96** (2002), 169–186.

[11] P. González-Vera, J. C. Santos-León and O. Njåstad, Some results about numerical quadrature on the unit circle, *Adv. Comput. Math.*, **5** (1996), 297-328.

[12] W. B. Gragg, Positive definite Toeplitz matrics, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle, *J. Comput. Appl. Math.*, **46** (1993), 183-198. Russian original in Numerical methods in linear algebra, E. S. Nikolaev (ed.), pp. 16-32, Moscow University Press, 1982.

[13] M. E. H. Ismail, "Classical and Quantum Orthogonal Polynomials in One Variable", Encyclopedia of Mathematics and its Applications Vol. **98**, Cambridge Univ. Press, Cambridge, 2005.

[14] M. E. H. Ismail and Xin Li, On sieved orthogonal polynomials IX: orthogonality on the unit circle, *Pacific J. Math.*, **152** (1992), 289-297.

[15] W. B. Jones, O. Njåstad and W. J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, *Bull. Lond. Math. Soc.*, **21** (1989), 113-152.

[16] A. Kheifets, L. Golinskii, F. Peherstorfer and P. Yuditskii, Scattering Theory for CMV Matrices: Uniqueness, Helson-Szegő and Strong Szegő Theorems, *Integr. Equ. Oper. Theory*, **69** (2011), 479-508.

[17] F. Marcellán and G. Sansigre, Orthogonal polynomials on the unit circle: Symmetrization and quadratic decomposition, *J. Approx. Theory*, **65** (1991), 109-119.

[18] F. Marcellán and G. Sansigre, Symmetrization, quadratic decomposition and cubic transformations of orthogonal polynomials on the unit circle. In "Orthogonal polynomials and their applications (Erice, 1990)", C. Brezinski, L. Gori and A. Ronveaux Editors, *IMACS Ann. Comput. Appl. Math.*, **9**, Baltzer, Basel, 1991. 341-345.

[19] F. Peherstorfer, Positive trigonometric quadrature formulas and quadrature on the unit circle, *Math. Comp.*, **80** (2011), 1685-1701.

[20] F. Peherstorfer, A. Volberg and P. Yuditskii, CMV matrices with asymptotically constant coefficients. Szegő-Blaschke class, scattering theory, *J. Funct. Anal.*, **256** (2009), 2157-2210.

[21] B. Simanek, Week convergence of CD kernels: A new approach on the circle and real line, *J. Approx. Theory*, **164** (2012), 204-209.

[22] B. Simon, "Orthogonal Polynomials on the Unit Circle. Part 1. Classical Theory", Amer. Math. Soc. Colloq. Publ., vol. **54**, part 1, Amer. Math. Soc., Providence, RI, 2005.

[23] B. Simon, "Orthogonal Polynomials on the Unit Circle. Part 2. Spectral Theory", Amer. Math. Soc. Colloq. Publ., vol. **54**, part 2, Amer. Math. Soc., Providence, RI, 2005.

[24] B. Simon, “Szegő‘s Theorem and Its Descendants: Spectral Theory for L^2 perturbations of Orthogonal Polynomials”, Princeton Univ. Press, Princeton, 2011.

[25] A. Sri Ranga, Szegő polynomials from hypergeometric functions, *Proc. Amer. Math. Soc.*, **138** (2010), 4259-4270.

[26] G. Szegő, “Orthogonal Polynomials” Amer. Math. Soc. Colloq. Publ., vol. **23**, Amer. Math. Soc., Providence, RI, 1975. Fourth Edition.

[27] S. Tsujimoto and A. Zhedanov, Elliptic hypergeometric Laurent biorthogonal polynomials with a dense point spectrum on the unit circle, *SIGMA Symmetry Integrability Geom. Methods Appl.*, **5** (2009), 30p.

[28] M. L. Wong, First and second kind paraorthogonal polynomials and their zeros, *J. Approx. Theory*, **146** (2007), 282-293.